



HAL
open science

Maximal analytic extensions of the Emparan-Reall black ring

Piotr T. Chrusciel, Julien Cortier

► **To cite this version:**

Piotr T. Chrusciel, Julien Cortier. Maximal analytic extensions of the Emparan-Reall black ring. *Journal of Differential Geometry*, 2010, 85 (3), pp.425-459. 10.4310/jdg/1292940690 . hal-00674854

HAL Id: hal-00674854

<https://hal.science/hal-00674854>

Submitted on 28 Feb 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MAXIMAL ANALYTIC EXTENSIONS OF THE EMPARAN-REALL BLACK RING

PIOTR T. CHRUSCIEL & JULIEN CORTIER

Abstract

We construct a Kruskal-Szekeres-type analytic extension of the Emparan-Reall black ring, and investigate its geometry. We prove that the extension is maximal, globally hyperbolic, and unique within a natural class of extensions. The key to those results is the proof that causal geodesics are either complete, or approach a singular boundary in finite affine time. Alternative maximal analytic extensions are also constructed.

CONTENTS

1. Introduction	1
2. The Emparan-Reall space-time	2
3. The extension	3
4. Global structure	7
4.1. The event horizon has $S^2 \times S^1 \times \mathbb{R}$ topology	7
4.2. Inextendibility at $z = \xi_F$	8
4.3. Maximality	8
4.4. Global hyperbolicity	10
4.5. \mathcal{I}	12
4.6. Uniqueness?	13
5. Geodesics	18
5.1. Geodesics in the domain of outer communications away from the horizon	18
5.2. Geodesics in the region $\{\xi_F < z < \xi_3\}$	21
5.3. Geodesics in the region $\{\xi_3 < z \leq 2\xi_3\}$	28
5.4. The Killing horizon	30
References	30

1. Introduction

The Emparan-Reall [13] metrics form a remarkable class of vacuum black hole solutions of Einstein equations in dimension $4 + 1$. Some aspects of their

The authors are grateful to the Mittag-Leffler Institute, Djursholm, Sweden, for hospitality and financial support during a significant part of work on this paper.

global properties have been studied in [13], where it was shown that the solution contains a *Killing horizon* with $S^2 \times S^1 \times \mathbb{R}$ topology. The aim of this work is to point out that the *event horizon* coincides with the Killing horizon, and therefore also has this topology; and to construct an analytic extension with a bifurcate Killing horizon; and to establish some global properties of the extended space-time. The extension resembles closely the Kruskal-Szekeres extension of the Schwarzschild space-time, with a bifurcate Killing horizon, a black hole singularity, a white hole singularity, and two asymptotically flat regions. We show that causal geodesics in the extended space-time are either complete or reach a singularity in finite time. This implies maximality of our extension. We also show global hyperbolicity, present families of alternative extensions, establish uniqueness of our extension within a natural class, and verify existence of a conformal completion at null infinity.

2. The Emparan-Reall space-time

In local coordinates the Emparan-Reall metric can be written in the form

$$(2.1) \quad g = -\frac{F(x)}{F(z)} \left(dt + \sqrt{\frac{\nu}{\xi_F}} \frac{\xi_1 - z}{A} d\psi \right)^2 + \frac{F(z)}{A^2(x-z)^2} \times \left[-F(x) \left(\frac{dz^2}{G(z)} + \frac{G(z)}{F(z)} d\psi^2 \right) + F(z) \left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 \right) \right],$$

where $A > 0$, ν et ξ_F are constants, and

$$(2.2) \quad F(\xi) = 1 - \frac{\xi}{\xi_F},$$

$$(2.3) \quad G(\xi) = \nu\xi^3 - \xi^2 + 1 = \nu(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3),$$

are polynomials, with ν chosen so that $\xi_1 < 0 < \xi_2 < \xi_3$. The study of the coordinate singularities at $x = \xi_1$ and $x = \xi_2$ leads to the determination of ξ_F as:

$$(2.4) \quad \xi_F = \frac{\xi_1\xi_2 - \xi_3^2}{\xi_1 + \xi_2 - 2\xi_3} \in (\xi_2, \xi_3).$$

Emparan and Reall have established the asymptotically flat character of (2.1), as well as existence of an analytic extension across an analytic Killing horizon¹ at $z = \xi_3$. The extension given in [13] is somewhat similar of the extension of the Schwarzschild metric that one obtains by going to Eddington-Finkelstein coordinates, and is *not* maximal. Now, existence of an analytic extension with a bifurcate horizon, *à la* Kruskal-Szekeres, is guaranteed from this by an analytic version of the analysis of Rácz and Wald in [21]. But the global properties of an extension so constructed are not clear. It is therefore of interest to present an explicit extension with good properties. This extension is constructed in Section 3, and its global properties are studied in the remaining sections.

As in [13] we assume throughout that

$$(2.5) \quad \xi_1 \leq x \leq \xi_2.$$

¹We follow the terminology of [11].

As discussed in [13], the extremities correspond to a north and south pole of S^2 , with a function θ defined by $d\theta = dx/\sqrt{G(x)}$ providing a latitude on S^2 , except for the limit $x - z \rightarrow 0$, $x \rightarrow \xi_1$, which corresponds to an asymptotically flat region, see [13]; a detailed proof of asymptotic flatness can be found in [10]. The surface “ $\{z = \infty\}$ ” can be identified with “ $\{z = -\infty\}$ ” by introducing a coordinate $Y = -1/z$, with the metric extending analytically across $\{Y = 0\}$, see [13] for details.

We will denote by $(\mathcal{M}_{I \cup II}, g)$ the space-time constructed by Emparan and Reall, as outlined above, where the coordinate z runs then over $(\xi_F, \infty] \cup [-\infty, \xi_1]$. We will denote by (\mathcal{M}_I, g) the subset of $(\mathcal{M}_{I \cup II}, g)$ in which the coordinate z runs over $(\xi_3, \infty] \cup [-\infty, \xi_1]$; see Figure 3.1.

Strictly speaking, in the definitions of $(\mathcal{M}_{I \cup II}, g)$ and (\mathcal{M}_I, g) we should have used different symbols for the metric g ; we hope that this will not lead to confusions.

3. The extension

We start by working in the range $z \in (\xi_3, \infty)$; there we define new coordinates w, v by the formulae

$$(3.1) \quad dv = dt + \frac{bdz}{(z - \xi_3)(z - \xi_2)},$$

$$(3.2) \quad dw = dt - \frac{bdz}{(z - \xi_3)(z - \xi_2)},$$

where b is a constant to be chosen shortly. (Our coordinates v and w are closely related to, but not identical, to the coordinates v and w used in [13] when extending the metric through the Killing horizon $z = \xi_3$). Similarly to the construction of the extension of the Kerr metric in [4, 5], we define a new angular coordinate $\hat{\psi}$ by:

$$(3.3) \quad d\hat{\psi} = d\psi - a dt,$$

where a is a constant to be chosen later. Let

$$(3.4) \quad \sigma := \frac{1}{A} \sqrt{\frac{\nu}{\xi_F}}.$$

Using (3.1)–(3.3), we obtain

$$(3.5) \quad dt = \frac{1}{2}(dv + dw),$$

$$(3.6) \quad dz = \underbrace{\frac{(z - \xi_3)(z - \xi_2)}{2b}}_{=: H(z)/2} (dv - dw),$$

$$(3.7) \quad d\psi = d\hat{\psi} + \frac{a}{2}(dv + dw),$$

which leads to

$$(3.8) \quad g_{vv} = g_{ww} = -\frac{F(x)}{4F(z)} \left(1 + a\sigma(\xi_1 - z)\right)^2 - \frac{F(x)F(z)}{4A^2(x - z)^2} \left(\frac{a^2 G(z)}{F(z)} + \frac{H^2(z)}{G(z)}\right),$$

$$(3.9) \quad g_{vw} = -\frac{F(x)}{4F(z)} \left(1 + a\sigma(\xi_1 - z)\right)^2 - \frac{F(x)F(z)}{4A^2(x-z)^2} \left(\frac{a^2G(z)}{F(z)} - \frac{H^2(z)}{G(z)}\right),$$

$$(3.10) \quad g_{v\hat{\psi}} = g_{w\hat{\psi}} = -\frac{F(x)}{2F(z)}\sigma(\xi_1 - z) \left(1 + a\sigma(\xi_1 - z)\right) - \frac{F(x)G(z)a}{2A^2(x-z)^2},$$

$$(3.11) \quad g_{\hat{\psi}\hat{\psi}} = -\frac{F(x)}{F(z)}\sigma^2(\xi_1 - z) - \frac{F(x)G(z)}{A^2(x-z)^2}.$$

The Jacobian of the coordinate transformation is

$$\frac{\partial(w, v, \hat{\psi}, x, \varphi)}{\partial(t, z, \psi, x, \varphi)} = 2 \frac{\partial v}{\partial z} = \frac{2b}{(z - \xi_2)(z - \xi_3)}.$$

In the original coordinates (t, z, ψ, x, φ) the determinant of g was

$$(3.12) \quad \det(g_{(t,z,\psi,x,\varphi)}) = -\frac{F^2(x)F^4(z)}{A^8(x-z)^8},$$

so that in the new coordinates it reads

$$(3.13) \quad \det(g_{(w,v,\hat{\psi},x,\varphi)}) = -\frac{F^2(x)F^4(z)(z - \xi_2)^2(z - \xi_3)^2}{4A^8b^2(x-z)^8}.$$

This last expression is negative on $(\xi_F, \infty) \setminus \{\xi_3\}$, and has a second order zero at $z = \xi_3$. In order to remove this degeneracy one introduces

$$(3.14) \quad \hat{v} = \exp(cv), \quad \hat{w} = -\exp(-cw),$$

where c is some constant to be chosen. Hence we have

$$(3.15) \quad d\hat{v} = c\hat{v}dv, \quad d\hat{w} = -c\hat{w}dw,$$

and the determinant in the coordinates $(\hat{w}, \hat{v}, \hat{\psi}, x, \varphi)$ reads

$$(3.16) \quad \det(g_{(\hat{w},\hat{v},\hat{\psi},x,\varphi)}) = -\frac{F^2(x)F^4(z)(z - \xi_2)^2(z - \xi_3)^2}{4A^8b^2(x-z)^8c^4\hat{v}^2\hat{w}^2}.$$

But one has $\hat{v}^2\hat{w}^2 = \exp(2c(v-w))$, so that

$$(3.17) \quad \begin{aligned} \hat{v}^2\hat{w}^2 &= \exp\left(4cb \int \frac{1}{(z - \xi_2)(z - \xi_3)} dz\right) \\ &= \exp\left(\frac{4cb}{(\xi_3 - \xi_2)}(\ln(z - \xi_3) - \ln(z - \xi_2))\right). \end{aligned}$$

Taking into account (3.17), and the determinant (3.16), we choose the constant c to satisfy:

$$(3.18) \quad \frac{2cb}{(\xi_3 - \xi_2)} = 1.$$

We obtain

$$(3.19) \quad \hat{v}\hat{w} = -\frac{z - \xi_3}{z - \xi_2},$$

and

$$(3.20) \quad \det(g_{(\hat{w},\hat{v},\hat{\psi},x,\varphi)}) = -\frac{F^2(x)F^4(z)(z - \xi_2)^4}{4A^8b^2(x-z)^8c^4}.$$

With this choice, the determinant of g in the $(\hat{w}, \hat{v}, \hat{\psi}, x, \varphi)$ coordinates extends to a strictly negative analytic function on $\{z \in (\xi_F, \infty)\}$. In fact, z as an analytic function of $\hat{v}\hat{w}$ on $\{\hat{v}\hat{w} \neq -1\}$ (that last set corresponds to $z = \infty \Leftrightarrow Y = 0$, we will return to this shortly):

$$(3.21) \quad z = \frac{\xi_3 + \xi_2 \hat{v}\hat{w}}{1 + \hat{v}\hat{w}}.$$

In the $(\hat{w}, \hat{v}, \hat{\psi}, x, \varphi)$ coordinates, one obtains the coefficients of the metric from (3.15) using

$$(3.22) \quad \begin{aligned} g_{\hat{v}\hat{v}} &= \frac{1}{c^2 \hat{v}^2} g_{vv}, & g_{\hat{w}\hat{w}} &= \frac{1}{c^2 \hat{w}^2} g_{ww}, \\ g_{\hat{v}\hat{w}} &= -\frac{1}{c^2 \hat{v}\hat{w}} g_{vw}, & g_{\hat{v}\hat{\psi}} &= \frac{1}{c\hat{v}} g_{v\hat{\psi}}, & g_{\hat{w}\hat{\psi}} &= -\frac{1}{c\hat{w}} g_{w\hat{\psi}}. \end{aligned}$$

In order to show that the coefficients of the metric are analytic on the set

$$(3.23) \quad \left\{ \hat{w}, \hat{v} \mid z(\hat{v}\hat{w}) > \xi_F \right\} = \left\{ \hat{w}, \hat{v} \mid -1 < \hat{v}\hat{w} < \frac{\xi_3 - \xi_F}{\xi_F - \xi_2} \right\}$$

it is convenient to write

$$(3.24) \quad \begin{aligned} g_{\hat{v}\hat{v}} &= \frac{1}{c^2 \hat{v}^2 \hat{w}^2} \hat{w}^2 g_{vv}, & g_{\hat{w}\hat{w}} &= \frac{1}{c^2 \hat{v}^2 \hat{w}^2} \hat{v}^2 g_{ww}, \\ g_{\hat{v}\hat{w}} &= -\frac{1}{c^2 \hat{v}\hat{w}} g_{vw}, & g_{\hat{v}\hat{\psi}} &= \frac{1}{c\hat{v}\hat{w}} \hat{w} g_{v\hat{\psi}}, & g_{\hat{w}\hat{\psi}} &= -\frac{1}{c\hat{v}\hat{w}} \hat{v} g_{w\hat{\psi}}. \end{aligned}$$

Hence, to make sure that all the coefficients of metric are well behaved at $\{\hat{w}, \hat{v} \in \mathbb{R} \mid z = \xi_3\}$ (i.e. $\hat{v} = 0$ or $\hat{w} = 0$), it suffices to check that there is a multiplicative factor $(z - \xi_3)^2$ in $g_{vv} = g_{ww}$, as well as a multiplicative factor $(z - \xi_3)$ in g_{vw} and in $g_{v\hat{\psi}} = g_{w\hat{\psi}}$. In view of (3.8)–(3.11), one can see that this will be the case if, first, a is chosen so that $1 + a\sigma(\xi_1 - z) = a\sigma(\xi_3 - z)$, that is

$$(3.25) \quad a = \frac{1}{\sigma(\xi_3 - \xi_1)},$$

and then, if b is chosen such that

$$(3.26) \quad 0 = -\frac{a^2 \nu \xi_F (\xi_3 - \xi_1)}{\xi_3 - \xi_F} + \frac{1}{\nu b^2 (\xi_3 - \xi_1)}.$$

Equation (3.26) will hold if we set

$$(3.27) \quad b^2 = \frac{(\xi_3 - \xi_F)}{\nu^2 a^2 \xi_F (\xi_3 - \xi_1)^2}.$$

So far we have been focussing on the region $z \in (\xi_F, \infty)$, which overlaps only with part of the manifold “ $\{z \in (\xi_3, \infty] \cup [-\infty, \xi_1]\}$ ”. A well behaved coordinate on that last region is $Y = -1/z$. This allows one to go smoothly through $Y = 0$ in (3.19):

$$(3.28) \quad \hat{v}\hat{w} = -\frac{1 + \xi_3 Y}{1 + \xi_2 Y} \iff Y = -\frac{1 + \hat{v}\hat{w}}{\xi_3 + \xi_2 \hat{v}\hat{w}}.$$

In other words, $\hat{v}\hat{w}$ extends analytically to the region of interest, $0 \leq Y \leq -1/\xi_1$ (and in fact beyond, but this is irrelevant to us). Similarly, the determinant

$\det(g_{(\hat{w}, \hat{v}, \hat{\psi}, x, \varphi)})$ extends analytically across $Y = 0$, being the ratio of two polynomials of order eight in z (equivalently, in Y), with limit

$$(3.29) \quad \det(g_{(\hat{w}, \hat{v}, \hat{\psi}, x, \varphi)}) \xrightarrow{z \rightarrow \infty} -\frac{F^2(x)}{4A^8 b^2 c^4 \xi_F^4}.$$

We conclude that the construction so far produces an analytic Lorentzian metric on the set

$$(3.30) \quad \hat{\Omega} := \left\{ \hat{w}, \hat{v} \mid -\frac{\xi_3 - \xi_1}{\xi_2 - \xi_1} \leq \hat{v}\hat{w} < \frac{\xi_3 - \xi_F}{\xi_F - \xi_2} \right\} \times S_{\hat{\psi}}^1 \times S_{(x, \varphi)}^2,$$

Here a subscript on S^k points to the names of the corresponding local variables.

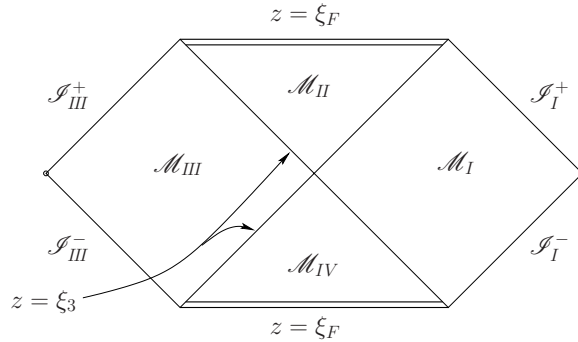


Figure 3.1. $\hat{\mathcal{M}}$ with its various subsets. For example, $\mathcal{M}_{I \cup II}$ is the union of \mathcal{M}_I and of \mathcal{M}_{II} and of that part of $\{z = \xi_3\}$ which lies in the intersection of their closures; this is the manifold constructed in [13]. Very roughly speaking, the various \mathcal{S} 's correspond to $x = z = \xi_1$. It should be stressed that this is neither a *conformal* diagram, nor is the space-time a product of the figure times $S^2 \times S^1$: \mathcal{M}_I cannot be the product of the depicted diamond with $S^2 \times S^1$, as this product is not simply connected, while \mathcal{M}_I is. But the diagram represents accurately the causal relations between the various \mathcal{M}_N 's, as well as the geometry near the bifurcate horizon $z = \xi_F$, as the manifold *does have* a product structure there.

The map

$$(3.31) \quad (\hat{w}, \hat{v}, \hat{\psi}, x, \varphi) \mapsto (-\hat{w}, -\hat{v}, -\hat{\psi}, x, -\varphi)$$

is an orientation-preserving analytic isometry of the analytically extended metric on $\hat{\Omega}$. It follows that the manifold

$\hat{\mathcal{M}}$

obtained by gluing together $\hat{\Omega}$ and two isometric copies of (\mathcal{M}_I, g) can be equipped with the obvious Lorentzian metric, still be denoted by g , which is furthermore analytic. The second copy of (\mathcal{M}_I, g) will be denoted by (\mathcal{M}_{III}, g) ; compare Figure 3.1. The reader should keep in mind the polar character of the

coordinates around the relevant axes of rotation, and the special character of the “point at infinity” $z = \xi_1 = x$.

4. Global structure

4.1. The event horizon has $S^2 \times S^1 \times \mathbb{R}$ topology. As shortly reviewed in Section 2, it is shown in [13] how to extend the metric (2.1) across

$$\mathcal{E} := \{z = \xi_3\}$$

to an analytic metric on $\mathcal{M}_{I \cup II}$. Further, we have shown in Section 3 how to extend g to an analytic metric on $\widehat{\mathcal{M}}$. Now,

$$(4.1) \quad g(\nabla z, \nabla z) = g^{zz} = -\frac{A^2(x-z)^2 G(z)}{F(x)F(z)}$$

in the region $\{z > \xi_3\}$, and by analyticity this equation remains valid on $\{z > \xi_F\}$. Equation (4.1) shows that \mathcal{E} is a null hypersurface, with z being a time function on $\{\xi_F < z < \xi_3\}$. The usual choice of time orientation implies that z is strictly decreasing along future directed causal curves in the region $\{\hat{v} > 0, \hat{w} > 0\}$, and strictly increasing along such curves in the region $\{\hat{v} < 0, \hat{w} < 0\}$. In particular no causal future directed curve can leave the region $\{\hat{v} > 0, \hat{w} > 0\}$. Hence the space-time contains a black hole region.

However, it is not clear that \mathcal{E} is the *event horizon* within the Emparan-Reall space-time $(\mathcal{M}_{I \cup II}, g)$, because the actual event horizon could be enclosing the region $z < \xi_3$. To show that this is not the case, consider the “area function”, defined as the determinant, say W , of the matrix

$$g(K_i, K_j),$$

where the K_i 's, $i = 1, 2, 3$, are the Killing vectors equal to $\partial_t, \partial_\psi$, and ∂_φ in the asymptotically flat region. In the original coordinates of (2.1) this equals

$$(4.2) \quad \frac{F(x)G(x)F(z)G(z)}{A^4(x-z)^4}.$$

Analyticity implies that this formula is valid throughout $\mathcal{M}_{I \cup II}$, as well as $\widehat{\mathcal{M}}$. Now,

$$F(z)G(z) = \frac{\nu}{\xi_F}(\xi_F - z)(z - \xi_1)(z - \xi_2)(z - \xi_3),$$

and, in view of the range (2.5) of the variable x , the sign of (4.2) depends only upon the values of z . Since $F(z)G(z)$ behaves as $-\nu z^4/\xi_F$ for large z , W is negative both for $z < \xi_1$ and for $z > \xi_3$. Hence, at each point p of those two regions the set of vectors in $T_p \mathcal{M}$ spanned by the Killing vectors is timelike. So, suppose for contradiction, that the event horizon \mathcal{H} intersects the region $\{z \in (\xi_3, \infty)\} \cup \{z \in [-\infty, \xi_1)\}$; here “ $z = \pm\infty$ ” should be understood as $Y = 0$, as already mentioned in the introduction. Since \mathcal{H} is a null hypersurface invariant under isometries, every Killing vector is tangent to \mathcal{H} . However, at each point at which W is negative there exists a linear combination of the Killing vectors which is timelike. This gives a contradiction because no timelike vectors are tangent to a null hypersurface.

We conclude that $\{z = \xi_3\}$ forms indeed the event horizon in the space-time $(\mathcal{M}_{I \cup II}, g)$ (as defined at the end of Section 2), with topology $\mathbb{R} \times S^1 \times S^2$.

The argument just given also shows that the domain of outer communications within (\mathcal{M}_I, g) coincides with $(\widehat{\mathcal{M}}, g)$.

Similarly, one finds that the domain of outer communications within $(\widehat{\mathcal{M}}, g)$, or that within $(\mathcal{M}_{I \cup II}, g)$, associated with an asymptotic region lying in (\mathcal{M}_I, g) , is (\mathcal{M}_I, g) . The boundary of the d.o.c. in $(\widehat{\mathcal{M}}, g)$ is a subset of the set $\{z = \xi_3\}$, which can be found by inspection of Figure 3.1.

4.2. Inextendibility at $z = \xi_F$. The obvious place where $(\widehat{\mathcal{M}}, g)$ could be enlarged is at $z = \xi_F$. To show that no extension is possible there, consider² the norm of the Killing vector field ∂_t :

$$(4.3) \quad g(\partial_t, \partial_t) = -\frac{F(x)}{F(z)} \rightarrow_{\xi_F < z \rightarrow \xi_F} \infty \quad (\text{recall that } F(x) \geq 1 - \frac{\xi_2}{\xi_F} > 0).$$

Suppose, for contradiction, that there exists a C^2 extension of the metric through $\{z = \xi_F\}$. Recall that any Killing vector field X satisfies the set of equations

$$(4.4) \quad \nabla_\alpha \nabla_\beta X_\sigma = R_{\lambda\alpha\beta\sigma} X^\lambda.$$

But the overdetermined set of linear equations (4.4) together with existence of a C^2 extension implies that ∂_t extends, in C^2 , to $\{z = \xi_F\}$, contradicting (4.3).

An alternative way, demanding somewhat more work, of proving that the Emparan-Reall metric is C^2 -inextendible across $\{z = \xi_F\}$, is to notice that $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ is unbounded along any curve along which z approaches ξ_F . This has been pointed out to us by Harvey Reall (private communication), and has been further verified by Alfonso Garcia-Parrado and José María Martín García using the symbolic algebra package xACT [17]:

$$(4.5) \quad R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{12A^4 \xi_F^4 G(\xi_F)^2 (x-z)^4 (1 + O(z - \xi_F))}{(\xi_F - x)^2 (z - \xi_F)^6}.$$

We are grateful to Alfonso and José María for carrying out the calculation.

4.3. Maximality. Let $k \in \mathbb{R} \cup \{\infty\} \cup \{\omega\}$. The $(n+1)$ -dimensional space-time $(\widetilde{\mathcal{M}}, \widetilde{g})$ is said to be a C^k -extension of an $(n+1)$ -dimensional space-time (\mathcal{M}, g) if there exists a C^k -immersion $\psi : \mathcal{M} \rightarrow \widetilde{\mathcal{M}}$ such that $\psi^* \widetilde{g} = g$, and such that $\psi(\mathcal{M}) \neq \widetilde{\mathcal{M}}$. A space-time (\mathcal{M}, g) is said to be C^k -maximal, or C^k -inextendible, if no C^k -extensions of (\mathcal{M}, g) exist.

A *scalar invariant* is a function which can be calculated using the geometric objects at hand and which is invariant under isometries. For instance, a function α_g which can be calculated in local coordinates from the metric g and its derivatives will be a scalar invariant if, for any local diffeomorphism ψ we have

$$(4.6) \quad \alpha_g(p) = \alpha_{\psi^*g}(\psi^{-1}(p)).$$

²This inextendibility criterion has been introduced in [2] (see the second part of Proposition 5, p. 139 there).

In the application of our Theorem 4.6 below to the Emparan-Reall space-time one can use the scalar invariant $g(X, X)$, calculated using a metric g and a Killing vector X . In this case the invariance property (4.6) reads instead

$$(4.7) \quad \alpha_{g,X}(p) = \alpha_{\psi^*g,(\psi^{-1})^*X}(\psi^{-1}(p)) .$$

A scalar invariant f on (\mathcal{M}, g) will be called a C^k -compatibility scalar if f satisfies the following property: For every C^k -extension $(\tilde{\mathcal{M}}, \tilde{g})$ of (\mathcal{M}, g) and for any bounded timelike geodesic segment γ in \mathcal{M} such that $\psi(\gamma)$ accumulates at the boundary $\partial(\psi(\mathcal{M}))$ (where ψ is the immersion map $\psi : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$), the function f is bounded along γ .

An example of a C^2 -compatibility scalar is the Kretschmann scalar, which writes $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. As explained in Section 4.2, another example is provided by the norm $g(X, X)$ of a Killing vector X of g . Any constant function is a compatibility scalar in this terminology, albeit not very useful in practice.

We shall need the following generalisation of a maximality criterion of [7, Appendix C]:³

PROPOSITION 4.1. *Suppose that every geodesic γ in (\mathcal{M}, g) is either complete, or some C^k -compatibility scalar is unbounded on γ . Then (\mathcal{M}, g) is C^k -inextendible.*

Proof. Suppose that there exists a C^k -extension $(\tilde{\mathcal{M}}, \tilde{g})$ of (\mathcal{M}, g) , with immersion $\psi : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$. We identify \mathcal{M} with its image $\psi(\mathcal{M})$ in $\tilde{\mathcal{M}}$.

Let $p \in \partial\mathcal{M}$ and let \mathcal{O} be a globally hyperbolic neighborhood of p . Let $q_n \in \mathcal{M}$ be a sequence of points approaching p , thus $q_n \in \mathcal{O}$ for n large enough. Suppose, first, that there exists n such that $q_n \in I^+(p) \cup I^-(p)$. By global hyperbolicity of \mathcal{O} there exists a timelike geodesic segment γ from q_n to p . Then the part of γ which lies within \mathcal{M} is inextendible and has finite affine length. Furthermore every C^k -compatibility scalar is bounded on γ . But there are no such geodesics through q_n by hypothesis. We conclude that

$$(4.8) \quad (I^+(p) \cup I^-(p)) \cap \mathcal{M} = \emptyset .$$

Let $q \in (I^+(p) \cup I^-(p)) \cap \mathcal{O}$, thus $q \notin \mathcal{M}$ by (4.8). Since $I^+(q) \cup I^-(q)$ is open, and $p \in I^+(q) \cup I^-(q)$, we have $q_n \in I^+(q) \cup I^-(q)$ for all n sufficiently large, say $n \geq n_0$. Let γ be a timelike geodesic segment from q_{n_0} to q . Since q is not in \mathcal{M} , the part of γ that lies within \mathcal{M} is inextendible within \mathcal{M} and has finite affine length, with all C^k -compatibility scalars bounded. This is again incompatible with our hypotheses, and the result is established. q.e.d.

In Section 5 below (see Theorem 5.1) we show that all maximally extended causal geodesics of our extension $(\hat{\mathcal{M}}, g)$ of the Emparan-Reall space-time are either complete, or reach the singular boundary $\{z = \xi_F\}$ in finite affine time. This, together with Section 4.2 and Proposition 4.1 gives:

THEOREM 4.2. *$(\hat{\mathcal{M}}, g)$ is maximal within the class of C^2 Lorentzian manifolds.*

³JC acknowledges useful discussions with M. Herzlich concerning the problem at hand.

4.4. Global hyperbolicity. In this section we show that $(\widehat{\mathcal{M}}, g)$ is globally hyperbolic. We shall need the following standard fact (see, e.g., [19, Lemma 13, p.408]):

LEMMA 4.3. *Let α be a maximally extended causal curve in a strongly causal space-time (\mathcal{M}, g) meeting a compact set K . Then α eventually leaves K , never to return, both to the future and to the past.*

Recall that in the region \mathcal{M}_I , the time-coordinate t is a time-function since ∇t is timelike; hence $\dot{t} > 0$ along any future-directed causal curve. In terms of the coordinates of Section 3 we have:

$$(4.9) \quad t = \frac{1}{2c} \ln \left(-\frac{\hat{v}}{\hat{w}} \right).$$

Letting \hat{v} and \hat{w} be the global coordinates of Section 3, we define the hypersurface

$$(4.10) \quad \mathcal{S} := \{\hat{v} + \hat{w} = 0\}.$$

(Thus, \mathcal{S} extends smoothly the hypersurface $\{t = 0\} \subset \mathcal{M}_I$, across the bifurcation surface $\{\hat{w} = \hat{v} = 0\}$, to its image in \mathcal{M}_{III} under the map (3.31).) The hypersurface \mathcal{S} is spacelike, and we wish to show that it is Cauchy. We start by noting that it is achronal: Indeed, on \mathcal{M}_I the function t , as well as its mirror counterpart on \mathcal{M}_{III} , are time functions on $\mathcal{M}_I \cup \mathcal{M}_{III}$, so any connected causal curve through those regions can meet \mathcal{S} at most once. Next, since z is a time function on $\mathcal{M}_{II} \cup \mathcal{M}_{IV}$, any causal curve entering $\mathcal{M}_{II} \cup \mathcal{M}_{IV}$ from $\mathcal{M}_I \cup \mathcal{M}_{III}$ cannot leave $\mathcal{M}_{II} \cup \mathcal{M}_{IV}$, and therefore cannot intersect again. A similar argument applies to those causal curves which enter \mathcal{M}_{II} from \mathcal{M}_{IV} , or vice-versa, and achronality follows.

So, from [15, Property 6], since \mathcal{S} is closed, spacelike, and achronal, it suffices to prove that every inextendible null geodesic in $\widehat{\mathcal{M}}$ intersects \mathcal{S} . For this, let γ be such a geodesic, say future-directed.

Suppose, first, that $\gamma(0) \in \mathcal{M}_I$; the argument in \mathcal{M}_{III} follows from isometry invariance. Choose $\varepsilon > 0$ sufficiently small so that the set K_ε , defined using the coordinates \hat{v}, \hat{w} of Section 3",

$$(4.11) \quad K_\varepsilon := \left\{ \hat{v} \geq \varepsilon, \hat{w} \leq -\varepsilon, \hat{v}\hat{w} \geq -\frac{\xi_3 - \xi_1}{\xi_2 - \xi_1} + \varepsilon \right\} \times S^1 \times S^2,$$

contains $\gamma(0)$ (see Figure 4.1). Suppose, moreover, that $t(\gamma(0)) > 0$. Since K_ε is compact, and (\mathcal{M}_I, g) is strongly causal (as it has a time function), Lemma 4.3 implies that γ must leave K_ε , and never reenter, except perhaps after exiting from (\mathcal{M}_I, g) and returning again. But we have already seen that γ cannot return to \mathcal{M}_I once it exited, so γ will indeed eventually never reenter K_ε . Choose any sequence ε_i tending to zero, then there exists a decreasing sequence s_i such that γ leaves K_{ε_i} at $\gamma(s_i)$, and never reenters.

Now, the ratio $-\hat{v}/\hat{w}$ is a time function on \mathcal{M}_I , and therefore increasing along γ to the future, and decreasing to the past. If γ crosses the hypersurface $\{-\hat{v}/\hat{w} = 1\}$ we are done. So suppose that γ does not. Then:

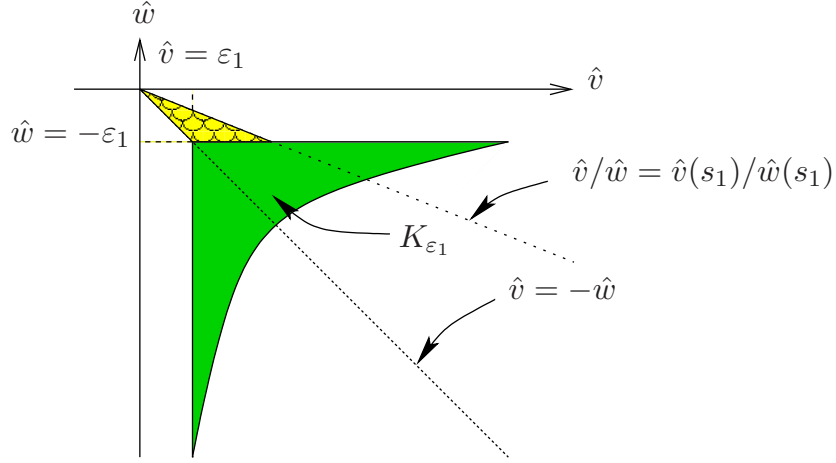


Figure 4.1. The set K_{ε_1} . Here \hat{w} and \hat{v} are the global coordinates of Section 3.

- 1) Either $\gamma(s_1)$ definitively leaves $\partial K_{\varepsilon_1}$ through that part of the boundary $\partial K_{\varepsilon_1}$ on which $\hat{w} = -\varepsilon_1$. Then γ remains in the tiled yellow triangle in Figure 4.1 until it leaves \mathcal{M}_I , and thus all subsequent definitive exit points $\gamma(s_i) \in \partial K_{\varepsilon_i}$ lie on $\hat{w} = -\varepsilon_i$. So $\hat{w}(s_i) \rightarrow 0$ and $\hat{v}(s_i) \rightarrow 0$. The argument around equation (5.66) below shows that $\hat{w}(s) \rightarrow 0$ and $\hat{v}(s) \rightarrow 0$ in finite time. By the analysis of Section 5, γ smoothly extends through $\{\hat{v} = \hat{w} = 0\}$. But this is impossible since γ is null, while the only null geodesics meeting $\{\hat{v} = \hat{w} = 0\}$ are the generators of the Killing horizon, entirely contained within $\{\hat{v} = \hat{w} = 0\}$.
- 2) Or $\gamma(s_1)$ definitively leaves $\partial K_{\varepsilon_1}$ through that part of the boundary $\partial K_{\varepsilon_1}$ on which

$$\hat{w}\hat{v} = -\frac{\xi_3 - \xi_1}{\xi_2 - \xi_1} + \varepsilon_1 .$$

From what has been said, all subsequent definitive exit points take place on

$$\hat{w}\hat{v} = -\frac{\xi_3 - \xi_1}{\xi_2 - \xi_1} + \varepsilon_i .$$

This implies that $z(s) \rightarrow \xi_1$. We need to consider two cases:

Suppose, first, that the constant of motion c_ψ vanishes. Then $c_t = 0$ is not possible by (5.12) (recall that $\lambda = 0$), while from (5.7)

$$(4.12) \quad \dot{t}(s) \text{ approaches } -\frac{F(\xi_1)}{F(x(s))} c_t ,$$

which is strictly bounded away from zero. So \dot{t} is positive and uniformly bounded away from zero sufficiently far to the past. This, together with Theorem 5.6, shows that γ crosses $\{-\hat{v} = \hat{w}\} \subset \mathcal{S}$, which contradicts our assumption.

It remains to consider the possibility that $c_\psi \neq 0$. Equation (5.12) implies that $x(s) \rightarrow \xi_1$. Hence γ enters, and remains, within the asymptotically flat region. There γ is forced to cross $\{\frac{-\hat{v}}{\hat{w}} = 1\}$ by arguments known in principle, contradicting again our assumption. q.e.d.

We conclude that γ 's intersecting \mathcal{M}_I or \mathcal{M}_{III} , with $t(\gamma(0)) > 0$, meet \mathcal{S} when followed to the past.

The proof that future directed causal geodesics γ intersecting \mathcal{M}_I or \mathcal{M}_{III} , with $t(\gamma(0)) < 0$, meet \mathcal{S} , is identical: one needs instead to follow γ to the future rather than to the past. Alternatively one can invoke the existence of isometries of those regions which map t to $-t$.

Suppose, finally, that $\gamma(0) \in \mathcal{M}_{II} \cup \mathcal{M}_{IV}$. By Proposition 5.4, the null geodesic γ exits in finite time through the bifurcate horizon $\{\hat{w}\hat{v} = 0\}$. If γ exits through $\{\hat{w} = \hat{v} = 0\}$ it intersects \mathcal{S} there, and we are done; otherwise it enters $\mathcal{M}_I \cup \mathcal{M}_{III}$. But we have just seen that γ must then intersect \mathcal{S} , and the proof is complete. q.e.d.

4.5. \mathcal{S} . In this section we address the question of existence of conformal completions at null infinity *à la Penrose*, for a class of higher dimensional stationary space-times that includes the Emparan-Reall metrics; see the Appendix in [12] for the 3 + 1 dimensional case.

We start by noting that any stationary asymptotically flat space-time which is vacuum, or electro-vacuum, outside of a spatially compact set is necessarily *asymptotically Schwarzschildian*, in the sense that there exists a coordinate system in which the leading order terms of the metric have the Schwarzschild form, with the error terms falling-off one power of r faster:

$$(4.13) \quad g = g_m + O(r^{-(n-1)}),$$

where g_m is the Schwarzschild metric of mass m , and the size of the decay of the error terms in (4.13) is measured in a manifestly asymptotically Minkowskian coordinate system. The proof of this fact is outlined briefly in [1, Section 2]. In that last reference it is also shown that the remainder term has a full asymptotic expansion in terms of inverse powers of r in dimension $2k + 1$, $k \geq 3$, or in dimension $4 + 1$ for static metrics. Otherwise, the remainder is known to have an asymptotic expansion in terms of inverse powers of r and of $\ln r$, and whether or not there will be non-trivial logarithmic terms in the expansion is not known in general.

In higher dimensions, the question of existence of a conformal completion at null infinity is straightforward: We start by writing the $(n + 1)$ -dimensional Minkowski metric as

$$(4.14) \quad \eta = -dt^2 + dr^2 + r^2h,$$

where h is the round unit metric on an $(n - 2)$ -dimensional sphere. Replacing t by the standard retarded time $u = t - r$, one is led to the following form of the metric g :

$$(4.15) \quad g = -du^2 - 2du dr + r^2h + O(r^{-(n-2)})dx^\mu dx^\nu,$$

where the dx^μ 's are the manifestly Minkowskian coordinates (t, x^1, \dots, x^n) coordinates for η . Setting $x = 1/r$ in (4.15) one obtains

$$(4.16) \quad g = \frac{1}{x^2} \left(-x^2 du^2 + 2du dx + h + O(x^{n-4}) dy^\alpha dy^\beta \right),$$

with correction terms in (4.16) which will extend smoothly to $x = 0$ in the coordinate system $(y^\mu) = (u, x, v^A)$, where the v^A 's are local coordinates on S^{n-2} . For example, a term $O(r^{-2}) dx^i dx^j$ in g will contribute a term

$$O(r^{-2}) dr^2 = O(r^{-2}) x^{-4} dx^2 = x^{-2} (O(1) dx^2),$$

which is bounded up to $x = 0$ after a rescaling by x^2 . The remaining terms in (4.16) are analyzed similarly.

In dimension $4 + 1$, care has to be taken to make sure that the correction terms do not affect the signature of the metric so extended; in higher dimension this is already apparent from (4.16).

So, to construct a conformal completion at null infinity for the Emparan Reall metric it suffices to verify that the determinant of the conformally rescaled metric, when expressed in the coordinates described above, does not vanish at $x = 0$. This is indeed the case, and can be seen by calculating the Jacobian of the map

$$(t, z, \psi, x, \varphi) \mapsto (u, x, v^A);$$

the result can then be used to calculate the determinant of the metric in the new coordinates, making use of the formula for the determinant of the metric in the original coordinates.

For a general stationary vacuum $4 + 1$ dimensional metric one can always transform to the coordinates, alluded to above, in which the metric is manifestly Schwarzschildian in leading order. Instead of using $(u = t - r, x = 1/r)$ one can use coordinates $(u_m, x = 1/r)$, where u_m is the corresponding null coordinate u for the $4 + 1$ dimensional Schwarzschild metric. This will lead to a conformally rescaled metric with the correct signature on the conformal boundary. Note, however, that this transformation might introduce log terms in the metric, even if there were none to start with; this is why we did not use this above.

In summary, whenever a stationary, vacuum, asymptotically flat, $(n + 1)$ -dimensional metric, $4 \neq n \geq 3$, has an asymptotic expansion in terms of inverse powers of r , one is led to a smooth \mathcal{I} . This is the case for any such metric in dimensions $3 + 1$ or $2k + 1$, $k \geq 3$. In the remaining dimensions one always has a polyhomogeneous conformal completion at null infinity, with a conformally rescaled metric which is C^{n-4} up-to-boundary. For the Emparan-Reall metric there exists a completion which has no logarithmic terms, and is thus C^∞ up-to-boundary.

4.6. Uniqueness?

4.6.1. Distinct extensions. We start by noting that maximal analytic extensions of manifolds are not unique. The simplest counterexamples are as follows: remove a subset Ω from a maximally extended manifold \mathcal{M} so that $\mathcal{M} \setminus \Omega$ is not simply connected, and pass to the universal cover; extend maximally the

space-time so obtained, if further needed. This provides a new maximal extension. Whether or not such constructions can be used to classify all maximal analytic extensions remains to be seen.

One can likewise ask the question, whether it is true that $(\widehat{\mathcal{M}}, g)$ is unique within the class of simply connected analytic extensions of (\mathcal{M}_I, g) which are inextendible and globally hyperbolic. The following variation of the construction gives a negative answer, when “inextendible” is meant as “inextendible within the class of globally hyperbolic manifolds”: Let \mathcal{S} be the Cauchy surface $\{t = 0\}$ in $\widehat{\mathcal{M}}$, as described in Section 4.4 (see (4.10)), and remove from \mathcal{S} a closed subset Ω so that $\mathcal{S} \setminus \Omega$ is not simply connected. Let $\widetilde{\mathcal{S}}$ be a maximal analytic extension of the universal covering space of $\mathcal{S} \setminus \Omega$, with the obvious Cauchy data inherited from \mathcal{S} , and let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be the maximal globally hyperbolic development thereof. Then $(\widetilde{\mathcal{M}}, \widetilde{g})$ is a globally hyperbolic analytic extension of (\mathcal{M}_I, g) which is maximal in the class of globally hyperbolic manifolds, and distinct from $(\widehat{\mathcal{M}}, g)$.

The examples just discussed will exhibit the following undesirable feature: existence of maximally extended geodesics of affine length near which the space-time is locally extendible in the sense of [20]. This does not happen in $(\widehat{\mathcal{M}}, g)$. It turns out that there exists at least one more maximal extension of the Emparan-Reall space-time (\mathcal{M}_I, g) which does not suffer from this *local extendibility* pathology. This results from a general construction which proceeds as follows:

Consider any spacetime (\mathcal{M}, g) , and let \mathcal{M}_I be an open subset of \mathcal{M} . Suppose that there exists an isometry Ψ of (\mathcal{M}, g) satisfying: a) Ψ has no fixed points; b) $\Psi(\mathcal{M}_I) \cap \mathcal{M}_I = \emptyset$; and c) Ψ^2 is the identity map. Then, by a) and c), \mathcal{M}/Ψ equipped with the obvious metric (still denoted by g) is a Lorentzian manifold. Furthermore, by b), \mathcal{M}_I embeds diffeomorphically into \mathcal{M}/Ψ in the obvious way. It follows from the results in [18] that $(\mathcal{M}/\Psi, g)$ is analytic if (\mathcal{M}, g) was (compare [8, Appendix A]).

Keeping in mind that a space-time is time-oriented by definition, \mathcal{M}/Ψ will be a space-time if and only if Ψ preserves time-orientation. If \mathcal{M} is simply connected, then $\pi_1(\mathcal{M}/\Psi) = \mathbb{Z}_2$.

As an example of this construction, consider the Kruskal-Szekeres extension (\mathcal{M}, g) of the Schwarzschild space-time (\mathcal{M}_I, g) ; by the latter we mean a connected component of the set $\{r > 2m\}$ within \mathcal{M} . Let (T, X) the global coordinates on \mathcal{M} as defined on p. 153 of [22]. Let $\mathring{\Psi} : S^2 \rightarrow S^2$ be the antipodal map. Consider the four isometries $\Psi_{\pm\pm}$ of the Kruskal-Szekeres space-time defined by the formula, for $p \in S^2$,

$$\Psi_{\pm\pm}(T, X, p) = (\pm T, \pm X, \mathring{\Psi}(p)) .$$

Set $\mathcal{M}_{\pm\pm} := \mathcal{M}/\Psi_{\pm\pm}$. Since Ψ_{++} is the identity, $\mathcal{M}_{++} = \mathcal{M}$ is the Kruskal-Szekeres manifold. Next, both $\mathcal{M}_{-,+}$ and $\mathcal{M}_{+,-}$ are smooth maximal analytic Lorentzian extensions of (\mathcal{M}_I, g) , but are not space-times. Finally, \mathcal{M}_{+-} is a maximal globally hyperbolic analytic extension of the Schwarzschild manifold distinct from \mathcal{M} . This is the “ $\mathbb{R}\mathbb{P}^3$ geon”, discussed in [14].

Similar examples can be constructed for the black ring solution; we restrict attention to orientation, and time-orientation, preserving maps. So, let $(\widehat{\mathcal{M}}, g)$ be our extension, as constructed above, of the domain of outer communication (\mathcal{M}_I, g) within the Emparan-Reall space-time $(\mathcal{M}_{I \cup II}, g)$, and let $\Psi : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}$ be defined as

$$(4.17) \quad \Psi(\hat{v}, \hat{w}, \hat{\psi}, x, \varphi) = (\hat{w}, \hat{v}, \hat{\psi} + \pi, x, -\varphi) .$$

By inspection of (3.8)-(3.10) and (3.22), the map Ψ is an isometry, and clearly satisfies conditions a), b) and c) above. Then $\widehat{\mathcal{M}}/\Psi$ is a maximal, orientable, time-orientable, analytic extension of \mathcal{M}_I distinct from $\widehat{\mathcal{M}}$.

4.6.2. A uniqueness theorem. Our aim in this section is to prove a uniqueness result for our extension $(\widehat{\mathcal{M}}, g)$ of the Emparan-Reall space-time (\mathcal{M}_I, g) . The examples of the previous section show that the hypotheses are optimal:

THEOREM 4.4. *$(\widehat{\mathcal{M}}, g)$ is unique within the class of simply connected analytic extensions of (\mathcal{M}_I, g) which have the property that all maximally extended causal geodesics on which $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded are complete.*

Uniqueness is understood up to isometry. Theorem 4.4 follows immediately from Theorem 4.6, which we are about to prove, and from our analysis of causal geodesics of $(\widehat{\mathcal{M}}, g)$ in Section 5 below, see Theorem 5.1 there.

For the record we state the corresponding result for the Schwarzschild space-time, with identical (but simpler, as in this case the geodesics are simpler to analyse) proof:

THEOREM 4.5. *The Kruskal-Szekeres space-time is the unique extension, within the class of simply connected analytic extensions of the Schwarzschild region $r > 2m$, with the property that all maximally extended causal geodesics on which $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded are complete.*

We continue with some terminology. A maximally extended geodesic ray $\gamma : [0, s_+) \rightarrow \mathcal{M}$ will be called *s-complete* if $s_+ = \infty$ unless there exists some polynomial scalar invariant α such that

$$\limsup_{s \rightarrow s_+} |\alpha(\gamma(s))| = \infty .$$

A similar definition applies to maximally extended geodesics $\gamma : (s_-, s^+) \rightarrow \mathcal{M}$, with some polynomial scalar invariant (not necessarily the same) unbounded in the incomplete direction, if any. Here, by a *polynomial scalar invariant* we mean a scalar function which is a polynomial in the metric, its inverse, the Riemann tensor and its derivatives. It should be clear how to include in this notion some other objects of interest, such as the norm $g(X, X)$ of a Killing vector X , or of a Yano-Killing tensor, etc. But care should be taken not to take scalars such as $\ln(R_{ijkl}R^{ijkl})$ which could blow up even though the geometry remains regular; this is why we restrict attention to polynomials.

A Lorentzian manifold (\mathcal{M}, g) will be said to be *s-complete* if every maximally extended geodesic is *s-complete*. The notions of *timelike s-completeness*, or *causal s-completeness* are defined similarly, by specifying the causal type of the geodesics in the definition above.

We have the following version of [16, Theorem 6.3, p. 255] (compare also the *Remark* on p. 256 there), where geodesic completeness is weakened to *timelike s -completeness*:

THEOREM 4.6. *Let (\mathcal{M}, g) , (\mathcal{M}', g') be analytic Lorentzian manifolds of dimension $n + 1$, $n \geq 1$, with \mathcal{M} connected and simply connected, and \mathcal{M}' timelike s -complete. Then every isometric immersion $f_U : U \subset \mathcal{M} \hookrightarrow \mathcal{M}'$, where U is an open subset of \mathcal{M} , extends uniquely to an isometric immersion $f : \mathcal{M} \hookrightarrow \mathcal{M}'$.*

Proof. We need some preliminary lemmas which are proved as in [16], by replacing “affine mappings” there by “isometric immersions”:

LEMMA 4.7. [16, Lemma 1, p. 252] *Let \mathcal{M} , \mathcal{M}' be analytic manifolds, with \mathcal{M} connected. Let f, g be analytic mappings $\mathcal{M} \rightarrow \mathcal{M}'$. If f and g coincide on a nonempty open subset of \mathcal{M} , then they coincide everywhere.*

LEMMA 4.8. [16, Lemma 4, p. 254] *Let (\mathcal{M}, g) and (\mathcal{M}', g') be pseudo-Riemannian manifolds of same dimension, with \mathcal{M} connected, and let f and g be isometric immersions of \mathcal{M} into \mathcal{M}' . If there exists some point $x \in \mathcal{M}$ such that $f(x) = g(x)$ and $f_*(X) = g_*(X)$ for every vector X of $T_x\mathcal{M}$, then $f = g$ on \mathcal{M} .*

We can turn our attention now to the proof of Theorem 4.6. Similarly to the proof of Theorem 6.1 in [16], we define an *analytic continuation of f_U along a continuous path $c : [0, 1] \rightarrow \mathcal{M}$* to be a set of mappings f_s , $0 \leq s \leq 1$, together with a family of open subsets U_s , $0 \leq s \leq 1$, satisfying the properties:

- $f_0 = f_U$ on $U_0 = U$;
- for every $s \in [0, 1]$, U_s is a neighborhood of the point $c(s)$ of the path c , and f_s is an isometric immersion $f_s : U_s \subset \mathcal{M} \hookrightarrow \mathcal{M}'$;
- for every $s \in [0, 1]$, there exists a number $\delta_s > 0$ such that for all $s' \in [0, 1]$, $(|s' - s| < \delta_s) \Rightarrow (c(s') \in U_s \text{ and } f_{s'} = f_s \text{ in a neighborhood of } c(s'))$;

We need to prove that, under the hypothesis of s -completeness, such an analytic continuation does exist along any curve c . The argument is simplest for timelike curves, so let us first assume that c is timelike. To do so, we consider the set:

$$(4.18) \quad A := \{s \in [0, 1] \mid \text{an analytic continuation exists along } c \text{ on } [0, s]\}$$

A is nonempty, as it contains a neighborhood of 0. Hence $\bar{s} := \sup A$ exists and is positive. We need to show that in fact, $\bar{s} = 1$ and can be reached. Assume that this is not the case. Let W be a normal convex neighborhood of $c(\bar{s})$ such that every point x in W has a normal neighborhood containing W . (Such a W exists from Theorem 8.7, chapter III of [16].) We can choose $s_1 < \bar{s}$ such that $c(s_1) \in W$, and we let V be a normal neighborhood of $c(s_1)$ containing W . Since $s_1 \in A$, f_{s_1} is well defined, and is an isometric immersion of a neighborhood of $c(s_1)$ into \mathcal{M}' ; we will extend it to $V \cap I^\pm(c(s_1))$. To do so, we know that $\exp : V^* \rightarrow V$ is a diffeomorphism, where V^* is a neighborhood of 0 in $T_{c(s_1)}\mathcal{M}$, hence, in particular, for $y \in V \cap I^\pm(c(s_1))$, there exists a unique $X \in V^*$ such that $y = \exp X$. Define $X' := f_{s_1*}X$. Then X' is a vector tangent to \mathcal{M}' at the point $f_{s_1}(c(s_1))$. Since y is in the timelike cone of $c(\bar{s})$, X is timelike, and so is X' , as f_{s_1} is isometric. We now need to prove the following:

LEMMA 4.9. *The geodesic $s \mapsto \exp(sX')$ of \mathcal{M}' is well defined for $0 \leq s \leq 1$.*

Proof. Let

$$(4.19) \quad s^* := \sup\{s \in [0, 1] \mid \exp(s'X') \text{ exists } \forall s' \in [0, s]\}.$$

First, such a s^* exists, is positive, and we notice that if $s^* < 1$, then it is not reached. We wish to show that $s^* = 1$ and is reached. Hence, it suffices to show that “ s^* is not reached” leads to a contradiction. Indeed, in such a case the timelike geodesic $s \mapsto \exp(sX')$ ends at finite affine parameter, thus, there exists a scalar invariant φ such that $\varphi(\exp(sX'))$ is unbounded as $s \rightarrow s^*$. Now, for all $s < s^*$, we can define $h(\exp(sX)) := \exp(sX')$, and this gives an extension h of f_{s_1} which is analytic (since it commutes with the exponential maps, which are analytic). By Lemma 4.8, h is in fact an isometric immersion. By definition of scalar invariants we have

$$\varphi(\exp(sX')) = \tilde{\varphi}(\exp(sX)),$$

where $\tilde{\varphi}$ is the invariant in (\mathcal{M}, g) corresponding to φ . But this is not possible since $\tilde{\varphi}(\exp(sX))$ has a finite limit when $s \rightarrow s^*$, and provides the desired contradiction. q.e.d.

From the last lemma we deduce that there exists a unique element, say $h(y)$, in a normal neighborhood of $f_{s_1}(c(s_1))$ in \mathcal{M}' such that $h(y) = \exp(X')$. Hence, we have extended f_{s_1} to a map h defined on $V \cap I^\pm(c(s_1))$. In fact, h is also an isometric immersion, by the same argument as above, since it commutes with the exponential maps of \mathcal{M} and \mathcal{M}' . Then, since the curve c is timelike, this is sufficient to conclude that we can do the analytic continuation beyond $c(\bar{s})$, since $V \cap I^\pm(c(s_1))$ is an open set, and thus contains a segment of the geodesic $c(s)$, for s in a neighborhood of \bar{s} .

Let us consider now a general, not necessarily timelike, continuous curve $c(s), 0 \leq s \leq 1$, with $c(0) \in U$. As before, we consider the set:

$$(4.20) \quad \{s \in [0, 1] \mid \text{there exists an analytic continuation of } f_U \text{ along } c(s'), 0 \leq s' \leq s\},$$

and its supremum \tilde{s} . Assume that \tilde{s} is not reached. Let again W be a normal neighborhood of $c(\tilde{s})$ such that every point of W contains a normal neighborhood which contains W . Then, let z be an element of the set $I^+(c(\tilde{s})) \cap W$. $I^-(z) \cap W$ is therefore an open set in W containing $c(\tilde{s})$. Hence we can choose $s_1 < \tilde{s}$ such that the curve segment $c([s_1, \tilde{s}])$ is included in $I^-(z) \cap W$, see Figure 4.2. In particular, $z \in I^+(c(s_1)) \cap W$. Since there exists an analytic continuation up to $c(s_1)$, we have an isometric immersion f_{s_1} defined on a neighborhood U_{s_1} of $c(s_1)$, which can be assumed to be included in W . Hence, from what has been seen previously, f_{s_1} can be extended as an isometric immersion, ψ_1 , on $U_z := U_{s_1} \cup (I^+(c(s_1)) \cap W)$, which contains z . We now do the same operation for ψ_1 on U_z : we can extend it by analytic continuation to an isometric immersion ψ_2 defined on $U_z \cup (I^-(z) \cap W)$, which is an open set containing the entire segment of the curve x between $c(s_1)$ and $c(\tilde{s})$. In particular, ψ_1 and ψ_2 coincide on U_z , i.e. on their common domain of definition; thus we obtain an analytic continuation of f_{s_1} along the curve $c(s)$, for $s_1 \leq s \leq \tilde{s}$; this continuation also coincides with the continuation $f_s, s \in [s_1, \tilde{s}]$. This

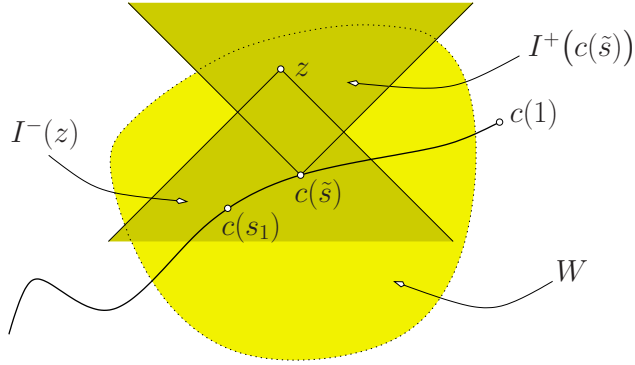


Figure 4.2. The analytic continuation at $c(\tilde{s})$.

is in contradiction with the assumption that \tilde{s} is not reached by any analytic continuation from f_U along x . Hence $\tilde{s} = 1$ and is reached, that is to say we have proved the existence of an analytic continuation of f_U along all the curve x .

The remaining arguments are as in [16].

q.e.d.

5. Geodesics

We continue with a study of geodesics in $(\widehat{\mathcal{M}}, g)$. Our aim is to prove:

THEOREM 5.1. *All maximally extended causal geodesics in $(\widehat{\mathcal{M}}, g)$ are either complete, or reach a singular boundary $\{z = \xi_F\}$ in finite affine time.*

The proof of Theorem 5.1 will occupy the remainder of this section. We will analyze separately the behavior of the geodesics in various regions of interest, using coordinates suited for the region at hand.

5.1. Geodesics in the domain of outer communications away from the horizon. Whether in \mathcal{M}_I or in $\widehat{\mathcal{M}}$, the *domain of outer communications*¹ $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle$ coincides with the set

$$\{z \in (\xi_3, \infty] \cup [-\infty, \xi_1]\}.$$

We continue by showing that all geodesic segments in $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle$ of finite affine length which do not approach the boundary $\{z = \xi_3\}$ remain within compact sets of \mathcal{M} , with uniform bounds on the velocity vector. This holds regardless of the causal nature of the geodesic. To see this, let $s \mapsto \gamma(s)$ be an affinely parameterized geodesic,

$$\gamma(s) = (t(s), \psi(s), z(s), x(s), \varphi(s)).$$

We have four constants of motion,

$$(5.1) \quad \lambda := g(\dot{\gamma}, \dot{\gamma}), \quad c_t := g(\partial_t, \dot{\gamma}), \quad c_\psi := g(\partial_\psi, \dot{\gamma}), \quad c_\varphi := g(\partial_\varphi, \dot{\gamma}).$$

Written out in detail, keeping in mind that $\sigma = \frac{1}{A} \sqrt{\frac{\nu}{\xi_F}}$,

$$(5.2) \quad \lambda = -\frac{F(x)}{F(z)} \left(\dot{t} + \sigma(\xi_1 - z)\dot{\psi} \right)^2 + \frac{F(z)}{A^2(x-z)^2} \left[-F(x) \left(\frac{\dot{z}^2}{G(z)} + \frac{G(z)}{F(z)} \dot{\psi}^2 \right) + F(z) \left(\frac{\dot{x}^2}{G(x)} + \frac{G(x)}{F(x)} \dot{\varphi}^2 \right) \right] ;$$

$$(5.3) \quad c_t = -\frac{F(x)}{F(z)} \left(\dot{t} + \sigma(\xi_1 - z)\dot{\psi} \right) ;$$

$$(5.4) \quad c_\psi = \sigma(\xi_1 - z)c_t - \frac{G(z)F(x)}{A^2(x-z)^2} \dot{\psi} ;$$

$$(5.5) \quad c_\varphi = \frac{F^2(z)G(x)}{A^2(x-z)^2 F(x)} \dot{\varphi} .$$

This leads to

$$(5.6) \quad \dot{\psi} = \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi) ,$$

$$(5.7) \quad \dot{t} = -\frac{F(z)}{F(x)} c_t - \sigma(\xi_1 - z) \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi) ,$$

$$(5.8) \quad G(x)\dot{\varphi} = \frac{A^2(x-z)^2 F(x)}{F(z)^2} c_\varphi ,$$

$$(5.9) \quad \lambda = -\frac{F(z)}{F(x)} c_t^2 - \frac{F(z)F(x)}{A^2(x-z)^2 G(z)} \dot{z}^2 - \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 + \frac{F^2(z)}{A^2(x-z)^2} \frac{\dot{x}^2}{G(x)} + \frac{A^2(x-z)^2 F(x)}{G(x)F^2(z)} c_\varphi^2 .$$

We have:

- 1) Those geodesics for which $\liminf_{s \rightarrow \infty} (x(s) - z(s)) = 0$ can be studied by transforming the metric to explicitly asymptotically flat coordinates as in [13], and using known methods (see, e.g., [6, Appendix B] and [9, Appendix]). Such geodesics eventually remain in the asymptotically flat region and are complete in the relevant direction. So, without loss of generality we can assume in the remainder of our analysis that

$$(5.10) \quad |x - z| \geq \epsilon_0$$

for some $0 < \epsilon_0 < 1$.

- 2) Consider those geodesic segments for which

$$2\xi_1 \leq z(s) \leq \xi_1 .$$

In this region the functions z and x are related to polar-type coordinates near axes of rotation $G(z) = 0$ and $G(x) = 0$; in fact, well behaved polar-type coordinates (θ, μ) are obtained by introducing

$$(5.11) \quad d\theta = \frac{dx}{\sqrt{G(x)}} , \quad d\mu = \frac{dz}{\sqrt{|G(z)|}} .$$

We then rewrite (5.9) as

$$(5.12) \quad F(x)\dot{\mu}^2 + F(z)\dot{\theta}^2 + \frac{A^4(x-z)^4 F(x)}{G(x)F^3(z)} c_\varphi^2 \\ + \frac{A^4(x-z)^4}{F(x)F(z)|G(z)|} (\sigma(\xi_1 - z)c_t - c_\psi)^2 = \frac{A^2(x-z)^2}{F(z)} \left[\lambda + \frac{F(z)}{F(x)} c_t^2 \right].$$

The right-hand-side is bounded by a constant C , while the coefficients $F(x)$ of $\dot{\mu}^2$ and $F(y)$ of $\dot{\theta}^2$ are bounded from above and away from zero, so there exists a constant C_1 such that

$$(5.13) \quad \dot{\mu}^2 + \dot{\theta}^2 \leq C_1.$$

Inspecting (5.6)-(5.8), and noting that the zero of $G(z)$ in the denominator of the right-hand-side of (5.7) is canceled by the $z - \xi_1$ factor in the numerator, we find that there exists a constant C_2 such that

$$(5.14) \quad \dot{t}^2 + \dot{\theta}^2 + \dot{\mu}^2 + G^2(z)\dot{\psi}^2 + G^2(x)\dot{\varphi}^2 \leq C_2.$$

It follows from (5.12) that a non-zero c_φ prevents x from approaching ξ_1 and ξ_2 unless $x - z \rightarrow 0$, similarly a non-zero c_ψ prevents z from approaching ξ_1 unless $x - z \rightarrow 0$. So, under (5.10), we find a bound on $|\dot{\psi}|$ from (5.6) when c_ψ is zero (since then a factor $z - \xi_1$ in $G(z)$ is cancelled by a similar factor in the numerator), or from (5.14) otherwise. A similar analysis of $\dot{\varphi}$ allows us to conclude that

$$(5.15) \quad \dot{t}^2 + \dot{\theta}^2 + \dot{\mu}^2 + \dot{\psi}^2 + \dot{\varphi}^2 \leq C_3.$$

3) Consider, next, geodesic segments for which

$$-\infty \leq z \leq 2\xi_1 \quad \text{or} \quad \xi_3 + \epsilon \leq z \leq \infty,$$

where ϵ is some strictly positive number. Introducing $Y = -1/z$, from (5.9) we find

$$(5.16) \quad \frac{|F(z)|F(x)}{A^2(x-z)^2|G(z)|Y^4} \dot{Y}^2 + \frac{F^2(z)}{A^2(x-z)^2} \dot{\theta}^2 + \frac{A^2(x-z)^2 F(x)}{G(x)F^2(z)} c_\varphi^2 \\ = \lambda + \frac{F(z)}{F(x)} c_t^2 + \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2.$$

By an argument similar to the above, but simpler, we obtain

$$(5.17) \quad \dot{t}^2 + \dot{\theta}^2 + \dot{Y}^2 + \dot{\psi}^2 + \dot{\varphi}^2 \leq C_4.$$

Here one has to use a cancelation in the coefficient of c_t^2 in (5.16), as well as in the coefficient of c_t in (5.7), keeping in mind that $\sigma = \frac{1}{A} \sqrt{\frac{\nu}{\xi_F}}$; e.g.,

$$(5.18) \quad \dot{t} = -\frac{1}{F(x)} \underbrace{\left(\underbrace{F(z)}_{=1/(Y\xi_F)+O(1)} + \underbrace{\sigma^2(\xi_1 - z)^2 \frac{A^2(x-z)^2}{G(z)}}_{=-\sigma^2 A^2/(\nu Y)+O(1)} \right)}_{=O(1)} c_t + O(1),$$

where $O(1)$ denotes terms which are bounded as $Y \rightarrow 0$.

Usual considerations about maximally extended geodesics show now that:

PROPOSITION 5.2. *For any $\epsilon > 0$, the geodesics which are maximally extended within the region $z \in [\xi_3 + \epsilon, \infty] \cup [-\infty, \xi_1]$ are either complete, or acquire a smooth end point at $\{z = \xi_3 + \epsilon\}$.*

5.2. Geodesics in the region $\{\xi_F < z < \xi_3\}$. In this coordinate range both $F(z)$ and $G(z)$ are negative, and we rewrite (5.9) as

$$(5.19) \quad \frac{|F(z)|}{|G(z)|} \dot{z}^2 - \frac{F^2(z)}{F(x)} \dot{\theta}^2 = \dot{\chi} + \frac{A^4(x-z)^4}{G(x)F^2(z)} c_\varphi^2,$$

where we have set

$$(5.20) \quad \dot{\chi} := \frac{A^4(x-z)^4}{F^2(x)(-G(z))} (\sigma(\xi_1 - z)c_t - c_\psi)^2 + \frac{A^2(x-z)^2}{F(x)} \left[-\lambda + \frac{|F(z)|}{F(x)} c_t^2 \right].$$

5.2.1. Timelike geodesic incompleteness. The extended space-time will not be geodesically complete if one can find a maximally extended geodesic with finite affine length. Consider, thus any future directed, affinely parameterized timelike geodesic γ entirely contained in the region $\{\xi_F < z < \xi_3\} \cap \{\hat{v} > 0, \hat{w} > 0\}$, and maximally extended there; an identical argument applies to past directed timelike geodesics in the region $\{\xi_F < z < \xi_3\} \cap \{\hat{v} < 0, \hat{w} < 0\}$. Since z is a time function in this region, z is strictly decreasing along γ . From (5.19) we have

$$\frac{F(z)F(x)}{A^2(x-z)^2G(z)} \dot{z}^2 \geq -\lambda,$$

which gives $\sqrt{|F(z)|}|\dot{z}| \geq \epsilon\sqrt{|\lambda|}\sqrt{|G(z)|} > 0$ for some constant ϵ . The proper time parameterization is obtained by choosing $\lambda = -1$. Let $L(\gamma)$ denote the proper length along γ ; keeping in mind that $\dot{z} = dz/ds$ we obtain

$$L(\gamma) = \int_{\xi_F}^{\xi_3} \left| \frac{ds}{dz} \right| dz \leq \frac{1}{\epsilon} \int_{\xi_F}^{\xi_3} \sqrt{\frac{F(z)}{G(z)}} dz < \infty.$$

Hence every such geodesic reaches the singular boundary $\{z = \xi_F\}$ in finite proper time *unless* $(\dot{z}, \dot{\theta})$ becomes unbounded before reaching that set. We will see shortly that this second possibility cannot occur.

5.2.2. Uniform bounds. We wish, now, to derive uniform bounds on timelike geodesic segments contained in the region $\{\xi_F + \epsilon < z < \xi_3\}$, with any $\epsilon > 0$.

We start by noting that

$$(5.21) \quad \frac{d\hat{\psi}}{ds} = \frac{A^2(x-z)^2(\xi_3 - z)}{F(x)G(z)(\xi_3 - \xi_1)} (\sigma(\xi_1 - z)c_t - c_\psi) + \frac{F(z)}{\sigma(\xi_3 - \xi_1)F(x)} c_t.$$

which is well behaved throughout the region of current interest.

In the region $\{\hat{v} > 0, \hat{w} > 0\}$ we can introduce coordinates v and w using the formulae

$$v = \frac{\ln \hat{v}}{c}, \quad w = -\frac{\ln \hat{w}}{c},$$

and then define t and z using (3.5)-(3.6). With those definitions one recovers the form (2.1) of the metric, so that we can use the previous formulae for geodesics:

$$(5.22) \quad \frac{d\hat{v}}{ds} = c \left\{ -\hat{v} \frac{F(z)}{F(x)} c_t - \frac{\sigma}{\nu \hat{w} (z - \xi_2)^2} \left[\beta(x, z) + \sqrt{-F(\xi_3)} \frac{dz}{ds} \right] \right\},$$

$$(5.23) \quad \frac{d\hat{w}}{ds} = c \left\{ \frac{F(z)}{F(x)} c_t \hat{w} + \frac{\sigma}{\nu \hat{v} (z - \xi_2)^2} \left[\beta(x, z) - \sqrt{-F(\xi_3)} \frac{dz}{ds} \right] \right\},$$

where

$$\beta(x, z) := \frac{A^2(x-z)^2}{F(x)} (\sigma(\xi_1 - z) c_t - c_\psi),$$

and we note that both right-hand-sides have a potential problem at $\{z = \xi_3\}$, where $\hat{v}\hat{w}$ vanishes. Next, from (5.22) and (5.23),

$$(5.24) \quad \underbrace{\sqrt{-F(z)} \frac{dz}{ds}}_{=: \alpha} = -\sqrt{\left| \frac{F(z)}{F(\xi_3)} \right|} \frac{\nu(z - \xi_2)^2}{2c\sigma} \left(\hat{w} \frac{d\hat{v}}{ds} + \hat{v} \frac{d\hat{w}}{ds} \right),$$

while from (5.22)-(5.23) we further have

$$(5.25) \quad \hat{w} \frac{d\hat{v}}{ds} - \hat{v} \frac{d\hat{w}}{ds} = -\frac{2cF(z)}{F(x)} c_t \hat{v}\hat{w} - \frac{2c\sigma}{\nu(\xi_2 - z)^2} \frac{A^2(x-z)^2}{F(x)} (\sigma(\xi_1 - z) c_t - c_\psi).$$

We continue by rewriting (5.9) so that the problematic factors in (5.22)-(5.23) are grouped together

$$(5.26) \quad \frac{F(x)}{|G(z)|A^2(x-z)^2} \left[\underbrace{|F(z)| \left(\frac{dz}{ds} \right)^2 - \frac{A^4(x-z)^4}{F(x)^2} (\sigma(\xi_1 - z) c_t - c_\psi)^2}_{\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)} \right] \\ = -\lambda + \frac{|F(z)|}{F(x)} c_t^2 + \frac{F^2(z)}{A^2(x-z)^2} \left(\frac{d\theta}{ds} \right)^2 + \frac{A^2(x-z)^2 F(x)}{G(x) F^2(z)} c_\varphi^2.$$

By Section 5.2.1 any causal geodesic will either reach $\{z = \xi_3\}$ in finite affine time, say $s = \hat{s}$, or will cease to exist before that time. In what follows we therefore assume $0 \leq s < \hat{s}$.

We continue by writing down the evolution equations for x and z , which can easily be obtained from the Lagrangean $L = g(\dot{\gamma}, \dot{\gamma})$, and read

$$(5.27) \quad 2 \frac{d}{ds} \left(\frac{F^2(z)}{A^2(x-z)^2 G(x)} \frac{dx}{ds} \right) \\ = -\frac{F'(x)F(z)}{F^2(x)} c_t^2 - \frac{1}{A^2 G(z)} \frac{\partial}{\partial x} \left(\frac{F(x)}{(x-z)^2} \right) \left(F(z) \dot{z}^2 + \frac{A^4(x-z)^4}{F^2(x)} (\sigma(\xi_1 - z) c_t - c_\psi)^2 \right) \\ + \frac{F^2(z)}{A^2} \left[\frac{\partial}{\partial x} \left(\frac{1}{G(x)(x-z)^2} \right) \dot{x}^2 + \frac{\partial}{\partial x} \left(\frac{G(x)}{F(x)(x-z)^2} \right) \frac{A^4(x-z)^4 F^2(x)}{F^4(z) G^2(x)} c_\varphi^2 \right].$$

(5.28)

$$\begin{aligned}
 & -2 \frac{d}{ds} \left(\frac{F(x)F(z)}{A^2(x-z)^2 G(z)} \frac{dz}{ds} \right) = \frac{F'(z)}{F(x)} c_t^2 \\
 & -2\sigma \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)c_t + \frac{\partial}{\partial z} \left(\frac{F^2(z)}{A^2(x-z)^2} \right) \left[\dot{\theta}^2 + \frac{A^4(x-z)^4 F(x)}{F^4(z)G(x)} c_\varphi^2 \right] \\
 & - \frac{F(x)}{A^2} \left[\frac{\partial}{\partial z} \left(\frac{F(z)}{G(z)(x-z)^2} \right) \dot{z}^2 + \frac{\partial}{\partial z} \left(\frac{G(z)}{(x-z)^2} \right) \frac{A^4(x-z)^4}{F^2(x)G^2(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 \right] .
 \end{aligned}$$

Since z is a time function, the derivative \dot{z} has constant sign. So z can be used as a parameter along γ , and we can view (5.28) as an evolution equation in z for \dot{z} . For this we multiply by ds/dz , obtaining

(5.29)

$$\begin{aligned}
 & -2 \frac{d}{dz} \left(\frac{F(x)F(z)}{A^2(x-z)^2 G(z)} \frac{dz}{ds} \right) \\
 & = \left[\frac{F'(z)}{F(x)} c_t^2 - 2\sigma \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)c_t \right] \frac{ds}{dz} \\
 & + \frac{\partial}{\partial z} \left(\frac{F^2(z)}{A^2(x-z)^2} \right) \left[\left(\frac{d\theta}{dz} \right)^2 \frac{dz}{ds} + \frac{A^4(x-z)^4 F(x)}{F^4(z)G(x)} c_\varphi^2 \frac{ds}{dz} \right] \\
 & - \frac{F(x)}{A^2} \left[\frac{\partial}{\partial z} \left(\frac{F(z)}{G(z)(x-z)^2} \right) \frac{dz}{ds} + \frac{\partial}{\partial z} \left(\frac{G(z)}{(x-z)^2} \right) \frac{A^4(x-z)^4}{F^2(x)G^2(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 \frac{ds}{dz} \right] .
 \end{aligned}$$

Yet another variation on (5.19) reads

$$(5.30) \quad \frac{|F(z)|}{|G(z)|} = \frac{F^2(z)}{F(x)} \left(\frac{d\theta}{dz} \right)^2 + \left(\frac{ds}{dz} \right)^2 \left(\dot{\chi} + \frac{A^4(x-z)^4}{G(x)F^2(z)} c_\varphi^2 \right) .$$

To obtain uniform bounds as z approaches ξ_3 , one can proceed as follows: Let $f > 0$ be defined by the formula

$$(5.31) \quad f := \frac{F(x)F(z)}{A^2(x-z)^2 G(z)} \frac{dz}{ds} .$$

Using the identity

$$(5.32) \quad -\frac{2}{\sqrt{-G(z)}} \frac{d}{dz} \left(\sqrt{-G(z)} f \right) = -2 \frac{df}{dz} - \frac{G'(z)}{G(z)} f$$

we rewrite (5.29) as

(5.33)

$$\begin{aligned}
 & -\frac{2}{\sqrt{-G(z)}} \frac{d}{dz} \left(\frac{F(x)|F(z)|}{A^2(x-z)^2 \sqrt{-G(z)}} \frac{dz}{ds} \right) \\
 & = \left[\frac{F'(z)}{F(x)} c_t^2 - 2\sigma \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)c_t \right] \frac{ds}{dz} \\
 & + \frac{\partial}{\partial z} \left(\frac{F^2(z)}{A^2(x-z)^2} \right) \left[\left(\frac{d\theta}{dz} \right)^2 \frac{dz}{ds} + \frac{A^4(x-z)^4 F(x)}{F^4(z)G(x)} c_\varphi^2 \frac{ds}{dz} \right] \\
 & - \frac{F(x)}{A^2 G(z)} \left[\frac{\partial}{\partial z} \left(\frac{F(z)}{(x-z)^2} \right) \frac{dz}{ds} + \frac{\partial}{\partial z} \left(\frac{G(z)}{(x-z)^2} \right) \frac{A^4(x-z)^4}{F^2(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 \frac{ds}{dz} \right] .
 \end{aligned}$$

Equation (5.30) gives

$$(5.34) \quad \begin{aligned} & \frac{F(x)}{F^2(z)} \frac{dz}{ds} \left[\frac{F^2(z)}{F(x)} \left(\frac{d\theta}{dz} \right)^2 + \left(\frac{ds}{dz} \right)^2 \frac{A^4(x-z)^4}{G(x)F^2(z)} c_\varphi^2 \right] \\ &= \frac{F(x)}{|F(z)G(z)|} \frac{dz}{ds} - \dot{\chi} \frac{F(x)}{F^2(z)} \frac{ds}{dz}, \end{aligned}$$

leading to

$$(5.35) \quad \begin{aligned} & \frac{2}{\sqrt{-G(z)}} \frac{d}{dz} \left(\frac{F(x)|F(z)|}{A^2(x-z)^2 \sqrt{-G(z)}} \frac{dz}{ds} \right) \\ &= \frac{F(x)}{A^2G(z)} \left\{ \underbrace{\frac{\partial}{\partial z} \left(\frac{F(z)}{(x-z)^2} \right) - \frac{\partial}{\partial z} \left(\frac{F^2(z)}{(x-z)^2} \right) \frac{1}{F(z)}}_{-F'(z)/(x-z)^2} \right\} \frac{dz}{ds} - \frac{ds}{dz} \chi, \end{aligned}$$

where

$$(5.36) \quad \begin{aligned} \chi := & \frac{F'(z)}{F(x)} c_t^2 - 2\sigma \frac{A^2(x-z)^2}{F(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)c_t \\ & - \frac{\partial}{\partial z} \left(\frac{G(z)}{(x-z)^2} \right) \frac{A^2(x-z)^4}{F(x)G^2(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 \\ & + \frac{\partial}{\partial z} \left(\frac{F^2(z)}{A^2(x-z)^2} \right) \left[\frac{A^4(x-z)^4}{F(x)F^2(z)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 \right. \\ & \left. + \frac{A^2(x-z)^2}{F^2(z)} \left(\lambda + \frac{F(z)}{F(x)} c_t^2 \right) \right]. \end{aligned}$$

Using an identity similar to (5.32) with G replaced by F , this can also be rewritten as

$$(5.37) \quad 2 \frac{1}{\sqrt{|F(z)G(z)|}} \frac{d}{dz} \left(\frac{F(x)(|F(z)|)^{3/2}}{A^2(x-z)^2 \sqrt{|G(z)|}} \frac{dz}{ds} \right) = -\eta \chi \frac{ds}{dz};$$

here, we have written the general formula which holds whatever the sign of $F(z)G(z)$, with $\eta = \pm 1$ being that sign; in the current context, $\eta = 1$. Setting

$$h := \frac{F(x)(-F(z))^{3/2}}{A^2(x-z)^2 \sqrt{|G(z)|}} \frac{dz}{ds},$$

we obtain

$$(5.38) \quad 2h \frac{dh}{dz} = \frac{dh^2}{dz} = -\frac{F(x)F^2(z)}{A^2(x-z)^2} \eta \chi.$$

From (5.30) one has

$$(5.39) \quad \left| \frac{d\theta}{dz} \right| \leq \sqrt{\frac{F(x)}{|F(z)G(z)|}}.$$

Since the right-hand-side is integrable in z on $[\xi_F, \xi_3]$, we infer that θ has finite limits both as $z \rightarrow \xi_F$ and $z \rightarrow \xi_3$,

$$(5.40) \quad \theta \rightarrow_{z \rightarrow \xi_3} \theta_3, \quad x \rightarrow_{z \rightarrow \xi_3} x_3, \quad \theta \rightarrow_{z \rightarrow \xi_F} \theta_F, \quad x \rightarrow_{z \rightarrow \xi_F} x_F.$$

In everything that follows we choose some small $\epsilon > 0$ and assume that $z \in [\xi_F + \epsilon, \xi_3]$.

Suppose, first, that

$$(5.41) \quad \sigma(\xi_1 - \xi_3)c_t - c_\psi = 0 \implies \sigma(\xi_1 - z)c_t - c_\psi = \sigma(\xi_3 - z)c_t.$$

Then the right-hand-side of (5.38) is bounded, which implies that h is bounded, and has a limit as z approaches ξ_3 . From the definition of h we conclude that

$$|\dot{z}| \leq C\sqrt{-G(z)}.$$

Equation (5.19) implies, for z near ξ_3 ,

$$(5.42) \quad \dot{\theta}^2 + \frac{c_\varphi^2}{G(x)} \leq C.$$

We rewrite (5.22)-(5.23) as an evolution equations in z :

$$(5.43) \quad \frac{d\hat{v}}{dz} = c\hat{v} \left\{ -\frac{F(z)}{F(x)}c_t \frac{ds}{dz} + \frac{\sigma(z - \xi_1)}{G(z)} \left[\beta(x, z) \frac{ds}{dz} + \sqrt{-F(\xi_3)} \right] \right\},$$

$$(5.44) \quad \frac{d\hat{w}}{dz} = c\hat{w} \left\{ \frac{F(z)}{F(x)}c_t \frac{ds}{dz} - \frac{\sigma(z - \xi_1)}{G(z)} \left[\beta(x, z) \frac{ds}{dz} - \sqrt{-F(\xi_3)} \right] \right\},$$

where $\beta(x, z)$ has been defined as:

$$\beta(x, z) := \frac{A^2(x - z)^2}{F(x)} (\sigma(\xi_1 - z)c_t - c_\psi).$$

Note that (5.41) implies that all prefactors of $\frac{ds}{dz}$ are bounded near $z = \xi_3$.

Suppose, first, that $c_\psi = 0$, by (5.41) this happens if and only if $c_t = 0$. Comparing (5.43) with (5.44) one finds that

$$\frac{d(\ln \hat{w})}{dz} = \frac{d(\ln \hat{v})}{dz} \iff \frac{d(\ln(\hat{w}/\hat{v}))}{dz} = 0.$$

Thus there exists a constant ρ , different from 0 since $\hat{v}\hat{w} \neq 0$ in the region of interest, such that

$$\hat{v} = \rho\hat{w}.$$

Inserting back into (5.22) one finds

$$\left| \frac{d\hat{v}}{ds} \right| \leq C \frac{|\dot{z}|}{\hat{w}} \leq C \frac{\sqrt{\hat{v}\hat{w}}}{\hat{w}} = C \frac{\sqrt{\hat{v}}}{\sqrt{\hat{w}}} \leq C,$$

with a similar calculation for $d\hat{w}/ds$ using (5.23). Thus

$$(5.45) \quad \left| \frac{d\hat{v}}{ds} \right| + \left| \frac{d\hat{w}}{ds} \right| \leq C.$$

From (5.30), $\dot{\theta}$ is bounded, as well as derivatives of all coordinates functions along γ , and smooth extendibility of the geodesic across $\hat{w} = \hat{v} = 0$ readily follows.

So, still assuming (5.41), we suppose instead that c_t is different from zero. From (5.30), and since $\dot{\chi}$ is positive, we get

$$(5.46) \quad \left| \frac{ds}{dz} \right| \leq \frac{C}{\sqrt{|G(z)|}} .$$

Equation (5.43) with (5.44) gives now

$$\left| \frac{d(\ln(\hat{w}/\hat{v}))}{dz} \right| \leq \frac{C}{\sqrt{|G(z)|}} ,$$

which is integrable in z , so

$$\hat{v} = \rho(z)\hat{w} ,$$

for a function ρ which has a finite limit as $z \rightarrow \xi_3$. One concludes as when $c_t = 0$.

We continue with the general case,

$$(5.47) \quad \sigma(\xi_1 - \xi_3)c_t - c_\psi \neq 0 .$$

Near ξ_3 , the x -dependence of the most singular terms in (5.38) cancels out, leading to

$$(5.48) \quad \begin{aligned} \frac{dh^2}{dz} &= \frac{F^2(\xi_3) \left(\sigma(\xi_1 - \xi_3)c_t - c_\psi \right)^2 + O(z - \xi_3)}{\nu(\xi_3 - \xi_2)(\xi_3 - \xi_1)} \times \frac{1}{(z - \xi_3)^2} \\ &=: \frac{a^2 + O(z - \xi_3)}{(z - \xi_3)^2} . \end{aligned}$$

By integration,

$$(5.49) \quad h^2 = -\frac{a^2}{z - \xi_3} + O(\ln|z - \xi_3|) .$$

Then, from the definition of h , we eventually find

$$(5.50) \quad \frac{dz}{ds} = \frac{A^2(x_3 - \xi_3)^2}{\underbrace{\sqrt{|F(\xi_3)|F(x_3)}}_{=: \hat{a}}} (\sigma(\xi_1 - \xi_3)c_t - c_\psi) + O((z - \xi_3) \ln|z - \xi_3|) .$$

Next, we need to know how the x -limit is attained. For this, integration of (5.39) gives

$$|\theta(z) - \theta(\xi_3)| \leq C\sqrt{|z - \xi_3|} .$$

But, by the definition (5.11) of θ , we have, for θ_1 close to θ_2 ,

$$|\theta_1 - \theta_2| \approx \begin{cases} \sqrt{|x_1 - x_2|}, & G(x_2) = 0; \\ |x_1 - x_2|, & \text{otherwise.} \end{cases}$$

Hence

$$|x(z) - x(\xi_3)| \leq \begin{cases} C|z - \xi_3|, & G(x_3) = 0; \\ C\sqrt{|z - \xi_3|}, & \text{otherwise.} \end{cases} .$$

Inserting this into (5.19), the leading order singularity cancels out:

$$(5.51) \quad \dot{\theta}^2 + \frac{c_\varphi^2}{G(x)} \leq C|z - \xi_3|^{-1/2} .$$

We pass now to the equation satisfied by \dot{x} . In terms of θ , (5.27) can be rewritten as

$$(5.52) \quad \begin{aligned} & \frac{2}{\sqrt{G(x)}} \frac{d}{ds} \left(\frac{F^2(z)}{A^2(x-z)^2} \frac{d\theta}{ds} \right) \\ &= -\frac{F'(x)F(z)}{F^2(x)} c_t^2 - \frac{1}{A^2 G(z)} \frac{\partial}{\partial x} \left(\frac{F(x)}{(x-z)^2} \right) \left(F(z)z^2 + \frac{A^4(x-z)^4}{F^2(x)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 \right) \\ & \quad + \frac{F^2(z)}{A^2} \left[\frac{\partial}{\partial x} \left(\frac{1}{(x-z)^2} \right) \dot{\theta}^2 + \frac{\partial}{\partial x} \left(\frac{G(x)}{F(x)(x-z)^2} \right) \frac{A^4(x-z)^4 F^2(x)}{F^4(z)G^2(x)} c_\varphi^2 \right]. \end{aligned}$$

We can use (5.19) to eliminate z^2 from this equation, obtaining

$$(5.53) \quad \frac{2}{\sqrt{G(x)}} \frac{d}{ds} \left(\frac{F^2(z)}{A^2(x-z)^2} \frac{d\theta}{ds} \right) = \tilde{\chi} - \frac{F'(x)F^2(z)}{A^2 F(x)(x-z)^2} \dot{\theta}^2,$$

where

$$(5.54) \quad \tilde{\chi} := \left[-\frac{\partial}{\partial x} \left(\frac{F(x)}{(x-z)^2} \right) + \frac{F^2(x)}{G(x)} \frac{\partial}{\partial x} \left(\frac{G(x)}{F(x)(x-z)^2} \right) \right] \frac{A^2(x-z)^4}{G(x)F^2(z)} c_\varphi^2 - \frac{F'(x)F(z)}{F^2(x)} c_t^2 + \frac{\partial}{\partial x} \left(\frac{F(x)}{(x-z)^2} \right) \frac{(x-z)^2}{F(x)} \left(\lambda \frac{F(z)}{F(x)} c_t^2 \right).$$

Equivalently,

$$(5.55) \quad \frac{2}{\sqrt{F(x)G(x)}} \frac{d}{ds} \left(\frac{F^2(z)\sqrt{F(x)}}{A^2(x-z)^2} \frac{d\theta}{ds} \right) = \tilde{\chi}.$$

Multiplying by ds/dz , taking into account that $\dot{a} \neq 0$ by (5.47) and (5.50), from (5.51) we are led to an evolution equation of the form

$$(5.56) \quad \frac{d}{dz} \left(\frac{F^2(z)\sqrt{F(x)}}{A^2(x-z)^2} \frac{d\theta}{ds} \right) = O(|z - \xi_3|^{-3/4}).$$

We note that the right-hand side of (5.56) is integrable in z near ξ_3 , hence $|\dot{\theta}|$ is bounded near ξ_3 .

Using arguments already given, it is straightforward to obtain now the following:

THEOREM 5.3. *Causal geodesics maximally extended in the region $\{\xi_F < z < \xi_3\}$ and directed towards $z = \xi_F$ reach this last set in finite affine time.*

In fact the conclusion holds true as well for those spacelike geodesics for which \dot{z} does not change sign.

To get uniformity towards ξ_3 , we first note that, from the integrability of the right-hand-side of (5.56),

$$(5.57) \quad \lim_{z \rightarrow \xi_3} \dot{\theta} \text{ exists, in particular } |\dot{\theta}| \leq C_\epsilon \text{ on } [\xi_F + \epsilon, \xi_3].$$

We return now to (5.22)-(5.23). From what has been said, the limit $\lim_{z \rightarrow \xi_3} \beta$ exists. By (5.26) multiplied by $|G(z)|$, or otherwise, we find that the limit

$\lim_{z \rightarrow \xi_3} \alpha$ exists, and

$$(5.58) \quad \lim_{z \rightarrow \xi_3} \alpha = \pm \lim_{z \rightarrow \xi_3} \beta .$$

We have $\lim_{z \rightarrow \xi_3} \beta \neq 0$ by (5.47). Suppose, first, that (5.58) holds with the plus sign. We write $(\alpha - \beta)/G$ as $[(\alpha^2 - \beta^2)/G]/(\alpha + \beta)$, and use (5.26) to obtain that the limit

$$\lim_{z \rightarrow \xi_3} \frac{\alpha - \beta}{G(z)}$$

exists. This implies that (5.23) can be written in the form

$$(5.59) \quad \frac{d\hat{w}}{ds} = \phi(s)\hat{w}$$

for some continuous function ϕ . By integration \hat{w} has a non-zero limit as $z \rightarrow \xi_3$, and (5.59) implies that $\frac{d\hat{w}}{ds}$ has a limit as well. It follows that \hat{v} tends to be zero, since $\hat{v}\hat{w}$ does. Equation (5.22) shows now that $\frac{d\hat{w}}{ds}$ has a limit. Hence

$$(5.60) \quad \hat{v} + \hat{w} + \left| \frac{d\hat{w}}{ds} \right| + \left| \frac{d\hat{v}}{ds} \right| + \left| \frac{dz}{ds} \right| + \left| \frac{d\theta}{ds} \right| \leq C .$$

From what has been said so far we conclude that those geodesics smoothly extend across the Killing horizon $\{\hat{w}\hat{v} = 0\}$.

A similar argument, with \hat{v} interchanged with \hat{w} , applies when the minus sign occurs in (5.58).

Summarising, we have proved:

PROPOSITION 5.4. *Causal geodesics in the region $\{\xi_F < z < \xi_3\}$ reach the bifurcate Killing horizon $\{\hat{w}\hat{v} = 0\}$ in finite affine time, and are smoothly extendible there.*

5.3. Geodesics in the region $\{\xi_3 < z \leq 2\xi_3\}$. In this coordinate range $F(z)$ is strictly negative and $G(z)$ is positive, so we rewrite (5.9) as

$$(5.61) \quad \begin{aligned} & \frac{|F(z)|}{G(z)} \dot{z}^2 + \frac{F^2(z)}{F(x)} \dot{\theta}^2 + \frac{A^4(x-z)^4}{G(x)F^2(z)} c_\varphi^2 \\ &= \frac{A^4(x-z)^4}{F^2(x)G(z)} (\sigma(\xi_1 - z)c_t - c_\psi)^2 - \frac{A^2(x-z)^2}{F(x)} \left[-\lambda + \frac{|F(z)|}{F(x)} c_t^2 \right] . \end{aligned}$$

From this one immediately obtains a uniform bound on \dot{z} , as well as

$$(5.62) \quad \left| \frac{d\theta}{ds} \right| + \frac{c_\varphi^2}{\sqrt{G(x)}} \leq \frac{C}{\sqrt{|G(z)|}} .$$

Geodesics on which z stays bounded away from ξ_3 have already been taken care of in Section 5.1. So we consider a geodesic segment $\gamma : [0, \hat{s}] \rightarrow \{\xi_3 < z \leq 2\xi_3\}$, with $\hat{s} < \infty$, such that there exists a sequence $s_i \rightarrow \hat{s}$ with $z(s_i) \rightarrow \xi_3$. (If $\hat{s} = \infty$ there is nothing to prove.) But we have just seen that the function $s \mapsto z(s)$ is uniformly Lipschitz, hence $z(s) \rightarrow \xi_3$ as $s \rightarrow \hat{s}$.

The Killing vector field

$$(5.63) \quad X := \frac{\partial}{\partial t} + \frac{A\sqrt{\xi_F}}{\sqrt{\nu}(\xi_3 - \xi_1)} \frac{\partial}{\partial \psi}$$

is tangent to the generators of the Killing horizon \mathcal{E} , thus light-like at \mathcal{E} . As the horizon is non-degenerate, X is timelike near \mathcal{E} for small negative values of $\hat{v}\hat{w}$. But then $g(X, \dot{\gamma}) < 0$ for causal future directed geodesics in the domain of outer communications near \mathcal{E} , which shows that for causal geodesics through the region of current interest we must have

$$(5.64) \quad g(X, \dot{\gamma}) = c_t + \frac{A\sqrt{\xi_F}}{\sqrt{\nu}(\xi_3 - \xi_1)}c_\psi \neq 0 \iff \sigma(\xi_1 - \xi_3)c_t - c_\psi \neq 0.$$

It follows that causal geodesics intersecting $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle$ for which (5.64) does *not* hold stay away from a neighborhood of \mathcal{E} , and are therefore complete by the results in Section 5.1.

From what has been said we conclude that:

PROPOSITION 5.5. *Causal geodesics crossing the event horizon $\{\hat{v}\hat{w} = 0\}$ and satisfying*

$$\sigma(\xi_1 - \xi_3)c_t - c_\psi = 0$$

are entirely contained within $\{\hat{v}\hat{w} > 0\} \cup \{\hat{v} = 0 = \hat{w}\}$.

We continue by noting that (5.37) remains true with χ as given by (5.36) on every interval of values of s on which \dot{z} does not change sign. We want to show, using a contradiction argument, that \dot{z} will be eventually negative for s close enough to \hat{s} . So suppose not, then there will be increasing sequences $\{s_i^\pm\}_{i \in \mathbb{N}}$, $s_i^\pm \rightarrow \hat{s}$, with $s_i^- < s_i^+$, such that

$$\dot{z}(s_i^\pm) = 0, \quad \dot{z}(s) < 0 \text{ on } I_i := (s_i^-, s_i^+), \quad z_i^\pm := z(s_i^\pm) \searrow \xi_3, \quad z_i^- > z_i^+.$$

By inspection of (5.36) and (5.38), there exists $z_* > \xi_3$ and $\epsilon > 0$ such that for all $z \in (\xi_3, z_*)$ we have

$$(5.65) \quad \frac{dh^2}{dz} \leq -\frac{\epsilon}{(z - \xi_3)^2}.$$

Integrating,

$$(5.66) \quad h^2(z_i^-) - h^2(z_i^+) \leq -\int_{z_i^+}^{z_i^-} \frac{\epsilon}{(z - \xi_3)^2} dz < 0,$$

which contradicts the fact that $\dot{z}(s_i^\pm) = 0$, and shows that \dot{z} is indeed strictly negative sufficiently close to the event horizon.

As in Section 5.2.2 we obtain now (5.49). This, together with the definition of h , implies that dz/ds is strictly bounded away from zero; equivalently,

$$\left| \frac{ds}{dz} \right| \leq C.$$

In particular $\{\hat{v}\hat{w} = 0\}$ is reached in finite affine time. Moreover, from (5.62) we now find

$$(5.67) \quad \left| \frac{d\theta}{dz} \right| \leq \frac{C}{\sqrt{|G(z)|}}.$$

Keeping in mind (5.64), one can now repeat the arguments of Section 5.2.2 (after (5.47), with (5.39) replaced by (5.67)), to conclude that:

THEOREM 5.6. *All maximally extended causal geodesics through $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle$ are either complete, or can be smoothly extended across the horizon $\{\hat{v}\hat{w} = 0\}$.*

5.4. The Killing horizon. Consider a causal geodesic γ such that $\gamma(s_0) \in \{\hat{v}\hat{w} = 0\}$. If $d\hat{v}/ds$ or $d\hat{w}/ds$ are both different from zero at s_0 , then γ immediately enters the regions already covered. If γ enters $\{z < \xi_3\}$ it will stay there by monotonicity of z , so Theorem 5.3 applies. Otherwise it enters the region $\{z > \xi_3\}$; then either it approaches $z = \xi_3$ again, in which case it crosses back to $\{z < \xi_3\}$ by Theorem 5.6, and Theorem 5.3 applies; or it stays away from $z = \xi_3$ for all subsequent times, in which case it is complete, again by Theorem 5.6.

So it remains to consider geodesics for which

$$\forall s \quad \hat{v}(s)\hat{w}(s) = 0 = \frac{d\hat{v}}{ds}(s)\frac{d\hat{w}}{ds}(s).$$

Since the bifurcation ring $S := \{\hat{w} = \hat{v} = 0\}$ is spacelike, those causal geodesics which pass through S immediately leave the bifurcate horizon, except for the generators of the latter. But those generators are complete by standard results [3]. Since the only causal curves on a null hypersurface are its generators, the analysis is complete.

This achieves the proof of Theorem 5.1.

q.e.d.

References

1. R. Beig and P.T. Chruściel, *The asymptotics of stationary electro-vacuum metrics in odd space-time dimensions*, *Class. Quantum Grav.* **24** (2007), 867–874, Zbl pre05131275. MR MR2297271
2. B. Berger, P.T. Chruściel, J. Isenberg, and V. Moncrief, *Global foliations of vacuum space-times with T^2 isometry*, *Ann. Phys. (NY)* **260** (1997), 117–148, Zbl 0929.58013. MR MR1474313 (98j:83005)
3. R.H. Boyer, *Geodesic Killing orbits and bifurcate Killing horizons*, *Proc. Roy. Soc. London A* **311** (1969), 245–252, Zbl 0193.27092. MR MR0244911 (39 #6224)
4. R.H. Boyer and R.W. Lindquist, *Maximal analytic extension of the Kerr metric*, *Jour. Math. Phys.* **8** (1967), 265–281, Zbl 0149.23503. MR MR0218093 (36 #1182)
5. B. Carter, *Global structure of the Kerr family of gravitational fields*, *Phys. Rev.* **174** (1968), 1559–1571.
6. P.T. Chruściel, *On the invariant mass conjecture in general relativity*, *Commun. Math. Phys.* **120** (1988), 233–248, Zbl 0661.53060. MR MR973533 (89m:83033)
7. ———, *On uniqueness in the large of solutions of Einstein equations (“Strong Cosmic Censorship”)*, Australian National University Press, Canberra, 1991.
8. ———, *Uniqueness of black holes revisited*, *Helv. Phys. Acta* **69** (1996), 529–552, *Proceedings of Journées Relativistes 1996*, Ascona, May 1996, N. Straumann, Ph. Jetzer and G. Lavrelashvili (Eds.), arXiv:gr-qc/9610010, Zbl 0867.53072. MR MR1435995 (97m:83052)
9. ———, *A poor man’s positive energy theorem: II. Null geodesics*, *Class. Quantum Grav.* **21** (2004), 4399–4416, arXiv:gr-qc/0406077, Zbl 1061.83023.
10. ———, *Beijing lecture notes on mathematical relativity*, www.phys.univ-tours.fr/~piotr/papers/BeijingAll.pdf, 2006.

11. P.T. Chruściel and J. Costa, *On uniqueness of stationary black holes*, Astérisque (2008), 195–265, arXiv:0806.0016v2 [gr-qc]. MR MR2521649
12. T. Damour and B. Schmidt, *Reliability of perturbation theory in general relativity*, Jour. Math. Phys. **31** (1990), 2441–2453.
13. R. Emparan and H.S. Reall, *A rotating black ring solution in five dimensions*, Phys. Rev. Lett. **88** (2002), 101101, arXiv:hep-th/0110260. MR **MR1901280 (2003e:83060)**
14. J.L. Friedman, K. Schleich, and D.M. Witt, *Topological censorship*, Phys. Rev. Lett. **71** (1993), 1486–1489, erratum **75** (1995) 1872, Zbl 0934.83033.
15. R. Geroch, *Domain of dependence*, Jour. Math. Phys. **11** (1970), 437–449, Zbl 0189.27602. MR MR0270697 (42 #5585)
16. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Interscience Publishers, New York, 1963, Zbl 0119.37502. MR **MR1393941 (97c:53001b)**
17. J.M. Martín-García, *xAct: Efficient Tensor Computer Algebra*, <http://metric.iem.csic.es/Martin-Garcia/xAct>.
18. K. Nomizu, *On local and global existence of Killing vector fields*, Ann. Math. **72** (1960), 105–120, Zbl 0093.35103. MR MR0119172 (22 #9938)
19. B. O’Neill, *Semi-riemannian geometry: with application to relativity*, Academic Press, 1984, Zbl 0531.53051. MR **MR719023 (85f:53002)**
20. I. Rácz, *Space-time extensions. I*, Jour. Math. Phys. **34** (1993), 2448–2464, Zbl 0791.53062. MR **MR1219001 (94h:53094)**
21. I. Rácz and R.M. Wald, *Global extensions of space-times describing asymptotic final states of black holes*, Class. Quantum Grav. **13** (1996), 539–552, arXiv:gr-qc/9507055, Zbl 0866.53071. MR **MR1385315 (97a:83071)**
22. R.M. Wald, *General relativity*, University of Chicago Press, Chicago, 1984.

LMPT, FÉDÉRATION DENIS POISSON, TOURS
 MATHEMATICAL INSTITUTE AND HERTFORD COLLEGE, OXFORD

INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER
 UNIVERSITÉ MONTPELLIER 2