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A Direct Algorithm for 1D Total Variation Denoising

Laurent Condat

Abstract—A very fast noniterative algorithm is proposed for denoising or smoothing one-dimensional discrete signals, by solving the total variation regularized least-squares problem or the related fused lasso problem. A C code implementation is available on the web page of the author.

Index Terms—Total variation, denoising, nonlinear smoothing, fused lasso, regularized least-squares, nonparametric regression, convex nonsmooth optimization

I. INTRODUCTION

The problem of smoothing a signal, to remove or at least attenuate the noise it contains, has numerous applications in communications, control, machine learning, and many other fields of engineering and science [1]. In this paper, we focus on the numerical implementation of total variation (TV) denoising for one-dimensional (1D) discrete signals; that is, we are given a (noisy) signal $y = (y[1], \dots, y[N]) \in \mathbb{R}^N$ of size $N \geq 1$, and we want to efficiently compute the denoised signal $x \in \mathbb{R}^N$, defined implicitly as the solution to the minimization problem

$$\underset{x \in \mathbb{R}^N}{\text{minimize}} \frac{1}{2} \sum_{k=1}^N |y[k] - x[k]|^2 + \lambda \sum_{k=1}^{N-1} |x[k+1] - x[k]|, \quad (1)$$

for some regularization parameter $\lambda \geq 0$ (whose choice is a difficult problem by itself [2]). We recall that, as the functional to minimize is strongly convex, the solution x to the problem exists and is unique, whatever the data y . The TV denoising problem has received large attention in the communities of signal and image processing, inverse problems, sparse sampling, statistical regression analysis, optimization theory, among others. It is not the purpose of this paper to review the properties of the nonlinear TV denoising filter, since numerous papers can be found on this vast topic; see, e.g., [3]–[6] for various insights.

To solve the convex nonsmooth optimization problem (1), we mostly find in the literature iterative fixed-point methods [7]. Until not so long ago, such methods applied to TV regularization had rather high computational complexity [8]–[12], but the growing interest for related ℓ_1 -norm problems in compressed sensing or sparse recovery [13], [14] has yielded advances in the field. Recent iterative methods based on operator splitting, which exploit both the primal and dual formulations of the problems and use variable stepsize strategies or Nesterov-style accelerations, are quite efficient when applied to TV-based problems [15]–[17]. Graph cuts methods can also be used to solve (1) or

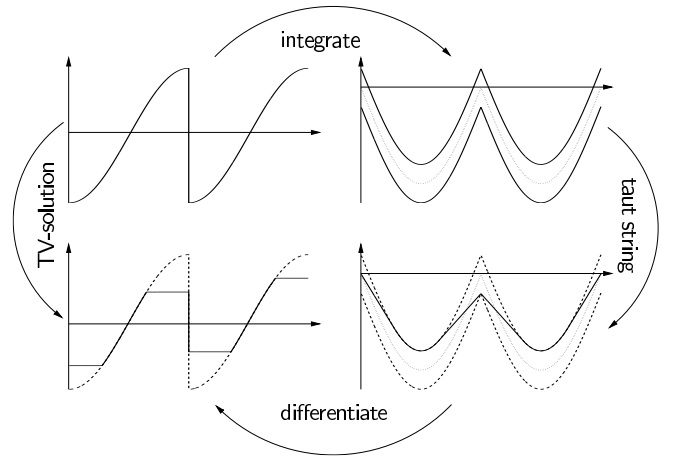


Fig. 1. Total variation denoising can be interpreted as pulling the antiderivative of the signal taut in a tube around it. The taut string algorithm is based on this idea, but not the proposed algorithm. This figure is borrowed from a PDF slide of a talk by M. Grasmair in 2007, entitled “dual settings for total variation regularization”.

its extension on graphs [18]; they actually solve a quantized version of (1): the minimizer x is not searched in \mathbb{R}^N but in $\varepsilon \mathbb{Z}^N$, for some $\varepsilon > 0$, with complexity $O(\log_2(1/\varepsilon)N)$. If ε is small enough, the exact solution in \mathbb{R}^N can be obtained from the quantized one, as shown by Hochbaum [19], [20]. In this paper, we present a novel and very fast algorithm to compute the denoised signal x solution to (1), exactly, in a direct, noniterative, way, possibly in-place. It is appropriate for real-time processing of an incoming stream of data, as it locates the jumps in x one after the other by forward scans, almost online. The possibility of such an algorithm sheds light on the relatively local nature of the TV denoising filter [21].

After this work was completed, the author found that there already exists a direct, linear time, method for 1D TV denoising, called the *taut string algorithm* [22], see also [23]–[27]. Although known by some statisticians, this method seems to have been largely ignored, at least in the signal processing community. Evidence of this is that iterative methods are regularly proposed for 1D TV denoising [28]–[32]. To understand the principle of the taut string method, define the sequence of running sums r by $r[k] = \sum_{i=1}^k y[i]$ for $1 \leq k \leq N$, and consider the problem:

$$\underset{s \in \mathbb{R}^{N+1}}{\text{minimize}} \sum_{k=1}^N \sqrt{1 + |s[k] - s[k-1]|^2} \quad \text{subject to} \\ s[0] = 0, \quad s[N] = r[N], \quad \text{and} \quad \max_{1 \leq k \leq N} |s[k] - r[k]| \leq \lambda. \quad (2)$$

1D TV Denoising Algorithm

Input: integer size $N \geq 1$, real sequence $(y[1], \dots, y[N])$, real parameter $\lambda > 0$. Output: real sequence $(x[1], \dots, x[N])$.

1. Set $k = k_0 = k_- = k_+ \leftarrow 1$, $v_{\min} \leftarrow y[1] - \lambda$, $v_{\max} \leftarrow y[1] + \lambda$, $u_{\min} \leftarrow \lambda$, $u_{\max} \leftarrow -\lambda$.
 2. If $k = N$, set $x[N] \leftarrow v_{\min} + u_{\min}$ and terminate.
 3. If $y[k+1] + u_{\min} < v_{\min} - \lambda$, set $x[k_0] = \dots = x[k_-] \leftarrow v_{\min}$, $k = k_0 = k_- = k_+ \leftarrow k_- + 1$, $v_{\min} \leftarrow y[k]$,
 $\quad \left[\quad v_{\max} \leftarrow y[k] + 2\lambda, u_{\min} \leftarrow \lambda, u_{\max} \leftarrow -\lambda. \right.$
 4. Else, if $y[k+1] + u_{\max} > v_{\max} + \lambda$, set $x[k_0] = \dots = x[k_+] \leftarrow v_{\max}$, $k = k_0 = k_- = k_+ \leftarrow k_+ + 1$, $v_{\min} \leftarrow y[k] - 2\lambda$,
 $\quad \left[\quad v_{\max} \leftarrow y[k], u_{\min} \leftarrow \lambda, u_{\max} \leftarrow -\lambda. \right.$
 5. Else, set $k \leftarrow k + 1$, $u_{\min} \leftarrow u_{\min} + y[k] - v_{\min}$ and $u_{\max} \leftarrow u_{\max} + y[k] - v_{\max}$.
 $\quad \left[\quad \text{If } u_{\min} \geq \lambda, \text{ set } v_{\min} \leftarrow v_{\min} + (u_{\min} - \lambda)/(k - k_0 + 1), u_{\min} \leftarrow \lambda, k_- \leftarrow k. \right.$
 $\quad \left[\quad \text{If } u_{\max} \leq -\lambda, \text{ set } v_{\max} \leftarrow v_{\max} + (u_{\max} + \lambda)/(k - k_0 + 1), u_{\max} \leftarrow -\lambda, k_+ \leftarrow k. \right.$
 6. If $k < N$, go to 3.
 7. If $u_{\min} < 0$, set $x[k_0] = \dots = x[k_-] \leftarrow v_{\min}$, $k = k_0 = k_- \leftarrow k_- + 1$, $v_{\min} \leftarrow y[k]$, $u_{\min} \leftarrow \lambda$, $u_{\max} \leftarrow y[k] + \lambda - v_{\max}$.
 $\quad \left[\quad \text{Then go to 2.} \right.$
 8. Else, if $u_{\max} > 0$, set $x[k_0] = \dots = x[k_+] \leftarrow v_{\max}$, $k = k_0 = k_+ \leftarrow k_+ + 1$, $v_{\max} \leftarrow y[k]$, $u_{\max} \leftarrow -\lambda$, $u_{\min} \leftarrow y[k] - \lambda - v_{\min}$.
 $\quad \left[\quad \text{Then go to 2.} \right.$
 9. Else, set $x[k_0] = \dots = x[N] \leftarrow v_{\min} + u_{\min}/(k - k_0 + 1)$ and terminate.
-

Then, the problems (1) and (2) are equivalent, in the sense that their respective solutions x and s are related by $x[k] = s[k] - s[k-1]$, for $1 \leq k \leq N$ [22], [33]. Thus, the formulation (2) allows to express the TV solution x as the discrete derivative of a string threaded through a tube around the discrete primitive of the data, and pulled taut such that its length is minimized. This principle is illustrated in Fig. 1. The proposed algorithm is different, as it does not manipulate any running sum and only performs forward scans.

The paper is organized as follows. In Sect. II, we describe and discuss the new algorithm. In Sect. III, we show that it can be used to solve the fused lasso problem.

II. PROPOSED METHOD

We first introduce the (Fenchel-Moreau-Rockafellar) *dual* problem to the *primal* problem (1) [7]:

$$\begin{aligned} & \underset{u \in \mathbb{R}^{N+1}}{\text{minimize}} \quad \sum_{k=1}^N |y[k] - u[k] + u[k-1]|^2 \quad \text{s.t.} \\ & |u[k]| \leq \lambda, \quad \forall k = 1, \dots, N-1, \quad \text{and } u[0] = u[N] = 0. \end{aligned} \quad (3)$$

Once the solution u to the dual problem is found, one recovers the primal solution x by

$$x[k] = y[k] - u[k] + u[k-1], \quad \forall k = 1, \dots, N. \quad (4)$$

Actually, the method of [10] and its accelerated version [15] solve (3) iteratively, using forward-backward splitting [7].

The Karush-Kuhn-Tucker conditions characterize the unique solutions x and u [7]. They yield, in addition to (4),

$$\begin{aligned} & u[0] = u[N] = 0 \quad \text{and} \quad \forall k = 1, \dots, N-1, \\ & \begin{cases} u[k] \in [-\lambda, \lambda] & \text{if } x[k] = x[k+1], \\ u[k] = -\lambda & \text{if } x[k] < x[k+1], \\ u[k] = \lambda & \text{if } x[k] > x[k+1]. \end{cases} \end{aligned} \quad (5)$$

Hence, the proposed algorithm consists in running forwardly through the samples $y[k]$; at location k , it tries

to prolongate the current segment of x by $x[k+1] = x[k]$. If this is not possible without violating (4) and (5), it goes back to the last location where a jump can be introduced in x , validates the current segment until this location, starts a new segment, and continues. In more details, the proposed algorithm, given at the top of the page, works as follows. The variable are initialized at Step 1. At Step 2., we are at some location k and we are building a segment starting at k_0 , with value $v = x[k_0] = \dots = x[k]$. v is unknown but we know the values v_{\min} and v_{\max} such that $v \in [v_{\min}, v_{\max}]$. The auxiliary values u_{\min} and u_{\max} are the values of $u[k]$ in the hypothetic cases $v = v_{\min}$ and $v = v_{\max}$, respectively. Now, we are trying to prolongate the segment with $x[k+1] = v$, by updating the four variables v_{\min} , v_{\max} , u_{\min} , u_{\max} , for the location $k+1$. There are three possible cases, corresponding to Steps 3., 4., 5. If the test at Step 3. is satisfied, we cannot update u_{\min} without violating (4) and (5), because v_{\min} is too high. This means that the assumption $x[k_0] = \dots = x[k+1]$ was wrong, so that the segment must be broken, and the negative jump necessarily takes place at the last location k_- where u_{\min} was equal to λ . By the same reasoning, if the test at Step 4. is satisfied, a positive jump must be introduced at the last location k_+ where u_{\max} was equal to $-\lambda$. Else, at Step 5., no jump is necessary yet and we can continue. It may be necessary to update the bounds v_{\min} and v_{\max} ; this is done at Step 6. Once the end of the signal is reached, with $k = N$, we must test if the hypothesis of a segment $x[k_0] = \dots = x[N]$ does not violate the condition $u[N] = 0$. The three possible cases correspond to Steps 8., 9., 10. If the test at Step 8. is satisfied, v_{\min} is too high and a negative jump is necessary. Similarly, if the test at Step 9. is satisfied, v_{\max} is too low and a positive jump is necessary. Else, a segment is constructed until the end of the signal and the algorithm terminates, either at Step 2. or at Step 10.

We note that the dual solution u is not computed. We can still recover it recursively from x using (4).

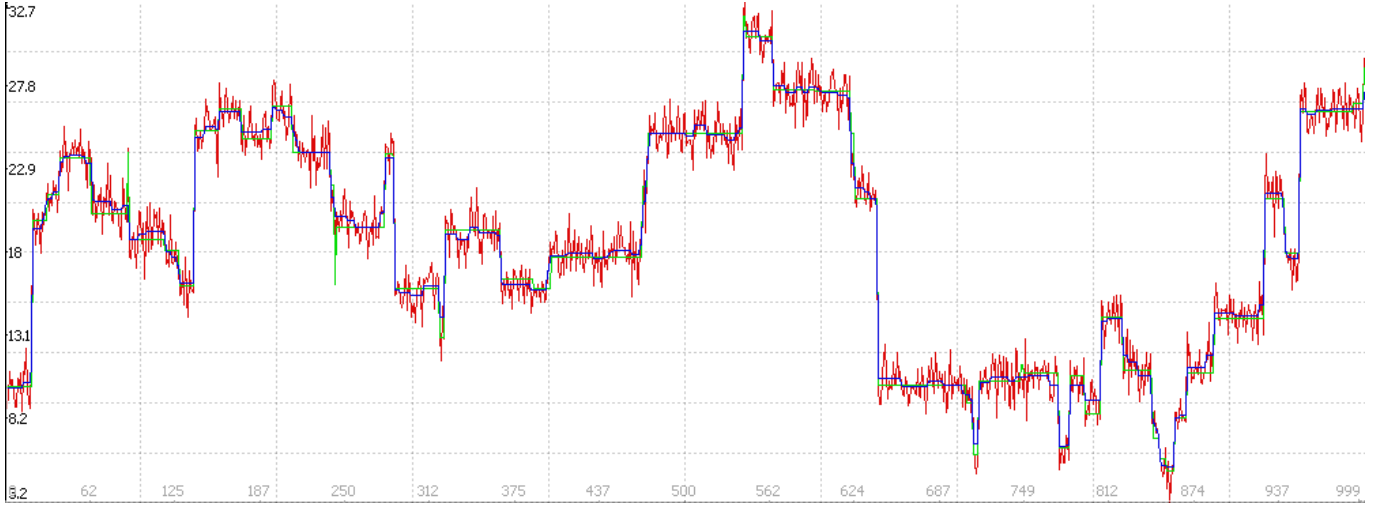


Fig. 2. In this example, y (in red) is a piecewise constant process of size $N=1000$ (unknown ground truth, in green) corrupted by additive Gaussian noise of unit variance. The TV-denoised signal x (in blue), solving (1) with $\lambda=2$ exactly, was computed by the proposed algorithm in 30 microseconds.

The worst case complexity of the algorithm is $O(N + (N-1) + \dots + 1) = O(N^2)$. Indeed, every added segment has size at least one, but the algorithm may have to scan all the remaining samples to validate it in one of the steps 3., 4., 8., 9. However, this worst case scenario is encountered only when x is a ramp with *very small* slope of order N^{-2} , except at the boundaries; for instance, consider that $\lambda = 1$ and $y[1] = -2$, $y[k] = \alpha(k-2)$ for $2 \leq k \leq N-1$, $y[N] = \alpha(N-3)+2$, where $\alpha = 4/((N-2)(N-3))$. The solution x is such that $x[1] = y[1] + 1$, $x[k] = y[k]$ for $2 \leq k \leq N-1$, $x[N] = y[N] - 1$. Actually, such a pathological case, for which there is no interest in applying TV denoising, is only a curiosity; the complexity is $O(N)$ in all practical situations, because the segments of x are validated with a delay which does not depend on N .

The algorithm was implemented in C and run on a Apple laptop with a 2.3 GHz Intel Core i7 processor. The computation time was around 30ms with $N = 10^6$, for various test signals and noise levels. Importantly, the computation time is insensitive to the value of λ . The taut string algorithm was implemented in C as well, by adapting Matlab code written by Lutz Dümbgen and available online¹. In the same conditions, the computation time was around 50ms. Thus, the taut string algorithm is efficient, but the proposed algorithm outperforms it by a constant factor consistently.

For illustration purpose, we consider the example of a discrete Lévy process, which is a stochastic process with independent increments [6], corrupted by additive white Gaussian noise. More precisely, $y[k] = x_*[k] + e[k]$ for $1 \leq k \leq N = 1000$, where the $e[k] \sim \mathcal{N}(0, 1)$ are independent and identically distributed (i.i.d.), and the ground truth x_* has a fixed value $x_*[1]$ and i.i.d. random increments $d[k] = x_*[k] - x_*[k-1]$ for $2 \leq k \leq N$. We chose a sparse Bernoulli-Gaussian law for the increments, since TV denoising approaches the optimal minimum mean squared error estimator for such

piecewise constant signals [5], [6]; that is, the probability density function of $d[k]$ is

$$p\delta(t) + \frac{(1-p)}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}}, \quad t \in \mathbb{R}, \quad (6)$$

where $p = 0.95$, $\sigma = 4$, and $\delta(t)$ is the Dirac distribution. We found empirically that the mean squared error $\|x_* - x\|_2^2/N$ is minimized for $\lambda = 2$. The computation time of x , averaged over several runs and realizations, was 30 microseconds. One realization of the experiment is depicted in Fig. 2.

III. APPLICATION TO THE FUSED LASSO PROBLEM

Besides denoising of 1D signals, the proposed algorithm can be used as a black box to solve other problems. Of particular interest is the *fused lasso signal approximator*, introduced in [34], which yields a solution that has sparsity in both the coefficients and their successive differences. It consists in solving the problem

$$\underset{z \in \mathbb{R}^N}{\text{minimize}} \frac{1}{2} \sum_{k=1}^N |z[k] - y[k]|^2 + \lambda \sum_{k=1}^{N-1} |z[k+1] - z[k]| + \mu \sum_{k=1}^N |z[k]|, \quad (7)$$

for some $\lambda \geq 0$ and $\mu \geq 0$. The fused lasso has many applications, e.g. in bioinformatics [35]–[37]. As shown in [36], the complexity of the fused lasso is the same as TV denoising, since the solution z can be obtained by simple soft-thresholding from the solution x of (1):

$$z[k] = \begin{cases} x[k] - \mu \cdot \text{sign}(x[k]) & \text{if } |x[k]| > \mu \\ 0 & \text{otherwise} \end{cases}. \quad (8)$$

It is straightforward to add soft-thresholding steps to the proposed algorithm to solve the generalization (7) of (1), for essentially the same computation time.

IV. CONCLUSION

In this article, we proposed a direct and very fast algorithm for denoising 1D signals by total variation (TV) minimization or fused lasso approximation. Since the algorithm

¹http://www.imsv.unibe.ch/content/staff/personalhomepages/duebngen/software/multiscale_densities/index_eng.html

computes the proximity operator [7] of the 1D TV, it can be used as a basic unit within iterative splitting methods, to solve inverse problems in signal processing and imaging. This approach will be developed in a forthcoming paper.

It would be worth investigating the possibility of extending the algorithm to generalizations of the TV, where the two-tap finite difference is replaced by another discrete differential operator [30], [33], to complex-valued or multi-valued signals [37], and to data of higher dimensions, like 2D images or graphs [25]. Furthermore, we should consider replacing the quadratic data fidelity term by other penalties, like the anti-log-likelihood of Poisson noise [27].

Besides, path-following, a.k.a. homotopy, algorithms have been proposed for ℓ_1 penalized problems; they can find the smallest value of λ and the associated x in (1), such that x has at most m segments, with complexity $O(mN)$ [14], [23], [36], [38]–[40]. The relationship between homotopy algorithms, the approach in [41] and the proposed algorithm should be studied. This is left for future work.

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