HAL open science

High-frequency averaging in the semi-classical singular Hartree equation

Lounes Mouzaoui

To cite this version:

Lounes Mouzaoui. High-frequency averaging in the semi-classical singular Hartree equation. 2012. hal-00675678

HAL Id: hal-00675678
https://hal.science/hal-00675678

Preprint submitted on 1 Mar 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HIGH-FREQUENCY AVERAGING IN THE SEMI-CLASSICAL SINGULAR HARTREE EQUATION

LOUNES MOUZAOUI

Abstract. We study the asymptotic behavior of the Schrödinger equation in the presence of a nonlinearity of Hartree type in the semi-classical regime. Our scaling corresponds to a weakly nonlinear regime where the nonlinearity affects the leading order amplitude of the solution without altering the rapid oscillations. We show the validity of the WKB-analysis when the potential in the nonlinearity is singular around the origin. No new resonant wave is created in our model, this phenomenon is inhibited due to the nonlinearity. The nonlocal nature of this latter leads us to build our result on a high-frequency averaging effects. In the proof we make use of the Wiener algebra and the space of square-integrable functions.

1. Introduction

In this paper we are interested in studying of the Schrödinger equation in the presence of a *Hartree type nonlinearity*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon\lambda(K * |u^\varepsilon|^2)u^\varepsilon, \tag{1.1}$$

with $\varepsilon > 0$, $u^\varepsilon \in \mathcal{S}'(I \times \mathbb{R}^d, \mathbb{C}), d \in \{1, 2, 3\}, I$ interval of $\mathbb{R}$, $\lambda \in \mathbb{R}$, and $K(x) = \frac{1}{|x|^\gamma}$ with $0 < \gamma < d$. This model is of physical application. It appears in many physical phenomena like in describing superfluids (see [4]). The presence of ε in this equation is to show the microscopic/macroscopic scale ratio. For small $\varepsilon > 0$, the scaling of (1.1) corresponds to the *semi-classical* regime, i.e. the regime of *high-frequency* solutions $u^\varepsilon(t, x)$. This system contains the Schrödinger-Poisson system ($d = 3, \gamma = 1$), a model which is used in studying the quantum transport in semi-conductor devices (see [2, 22, 25]). Many authors have interested to study the asymptotic behavior of (1.1) in this case like in [19, 21, 26] where it is made use of Wigner measure techniques. Several papers like [1, 10] are devoted to study the strong asymptotic limit of the solutions of (1.1) with Hartree nonlinearity in the case of a *single-phase* WKB initial data in the form

$$u_0^\varepsilon(x) = \alpha^\varepsilon(x)e^{i\varphi(x)/\varepsilon},$$

with $\varphi(x) \in \mathbb{C}$ ε-independent and $\alpha^\varepsilon(x) \in \mathbb{R}$.

Due to the small parameter ε in front of the nonlinearity, we consider a *weakly nonlinear regime.* This means that the nonlinearity does not affect the geometry of the propagation (see Section 3), and technically means that it does not show up in the eikonal equation, but only in the transport equations determining the

Date: March 1, 2012.

This work was supported by the French ANR project R.A.S. (ANR-08-JCJC-0124-01).

amplitudes. The object of this paper is to construct an approximate solution u^ε_{app} for the exact solution u^ε of (1.1) subject to an initial data of WKB type, given by a superposition of ε-oscillatory plane waves, i.e.

$$u^\varepsilon_{app}\big|_{t=0}(x) = u^\varepsilon_0(x) = \sum_{j\in\mathbb{N}} \alpha_j(x) e^{i\kappa_j\cdot x/\varepsilon}. \tag{1.2}$$

We begin the paper by showing the existence of an exact solution to (1.1) in $L^2(\mathbb{R}^d)\cap W(\mathbb{R}^d)$ where $W(\mathbb{R}^d)$ denotes the *Wiener algebra*

$$W(\mathbb{R}^d) = \mathcal{F}(L^1(\mathbb{R}^d)) = \{f\in\mathcal{S}'(\mathbb{R}^d,\mathbb{C}), \|f\|_W := \|\widehat{f}\|_{L^1(\mathbb{R}^d)} < +\infty\},$$

and where

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x.$$

In the next step, we construct an approximate solution in the form

$$u^\varepsilon_{app}(t,x) = \sum_{j\in\mathbb{N}} a_j(t,x)e^{i\phi_j(t,x)/\varepsilon},$$

in $L^2(\mathbb{R}^d)\cap W(\mathbb{R}^d)$ by determining the amplitudes a_j and the phases ϕ_j, then we proceed to study its stability to justify our construction.

Fourier transform of the potential K is found to be

$$\widehat{K}(\xi) = \frac{C_{d,\gamma}}{|\xi|^{d-\gamma}},$$

(see [3, Proposition 1.29]) so under the general assumptions of [15] where the considered kernel K is such that $(1+|\xi|)\widehat{K}(\xi)\in L^\infty(\mathbb{R}^d)$ we can not deduce the well posedness of (1.1). It is a critical case for our model because we do not know how to construct an exact local solution in $W(\mathbb{R}^d)$ due to the singularity of $\widehat{K}$ around the origin ($\widehat{K}\notin L^\infty$).

To define the framework of the amplitudes $(a_j)_{j\in\mathbb{N}}$ we introduce the following definitions.

Definition 1.1. *For $d\in\{1,2,3\}$ and $0<\gamma<d$ we define $n\in\mathbb{N}$ as follows*

$$n = \begin{cases} 2 & \text{if } d=1 \text{ or } 2\\ 2 & \text{if } d=3 \text{ and } \gamma\in[1,3[\\ 3 & \text{if } d=3 \text{ and } \gamma\in]0,1[. \end{cases}$$

We define the space

$$Y(\mathbb{R}^d) := \{f\in L^2\cap W, \partial^\eta f\in L^2\cap W, \forall\eta\in\mathbb{N}^d, |\eta|\leqslant n\},$$

equipped with the norm

$$\|f\|_{Y(\mathbb{R}^d)} = \sum_{0\leqslant|\eta|\leqslant n}\|\partial^\eta f\|_{L^2\cap W}.$$

We set

$$E(\mathbb{R}^d) = \{a=(a_j)_{j\in\mathbb{N}} \mid (a_j)_{j\in\mathbb{N}}\in\ell^1(\mathbb{N},Y)\},$$

which is a Banach space when it is equipped with the norm

$$\|a\|_{E(\mathbb{R}^d)} = \sum_{j\in\mathbb{N}}\|a_j\|_Y.$$

Now, we can state the main theorem of this work.

Theorem 1.1. *For $d \in \{1,2,3\}$ and $0<\gamma<d$, consider the Cauchy problem* (1.1)*, subject to initial data u_0^ε of the form* (1.2)*, with initial amplitudes $(\alpha_j)_{j\in\mathbb{N}} \in E(\mathbb{R}^d)$. We assume that there exists $\delta>0$ such that*

$$\inf\{|\kappa_k-\kappa_m|, k,m\in\mathbb{N}, k\neq m\} \geqslant \delta > 0.$$

Then, for all $T>0$ there exists $C, \varepsilon_0(T)>0$, such that for any $\varepsilon \in]0,\varepsilon_0]$, the exact solution to (1.1) *satisfies $u^\varepsilon \in L^\infty([0,T]; L^2\cap W)$ and can be approximated by*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^\infty([0,T];L^2\cap W)} \leqslant C\varepsilon^\beta,$$

where $\beta=\min\{1, d-\gamma\}$ and where u^ε_{app} is defined by

$$u^\varepsilon_{app}(t,x) = \sum_{j\in\mathbb{N}} \alpha_j(x-t\kappa_j)e^{iS_j(t,x)}e^{i\kappa_j\cdot x/\varepsilon - i|\kappa_j|^2/2\varepsilon},$$

with $S_j\in\mathbb{R}$ defined in (3.9)*.*

In particular, for $d=3$ and $\gamma=1$ we obtain an approximation result for Schrödinger-Poisson equation with $\beta=1$. This confirms in a general way what was guessed in [8] for the Schrödinger-Poisson system. An important feature of our justification consists in the absence of phenomena of phase resonances like we can see later.

2. Existence and uniqueness

We consider the following Cauchy problem

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda(K*|u|^2)u, \tag{2.1}$$

subject to an initial data $u_0\in L^2(\mathbb{R}^d)\cap W(\mathbb{R}^d)$, with $K(x)=\frac{1}{|x|^\gamma}, 0<\gamma<d$. For $0<\gamma<d$, the Fourier transform of K for $\xi\in\mathbb{R}^d$ is

$$\widehat{K}(\xi) = \frac{C_{d,\gamma}}{|\xi|^{d-\gamma}}.$$

In the following we denote $\mathcal{K}_1 = \widehat{K}(\xi)\mathbb{1}_{[|\xi|\leqslant 1]}$ and $\mathcal{K}_2=\widehat{K}(\xi)\mathbb{1}_{[|\xi|>1]}$, so, $\widehat{K}=\mathcal{K}_1+\mathcal{K}_2$ with $\mathcal{K}_1\in L^p(\mathbb{R}^d)$ for all $p\in[1,\frac{d}{d-\gamma}[$ and $\mathcal{K}_2\in L^q(\mathbb{R}^d)$ for all $q\in]\frac{d}{d-\gamma},+\infty]$. Let $g(u)=(K*|u|^2)u$. For $\varepsilon>0$ we set

$$U^\varepsilon(t) = e^{i\varepsilon\frac{t}{2}\Delta},$$

with $U(t):=U^1(t)$. We recall some important properties of $W(\mathbb{R}^d)$.

Lemma 2.1. *Wiener algebra space $W(\mathbb{R}^d)$ enjoys the following properties (see* [9, 13]*):*

i. $W(\mathbb{R}^d)$ is a Banach space, continuously embedded into $L^\infty(\mathbb{R}^d)$.
ii. $W(\mathbb{R}^d)$ is an algebra, in the sense that the mapping $(f,g)\mapsto fg$ is continuous from $W(\mathbb{R}^d)^2$ to $W(\mathbb{R}^d)$, and moreover

$$\forall f,g\in W(\mathbb{R}^d),\ \|fg\|_W \leqslant \|f\|_W\|g\|_W.$$

iii. For all $t\in\mathbb{R}$, $U^\varepsilon(t)$ is unitary on $W(\mathbb{R}^d)$.

iv. For all $s > \frac{d}{2}$ there exists a positive constant $C(s,d)$ such that for all $f \in H^s(\mathbb{R}^d)$

$$\|f\|_W \leqslant C(s,d)\|f\|_{H^s}.$$

The following theorem ensures the existence and uniqueness of the solution to (2.1):

Theorem 2.1. *We consider the above initial value problem* (2.1) *with $u_0 \in L^2(\mathbb{R}^d) \cap W(\mathbb{R}^d)$. Then there exists $T > 0$ depending on $\|u_0\|_{L^2\cap W}$ and a unique solution $u \in C([0,T]; L^2(\mathbb{R}^d) \cap W(\mathbb{R}^d))$ to* (2.1).

Proof. The following lemma will be useful to prove the above theorem :

Lemma 2.2. *Let $h \in L^1 \cap W, f_1, f_2 \in L^2 \cap W$ and K like defined as above. Then*

$$\|K * h\|_W \leqslant \|\mathcal{K}_1\|_{L^1}\|h\|_{L^1} + \|\mathcal{K}_2\|_{L^\infty}\|h\|_W, \tag{2.2}$$

in particular we have

$$\|g(f_1) - g(f_2)\|_{L^2\cap W} \leqslant C(\|f_1\|^2_{L^2\cap W} + \|f_2\|^2_{L^2\cap W})\|f_1 - f_2\|_{L^2\cap W},$$

for some positive constant C.

Proof. By definition of the W norm :

$$\begin{aligned}\|K * h\|_W &= C\|(\mathcal{K}_1 + \mathcal{K}_2)\widehat{h}\|_{L^1}\\ &\leqslant C\|\mathcal{K}_1\|_{L^1}\|\widehat{h}\|_{L^\infty} + C\|\mathcal{K}_2\|_{L^\infty}\|\widehat{h}\|_{L^1}\\ &\leqslant C\|\mathcal{K}_1\|_{L^1}\|h\|_{L^1} + C\|\mathcal{K}_2\|_{L^\infty}\|h\|_W.\end{aligned}$$

Moreover, for $f_1, f_2 \in L^2 \cap W$

$$\begin{aligned}&\|g(f_1)-g(f_2)\|_{L^2\cap W} =\\ &\quad= \|(K * (|f_1|^2 - |f_2|^2))f_1 + (K * |f_2|^2)(f_1 - f_2)\|_{L^2\cap W}\\ &\quad\leqslant \|(K * (|f_1|^2 - |f_2|^2))f_1\|_{L^2\cap W} + \|(K * |f_2|^2)(f_1 - f_2)\|_{L^2\cap W}.\end{aligned}$$

We control the last two norms. We have

$$\begin{aligned}&\|(K * |f_2|^2)(f_1 - f_2)\|_{L^2\cap W} =\\ &\quad= \|(K * |f_2|^2)(f_1 - f_2)\|_{L^2} + \|(K * |f_2|^2)(f_1 - f_2)\|_W\\ &\quad\leqslant \|K * |f_2|^2\|_{L^\infty}\|f_1 - f_2\|_{L^2}\|K * |f_2|^2\|_W\|f_1 - f_2\|_W\\ &\quad\leqslant C\|K * |f_2|^2\|_W\|f_1 - f_2\|_{L^2\cap W}\\ &\quad\leqslant C\|\mathcal{K}_1\|_{L^1}\|f_2\|^2_{L^2}\|f_1 - f_2\|_{L^2\cap W} + C\|\mathcal{K}_2\|_{L^\infty}\|f_2\|^2_W\|f_1 - f_2\|_{L^2\cap W},\end{aligned}$$

where we have used (2.2) and the embedding of $W(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$. Remark that

$$|f_1|^2 - |f_2|^2 = \frac{1}{2}(f_1 - f_2)(\overline{f_1} + \overline{f_2}) + \frac{1}{2}(\overline{f_1} - \overline{f_2})(f_1 + f_2).$$

From (2.2) and Lemma 2.1 iii, we have

$$
\begin{aligned}
&\|(K*(|f_1|^2-|f_2|^2))f_1\|_{L^2\cap W} \leqslant \\
&\quad \leqslant \|K*(|f_1|^2-|f_2|^2)\|_{L^\infty}\|f_1\|_{L^2\cap W} + \|K*(|f_1|^2-|f_2|^2)\|_W\|f_1\|_{L^2\cap W} \\
&\quad \leqslant C\|K*(|f_1|^2-|f_2|^2)\|_W\|f_1\|_{L^2\cap W} \\
&\quad \leqslant C\|K*((f_1-f_2)(\overline{f_1}+\overline{f_2}))\|_W\|f_1\|_{L^2\cap W} \\
&\qquad + C\|K*((\overline{f_1}-\overline{f_2})(f_1+f_2))\|_W\|f_1\|_{L^2\cap W} \\
&\quad \leqslant \|\mathcal{K}_1\|_{L^1}\|(f_1-f_2)(\overline{f_1}+\overline{f_2})\|_{L^1}\|f_1\|_{L^2\cap W} \\
&\qquad + C\|\mathcal{K}_2\|_{L^\infty}\|(f_1-f_2)(\overline{f_1}+\overline{f_2})\|_W\|f_1\|_{L^2\cap W} \\
&\qquad + C\|\mathcal{K}_1\|_{L^1}\|(\overline{f_1}-\overline{f_2})(f_1+f_2)\|_{L^1}\|f_1\|_{L^2\cap W} \\
&\qquad + C\|\mathcal{K}_2\|_{L^\infty}\|(\overline{f_1}-\overline{f_2})(f_1+f_2)\|_W\|f_1\|_{L^2\cap W} \\
&\quad \leqslant C\|f_1\|^2_{L^2\cap W}\|f_1-f_2\|_{L^2\cap W} + C\|f_1\|_{L^2\cap W}\|f_2\|_{L^2\cap W}\|f_1-f_2\|_{L^2\cap W} \\
&\qquad + C\|f_2\|^2_{L^2\cap W}\|f_1-f_2\|_{L^2\cap W} \\
&\quad \leqslant C(\|f_1\|^2_{L^2\cap W}+\|f_2\|^2_{L^2\cap W})\|f_1-f_2\|_{L^2\cap W},
\end{aligned}
$$

where we have used the Cauchy-Schwartz inequality. The desired control follows easily. □

Let $T>0$ to be specified later. We set

$$X = \{u \in C([0,T];L^2\cap W), \|u\|_{L^\infty([0,T];L^2\cap W)} \leqslant 2\|u_0\|_{L^2\cap W}\}.$$

Duhamel's formulation of (2.1) reads

$$u(t) = U(t)u_0 - i\lambda\int_0^t U(t-\tau)(K*|u|^2)u(\tau)\,\mathrm{d}\tau.$$

We denote by $\Phi(u)(t)$ the right hand side in the above formula and $G(u)=\Phi(u)(t)-U(t)u_0$. For $q\geq 1$ and a space S we define $\|u\|_{L^q_T S} := \|u\|_{L^q([0,T];S)}$ for $u\in L^q([0,T];S)$. Let $u\in X$. We have

$$\|\Phi(u)\|_{L^\infty_T L^2\cap L^\infty} \leqslant \|\Phi(u)(t)\|_{L^\infty_T L^2} + \|\Phi(u)(t)\|_{L^\infty_T W},$$

and

$$
\begin{aligned}
\|\Phi(u)\|_{L^\infty_T L^2} &\leqslant \|u_0\|_{L^2} + \|G(u)\|_{L^\infty_T L^2} \\
&\leqslant \|u_0\|_{L^2} + C\|g\|_{L^1_T L^2} \\
&\leqslant \|u_0\|_{L^2} + C\|K*|u|^2\|_{L^\infty_T L^\infty}\|u\|_{L^\infty_T L^2}T.
\end{aligned}
$$

We apply Lemma 2.2 after replacing h by $|u|^2$. We obtain by the embedding of W into L^∞

$$
\begin{aligned}
\|K*|u|^2\|_{L^\infty} &\leqslant C\|K*|u|^2\|_W \\
&\leqslant C\|\mathcal{K}_1\|_{L^1}\||u|^2\|_W + C\|\mathcal{K}_2\|_{L^\infty}\||u|^2\|_{L^1} \\
&\leqslant C\|u\|^2_{L^2\cap W}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\|\Phi(u)\|_{L^\infty_T L^2} &\leqslant \|u_0\|_{L^2} + C\|u\|^2_{L^\infty_T L^2\cap W}\|u\|_{L^\infty_T L^2}T \\
&\leqslant \|u_0\|_{L^2} + C\|u\|^3_{L^\infty_T L^2\cap W}T.
\end{aligned}
$$

Moreover, always by applying Lemma 2.2 with $h = |u|^2$ we obtain

$$\begin{aligned}\|\Phi(u)(t)\|_W &\leqslant \|u_0\|_W + \int_0^t \|K * |u|^2(\tau)\|_W \|u(\tau)\|_W \,\mathrm{d}\tau \\ &\leqslant \|u_0\|_W + C\int_0^t \|u(\tau)\|^3_{L^2\cap W}\,\mathrm{d}\tau,\end{aligned}$$

and thus

$$\|\Phi(u)\|_{L^\infty_T W} \leqslant \|u_0\|_W + C\|u\|^3_{L^\infty_T L^2\cap W} T.$$

Finally we have

$$\|\Phi(u)\|_{L^\infty_T L^2\cap W} \leqslant \|u_0\|_{L^2\cap W} + C\|u_0\|^3_{L^2\cap W} T.$$

By reducing sufficiently T (depending on $\|u_0\|_{L^2\cap W}$) we get for all $u \in X$, $\|\Phi(u)(t)\|_{L^\infty_T L^2} \leqslant 2\|u_0\|_{L^2\cap W}$. Moreover, for $u, v \in X$ we have

$$\|\Phi(u) - \Phi(v)\|_{L^\infty_T L^2\cap W} \leqslant \|G(u) - G(v)\|_{L^\infty_T L^2} + \|G(u) - G(v)\|_{L^\infty_T W}.$$

For some positive constant C we have

$$\begin{aligned}\|G(u) - G(v)\|_{L^\infty_T L^2} &\leqslant C\|g(u) - g(v)\|_{L^1_T L^2} \\ &\leqslant C\|g(u) - g(v)\|_{L^\infty_T L^2} T \\ &\leqslant C\|g(u) - g(v)\|_{L^\infty_T L^2\cap W} T.\end{aligned}$$

Moreover

$$\|G(u) - G(v)\|_W \leqslant \int_0^t \|g(u)(\tau) - g(v)(\tau)\|_W \,\mathrm{d}\tau,$$

and

$$\begin{aligned}\|G(u) - G(v)\|_{L^\infty_T W} &\leqslant \|g(u) - g(v)\|_{L^\infty_T W} T \\ &\leqslant \|g(u) - g(v)\|_{L^\infty_T L^2\cap W} T.\end{aligned}$$

By replacing f_1, f_2 by u, v respectively in Lemma 2.2 we obtain

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{L^\infty_T L^2\cap W} &\leqslant C\|g(u) - g(v)\|_{L^\infty_T L^2\cap W} T \\ &\leqslant C\|u_0\|^2_{L^2\cap W}\|u - v\|_{L^\infty_T L^2\cap W} T.\end{aligned}$$

Choosing T possibly smaller (still depending on $\|u_0\|_{L^2\cap W}$) we deduce that Φ is a contraction from X to X. Thus Φ has a unique fixed point $u \in X$ and Theorem 2.1 follows. □

3. Derivation of the approximate solution

We consider the rescaled version of (2.1) :

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon\lambda(K * |u^\varepsilon|^2)u^\varepsilon. \tag{3.1}$$

We seek an approximation of solutions to (3.1) in the form

$$u^\varepsilon_{app}(t,x) = \sum_{j\in\mathbb{N}} a_j(t,x)e^{i\phi_j(t,x)/\varepsilon}.$$

We begin by proceeding formally. We plug the ansatz above into (3.1). This yields

$$i\varepsilon\partial_t u^\varepsilon_{app} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{app} - \varepsilon(K * |u^\varepsilon_{app}|^2)u^\varepsilon_{app} = \sum_{k=0}^{2}\varepsilon^k Z^\varepsilon_k + \varepsilon(W^\varepsilon + r^\varepsilon), \tag{3.2}$$

with

$$Z_0^\varepsilon = -\sum_{j\in\mathbb{N}}(\partial_t\phi_j + \frac{1}{2}|\nabla\phi_j|^2)a_j e^{i\phi_j/\varepsilon},$$

$$Z_1^\varepsilon = i\sum_{j\in\mathbb{N}}(\partial_t a_j + \nabla\phi_j\nabla a_j + \frac{1}{2}a_j\Delta\phi_j)e^{i\phi_j/\varepsilon},$$

and

$$Z_2^\varepsilon = \frac{1}{2}\sum_{j\in\mathbb{N}}\Delta a_j e^{i\phi_j/\varepsilon}. \tag{3.3}$$

Other terms appear due to the presence of the nonlinearity which are

$$W^\varepsilon = -\sum_{j\in\mathbb{N}}(K * \sum_{\ell\in\mathbb{N}}|a_\ell|^2)a_j e^{i\phi_j/\varepsilon}, \tag{3.4}$$

and

$$r^\varepsilon = -\sum_{j\in\mathbb{N}}(K * \sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}}(a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon}))a_j e^{i\phi_j/\varepsilon}. \tag{3.5}$$

We aim to eliminate all equal powers of ε. Hence, by setting $Z_0^\varepsilon = 0$ we obtain the eikonal equation, whose solution is explicitly given by

$$\phi_j(t,x) = \kappa_j\cdot x - \frac{t}{2}|\kappa_j|^2. \tag{3.6}$$

Next, we set $Z_1^\varepsilon + W^\varepsilon = 0$ without including r^ε. This latter will constitute the first term error, the second will be Z_2^ε. We obtain for all $j\in\mathbb{N}$

$$\partial_t a_j + \kappa_j\cdot\nabla a_j = -i(K * \sum_{\ell\in\mathbb{N}}|a_\ell|^2)a_j, \quad a_j(0) = \alpha_j, \tag{3.7}$$

where we have used the fact that $\Delta\phi_j = 0$.

Lemma 3.1. *The transport equation* (3.7) *with initial amplitudes* $(\alpha_j)_{j\in\mathbb{N}} \in E(\mathbb{R}^d)$ *admits a unique global-in-time solutions* $a = (a_j)_{j\in\mathbb{N}} \in C([0,\infty[; E(\mathbb{R}^d))$, *which can be written in the form*

$$a_j(t,x) = \alpha_j(x - t\kappa_j)e^{iS_j(t,x)}, \tag{3.8}$$

where

$$S_j(t,x) = -\int_0^t (K * \sum_{\ell\in\mathbb{N}}|\alpha_\ell(x + (\tau - t)\kappa_j - \tau\kappa_\ell)|^2)\,\mathrm{d}\tau. \tag{3.9}$$

Proof. We multiply (3.7) with $\overline{a_j}$. We obtain

$$\overline{a_j}\partial_t a_j + \kappa_j\cdot(\overline{a_j}\nabla a_j) = -i(K * \sum_{\ell\in\mathbb{N}}|a_\ell|^2)|a_j|^2 \in i\mathbb{R}. \tag{3.10}$$

But

$$2\mathrm{Re}(\overline{a_j}\partial_t a_j + \kappa_j\cdot(\overline{a_j}\nabla a_j)) = (\partial_t + \kappa_j\cdot\nabla)|a_j|^2.$$

So, we deduce from (3.10) that $(\partial_t + \kappa_j\cdot\nabla)|a_j|^2 = 0$, which yields

$$|a_j(t,x)|^2 = |\alpha_j(x - t\kappa_j)|^2.$$

and (3.8) follows for some real function S_j. To determine S_j we inject (3.8) into (3.7). We get

$$i(\partial_t + \kappa_j \cdot \nabla) S_j(t,x)\alpha_j(x - t\kappa_j) = -i(K * \sum_{\ell\in\mathbb{N}} |\alpha_j(x - t\kappa_\ell)|^2)\alpha_j(x - t\kappa_j).$$

It suffices to impose

$$\partial_t(S_j(t, x + t\kappa_j)) = -K * \sum_{\ell\in\mathbb{N}} |\alpha_j(x - t(\kappa_\ell - \kappa_j))|^2,$$

which yields

$$S_j(t, x + t\kappa_j) = -\int_0^t K * \sum_{\ell\in\mathbb{N}} |\alpha_j(x - \tau(\kappa_\ell - \kappa_j))|^2 \, \mathrm{d}\tau,$$

and finally we have

$$S_j(t,x) = -\int_0^t K * \sum_{\ell\in\mathbb{N}} |\alpha_j(x + (\tau - t)\kappa_j) - \tau\kappa_\ell|^2 \, \mathrm{d}\tau.$$

Global in time existence of a_j's is not trivial from their explicit formula like we can see in the following. We rewrite (3.7) in its integral form

$$a_j(t,x) = \alpha_j(x - t\kappa_j) + \int_0^t \mathcal{N}(a)_j \, (\tau, x + (\tau - t)\kappa_j)\mathrm{d}\tau, \tag{3.11}$$

where the nonlinearity $\mathcal{N}$ is given by

$$\mathcal{N}(a)_j = -i\Big(K * \sum_{\ell\in\mathbb{N}} |a_\ell|^2\Big)a_j.$$

For $a \in E(\mathbb{R}^d)$ we have by definition

$$\|\mathcal{N}(a)\|_E = \sum_{j\in\mathbb{N}} \|\mathcal{N}(a)_j\|_Y = \sum_{j\in\mathbb{N}} \sum_{|\eta|\leqslant n} \|\partial^\eta \mathcal{N}(a)_j\|_{L^2\cap W}.$$

Let $|\eta| \leqslant n$. Leibnitz formula yields

$$\begin{aligned}
\|\partial^\eta \mathcal{N}(a)_j\|_{L^2\cap W} =& \|\partial^\eta\Big((K * \sum_{\ell\in\mathbb{N}} |a_\ell|^2)a_j\Big)\|_{L^2\cap W} \\
\leqslant& \sum_{\theta\leqslant\eta} C_\eta^\theta \|(K * \sum_{\ell\in\mathbb{N}} \partial^\theta |a_\ell|^2) \, \partial^{\eta-\theta} a_j\|_{L^2\cap W} \\
\leqslant& C \sum_{|\theta|\leqslant n} \|(K * \sum_{\ell\in\mathbb{N}} \partial^\theta |a_\ell|^2)\|_W \|a_j\|_Y \\
\leqslant& C\|a_j\|_Y \Big(\sum_{\ell\in\mathbb{N}} \|\mathcal{K}_1\|_{L^1} \sum_{|\theta|\leqslant n} \|\partial^\theta |a_\ell|^2\|_{L^1} \\
&+ \|\mathcal{K}_2\|_{L^\infty} \sum_{|\theta|\leqslant n} \|\partial^\theta |a_\ell|^2\|_W\Big) \\
\leqslant& C\|a_j\|_Y \sum_{\ell\in\mathbb{N}} (\|\mathcal{K}_1\|_{L^1} + \|\mathcal{K}_2\|_{L^\infty})\|a_\ell\|_Y^2 \\
\leqslant& C\|a_j\|_Y \|a\|_E^2.
\end{aligned}$$

Finally we have

$$\|\mathcal{N}(a)\|_E \leqslant C\|a\|_E^3.$$

This shows that $\mathcal{N}(a)$ defines a continuous mapping from E^3 to E and by the standard Cauchy-Lipschitz theorem for the ordinary differential equations, a local-in-time existence results immediately follows. Now, we have to show that the solution $a(t) = (a_j(t))_{j\in\mathbb{N}}$ is global in time. Let $[0, T_{\max}[$ be the maximal time interval where $(a_j)_{j\in\mathbb{N}}$ is defined. From (3.11) we have for $|\eta| \leqslant n$ and $t \in [0, T_{\max}[$

$$\begin{aligned}\partial^\eta a_j(t) &= \partial^\eta \alpha_j(\cdot - t\kappa_j) + \int_0^t \partial^\eta \mathcal{N}(a)_j(\tau, \cdot + (\tau - t)\kappa_j)\, \mathrm{d}\tau \\ &= \partial^\eta \alpha_j(\cdot - t\kappa_j) \\ &\quad + \int_0^t \sum_{\theta\leqslant\eta} C_\eta^\theta (K * \sum_{\ell\in\mathbb{N}} \partial^\theta |\alpha_\ell(\cdot - t\kappa_j)|^2)\, \partial^{\eta-\theta} a_j(\tau, \cdot + (\tau - t)\kappa_j)\, \mathrm{d}\tau.\end{aligned}$$

Taking $L^2 \cap W$ norm yields

$$\|\partial^\eta a_j(t)\|_{L^2\cap W} \leqslant$$

$$\begin{aligned}&\leqslant \|\partial^\eta \alpha_j\|_{L^2\cap W} + C\sum_{\theta\leqslant\eta}\int_0^t \|K * \sum_{\ell\in\mathbb{N}} \partial^\theta |\alpha_\ell(\cdot - t\kappa_j)|^2\|_W \|\partial^{\eta-\theta} a_j(\tau)\|_{L^2\cap W}\, \mathrm{d}\tau \\ &\leqslant \|\partial^\eta \alpha_j\|_{L^2\cap W} + C\sum_{\theta\leqslant\eta}\int_0^t \|K * \sum_{\ell\in\mathbb{N}} \partial^\theta |\alpha_\ell|^2\|_W \|\partial^{\eta-\theta} a_j(\tau)\|_{L^2\cap W}\, \mathrm{d}\tau \\ &\leqslant \|\partial^\eta \alpha_j\|_{L^2\cap W} + C(\|\mathcal{K}_1\|_{L^1} + \|\mathcal{K}_2\|_{L^\infty})\sum_{\ell\in\mathbb{N}} \|\alpha_\ell\|_Y^2 \int_0^t \|a_j(\tau)\|_Y\, \mathrm{d}\tau \\ &\leqslant \|\partial^\eta \alpha_j\|_{L^2\cap W} + C\int_0^t \|a_j(\tau)\|_Y\, \mathrm{d}\tau,\end{aligned}$$

and so

$$\|a_j(t)\|_Y = \sum_{|\eta|\leqslant n} \|\partial^\eta a_j(t)\|_{L^2\cap W} \leqslant \|\alpha_j\|_Y + C\int_0^t \|a_j(\tau)\|_Y\, \mathrm{d}\tau.$$

By Gronwall lemma

$$\|a_j(t)\|_Y \leqslant \|\alpha_j\|_Y e^{Ct}.$$

for some positive constant C independent of j and t. After summing with respect to j we obtain

$$\|a(t)\|_E \leqslant \|\alpha\|_E e^{Ct}.$$

The growth of $\|a(t)\|_E$ is at most exponential, and thus can not explode at finite time. We deduce that $T_{\max} = \infty$ and $a = (a_j)_{j\in\mathbb{N}} \in C([0, \infty[; E(\mathbb{R}^d))$.

□

4. Estimations on the remainder

We will estimate in $L^2 \cap W$ the term r^ε and Z_2^ε. To this end, we assume $(\alpha_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}, Y(\mathbb{R}^d))$.

Proposition 4.1. *Let r^ε be defined by* (3.5) *with the plane-wave phases ϕ_j given by* (3.6)*. We have the following bound:*

$$\|r^\varepsilon\|_{L^2\cap W} \leqslant C\delta^{\gamma-d}\|a\|_E^3\varepsilon^{d-\gamma}, \tag{4.1}$$

$$\|Z_2^\varepsilon\|_{L^2\cap W} \leqslant \|a\|_E, \tag{4.2}$$

where δ is like defined in Theorem 1.1.

Proof. Inequality (4.2) is obvious. We recall the definition of r^ε:

$$r^\varepsilon = -\sum_{j\in\mathbb{N}}(K * \sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}} a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon})a_je^{i\phi_j/\varepsilon}.$$

We estimate W norm of r^ε, we obtain

$$\begin{aligned}
\|r^\varepsilon\|_W &= \|\sum_{j\in\mathbb{N}}(K * \sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}} a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon})a_je^{i\phi_j/\varepsilon}\|_W\\
&\leqslant \sum_{j\in\mathbb{N}}\|K * \sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}} a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon}\|_W\|a_j\|_W\\
&\leqslant \sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}}\|K * a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon}\|_W\sum_{j\in\mathbb{N}}\|a_j\|_W.
\end{aligned}$$

Let us look more closely to the W norm of the convolution above. For b sufficiently regular and $\omega \in \mathbb{R}^d$, we denote $I^\varepsilon(x) = K * (be^{i\omega\cdot x/\varepsilon})$. Remark that

$$\begin{aligned}
\widehat{I^\varepsilon} &= (2\pi)^{d/2}\widehat{K}\,\widehat{be^{i\omega\cdot x/\varepsilon}}\\
&= (2\pi)^{d/2}\widehat{K}\,\widehat{b}(\cdot - \frac{\omega}{\varepsilon}).
\end{aligned}$$

So, if we take the W norm of I^ε we obtain

$$\begin{aligned}
\|I^\varepsilon\|_W &= \|\widehat{I^\varepsilon}\|_{L^1}\\
&= C\|\widehat{K}\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^m\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^{-m}\widehat{b}(\cdot - \frac{\omega}{\varepsilon})\|_{L^1}\\
&\leqslant C\|\mathcal{K}_1\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^m\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^{-m}\widehat{b}(\cdot - \frac{\omega}{\varepsilon})\|_{L^1}\\
&\quad + C\|\mathcal{K}_2\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^m\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^{-m}\widehat{b}(\cdot - \frac{\omega}{\varepsilon})\|_{L^1}\\
&\leqslant C\|\mathcal{K}_1\|_{L^1}\|\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^m\widehat{b}(\cdot - \frac{\omega}{\varepsilon})\|_{L^\infty}\sup_{|\xi|\leqslant 1}\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^{-m}\\
&\quad + C\|\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^m\widehat{b}(\cdot - \frac{\omega}{\varepsilon})\|_{L^1}\sup_{|\xi|>1}\left(|\xi|^{\gamma-d}\left\langle\xi - \frac{\omega}{\varepsilon}\right\rangle^{-m}\right).
\end{aligned}$$

By Peetre inequality we have for all $m \leqslant d-\gamma$ and $|\xi|>1$

$$|\xi|^{d-\gamma}\left\langle \xi-\frac{\omega}{\varepsilon}\right\rangle^{m} \geqslant |\xi|^{d-\gamma}\frac{\left\langle \frac{\omega}{\varepsilon}\right\rangle^{m}}{\langle\xi\rangle^{m}} \geqslant C\left\langle \frac{\omega}{\varepsilon}\right\rangle^{m} \geqslant C\varepsilon^{-m},$$

and thus

$$\sup_{|\xi|>1}\left(|\xi|^{\gamma-d}\left\langle \xi-\frac{\omega}{\varepsilon}\right\rangle^{-m}\right) \leqslant C\varepsilon^{m} \leqslant C\varepsilon^{d-\gamma},$$

Moreover

$$\left\langle \xi-\frac{\omega}{\varepsilon}\right\rangle^{m} \geqslant \frac{\left\langle \frac{\omega}{\varepsilon}\right\rangle^{m}}{\langle\xi\rangle^{m}} \geqslant \left\langle \frac{\omega}{\varepsilon}\right\rangle^{m} \geqslant C\varepsilon^{-m},$$

and

$$\sup_{|\xi|\leqslant 1}\left\langle \xi-\frac{\omega}{\varepsilon}\right\rangle^{-m} \leqslant C\varepsilon^{d-\gamma}.$$

So, for $m=d-\gamma$

$$\|I^{\varepsilon}\|_{W} \leqslant C\Big(\|\mathcal{K}_1\|_{L^1}\|\langle\xi\rangle^{d-\gamma}\widehat{b}\|_{L^\infty} + C\|\mathcal{K}_2\|_{L^\infty}\|\langle\xi\rangle\widehat{b}\|_{L^1}\Big)|\omega|^{d-\gamma}\varepsilon^{d-\gamma}.$$

Hence, for all $k,\ell\in\mathbb{N}, k\neq\ell$

$$\begin{aligned}
\|K * a_k\overline{a_\ell}e^{\frac{\phi_k-\phi_\ell}{\varepsilon}}\|_{W} &\leqslant C\|\mathcal{K}_1\|_{L^1}\|\langle\xi\rangle^{d-\gamma}\widehat{a_k\overline{a_\ell}}\|_{L^\infty}|\kappa_k-\kappa_\ell|^{d-\gamma}\varepsilon^{d-\gamma}\\
&\quad + C\|\mathcal{K}_2\|_{L^\infty}\|\langle\xi\rangle^{d-\gamma}\widehat{a_k\overline{a_\ell}}\|_{L^1}|\kappa_k-\kappa_\ell|^{d-\gamma}\varepsilon^{d-\gamma}\\
&\leqslant C\delta^{\gamma-d}\|\mathcal{K}_1\|_{L^1}\|\langle\xi\rangle^{d-\gamma}\widehat{a_k\overline{a_\ell}}\|_{L^\infty}\varepsilon^{d-\gamma}\\
&\quad + \ C\delta^{\gamma-d}\|\mathcal{K}_2\|_{L^\infty}\|\langle\xi\rangle^{d-\gamma}\widehat{a_k\overline{a_\ell}}\|_{L^1}\varepsilon^{d-\gamma}.
\end{aligned}$$

Remark that

$$\begin{aligned}
\|\langle\xi\rangle^{d-\gamma}\widehat{a_k\overline{a_\ell}}\|_{L^\infty} &\leqslant \|\langle\xi\rangle^{n}\widehat{a_k\overline{a_\ell}}\|_{L^\infty}\\
&\leqslant C\sum_{|\eta|\leqslant n}\|\widehat{\partial^\eta(a_k\overline{a_\ell})}\|_{L^\infty}\\
&\leqslant C\sum_{|\eta|\leqslant n}\|\partial^\eta(a_k\overline{a_\ell})\|_{L^1}\\
&\leqslant C\sum_{|\eta|\leqslant n}\|\partial^\eta a_k\|_{L^2}\sum_{|\eta|\leqslant n}\|\partial^\eta a_\ell\|_{L^2}\\
&\leqslant C\|a_k\|_Y\|a_\ell\|_Y.
\end{aligned}$$

Moreover

$$\begin{aligned}
\|\langle\xi\rangle^{d-\gamma}\widehat{a_k\overline{a_\ell}}\|_{L^1} &\leqslant \|\langle\xi\rangle^{n}\widehat{a_k\overline{a_\ell}}\|_{L^1}\\
&\leqslant C\sum_{|\eta|\leqslant n}\|\widehat{\partial^\eta(a_k\overline{a_\ell})}\|_{L^1}\\
&\leqslant C\sum_{|\eta|\leqslant n}\|\partial^\eta(a_k\overline{a_\ell})\|_{W}\\
&\leqslant C\sum_{|\eta|\leqslant n}\|\partial^\eta a_k\|_{W}\sum_{|\eta|\leqslant n}\|\partial^\eta a_\ell\|_{W}\\
&\leqslant C\|a_k\|_Y\|a_\ell\|_Y.
\end{aligned}$$

Thus we obtain

$$\|K * a_k\overline{a_\ell}e^{\frac{\phi_k-\phi_\ell}{\varepsilon}}\|_W \leqslant C\delta^{\gamma-d}(\|\mathcal{K}_1\|_{L^1}+C\|\mathcal{K}_2\|_{L^\infty})\|a_k\|_Y\|a_\ell\|_Y\varepsilon^{d-\gamma}.$$

In view of the above inequality, we deduce

$$\begin{aligned}\|r^\varepsilon\|_W &\leqslant \sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}} C\delta^{\gamma-d}(\|\mathcal{K}_1\|_{L^1}+C\|\mathcal{K}_2\|_{L^\infty})\|a_k\|_Y\|a_\ell\|_Y\varepsilon^{d-\gamma}\sum_{j\in\mathbb{N}}\|a_j\|_Y\\ &\leqslant C\delta^{\gamma-d}(\|\mathcal{K}_1\|_{L^1}+C\|\mathcal{K}_2\|_{L^\infty})\sum_{k\in\mathbb{N}}\|a_k\|_Y\sum_{\ell\in\mathbb{N}}\|a_\ell\|_Y\sum_{j\in\mathbb{N}}\|a_j\|_Y\varepsilon^{d-\gamma}\\ &\leqslant C\delta^{\gamma-d}(\|\mathcal{K}_1\|_{L^1}+C\|\mathcal{K}_2\|_{L^\infty})\|a\|_E^3\varepsilon^{d-\gamma}.\end{aligned}$$

To control L^2 norm of r^ε it suffices to remark that

$$\begin{aligned}\|r^\varepsilon\|_{L^2} &= \|\sum_{j\in\mathbb{N}}(K*\sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}} a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon})a_je^{i\frac{\phi_j}{\varepsilon}}\|_{L^2}\\ &\leqslant \sum_{j\in\mathbb{N}}\|K*\sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}} a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon}\|_{L^\infty}\|a_j\|_{L^2}\\ &\leqslant C\sum_{\substack{k,\,\ell\,\in\,\mathbb{N}\\ k\,\neq\,\ell}}\|K*a_k\overline{a_\ell}e^{i(\phi_k-\phi_\ell)/\varepsilon}\|_W\sum_{j\in\mathbb{N}}\|a_j\|_Y.\end{aligned}$$

and in the same way as previously we get

$$\|r^\varepsilon\|_{L^2}\leqslant C\delta^{\gamma-d}(\|\mathcal{K}_1\|_{L^1}+\|\mathcal{K}_2\|_{L^\infty})\|a\|_E^3\varepsilon^{d-\gamma}.$$

Finally

$$\|r^\varepsilon\|_{L^2\cap W}\leqslant C\delta^{\gamma-d}(\|\mathcal{K}_1\|_{L^1}+\|\mathcal{K}_2\|_{L^\infty})\|a\|_E^3\varepsilon^{d-\gamma}.$$

□

5. Justification of the approach

In this section we show that the solution u^ε_{app} is a good approximation of the exact solution in the sense of Theorem 1.1. We assume that assumptions of Theorem 1.1 are verified. At this stage we have constructed an approximate solution

$$u^\varepsilon_{app}=\sum_{j\in\mathbb{N}}a_j(t,x)e^{i\phi(t,x)/\varepsilon},$$

in $C([0,\infty[;E(\mathbb{R}^d))$ where the corresponding ϕ_j's are like constructed in (3.6) and the profiles are given by Lemma 3.1. Let $T^\varepsilon>0$ the time of local existence of the exact solution u^ε to (3.1) given by the fixed point argument. We consider a fixed $T>0$ and we introduce the error between the exact solution u^ε and the approximate solution u^ε_{app} :

$$w^\varepsilon=u^\varepsilon-u^\varepsilon_{app}.$$

We have in particular

$$u^\varepsilon_{app}\in C([0,T],L^2\cap W),$$

so there exists $R>0$ independent of ε such that

$$\|u^\varepsilon_{app}(t)\|_{L^2\cap W}\leqslant R,\ \forall t\in[0,T].$$

Since $u^\varepsilon \in C([0,T^\varepsilon], L^2\cap W)$ and $w^\varepsilon(0)=0$, there exists t^ε such that

$$\|w^\varepsilon(t)\|_{L^2\cap W} \leqslant R, \tag{5.1}$$

for $t\in[0,t^\varepsilon]$. So long as (5.1) holds, we infer

$$\begin{aligned}\|w^\varepsilon(t)\|_{L^2\cap W} &\leqslant |\lambda| \int_0^t \| \left(g(u^\varepsilon+w^\varepsilon)-g(u^\varepsilon_{app})\right)(\tau)\|_{L^2\cap W}\,\mathrm{d}\tau \\ &\quad + |\lambda|\varepsilon\int_0^t \|Z_2^\varepsilon(\tau)\|_{L^2\cap W}\,\mathrm{d}\tau + |\lambda|\int_0^t \|R(\tau)\|_{L^2\cap W}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|w^\varepsilon(\tau)\|_{L^2\cap W}\,\mathrm{d}\tau + C\varepsilon\int_0^t \mathrm{d}\tau + C\varepsilon^{d-\gamma}\int_0^t \mathrm{d}\tau \\ &\leqslant C\int_0^t \|w^\varepsilon(\tau)\|_{L^2\cap W}\,\mathrm{d}\tau + C\varepsilon^\beta\int_0^t \mathrm{d}\tau,\end{aligned}$$

where we have used Lemma 2.2 and Proposition 4.1, with $\beta=\min\{1,d-\gamma\}$. Gronwall lemma implies that so long as (5.1) holds,

$$\|w^\varepsilon(t)\|_{L^2\cap W} \leqslant C(e^{Ct}-1)\varepsilon^\beta,$$

for all $t\in[0,t^\varepsilon]$. Choosing $\varepsilon\in]0,\varepsilon_0]$ with ε_0 sufficiently small, we see that (5.1) remains true for $t\in[0,T]$ and Theorem 1.1 follows.

References

[1] T. Alazard and R. Carles. Semi-classical limit of Schrödinger-Poisson equations in space dimension $n\geq 3$. *J. Differential Equations*, 233(1):241–275, 2007.

[2] A. Arnold. Self-consistent relaxation-time models in quantum mechanics. *Comm. Partial Differential Equations*, 21(3-4):473–506, 1996.

[3] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[4] N.N.G. Berloff. Nonlocal nonlinear Schrödinger equation as model of superfluidity. *Journal of Low Temperature Physics*, 116:359–380, 1999.

[5] F. Brezzi and P. A. Markowich. The three-dimensional Wigner-Poisson problem: existence, uniqueness and approximation. *Math. Methods Appl. Sci.*, 14(1):35–61, 1991.

[6] R. Carles. WKB analysis for nonlinear Schrödinger equations with potential. *Comm. Math. Phys.*, 269(1):195–221, 2007.

[7] R. Carles. *Semi-classical analysis for nonlinear Schrödinger equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[8] R. Carles, E. Dumas, and C. Sparber. Geometric optics and instability for NLS and Davey-Stewartson models. *J. Eur. Math. Soc. (JEMS)*, June 2010.

[9] R. Carles, E. Dumas, and C. Sparber. Multiphase weakly nonlinear geometric optics for Schrödinger equations. *SIAM J. Math. Anal.*, 42(1):489–518, 2010.

[10] R. Carles and S. Masaki. Semi-classical analysis for Hartree equation. *Asymptot. Anal.*, 58(4):211–227, 2008.

[11] R. Carles, N. J. Mauser, and Hans P. Stimming. (Semi)classical limit of the Hartree equation with harmonic potential. *SIAM J. Appl. Math.*, 66(1):29–56 (electronic), 2005.

[12] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[13] M. Colin and D. Lannes. Short pulses approximations in dispersive media. *SIAM J. Math. Anal.*, 41(2):708–732, 2009.

[14] J. Giannoulis, A. Mielke, and C. Sparber. Interaction of modulated pulses in the nonlinear Schrödinger equation with periodic potential. *J. Differential Equations*, 245(4):939–963, 2008.

[15] J. Giannoulis, A. Mielke, and C. Sparber. High-frequency averaging in semi-classical Hartree-type equations. *Asymptot. Anal.*, 70(1-2):87–100, 2010.

[16] L. Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[17] J.-L. Joly, G. Métivier, and J. Rauch. Coherent nonlinear waves and the Wiener algebra. *Ann. Inst. Fourier (Grenoble)*, 44(1):167–196, 1994.

[18] P. Kirrmann, G. Schneider, and A. Mielke. The validity of modulation equations for extended systems with cubic nonlinearities. *Proc. Roy. Soc. Edinburgh Sect. A*, 122(1-2):85–91, 1992.

[19] P.-L. Lions and T Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.

[20] H. Liu and E. Tadmor. Semi-classical limit of the nonlinear Schrödinger-Poisson equation with subcritical initial data. *Methods Appl. Anal*, 122:85–91, 1992.

[21] P A. Markowich and N. J. Mauser. The classical limit of a self-consistent quantum-Vlasov equation in 3D. *Math. Models Methods Appl. Sci.*, 3(1):109–124, 1993.

[22] P.A. Markowich, C Ringhofer, and C. Schmeiser. *Semi-conductor Equations*. Spriger Verlag, Wien, New York, 1990.

[23] J. Rauch. *Hyperbolic partial differential equations and geometric optics*. American Mathematics Society Graduate Texts in Mathematics Series, To appear.

[24] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[25] T. Wada. Asymptotic behaviour of the solutions of the Hartree-type equation. *Adv. Math. Sci. Appl.*, 6(1), 1996.

[26] P. Zhang. Wigner measure and the semi-classical limit of Schrödinger-Poisson equation. *SIAM J. Math. Anal*, 34(3):700–718, 2002.

[27] P. Zhang, Y. Zheng, and N. J. Mauser. The limit from the Schrödinger-Poisson to the Vlasov-Poisson equations with general data in one dimension. *Comm. Pure Appl. Math.*, 55(5):582–632, 2002.

Univ. Montpellier 2, Mathématiques, UMR 5149, CC 051, 34095 Montpellier, France
E-mail address: `lounes.mouzaoui@univ-montp2.fr`