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Vector Addition Systems Reachability Problem (A Simpler Solution)

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Abstract. The reachability problem for Vector Addition Systems (VASs) is a central problem of net theory. The general problem is known to be decidable by algorithms based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney decomposition (KLMTS decomposition). Recently from this decomposition, we deduced that a final configuration is not reachable from an initial one if and only if there exists a Presburger inductive invariant that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes an inductive invariant, we deduce from this result that there exist checkable certificates of non-reachability in the Presburger arithmetic. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas. In another recent paper we provide the first proof of the VAS reachability problem that is not based on the KLMST decomposition. The proof is based on the notion of production relations that directly proves the existence of Presburger inductive invariants. In this paper we propose new intermediate results simplifying a bit more this last proof.

1 Introduction

Vector Addition Systems (VASs) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [2]. Their reachability problem is central since many computational problems (even outside the realm of parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [13] a partial proof of decidability of this problem. The proof was completed in 1981 by Mayr [11] and simplified by Kosaraju [7] from [13,11]. Ten years later [8], Lambert provided a further simplified version based on [7]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known to be non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VASs is still an open-problem. Even the existence of an elementary upper-bound complexity is open. In fact, the known general reachability algorithms are exclusively based on the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition.

Recently [9] we proved thanks to the KLMST decomposition that Parikh images of languages accepted by VASs are semi-pseudo-linear, a class that extends the Presburger sets. An application of this result was provided; we proved that a final configuration is not reachable from an initial one if and only if there exists a forward inductive invariant

definable in the Presburger arithmetic that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes a forward inductive invariant, we deduce that there exist checkable certificates of non-reachability in the Presburger arithmetic. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas.

In [10] we provided a new proof of the decidability of the reachability problem that does not introduce the KLMST decomposition. The proof is based on *transformer relations* and it proves directly that reachability sets are *almost semilinear*, a class of sets inspired by the class of semilinear sets [3] that extend the class of Presburger sets. This proof is based on a characterization of the conic sets definable in $\text{FO}(\mathbb{Q}, +, \leq)$ thanks to topological closures with vectors spaces. In this paper we provide a more direct proof that does not require this characterization. In fact we provide a direct proof that the conic sets generated by transformer relations are definable in $\text{FO}(\mathbb{Q}, +, \leq)$. As a direct consequence topological properties one conic sets are no longer used in this new version.

Outline of the paper: Section 2 recalls the definition of *almost semilinear sets*, a class of sets inspired by the decomposition of Presburger sets into semilinear sets. Section 3 introduces definitions related to *vector addition systems*. Section 4 introduces a well-order over the runs of vector addition systems. This well-order is central in the proof and it was first introduced by Petr Jančar in another context[5]. Based on the definition of this well-order we introduce in Section 5 the notion of transformer relations and we prove that conic relations generated by transformer relations are definable in $\text{FO}(\mathbb{Q}, +, \leq)$. Thanks to this result and the well-order introduced in the previous section we show in Section 6 that reachability sets of vector addition systems are almost semilinear. In Section 7 we introduce a dimension function for subsets of integer vectors. In Section 8 the almost semilinear sets are proved to be approximable by Presburger sets in a precise way based on the dimension function previously introduced. Thanks to this approximation and since reachability sets are almost semilinear we finally prove in Section 9 that the vector addition system reachability problem can be decided by inductive invariants definable in the Presburger arithmetic.

2 Almost Semilinear Sets

We denote by $\mathbb{N}, \mathbb{N}_{>0}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{>0}$ the set of *natural numbers, positive integers, integers, rational numbers, non negative rational numbers, and positive rational numbers*. Vectors and sets of vectors are denoted in bold face. The i th component of a vector $\mathbf{v} \in \mathbb{Q}^d$ is denoted by $\mathbf{v}(i)$. Given two sets $\mathbf{V}_1, \mathbf{V}_2 \subseteq \mathbb{Q}^d$ we denote by $\mathbf{V}_1 + \mathbf{V}_2$ the set $\{\mathbf{v}_1 + \mathbf{v}_2 \mid (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2\}$, and we denote by $\mathbf{V}_1 - \mathbf{V}_2$ the set $\{\mathbf{v}_1 - \mathbf{v}_2 \mid (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2\}$. In the same way given $T \subseteq \mathbb{Q}$ and $\mathbf{V} \subseteq \mathbb{Q}^d$ we let $T\mathbf{V} = \{t\mathbf{v} \mid (t, \mathbf{v}) \in T \times \mathbf{V}\}$. We also denote by $\mathbf{v}_1 + \mathbf{V}_2$ and $\mathbf{V}_1 + \mathbf{v}_2$ the sets $\{\mathbf{v}_1\} + \mathbf{V}_2$ and $\mathbf{V}_1 + \{\mathbf{v}_2\}$, and we denote by $t\mathbf{V}$ and $T\mathbf{v}$ the sets $\{t\}\mathbf{V}$ and $T\{\mathbf{v}\}$.

A *periodic set* is a subset $\mathbf{P} \subseteq \mathbb{Z}^d$ such that $\mathbf{0} \in \mathbf{P}$ and such that $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. A *conic set* is a subset $\mathbf{C} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{C}$, $\mathbf{C} + \mathbf{C} \subseteq \mathbf{C}$ and such that $\mathbb{Q}_{\geq 0}\mathbf{C} \subseteq \mathbf{C}$. A

periodic set \mathbf{P} is said to be *finitely generated* if there exist vectors $\mathbf{p}_1, \dots, \mathbf{p}_k \in \mathbf{P}$ such that $\mathbf{P} = \mathbb{N}\mathbf{p}_1 + \dots + \mathbb{N}\mathbf{p}_k$. A periodic set \mathbf{P} is said to be *asymptotically definable* if the conic set $\mathbb{Q}_{\geq 0}\mathbf{P}$ is definable in $\text{FO}(\mathbb{Q}, +, \leq)$. A *Presburger set* is a subset $\mathbf{Z} \subseteq \mathbb{Z}^d$ definable in $\text{FO}(\mathbb{Z}, +, \leq)$. Let us recall that $\mathbf{Z} \subseteq \mathbb{Z}^d$ is a Presburger set if and only if it is a finite union of *linear sets* $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{Z}^d$ and $\mathbf{P} \subseteq \mathbb{Z}^d$ is a finitely generated periodic set [3]. An *almost semilinear set* is a finite union of sets of the form $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{Z}^d$ and $\mathbf{P} \subseteq \mathbb{Z}^d$ is an asymptotically definable periodic set [10].

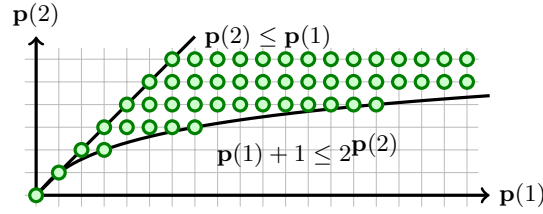


Fig. 1. An asymptotically definable periodic set.

Example 2.1. The periodic set $\mathbf{P} = \{\mathbf{p} \in \mathbb{N}^2 \mid \mathbf{p}(2) \leq \mathbf{p}(1) \leq 2\mathbf{p}(2) - 1\}$ is represented in Figure 1. Observe that $\mathbb{Q}_{\geq 0}\mathbf{P} = \{\mathbf{0}\} \cup \{\mathbf{c} \in \mathbb{Q}_{> 0}^2 \mid \mathbf{p}(2) \leq \mathbf{p}(1)\}$ is a conic set definable in $\text{FO}(\mathbb{Q}, +, \leq)$. Thus \mathbf{P} is an asymptotically definable periodic set.

3 Vector Addition Systems

A *Vector Addition System (VAS)* is given by a finite subset $\mathbf{A} \subseteq \mathbb{Z}^d$. A *configuration* is a vector $\mathbf{c} \in \mathbb{N}^d$. A *run* ρ is a non-empty word of configurations $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ such that the difference $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ is in \mathbf{A} for every $j \in \{1, \dots, k\}$. In that case we say that ρ is a run *labeled* by $w = \mathbf{a}_1 \dots \mathbf{a}_k$, the configurations \mathbf{c}_0 and \mathbf{c}_k are respectively called the *source* and the *target* and they are denoted by $\text{src}(\rho)$ and $\text{tgt}(\rho)$. Given a word $w \in \mathbf{A}^*$, we introduce the binary relation \xrightarrow{w} over the set of configurations by $\mathbf{x} \xrightarrow{w} \mathbf{y}$ if there exists a run ρ from \mathbf{x} to \mathbf{y} labeled by w . Observe that in this case ρ is unique. The *reachability relation* is the relation $\xrightarrow{*}$ over \mathbb{N}^d defined by $\mathbf{x} \xrightarrow{*} \mathbf{y}$ if there exists a run from \mathbf{x} to \mathbf{y} .

The *forward/backward reachability sets* $\text{post}^*(\mathbf{X})$ and $\text{pre}^*(\mathbf{Y})$ where $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$ are defined by:

$$\begin{aligned} \text{post}^*(\mathbf{X}) &= \{\mathbf{c} \in \mathbb{N}^d \mid \exists \mathbf{x} \in \mathbf{X} \mathbf{x} \xrightarrow{*} \mathbf{c}\} \\ \text{pre}^*(\mathbf{Y}) &= \{\mathbf{c} \in \mathbb{N}^d \mid \exists \mathbf{y} \in \mathbf{Y} \mathbf{c} \xrightarrow{*} \mathbf{y}\} \end{aligned}$$

A set $\mathbf{X} \subseteq \mathbb{N}^d$ is said to be a *forward inductive invariant* if $\text{post}^*(\mathbf{X}) \subseteq \mathbf{X}$. Symmetrically a set $\mathbf{Y} \subseteq \mathbb{N}^d$ is said to be a *backward inductive invariant* if $\text{pre}^*(\mathbf{Y}) \subseteq \mathbf{Y}$.

In this paper we prove that for every $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$ such that there does not exist a run from \mathbf{x} to \mathbf{y} , then there exists a partition of \mathbb{N}^d into (\mathbf{X}, \mathbf{Y}) such that \mathbf{X} is a Presburger forward inductive invariant that contains \mathbf{x} and \mathbf{Y} is a Presburger backward inductive invariant that contains \mathbf{y} . This result will provide directly the following corollary.

Corollary 3.1. *The reachability problem for vector addition systems is decidable.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$ be two configurations. Let us consider an algorithm that enumerates in parallel the runs ρ and the partitions (\mathbf{X}, \mathbf{Y}) of \mathbb{N}^d into Presburger sets. If the algorithm encounters a run from \mathbf{x} to \mathbf{y} then it returns “reachable” and if \mathbf{X} is a forward inductive invariant that contains \mathbf{x} and \mathbf{Y} is a Presburger backward inductive invariant that contains \mathbf{y} then it returns “unreachable”. Observe that this last condition can be effectively decided since \mathbf{X} is a forward inductive invariant if and only if $\mathbb{N}^d \cap (\mathbf{X} + \mathbf{A}) \setminus \mathbf{X}$ is empty and \mathbf{Y} is a backward inductive invariant if and only if $\mathbb{N}^d \cap (\mathbf{Y} - \mathbf{A}) \setminus \mathbf{Y}$ is empty. Note that this algorithm is correct. Moreover, it terminates thanks to the main result proved in this paper. \square

Remark 3.2. In general the reachability sets $\text{post}^*(\{\mathbf{x}\})$ and $\text{pre}^*(\{\mathbf{y}\})$ are not definable in the Presburger arithmetic[4].

4 Well-Order Over The Runs

We introduce the relation \trianglelefteq over the runs defined by $\rho \trianglelefteq \rho'$ if ρ is a run of the form $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$ and if there exists a sequence $(\mathbf{v}_j)_{0 \leq j \leq k+1}$ of vectors $\mathbf{v}_j \in \mathbb{N}^d$ such that ρ' is a run of the form $\rho' = \rho_0 \dots \rho_k$ where ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_{j+1} + \mathbf{v}_{j+1}$. A very similar relation was first introduced by Petr Jančar in [5].

Lemma 4.1 ([5]). *The relation \trianglelefteq is a well-order over the runs.*

Proof. A proof of this lemma with different notations can be retrieved from Section 6 of [5]. We prefer to give a direct proof of this important result rather than a reduction to the original one. To do so, we introduce a well-order \preceq over the runs based on the Higman’s Lemma and we show that \preceq and \trianglelefteq are equal.

We first associate to a run $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ the word $\alpha(\rho) = (\mathbf{a}_1, \mathbf{c}_1) \dots (\mathbf{a}_k, \mathbf{c}_k)$ over the set $S = \mathbf{A} \times \mathbb{N}^d$ where $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$. The set S is well-ordered by the relation \sqsubseteq defined by $(\mathbf{a}_1, \mathbf{c}_1) \sqsubseteq (\mathbf{a}_2, \mathbf{c}_2)$ if $\mathbf{a}_1 = \mathbf{a}_2$ and $\mathbf{c}_1 \leq \mathbf{c}_2$. The set of words S^* is well-ordered thanks to the Higman’s lemma by the relation \sqsubseteq^* defined by $s_1 \dots s_k \sqsubseteq^* w$ if w is a word in $S^* s'_1 S^* \dots s'_k S^*$ with $s_j \sqsubseteq s'_j$ for every $j \in \{1, \dots, k\}$. The well-order \preceq over the runs is defined by $\rho \preceq \rho'$ if $\text{src}(\rho) \leq \text{src}(\rho')$, $\text{tgt}(\rho) \leq \text{tgt}(\rho')$ and $\alpha(\rho) \sqsubseteq^* \alpha(\rho')$. Now, let us prove that \preceq and \trianglelefteq are equal. We consider a run $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ with $\mathbf{c}_j \in \mathbb{N}^d$ and we introduce the action $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ for each $j \in \{1, \dots, k\}$.

Assume first that $\rho \preceq \rho'$ for some run ρ' . Since $\alpha(\rho) = (\mathbf{a}_1, \mathbf{c}_1) \dots (\mathbf{a}_k, \mathbf{c}_k)$ and $\alpha(\rho) \sqsubseteq^* \alpha(\rho')$ we deduce a decomposition of $\alpha(\rho')$ into the following word where $\mathbf{c}'_j \geq \mathbf{c}_j$ for every $j \in \{1, \dots, k\}$:

$$w_0(\mathbf{a}_1, \mathbf{c}'_1)w_1 \dots (\mathbf{a}_k, \mathbf{c}'_k)w_k$$

In particular ρ' can be decomposed in $\rho' = \rho_0 \dots \rho_k$ where ρ_0 is a run from $\text{src}(\rho')$ to $\mathbf{c}'_1 - \mathbf{a}_1$, ρ_j is a run from \mathbf{c}'_j to $\mathbf{c}'_{j+1} - \mathbf{a}_{j+1}$ for every $j \in \{1, \dots, k-1\}$, and ρ_k is a run from \mathbf{c}'_k to $\text{tgt}(\rho')$. Let us introduce the sequence $(\mathbf{v}_j)_{0 \leq j \leq k+1}$ of vectors defined by $\mathbf{v}_0 = \text{src}(\rho') - \text{src}(\rho)$, $\mathbf{v}_j = \mathbf{c}'_j - \mathbf{c}_j$ for every $j \in \{1, \dots, k\}$ and $\mathbf{v}_{k+1} = \text{tgt}(\rho') - \text{tgt}(\rho)$. Note that $\mathbf{v}_j \in \mathbb{N}^d$ for every $j \in \{0, \dots, k+1\}$. Observe that for every $j \in \{0, \dots, k\}$ we have $\mathbf{c}'_{j+1} - \mathbf{a}_j = \mathbf{c}_{j+1} - \mathbf{a}_j + \mathbf{v}_{j+1} = \mathbf{c}_j + \mathbf{v}_{j+1}$. Hence ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$ for every $j \in \{0, \dots, k\}$. Therefore $\rho \preceq \rho'$.

Conversely, let us assume that $\rho \preceq \rho'$ for some run ρ' . We introduce a sequence $(\mathbf{v}_j)_{0 \leq j \leq k+1}$ of vectors in \mathbb{N}^d such that $\rho' = \rho_0 \dots \rho_k$ where ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$. We deduce the following equality where $\mathbf{c}'_j = \mathbf{c}_j + \mathbf{v}_j$ and $\mathbf{a}'_j \in \mathbf{A}$:

$$\alpha(\rho') = \alpha(\rho_0)(\mathbf{a}'_1, \mathbf{c}'_1)\alpha(\rho_1) \dots (\mathbf{a}'_k, \mathbf{c}'_k)\alpha(\rho_k)$$

Observe that $\mathbf{a}'_j = \text{tgt}(\rho_{j-1}) - \text{src}(\rho_j) = (\mathbf{c}_{j-1} + \mathbf{v}_j) - (\mathbf{c}_j + \mathbf{v}_j) = \mathbf{a}_j$. We deduce that $\alpha(\rho) \sqsubseteq^* \alpha(\rho')$. Moreover, since $\text{src}(\rho) \leq \text{src}(\rho')$ and $\text{tgt}(\rho) \leq \text{tgt}(\rho')$ we get $\rho \preceq \rho'$. \square

5 Transformer Relations

Based on the definition of the well-order \preceq , we introduce the *transformer relation with capacity* $\mathbf{c} \in \mathbb{N}^d$ as the binary relation $\overset{\mathbf{c}}{\curvearrowright}$ over \mathbb{N}^d defined by $\mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}$ if there exists a run from $\mathbf{c} + \mathbf{x}$ to $\mathbf{c} + \mathbf{y}$. We also associate to every run $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ with $\mathbf{c}_j \in \mathbb{N}^d$ the *transformer relation along the run* ρ denoted by $\overset{\rho}{\curvearrowright}$ is defined as the following composition:

$$\overset{\rho}{\curvearrowright} = \overset{\mathbf{c}_0}{\curvearrowright} \circ \dots \circ \overset{\mathbf{c}_k}{\curvearrowright}$$

In this section transformer relations are shown to be asymptotically definable. Since asymptotically definable periodic relations are stable by composition (see the following Lemma 5.1), it is sufficient to prove that $\overset{\mathbf{c}}{\curvearrowright}$ is in this class for every capacity $\mathbf{c} \in \mathbb{N}^d$.

Lemma 5.1. *Asymptotically definable periodic relations are stable by composition.*

Proof. Assume that $R, S \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ are two periodic relations and observe that $(\mathbf{0}, \mathbf{0}) \in R \circ S$. Moreover if $(\mathbf{x}_1, \mathbf{z}_1)$ and $(\mathbf{x}_2, \mathbf{z}_2)$ are in $R \circ S$ then there exists $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{Z}^d$ such that $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in R$ and $(\mathbf{y}_1, \mathbf{z}_1), (\mathbf{y}_2, \mathbf{z}_2) \in S$. As R and S are periodic we get $(\mathbf{x}, \mathbf{y}) \in R$ and $(\mathbf{y}, \mathbf{z}) \in S$ where $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ and $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$. Thus $(\mathbf{x}, \mathbf{z}) \in R \circ S$ and we have proved that $R \circ S$ is periodic. Now just observe that $\mathbb{Q}_{\geq 0}(R \circ S) = (\mathbb{Q}_{\geq 0}R) \circ (\mathbb{Q}_{\geq 0}S)$. Hence if R and S are asymptotically definable then $R \circ S$ is also asymptotically definable. \square

Lemma 5.2. *The transformer relation $\overset{\mathbf{c}}{\curvearrowright}$ is periodic.*

Proof. Assume that $\mathbf{x}_1 \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}_1$ and $\mathbf{x}_2 \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}_2$. There exist two words $w_1, w_2 \in \mathbf{A}^*$ such that $\mathbf{c} + \mathbf{x}_1 \xrightarrow{w_1} \mathbf{c} + \mathbf{y}_1$ and $\mathbf{c} + \mathbf{x}_2 \xrightarrow{w_2} \mathbf{c} + \mathbf{y}_2$. By monotony we deduce that $\mathbf{c} + \mathbf{x}_1 + \mathbf{x}_2 \xrightarrow{w_1 w_2} \mathbf{c} + \mathbf{y}_1 + \mathbf{y}_2$. \square

Example 5.3. Observe that if $\mathbf{c} = \mathbf{0}$ then $\overset{\mathbf{c}}{\sim}$ is the reachability relation. Let $\mathbf{A} = \{(-1, 1)\}$ and $\mathbf{c} = (0, 0)$. Observe that $\mathbb{Q}_{\geq 0} \overset{\mathbf{c}}{\sim} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{Q}_{\geq 0}^2 \times \mathbb{Q}_{\geq 0}^2 \mid \mathbf{v} - \mathbf{u} \in \mathbb{Q}_{\geq 0}(-1, 1)\}$.

For the reminder of this section, we fix a capacity $\mathbf{c} \in \mathbb{N}^d$. We associate to every couple $\gamma \in \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ the set Ω_γ of runs ρ such that $(\text{src}(\rho), \text{tgt}(\rho)) \in (\mathbf{c}, \mathbf{c}) + \mathbb{Q}_{\geq 0}\gamma$. We denote by \mathbf{Q}_γ the set of configurations $\mathbf{q} \in \mathbb{N}^d$ such that there exists a run $\rho \in \Omega_\gamma$ in which \mathbf{q} occurs. Note that \mathbf{Q}_γ is non empty since it contains \mathbf{c} . We denote by I_γ the set of indexes $i \in \{1, \dots, d\}$ such that $\{\mathbf{q}(i) \mid \mathbf{q} \in \mathbf{Q}_\gamma\}$ is finite. We consider the projection function $\pi_\gamma : \mathbf{Q}_\gamma \rightarrow \mathbb{N}^{I_\gamma}$ defined by $\pi_\gamma(\mathbf{q})(i) = \mathbf{q}(i)$ for every $\mathbf{q} \in \mathbf{Q}_\gamma$ and for every $i \in I_\gamma$. We introduce the finite set of *states* $S_\gamma = \pi_\gamma(\mathbf{Q}_\gamma)$ and the set T_γ of *transitions* $(\pi_\gamma(\mathbf{q}), \mathbf{q}' - \mathbf{q}, \pi_\gamma(\mathbf{q}'))$ where $\mathbf{q}\mathbf{q}'$ is a factor of a run in Ω_γ . Let us observe that for every $(s, \mathbf{a}, s') \in T_\gamma$ we have $\mathbf{a} \in \mathbf{A}$ and $s'(i) = s(i) + \mathbf{a}(i)$ for every $i \in I_\gamma$. In particular from $T_\gamma \subseteq S_\gamma \times \mathbf{A} \times S_\gamma$ we deduce that T_γ is finite. We introduce the graph $G_\gamma = (S_\gamma, T_\gamma)$.

Example 5.4. Assume that $\mathbf{A} = \{(-1, 1)\}$, $\mathbf{c} = (0, 0)$ and let $\gamma = ((1, 0), (0, 1))$. Note that $\Omega_\gamma = \{(n, 0)(n-1, 1) \dots (0, n) \mid n \in \mathbb{N}\}$, $\mathbf{Q}_\gamma = \mathbb{N}^2$, $I_\gamma = \emptyset$, $S_\gamma = \{(\star, \star)\}$, and $T_\gamma = \{(\star, \star), (-1, 1), (\star, \star)\}$ where \star denotes a projected component.

An *intraproduction* for γ is a vector $\mathbf{h} \in \mathbb{N}^d$ such that there exists $n \in \mathbb{Q}_{\geq 0}$ satisfying $n\mathbf{x} \overset{\mathbf{c}}{\sim} \mathbf{h} \overset{\mathbf{c}}{\sim} n\mathbf{y}$ where $(\mathbf{x}, \mathbf{y}) = \gamma$. We denote by \mathbf{H}_γ the set of intraproductions for γ . Since $\overset{\mathbf{c}}{\sim}$ is periodic we deduce that the set of intraproductions is periodic.

Lemma 5.5. *We have $\mathbf{Q}_\gamma + \mathbf{H}_\gamma \subseteq \mathbf{Q}_\gamma$.*

Proof. Assume that $\gamma = (\mathbf{x}, \mathbf{y})$. Let $\mathbf{q} \in \mathbf{Q}_\gamma$ and $\mathbf{h} \in \mathbf{H}_\gamma$. As $\mathbf{q} \in \mathbf{Q}_\gamma$, there exist $n \in \mathbb{Q}_{\geq 0}$ and words $u, v \in \mathbf{A}^*$ such that $\mathbf{c} + n\mathbf{x} \xrightarrow{u} \mathbf{q} \xrightarrow{v} \mathbf{c} + n\mathbf{y}$. Since $\mathbf{h} \in \mathbf{H}_\gamma$ there exist $n' \in \mathbb{Q}_{\geq 0}$ and words $u', v' \in \mathbf{A}^*$ such that $\mathbf{c} + n'\mathbf{x} \xrightarrow{u'} \mathbf{c} + \mathbf{h} \xrightarrow{v'} \mathbf{c} + n'\mathbf{y}$. Let $m = n + n'$. By monotony, we have $\mathbf{c} + m\mathbf{x} \xrightarrow{u'u} \mathbf{q} + \mathbf{h} \xrightarrow{vv'} \mathbf{c} + m\mathbf{y}$. Hence $\mathbf{q} + \mathbf{h} \in \mathbf{Q}_\gamma$. \square

Lemma 5.6. *Every intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ satisfies $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$.*

Proof. Since the set of intraproductions is periodic we get $\mathbb{N}\mathbf{h} \subseteq \mathbf{H}_\gamma$. Moreover as $\mathbf{c} \in \mathbf{Q}_\gamma$, Lemma 5.5 shows that $\mathbf{c} + \mathbb{N}\mathbf{h} \subseteq \mathbf{Q}_\gamma$. By definition of I_γ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. \square

Corollary 5.7. *We have $\pi_\gamma(\text{src}(\rho)) = \pi_\gamma(\mathbf{c}) = \pi_\gamma(\text{tgt}(\rho))$ for every run $\rho \in \Omega_\gamma$.*

Proof. Assume that $\gamma = (\mathbf{x}, \mathbf{y})$. Since $\rho \in \Omega_\gamma$ there exists $n \in \mathbb{Q}_{\geq 0}$ such that ρ is a run from $\mathbf{c} + n\mathbf{x}$ to $\mathbf{c} + n\mathbf{y}$. In particular $n\mathbf{x}$ and $n\mathbf{y}$ are two intraproductions for γ . From Lemma 5.6 we get $n\mathbf{x}(i) = 0 = n\mathbf{y}(i)$ for every $i \in I_\gamma$. Hence $\pi_\gamma(\text{src}(\rho)) = \pi_\gamma(\mathbf{c}) = \pi_\gamma(\text{tgt}(\rho))$. \square

A *path* in G_γ is a word $p = (s_0, \mathbf{a}_1, s_1) \dots (s_{k-1}, \mathbf{a}_k, s_k)$ of transitions $(s_{j-1}, \mathbf{a}_j, s_j)$ in T_γ . Such a path is called a path from s_0 to s_k labeled by $w = \mathbf{a}_1 \dots \mathbf{a}_k$. When $s_0 = s_k$ the path is called a *cycle*. The previous corollary shows that every run $\rho \in \Omega_\gamma$ labeled by a word w provides a cycle in G_γ on $\pi_\gamma(\mathbf{c})$ labeled by w .

Corollary 5.8. *The graph G_γ is strongly connected.*

Proof. Let us consider $s \in S_\gamma$. There exists $\mathbf{q} \in \mathbf{Q}_\gamma$ that occurs in a run $\rho \in \Omega_\gamma$ such that $s = \pi_\gamma(\mathbf{q})$. Hence there exist two words $u, v \in \mathbf{A}^*$ such that $\text{src}(\rho) \xrightarrow{u} \mathbf{q} \xrightarrow{v} \text{tgt}(\rho)$. Since $\pi_\gamma(\text{src}(\rho)) = \pi_\gamma(\mathbf{c}) = \pi_\gamma(\text{tgt}(\rho))$ we deduce that there exist in G_γ a path from $\pi_\gamma(\mathbf{c})$ to s labeled by u and a path from s to $\pi_\gamma(\mathbf{c})$ labeled by v . \square

Lemma 5.9. *for every $\mathbf{q} \leq \mathbf{q}'$ in \mathbf{Q}_γ there exists an intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ such that $\mathbf{q}' \leq \mathbf{q} + \mathbf{h}$.*

Proof. Assume that $\gamma = (\mathbf{x}, \mathbf{y})$. As $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_\gamma$ there exists $m, m' \in \mathbb{Q}_{\geq 0}$ and $u, v, u', v' \in \mathbf{A}^*$ such that:

$$\mathbf{c} + m\mathbf{x} \xrightarrow{u} \mathbf{q} \xrightarrow{v} \mathbf{c} + m\mathbf{y} \quad \text{and} \quad \mathbf{c} + m'\mathbf{x} \xrightarrow{u'} \mathbf{q}' \xrightarrow{v'} \mathbf{c} + m'\mathbf{y}$$

Let us introduce $\mathbf{p} = \mathbf{q}' - \mathbf{q}$. By monotonicity:

$$\begin{aligned} \mathbf{c} + (m + m')\mathbf{x} &\xrightarrow{u'} \mathbf{q}' + m\mathbf{x} & \text{and} & \quad \mathbf{q} + \mathbf{p} + m\mathbf{x} \xrightarrow{v} \mathbf{c} + m\mathbf{y} + \mathbf{p} + m\mathbf{x} \\ \mathbf{c} + m\mathbf{x} + \mathbf{p} + m\mathbf{y} &\xrightarrow{u} \mathbf{q} + \mathbf{p} + m\mathbf{y} & \text{and} & \quad \mathbf{q}' + m\mathbf{y} \xrightarrow{v'} \mathbf{c} + (m + m')\mathbf{y} \end{aligned}$$

Since $\mathbf{q}' + m\mathbf{x} = \mathbf{q} + \mathbf{p} + m\mathbf{x}$ and $\mathbf{q} + \mathbf{p} + m\mathbf{y} = \mathbf{q}' + m\mathbf{y}$, we have proved that $\mathbf{c} + n\mathbf{x} \xrightarrow{u'v} \mathbf{c} + \mathbf{h} \xrightarrow{uv'} \mathbf{c} + n\mathbf{y}$ with $n = m + m'$ and $\mathbf{h} = \mathbf{p} + m(\mathbf{x} + \mathbf{y})$. Observe that $\mathbf{q} + \mathbf{h} = \mathbf{q}' + m(\mathbf{x} + \mathbf{y}) \geq \mathbf{q}'$. \square

Lemma 5.10. *There exist intraproducts $\mathbf{h} \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$.*

Proof. Let $i \notin I_\gamma$. There exists a sequence $(\mathbf{q}_k)_{k \in \mathbb{N}}$ of configurations $\mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $(\mathbf{q}_k(i))_{k \in \mathbb{N}}$ is strictly increasing. Since (\mathbb{N}^d, \leq) is well-ordered there exists $k < k'$ such that $\mathbf{q}_k \leq \mathbf{q}_{k'}$. Lemma 5.9 shows that there exists an intraproduction \mathbf{h}_i for γ such that $\mathbf{q}_{k'} \leq \mathbf{q}_k + \mathbf{h}_i$. In particular $\mathbf{h}_i(i) > 0$. As the set of intraproduction for γ is periodic we deduce that $\mathbf{h} = \sum_{i \notin I} \mathbf{h}_i$ is an intraproduction for γ . By construction we have $\mathbf{h}(i) > 0$ for every $i \notin I_\gamma$. From Lemma 5.6 we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. Therefore $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$. \square

Example 5.11. Let us come back to the Example 5.4. The run $(2, 0)(1, 1)(0, 2)$ shows that $(1, 1)$ is an intraproduction for γ .

Corollary 5.12. *States in S_γ are incomparable.*

Proof. Let us consider $s \leq s'$ in S_γ . There exists $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_\gamma$ such that $s = \pi_\gamma(\mathbf{q})$ and $s' = \pi_\gamma(\mathbf{q}')$. Lemma 5.10 shows that there exists an intraproduction $\mathbf{h}' \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}'(i) = 0\}$. By replacing \mathbf{h}' by a vector in $\mathbb{N}_{>0}\mathbf{h}'$ we can assume without loss of generality that $\mathbf{q}(i) \leq \mathbf{q}'(i) + \mathbf{h}'(i)$ for every $i \notin I_\gamma$. As $\mathbf{q}(i) = s(i) \leq s'(i) =$

$\mathbf{q}'(i) = \mathbf{q}'(i) + \mathbf{h}'(i)$ for every $i \in I_\gamma$ we deduce that $\mathbf{q} \leq \mathbf{q}' + \mathbf{h}'$. Lemma 5.5 shows that $\mathbf{q}' + \mathbf{h}' \in \mathbf{Q}_\gamma$. Lemma 5.9 shows that there exists an intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ such that $\mathbf{q}' + \mathbf{h}' \leq \mathbf{q} + \mathbf{h}$. From Lemma 5.6 we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. In particular $\mathbf{q}'(i) \leq \mathbf{q}(i)$ for every $i \in I_\gamma$. Hence $s' \leq s$ and we have proved that $s = s'$. \square

Corollary 5.13. *The class of graphs G_γ where $\gamma \in \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ is finite.*

Proof. Given $I \subseteq \{1, \dots, d\}$ we introduce the state $s_I \in \mathbb{N}^I$ defined by $s_I(i) = \mathbf{c}(i)$. We also introduce the set Γ_I of $\gamma \in \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ such that $I_\gamma = I$. Assume by contradiction that $S_I = \bigcup_{\gamma \in \Gamma_I} S_\gamma$ is infinite. For every $s \in S_I$ there exists $\gamma \in \Gamma_I$ such that $s \in S_\gamma$. Hence there exists a path p_s in G_γ from s_I to s . Since the states in S_γ are incomparable, we can assume that the states occurring in p_s are incomparable. By putting the paths p_s in a tree rooted by s_I with transitions labeled by actions in \mathbf{A} we deduce an infinite tree such that each node has a finite number of children (at most $|\mathbf{A}|$). The Koenig's lemma shows that this tree has an infinite branch. Since (\mathbb{N}^I, \leq) is well-ordered, there exists two comparable distinct nodes in this branch. There exists $s \in S_I$ such that these two comparable states occurs in p_s . We get a contradiction. Thus S_I is finite. We deduce the corollary. \square

A function $f : T_\gamma \rightarrow \mathbb{Q}$ is called a *Kirchhoff function* for γ if for every $s \in S_\gamma$ the following equality holds:

$$\sum_{t \in T_\gamma \cap (\{s\} \times \mathbf{A} \times S_\gamma)} f(t) = \sum_{t \in T_\gamma \cap (S_\gamma \times \mathbf{A} \times \{s\})} f(t)$$

We associate to every Kirchhoff function f the *displacement* $\Delta(f)$:

$$\Delta(f) = \sum_{\mathbf{a} \in \mathbf{A}} \sum_{t \in T_\gamma \cap (S_\gamma \times \{\mathbf{a}\} \times S_\gamma)} f(t) \mathbf{a}$$

The *Parikh image* of a cycle θ in G_γ is the function $f : T_\gamma \rightarrow \mathbb{N}$ defined by $f(t)$ is the number of occurrences of the transitions t in p . Let us observe that f is a Kirchhoff function for γ and $\Delta(f) = \Delta(w)$ where w is the label of θ and $\Delta(w) = \sum_{j=1}^k \mathbf{a}_j$ is the *displacement* of the word $w = \mathbf{a}_1 \dots \mathbf{a}_k$ with $\mathbf{a}_j \in \mathbb{N}^d$.

Assume that $\gamma = (\mathbf{x}, \mathbf{y})$. We introduce the relation R_γ of couples $(\mathbf{u}, \mathbf{v}) \in \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ satisfying $\mathbf{u}(i) > 0$ iff $\mathbf{x}(i) > 0$ and $\mathbf{v}(i) > 0$ iff $\mathbf{y}(i) > 0$ and such that there exists a Kirchhoff function f for γ such that $\mathbf{v} - \mathbf{u} = \Delta(f)$ and $f(t) > 0$ for every $t \in T_\gamma$.

Example 5.14. Assume that $\mathbf{A} = \{(-1, 1)\}$ and $\mathbf{c} = (0, 0)$. The relation R_γ is equal to $\{((n, 0), (0, n)) \mid n \in \mathbb{Q}_{> 0}\}$ if $\gamma \in \mathbb{Q}_{> 0} \times ((1, 0), (0, 1))$, it is equal to $\{((0, 0), (0, 0))\}$ if $\gamma = ((0, 0), (0, 0))$ and it is empty otherwise.

Lemma 5.15. *We have $\gamma \in R_\gamma$ for every $\gamma \in \mathbb{Q}_{\geq 0}^d \curvearrowright \mathbf{c}$.*

Proof. Assume that the couple $\gamma = (\mathbf{x}, \mathbf{y})$ is in $\mathbb{Q}_{\geq 0} \stackrel{\mathbf{c}}{\sim}$. For every transition $t \in T_\gamma$ there exist a run $\rho_t \in \Omega_\gamma$ and a factor $\mathbf{q}_t \mathbf{q}'_t$ of ρ_t such that $t = (\pi_\gamma(\mathbf{q}), \mathbf{q}' - \mathbf{q}, \pi_\gamma(\mathbf{q}'))$. Since $\rho_t \in \Omega_\gamma$, there exists $n_t \in \mathbb{Q}_{\geq 0}$ such that ρ_t is a run from $\mathbf{c} + n_t \mathbf{x}$ to $\mathbf{c} + n_t \mathbf{y}$ labeled by a word w_t . Assume that $T_\gamma = \{t_1, \dots, t_k\}$. As $\gamma \in \mathbb{Q}_{\geq 0} \stackrel{\mathbf{c}}{\sim}$, there exists $n \in \mathbb{N}_{>0}$ and a run from $\mathbf{c} + n\mathbf{x}$ to $\mathbf{c} + n\mathbf{y}$ labeled by a word w . Let us consider $m = n + \sum_{j=1}^k n_{t_j}$ and $\sigma = w w_{t_1} \dots w_{t_k}$. Observe that σ is the label of a cycle θ on $\pi_\gamma(\mathbf{c})$ which every transition t_j occurs in. Hence the Parikh image of this cycle provides a Kirchhoff function f for γ such that $\Delta(\sigma) = \Delta(f)$ and such that $f(t) > 0$ for every $t \in T$. Observe that $\Delta(\sigma) = m(\mathbf{y} - \mathbf{x})$. Hence $\mathbf{y} - \mathbf{x} = \Delta(\frac{1}{m}f)$ and we have proved that $\gamma \in R_\gamma$. \square

Lemma 5.16. *For every $(\mathbf{u}, \mathbf{v}) \in R_\gamma$ there exist $k \in \mathbb{N}_{>0}$, two cycles in G_γ on the state $\pi_\gamma(\mathbf{c})$ labeled by words $u, v \in \mathbf{A}^*$, and a vector $\mathbf{h} \in \mathbb{N}^d$ satisfying $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$ such that:*

$$\mathbf{c} + k\mathbf{u} \xrightarrow{u} \mathbf{c} + \mathbf{h} \xrightarrow{v} \mathbf{c} + k\mathbf{v}$$

Proof. Lemma 5.10 shows that there exist $\mathbf{h}' \in \mathbb{N}^d$ such that $I_\gamma = \{i \mid \mathbf{h}'(i) = 0\}$, $n \in \mathbb{Q}_{\geq 0}$ and two words $w_1, w_2 \in \mathbf{A}^*$ such that $\mathbf{c} + n\mathbf{x} \xrightarrow{w_1} \mathbf{c} + \mathbf{h}' \xrightarrow{w_2} \mathbf{c} + n\mathbf{y}$. Hence there exist cycles θ_1, θ_2 on $\pi_\gamma(\mathbf{c})$ labeled by w_1, w_2 . We denote by f_1 and f_2 the Parikh images of these two cycles. Let $(\mathbf{u}, \mathbf{v}) \in R_\gamma$. By replacing (\mathbf{u}, \mathbf{v}) by a couple in $\mathbb{N}_{>0}(\mathbf{u}, \mathbf{v})$ we can assume without loss of generality that $\mathbf{u}' = \mathbf{u} - n\mathbf{x}$ and $\mathbf{v}' = \mathbf{v} - n\mathbf{y}$ are both in \mathbb{N}^d , and there exists a Kirchhoff function f such that $f(t) \in \mathbb{N}_{>0}$ and $f(t) > f_1(t) + f_2(t)$ for every $t \in T_\gamma$, and such that $\mathbf{v} - \mathbf{u} = \Delta(f)$. Since $g = f - (f_1 + f_2)$ is a Kirchhoff function satisfying $g(t) \in \mathbb{N}_{>0}$ for every $t \in T_\gamma$ and G_γ is strongly connected, the Euler's Lemma shows that there exists a cycle in G_γ on the state $\pi_\gamma(\mathbf{c})$ labeled by a word σ such that $\Delta(\sigma) = \Delta(g)$. Since $\mathbf{c} + n\mathbf{x} \xrightarrow{w_1} \mathbf{c} + \mathbf{h}' \xrightarrow{w_2} \mathbf{c} + n\mathbf{y}$ and $n\mathbf{x} \leq \mathbf{u}, n\mathbf{y} \leq \mathbf{v}$ we deduce by monotony that for every $k \in \mathbb{N}$ we have:

$$\mathbf{c} + k\mathbf{u} \xrightarrow{w_1^k} \mathbf{c} + k(\mathbf{h}' + \mathbf{u}') \quad \mathbf{c} + k(\mathbf{h}' + \mathbf{v}') \xrightarrow{w_2^k} \mathbf{c} + k\mathbf{v}$$

Since there exists a cycle in G_γ on $\pi_\gamma(\mathbf{c})$ labeled by σ and $I_\gamma = \{i \mid \mathbf{h}'(i) = 0\}$, there exists $k \in \mathbb{N}_{>0}$ large enough such there exists a run from $\mathbf{c} + k\mathbf{h}'$ labeled by σ . Let us consider $r \in \{0, \dots, k\}$ and let us introduce $\mathbf{z}_r = (k - r)\mathbf{u}' + r\mathbf{v}'$. Note that $\mathbf{z}_r \in \mathbb{N}^d$. By monotony there exists a run from $\mathbf{c} + k\mathbf{h}' + \mathbf{z}_r$ labeled by σ . Note that $\Delta(\sigma) = \Delta(g)$ and from $g = f - (f_1 + f_2)$ we deduce that $\Delta(g) = \Delta(f) - (\Delta(f_1) + \Delta(f_2))$. Hence $\Delta(\sigma) = \mathbf{v} - \mathbf{u} - (\Delta(w_1) + \Delta(w_2)) = \mathbf{v} - \mathbf{u} - (\mathbf{h}' - n\mathbf{x} + n\mathbf{y} - \mathbf{h}') = \mathbf{v}' - \mathbf{u}'$. Therefore $\mathbf{z}_r + \Delta(\sigma) = \mathbf{z}_{r+1}$. We deduce that $\mathbf{c} + k\mathbf{h}' + \mathbf{z}_r \xrightarrow{\sigma} \mathbf{c} + k\mathbf{h}' + \mathbf{z}_{r+1}$. Therefore:

$$\mathbf{c} + k(\mathbf{h}' + \mathbf{u}') \xrightarrow{\sigma^k} \mathbf{c} + k(\mathbf{h}' + \mathbf{v}')$$

We have proved the lemma with $u = w_1^k, v = \sigma^k w_2^k$ and $\mathbf{h} = k(\mathbf{h}' + \mathbf{u}')$. \square

Corollary 5.17. *Transformer relations are asymptotically definable periodic relations.*

Proof. Lemma 5.15 and Lemma 5.16 show that we have the following equality:

$$\mathbb{Q}_{\geq 0} \stackrel{\mathbf{c}}{\sim} = \bigcup_{\gamma \in \mathbb{Q}_{\geq 0} \stackrel{\mathbf{c}}{\sim}} R_\gamma$$

Since the class of graphs G_γ is finite we deduce that the class of relations R_γ is finite. As these relations are definable in $\text{FO}(\mathbb{Q}, +, \leq)$ the corollary is proved. \square

6 Reachability Sets Decomposition

In this section $\text{post}^*(\mathbf{X}) \cap \mathbf{Y}$ and $\text{pre}^*(\mathbf{Y}) \cap \mathbf{X}$ are proved to be almost semilinear for every Presburger sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$. Since Presburger sets are finite unions of linear sets, we can assume that $\mathbf{X} = \mathbf{u} + \mathbf{M}$ and $\mathbf{Y} = \mathbf{v} + \mathbf{N}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{N}^d$ and $\mathbf{M}, \mathbf{N} \subseteq \mathbb{N}^d$ are finitely generated periodic sets. We introduce the set Ω of runs from a configuration $\mathbf{x} \in \mathbf{X}$ to a configuration $\mathbf{y} \in \mathbf{Y}$. We consider the order \sqsubseteq over the runs in Ω defined by $\rho \sqsubseteq \rho'$ if $(\text{src}(\rho'), \text{tgt}(\rho')) \in (\text{src}(\rho), \text{tgt}(\rho)) + \mathbf{M} \times \mathbf{N}$ and $\rho \preceq \rho'$. Since \mathbf{M} and \mathbf{N} are finitely generated the Dickson's lemma shows that \sqsubseteq is a well-order. We introduce the following sets for every run $\rho \in \Omega$:

$$\begin{aligned} \mathbf{M}_\rho &= \{\mathbf{m} \in \mathbf{M} \mid \exists \mathbf{n} \in \mathbf{N} \mathbf{m} \xrightarrow{\rho} \mathbf{n}\} \\ \mathbf{N}_\rho &= \{\mathbf{n} \in \mathbf{N} \mid \exists \mathbf{m} \in \mathbf{M} \mathbf{m} \xrightarrow{\rho} \mathbf{n}\} \end{aligned}$$

Lemma 6.1. *The sets \mathbf{M}_ρ and \mathbf{N}_ρ are periodic and satisfy:*

$$\begin{aligned} \mathbb{Q}_{\geq 0} \mathbf{M}_\rho &= \{\mathbf{m} \in \mathbb{Q}_{\geq 0} \mathbf{M} \mid \exists \mathbf{n} \in \mathbb{Q}_{\geq 0} \mathbf{N} (\mathbf{m}, \mathbf{n}) \in \mathbb{Q}_{\geq 0} \xrightarrow{\rho}\} \\ \mathbb{Q}_{\geq 0} \mathbf{N}_\rho &= \{\mathbf{n} \in \mathbb{Q}_{\geq 0} \mathbf{N} \mid \exists \mathbf{m} \in \mathbb{Q}_{\geq 0} \mathbf{M} (\mathbf{m}, \mathbf{n}) \in \mathbb{Q}_{\geq 0} \xrightarrow{\rho}\} \end{aligned}$$

Proof. Since \mathbf{M}, \mathbf{N} and $\xrightarrow{\rho}$ are periodic we deduce that \mathbf{M}_ρ and \mathbf{N}_ρ are periodic. The two equalities are immediate. \square

Lemma 6.2. *We have:*

$$\begin{aligned} \text{post}^*(\mathbf{X}) \cap \mathbf{Y} &= \bigcup_{\rho \in \min_{\sqsubseteq}(\Omega)} \text{tgt}(\rho) + \mathbf{N}_\rho \\ \text{pre}^*(\mathbf{Y}) \cap \mathbf{X} &= \bigcup_{\rho \in \min_{\sqsubseteq}(\Omega)} \text{src}(\rho) + \mathbf{M}_\rho \end{aligned}$$

Proof. Let us first prove that $\text{tgt}(\rho) + \mathbf{N}_\rho \subseteq \text{post}^*(\mathbf{X}) \cap \mathbf{Y}$ for every $\rho \in \Omega$. Assume that $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ with $\mathbf{c}_j \in \mathbb{N}^d$ and let $\mathbf{n} \in \mathbf{N}_\rho$. As $\rho \in \Omega$ we deduce that $\mathbf{c}_0 \in \mathbf{X}$ and $\mathbf{c}_k \in \mathbf{Y}$. Moreover, as $\mathbf{n} \in \mathbf{N}_\rho$ there exists $\mathbf{m} \in \mathbf{M}$ such that $\mathbf{m} \xrightarrow{\rho} \mathbf{n}$. Hence there exists a sequence $(\mathbf{v}_j)_{0 \leq j \leq k+1}$ of vectors $\mathbf{v}_j \in \mathbb{N}^d$ such that $\mathbf{v}_0 = \mathbf{m}$, $\mathbf{v}_{k+1} = \mathbf{n}$ and such that $\mathbf{v}_j \xrightarrow{\mathbf{c}_j} \mathbf{v}_{j+1}$ for every $j \in \{0, \dots, k\}$. In particular there exists a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$ labeled by a word $w_j \in \mathbf{A}^*$. Now just observe that we have a run from $\mathbf{c}_0 + \mathbf{v}_0$ to $\mathbf{c}_k + \mathbf{v}_{k+1}$ labeled by $w_0 \mathbf{a}_1 w_1 \dots \mathbf{a}_k w_k$ where $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$. Since $\mathbf{c}_0 \in \mathbf{X}$, $\mathbf{X} = \mathbf{u} + \mathbf{M}$ and $\mathbf{v}_0 \in \mathbf{M}$ we deduce that $\mathbf{c}_0 + \mathbf{v}_0 \in \mathbf{X}$. Symmetrically we have $\mathbf{c}_k + \mathbf{v}_{k+1} \in \mathbf{Y}$. Hence $\text{tgt}(\rho) + \mathbf{n} \in \text{post}^*(\mathbf{X}) \cap \mathbf{Y}$.

Now, let us prove that for every $\mathbf{y} \in \text{post}^*(\mathbf{X}) \cap \mathbf{Y}$ there exists $\rho \in \min_{\sqsubseteq}(\Omega)$ such that $\mathbf{y} \in \text{tgt}(\rho) + \mathbf{N}_\rho$. There exists a run ρ' from a configuration $\mathbf{x} \in \mathbf{X}$ to

y. Since \sqsubseteq is a well-order, there exists a run $\rho \in \min_{\sqsubseteq}(\Omega)$ such that $\rho \sqsubseteq \rho'$. By definition of \sqsubseteq we deduce that $\text{tgt}(\rho') \in \text{tgt}(\rho) + \mathbf{N}_\rho$. Hence, we have proved the equality $\text{post}^*(\mathbf{X}) \cap \mathbf{Y} = \bigcup_{\rho \in \min_{\sqsubseteq}(\Omega)} \text{tgt}(\rho) + \mathbf{N}_\rho$. The second equality is obtained symmetrically. \square

We deduce the following corollary.

Corollary 6.3. *The sets $\text{post}^*(\mathbf{X}) \cap \mathbf{Y}$ and $\text{pre}^*(\mathbf{Y}) \cap \mathbf{X}$ are almost semilinear for every Presburger sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$.*

Proof. Corollary 5.17 shows that $\stackrel{\rho}{\sim}$ is an asymptotically definable periodic relation. From Lemma 6.1 we deduce that \mathbf{M}_ρ and \mathbf{N}_ρ are asymptotically definable periodic sets. From Lemma 6.2 we deduce that $\text{post}^*(\mathbf{X}) \cap \mathbf{Y}$ and $\text{pre}^*(\mathbf{Y}) \cap \mathbf{X}$ are almost semilinear. \square

7 Dimension

In this section we introduce a dimension function for the subsets of \mathbb{Z}^d and we characterize the dimension of periodic sets.

A *vector space* is a set $\mathbf{V} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{V}$, $\mathbf{V} + \mathbf{V} \subseteq \mathbf{V}$ and such that $\mathbb{Q}\mathbf{V} \subseteq \mathbf{V}$. Let $\mathbf{X} \subseteq \mathbb{Q}^d$. The following set \mathbf{V} is a vector space called the *vector space generated by \mathbf{X}* .

$$\mathbf{V} = \left\{ \sum_{j=1}^k \lambda_j \mathbf{x}_j \mid k \in \mathbb{N} \text{ and } (\lambda_j, \mathbf{x}_j) \in \mathbb{Q} \times \mathbf{X} \right\}$$

This vector space is the minimal for the inclusion among the vector spaces that contain \mathbf{X} . Let us recall that every vector space \mathbf{V} is generated by a finite set. The *rank* $\text{rank}(\mathbf{V})$ of a vector space \mathbf{V} is the minimal natural number $r \in \mathbb{N}$ such that there exists a finite set \mathbf{X} with r vectors that generates \mathbf{V} . Let us recall that $\text{rank}(\mathbf{V}) \leq d$ for every vector space $\mathbf{V} \subseteq \mathbb{Q}^d$ and $\text{rank}(\mathbf{V}) \leq \text{rank}(\mathbf{W})$ for every pair of vector spaces $\mathbf{V} \subseteq \mathbf{W}$. Moreover, if \mathbf{V} is strictly included in \mathbf{W} then $\text{rank}(\mathbf{V}) < \text{rank}(\mathbf{W})$.

Example 7.1. Vector spaces \mathbf{V} included in \mathbb{Q}^2 satisfy $\text{rank}(\mathbf{V}) \in \{0, 1, 2\}$. Moreover these vectors spaces can be classified as follow : $\text{rank}(\mathbf{V}) = 0$ if and only if $\mathbf{V} = \{\mathbf{0}\}$, $\text{rank}(\mathbf{V}) = 1$ if and only if $\mathbf{V} = \mathbb{Q}\mathbf{v}$ with $\mathbf{v} \in \mathbb{Q}^2 \setminus \{\mathbf{0}\}$, and $\text{rank}(\mathbf{V}) = 2$ if and only if $\mathbf{V} = \mathbb{Q}^2$.

The *dimension* of a set $\mathbf{X} \subseteq \mathbb{Z}^d$ is the minimal integer $r \in \{-1, \dots, d\}$ such that $\mathbf{X} \subseteq \bigcup_{j=1}^k \mathbf{b}_j + \mathbf{V}_j$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space satisfying $\text{rank}(\mathbf{V}_j) \leq r$ for every j . We denote by $\dim(\mathbf{X})$ the dimension of \mathbf{X} . Observe that $\dim(\mathbf{v} + \mathbf{X}) = \dim(\mathbf{X})$ for every $\mathbf{X} \subseteq \mathbb{Z}^d$ and for every $\mathbf{v} \in \mathbb{Z}^d$. Sets with a dimension equal to -1 or 0 are characterized as follow. We have $\dim(\mathbf{X}) = -1$ if and only if \mathbf{X} is empty and $\dim(\mathbf{X}) = 0$ if and only if \mathbf{X} is a non-empty finite set. Note that $\dim(\mathbf{X} \cup \mathbf{Y}) = \max\{\dim(\mathbf{X}), \dim(\mathbf{Y})\}$ for every subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Z}^d$.

Lemma 7.2. *Let $\mathbf{P} \subseteq \mathbb{Z}^d$ be a periodic set included in $\bigcup_{j=1}^k \mathbf{b}_j + \mathbf{V}_j$ where $k \in \mathbb{N}_{>0}$, $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space. There exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$ and $\mathbf{b}_j \in \mathbf{V}_j$.*

Proof. Let us first prove by induction over $k \in \mathbb{N}_{>0}$ that for every periodic set $\mathbf{P} \subseteq \mathbb{Z}^d$ included in $\bigcup_{j=1}^k \mathbf{V}_j$ where $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space, there exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. The rank $k = 1$ is immediate. Let us prove the rank $k + 1$ and assume that \mathbf{P} is included in $\bigcup_{j=1}^{k+1} \mathbf{V}_j$. If $\mathbf{P} \subseteq \mathbf{V}_{k+1}$ the induction is proved. So we can assume that there exists $\mathbf{p} \in \mathbf{P} \setminus \mathbf{V}_{k+1}$. Let $\mathbf{x} \in \mathbf{P}$. Since $\mathbf{p} + n\mathbf{x} \in \mathbf{P}$ for every $n \in \mathbb{N}$ there exists $j \in \{1, \dots, k+1\}$ such that $n\mathbf{p} + \mathbf{x} \in \mathbf{V}_j$. As $\{1, \dots, k+1\}$ is finite, there exists j in this set and $n < n'$ such that $n\mathbf{p} + \mathbf{x}$ and $n'\mathbf{p} + \mathbf{x}$ are both in \mathbf{V}_j . In particular the difference of this two vectors is in \mathbf{V}_j . Since this difference is $(n' - n)\mathbf{p}$ and $\mathbf{p} \notin \mathbf{V}_{k+1}$ we get $j \in \{1, \dots, k\}$. Observe that $n(n'\mathbf{p} + \mathbf{x}) - n'(n\mathbf{p} + \mathbf{x})$ is the difference of two vectors in \mathbf{V}_j . Thus this vector is in \mathbf{V}_j and we deduce that $\mathbf{x} \in \mathbf{V}_j$. We have shown that $\mathbf{P} \subseteq \bigcup_{j=1}^k \mathbf{V}_j$. By induction there exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. We have proved the induction.

Finally, assume that $\mathbf{P} \subseteq \mathbb{Z}^d$ is a periodic set included in $\bigcup_{j=1}^k \mathbf{b}_j + \mathbf{V}_j$ where $k \in \mathbb{N}_{>0}$, $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space. Let J be the set of $j \in \{1, \dots, k\}$ such that $\mathbf{b}_j \in \mathbf{V}_j$ and let us prove that $\mathbf{P} \subseteq \bigcup_{j \in J} \mathbf{V}_j$. Let $\mathbf{p} \in \mathbf{P}$. Since $n\mathbf{p} \in \mathbf{P}$ for every $n \in \mathbb{N}$, there exists $j \in \{1, \dots, k\}$ and $n < n'$ such that $n\mathbf{p}$ and $n'\mathbf{p}$ are both in $\mathbf{b}_j + \mathbf{V}_j$. The difference of these two vectors shows that $(n' - n)\mathbf{p}$ is in \mathbf{V}_j . Form $\mathbf{b}_j \in n\mathbf{p} - \mathbf{V}_j \subseteq \mathbf{V}_j$ we deduce that $j \in J$. Thus $\mathbf{P} \subseteq \bigcup_{j \in J} \mathbf{V}_j$. From the previous paragraph we deduce that there exists $j \in J$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. \square

Lemma 7.3. *We have $\dim(\mathbf{P}) = \text{rank}(\mathbf{V})$ for every periodic set \mathbf{P} where \mathbf{V} is the vector space generated by \mathbf{P} .*

Proof. Since $\mathbf{P} \subseteq \mathbf{V}$ we deduce that $\dim(\mathbf{P}) \leq \text{rank}(\mathbf{V})$. For the converse inequality, there exist $k \in \mathbb{N}$, $(\mathbf{b}_j)_{1 \leq j \leq k}$ a sequence of vectors $\mathbf{b}_j \in \mathbb{Z}^d$ and a sequence $(\mathbf{V}_j)_{1 \leq j \leq k}$ of vector spaces $\mathbf{V}_j \subseteq \mathbb{Q}^d$ such that $\mathbf{P} \subseteq \bigcup_{j=1}^k \mathbf{b}_j + \mathbf{V}_j$ and such that $\text{rank}(\mathbf{V}_j) \leq \dim(\mathbf{P})$ for every j . Since \mathbf{P} is non empty we deduce that $k \in \mathbb{N}_{>0}$. Lemma 7.2 proves that there exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$ and $\mathbf{b}_j \in \mathbf{V}_j$. By minimality of the vector space generated by \mathbf{P} we get $\mathbf{V} \subseteq \mathbf{V}_j$. Hence $\text{rank}(\mathbf{V}) \leq \text{rank}(\mathbf{V}_j)$. From $\text{rank}(\mathbf{V}_j) \leq \dim(\mathbf{P})$ we get $\text{rank}(\mathbf{V}) \leq \dim(\mathbf{P})$. \square

8 Linearizations

A *linearization* of an almost semilinear set \mathbf{X} is a set of the form $\bigcup_{j=1}^k \mathbf{b}_j + (\mathbf{P}_j - \mathbf{P}_j) \cap \mathbb{Q}_{\geq 0} \mathbf{P}_j$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{P}_j \subseteq \mathbb{Z}^d$ is an asymptotically definable periodic set such that $\mathbf{X} = \bigcup_{j=1}^k \mathbf{b}_j + \mathbf{P}_j$. Let us recall that every subgroup of $(\mathbb{Z}^d, +)$ is finitely generated [14]. Moreover, since $\text{FO}(\mathbb{Q}, +, \leq, 0)$ admits a quantifier elimination algorithm, we deduce that linearizations are definable in the Presburger arithmetic. In this section we show that if $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Z}^d$ are two non-empty almost semilinear sets with an empty intersection then every linearizations \mathbf{S}, \mathbf{T} of \mathbf{X}, \mathbf{Y} satisfy the following strict inequality:

$$\dim(\mathbf{S} \cap \mathbf{T}) < \dim(\mathbf{X} \cup \mathbf{Y})$$

Lemma 8.1. *Let \mathbf{P} be a periodic set and \mathbf{V} be the vector space generated by \mathbf{P} . Assume that $\mathbf{b} + \mathbb{N}\mathbf{m}_1 + \dots + \mathbb{N}\mathbf{m}_k \subseteq (\mathbf{P} - \mathbf{P}) \cap \mathbb{Q}_{\geq 0}\mathbf{P}$ where $\mathbf{b} \in \mathbb{Z}^d$ and $(\mathbf{m}_j)_{1 \leq j \leq k} \in \mathbb{Z}^d$ is a sequence of vectors that generated \mathbf{V} . There exists $k \in \mathbb{N}_{>0}$ such that $\mathbf{b} + k\mathbb{N}_{>0}\mathbf{a} \subseteq \mathbf{P}$ where $\mathbf{a} = \mathbf{m}_1 + \dots + \mathbf{m}_k$.*

Proof. Since $\mathbf{b} \in \mathbf{P} - \mathbf{P}$ there exists $\mathbf{p}_+, \mathbf{p}_- \in \mathbf{P}$ such that $\mathbf{b} = \mathbf{p}_+ - \mathbf{p}_-$. As $-\mathbf{p}_+ \in \mathbf{V}$ then $-\mathbf{p}_+ \in \sum_{j=1}^k \mathbb{Q}\mathbf{m}_j$. Hence there exists $r \in \mathbb{N}_{>0}$ such that $-\mathbf{p}_+ \in \sum_{j=1}^k \mathbb{Z}\mathbf{m}_j$. By definition of \mathbf{a} , there exists $n \in \mathbb{N}_{>0}$ such that $-\mathbf{p}_+ + n\mathbf{a} \in \sum_{j=1}^k \mathbb{N}\mathbf{m}_j$. Hence $\mathbf{b} - r\mathbf{p}_+ + n\mathbf{a} \in \mathbf{b} + \sum_{j=1}^k \mathbb{N}\mathbf{m}_j$. Since this set is included in $\mathbb{Q}_{\geq 0}\mathbf{P}$ and $(r-1)\mathbf{p}_+ \in \mathbf{P}$ we deduce that $-\mathbf{p}_- + n\mathbf{a} = (\mathbf{b} - r\mathbf{p}_+ + n\mathbf{a}) + (r-1)\mathbf{p}_+$ is in $\mathbb{Q}_{\geq 0}\mathbf{P}$. Hence, there exists $s \in \mathbb{N}_{>0}$ such that $s(-\mathbf{p}_- + n\mathbf{a}) \in \mathbf{P}$. Let $k = sn$ and observe that $-\mathbf{p}_- + k\mathbf{a} = s(-\mathbf{p}_- + n\mathbf{a}) + (s-1)\mathbf{p}_-$. Hence $-\mathbf{p}_- + k\mathbf{a} \in \mathbf{P}$. Since $\mathbf{b} + k\mathbf{a} = (-\mathbf{p}_- + k\mathbf{a}) + \mathbf{p}_+$ and $k\mathbf{a} = (-\mathbf{p}_- + k\mathbf{a}) + \mathbf{p}_-$ we deduce that $\mathbf{b} + k\mathbf{a}$ and $k\mathbf{a}$ are both in \mathbf{P} . In particular $\mathbf{b} + k\mathbb{N}_{>0}\mathbf{a} \subseteq \mathbf{P}$. \square

Corollary 8.2. *Let $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Z}^d$ be two non-empty almost semilinear sets with an empty intersection. For every linearizations \mathbf{S}, \mathbf{T} of \mathbf{X}, \mathbf{Y} we have:*

$$\dim(\mathbf{S} \cap \mathbf{T}) < \dim(\mathbf{X} \cup \mathbf{Y})$$

Proof. We can assume that $\mathbf{X} = \mathbf{u} + \mathbf{P}, \mathbf{Y} = \mathbf{v} + \mathbf{Q}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ and $\mathbf{P}, \mathbf{Q} \subseteq \mathbb{Z}^d$ are two asymptotically definable periodic sets such that $\mathbf{X} \cap \mathbf{Y} = \emptyset$ and we can assume that $\mathbf{S} = \mathbf{u} + \mathbf{P}'$ where $\mathbf{P}' = (\mathbf{P} - \mathbf{P}) \cap \mathbb{Q}_{\geq 0}\mathbf{P}$ and $\mathbf{T} = \mathbf{v} + \mathbf{Q}'$ where $\mathbf{Q}' = (\mathbf{Q} - \mathbf{Q}) \cap \mathbb{Q}_{\geq 0}\mathbf{Q}$. Let \mathbf{U} and \mathbf{V} be the vector spaces generated by \mathbf{P} and \mathbf{Q} . Lemma 7.3 shows that $\dim(\mathbf{X}) = \text{rank}(\mathbf{U})$ and $\dim(\mathbf{Y}) = \text{rank}(\mathbf{V})$. Note that $\mathbf{S} \cap \mathbf{T}$ is a Presburger set and in particular a finite union of linear sets. If this set is empty the corollary is proved. Otherwise there exists $\mathbf{b} \in \mathbb{Z}^d$ and a finitely generated periodic set $\mathbf{M} \subseteq \mathbb{Z}^d$ such that $\mathbf{b} + \mathbf{M} \subseteq \mathbf{S} \cap \mathbf{T}$ and such that $\dim(\mathbf{S} \cap \mathbf{T}) = \dim(\mathbf{b} + \mathbf{M})$. Let \mathbf{W} be the vector space generated by \mathbf{M} . Observe that $\mathbf{b} + \mathbf{M} \subseteq (\mathbf{u} + \mathbf{U}) \cap (\mathbf{v} + \mathbf{V})$. Hence for every $\mathbf{m} \in \mathbf{M}$ since $\mathbf{b} + \mathbf{m} - \mathbf{u}$ and $\mathbf{b} + 2\mathbf{m} - \mathbf{u}$ are both in \mathbf{U} the difference is also in \mathbf{U} . Hence $\mathbf{m} \in \mathbf{U}$. We deduce that $\mathbf{M} \subseteq \mathbf{U}$ and symmetrically $\mathbf{M} \subseteq \mathbf{V}$. As \mathbf{M} is included in the vector space $\mathbf{U} \cap \mathbf{V}$, by minimality of \mathbf{W} , we get $\mathbf{W} \subseteq \mathbf{U} \cap \mathbf{V}$. Assume by contradiction that $\mathbf{U} = \mathbf{W} = \mathbf{V}$. Since \mathbf{M} is finitely generated, there exists a sequence $(\mathbf{m}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{m}_j \in \mathbf{M}$ such that $\mathbf{M} = \mathbb{N}\mathbf{m}_1 + \dots + \mathbb{N}\mathbf{m}_k$. Let $\mathbf{a} = \mathbf{m}_1 + \dots + \mathbf{m}_k$. From $\mathbf{b} - \mathbf{u} + \mathbf{M} \subseteq (\mathbf{P} - \mathbf{P}) \cap \mathbb{Q}_{\geq 0}\mathbf{P}$ and Lemma 8.1 we deduce that there exists $k \in \mathbb{N}_{>0}$ such that $\mathbf{b} - \mathbf{u} + k\mathbb{N}_{>0}\mathbf{a} \subseteq \mathbf{P}$. From $\mathbf{b} - \mathbf{v} + \mathbf{M} \subseteq (\mathbf{Q} - \mathbf{Q}) \cap \mathbb{Q}_{\geq 0}\mathbf{Q}$ and Lemma 8.1 we deduce that there exists $k' \in \mathbb{N}_{>0}$ such that $\mathbf{b} - \mathbf{v} + k'\mathbb{N}_{>0}\mathbf{a} \subseteq \mathbf{Q}$. In particular $\mathbf{b} + k k' \mathbf{a} \in (\mathbf{u} + \mathbf{P}) \cap (\mathbf{v} + \mathbf{Q})$ and we get a contradiction. Hence, \mathbf{W} is strictly included in \mathbf{U} or in \mathbf{V} . Hence $\text{rank}(\mathbf{W}) < \max\{\text{rank}(\mathbf{U}), \text{rank}(\mathbf{V})\} = \dim(\mathbf{X} \cup \mathbf{Y})$. From Lemma 7.3 we get $\dim(\mathbf{M}) = \text{rank}(\mathbf{W})$ and since $\dim(\mathbf{M}) = \dim(\mathbf{S} \cap \mathbf{T})$ the corollary is proved. \square

9 Presburger Invariants

We introduce the notion of *separators*. A *separator* is a couple (\mathbf{X}, \mathbf{Y}) of Presburger sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$ such that there does not exist a run from a configuration in \mathbf{X} to a

configuration in \mathbf{Y} . In particular $\mathbf{X} \cap \mathbf{Y} = \emptyset$. The Presburger set $\mathbf{D} = \mathbb{N}^d \setminus (\mathbf{X} \cup \mathbf{Y})$ is called the *domain* of (\mathbf{X}, \mathbf{Y}) . We observe that a separator (\mathbf{X}, \mathbf{Y}) with an empty domain is a partition of \mathbb{N}^d such that \mathbf{X} is a Presburger forward inductive invariant and \mathbf{Y} is a Presburger backward inductive invariant.

Lemma 9.1. *Let $(\mathbf{X}_0, \mathbf{Y}_0)$ be a separator with a non-empty domain \mathbf{D}_0 . There exists a separator (\mathbf{X}, \mathbf{Y}) with a domain \mathbf{D} such that $\mathbf{X}_0 \subseteq \mathbf{X}$, $\mathbf{Y}_0 \subseteq \mathbf{Y}$ and $\dim(\mathbf{D}) < \dim(\mathbf{D}_0)$.*

Proof. We first observe that a couple (\mathbf{X}, \mathbf{Y}) of Presburger sets is a separator if and only if $\text{post}^*(\mathbf{X}) \cap \text{pre}^*(\mathbf{Y}) = \emptyset$ if and only if $\text{post}^*(\mathbf{X}) \cap \mathbf{Y} = \emptyset$ if and only if $\text{pre}^*(\mathbf{Y}) \cap \mathbf{X} = \emptyset$.

As $\mathbf{X}_0, \mathbf{D}_0$ are Presburger sets, Corollary 6.3 shows that $\mathbf{H} = \text{post}^*(\mathbf{X}_0) \cap \mathbf{D}_0$ is an almost semilinear set. We introduce a linearization \mathbf{S} of this set. Since $\text{post}^*(\mathbf{X}_0) \cap \mathbf{D}_0 \subseteq \mathbf{S}$, we deduce that the set $\mathbf{Y} = \mathbf{Y}_0 \cup (\mathbf{D}_0 \setminus \mathbf{S})$ is such that $\text{post}^*(\mathbf{X}_0) \cap \mathbf{Y} = \emptyset$. Hence $(\mathbf{X}_0, \mathbf{Y})$ is a separator. Symmetrically, as \mathbf{D}_0, \mathbf{Y} are Presburger sets, Corollary 6.3 shows that $\mathbf{K} = \text{pre}^*(\mathbf{Y}) \cap \mathbf{D}_0$ is an almost semilinear set. We introduce a linearization \mathbf{T} of this set. Since $\text{pre}^*(\mathbf{Y}) \cap \mathbf{D}_0 \subseteq \mathbf{T}$, we deduce that the set $\mathbf{X} = \mathbf{X}_0 \cup (\mathbf{D}_0 \setminus \mathbf{T})$ is such that $\text{pre}^*(\mathbf{Y}) \cap \mathbf{X} = \emptyset$. Hence (\mathbf{X}, \mathbf{Y}) is a separator.

Let us introduce the domain \mathbf{D} of (\mathbf{X}, \mathbf{Y}) and observe that $\mathbf{D} = \mathbf{D}_0 \cap \mathbf{S} \cap \mathbf{T}$. If \mathbf{H} or \mathbf{K} is empty then \mathbf{S} or \mathbf{T} is empty and in particular \mathbf{D} is empty and the lemma is proved. So we can assume that \mathbf{H} and \mathbf{K} are non empty. Since $\mathbf{H} \subseteq \text{post}^*(\mathbf{X}_0) \subseteq \text{post}^*(\mathbf{X})$ and $\mathbf{K} \subseteq \text{pre}^*(\mathbf{Y})$ and (\mathbf{X}, \mathbf{Y}) is a separator, we deduce that $\mathbf{H} \cap \mathbf{K} = \emptyset$. Moreover as $\mathbf{H}, \mathbf{K} \subseteq \mathbf{D}_0$ we deduce that $\dim(\mathbf{H} \cup \mathbf{K}) \leq \dim(\mathbf{D}_0)$. As \mathbf{S} and \mathbf{T} are linearizations of the non-empty almost semilinear sets \mathbf{H}, \mathbf{K} and $\mathbf{H} \cap \mathbf{K} = \emptyset$, Corollary 8.2 shows that $\dim(\mathbf{S} \cap \mathbf{T}) < \dim(\mathbf{H} \cup \mathbf{K})$. Therefore $\dim(\mathbf{D}) < \dim(\mathbf{D}_0)$. \square

We deduce the main theorem of this paper.

Theorem 9.2. *For every $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$ such that there does not exist a run from \mathbf{x} to \mathbf{y} , then there exists a partition of \mathbb{N}^d into (\mathbf{X}, \mathbf{Y}) such that \mathbf{X} is a Presburger forward inductive invariant that contains \mathbf{x} and \mathbf{Y} is a Presburger backward inductive invariant that contains \mathbf{y} .*

Proof. Observe that $(\{\mathbf{x}\}, \{\mathbf{y}\})$ is a separator. Thanks to Lemma 9.1 with an immediate induction over the dimension of the domains we deduce that there exists a separator (\mathbf{X}, \mathbf{Y}) with an empty domain such that $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{Y}$. As the domain is empty we deduce that \mathbf{X} is a forward inductive invariant and \mathbf{Y} is a backward inductive invariant. \square

10 Conclusion

The reachability problem for vector addition systems can be solved with a simple algorithm based on inductive invariants definable in the Presburger arithmetic. This algorithm does not require the classical KLMST decomposition. Note however that the

complexity of this algorithm is still open. In fact, the complexity depends on the minimal length of a run from x to y when such a run exists, or the minimal length of a Presburger formula denoting a forward inductive invariant X such that $x \in X$ and $y \notin X$ when such a formula exists. We left as an open question the problem of computing lower and upper bounds for these lengths. Note that the VAS exhibiting a large (Ackermann size) but finite reachability set given in [12] does not directly provide an Ackermann lower-bound for these sizes since Presburger forward invariants can over-approximate reachability sets. Note that the existence of a primitive recursive upper bound of complexity for the reachability problem is still open since the Zakaria Bouziane's paper[1] introducing such a bound was proved to be incorrect by Petr Jančar[6].

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