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Mireille Bousquet-Mélou, Guillaume Chapuy, Louis-François Préville-Ratelle

THE REPRESENTATION OF THE SYMMETRIC GROUP ON m-TAMARI INTERVALS

MIREILLE BOUSQUET-MÉLOU, GUILLAUME CHAPUY, AND LOUIS-FRANÇOIS PRÉVILLE-RATELLE

Abstract. An m-ballot path of size n is a path on the square grid consisting of north and east unit steps, starting at $(0,0)$, ending at (mn,n), and never going below the line $\{x=my\}$. The set of these paths can be equipped with a lattice structure, called the m-Tamari lattice and denoted by $\mathcal{T}_n^{(m)}$, which generalizes the usual Tamari lattice $\mathcal{T}_n$ obtained when $m=1$. This lattice was introduced by F. Bergeron in connection with the study of diagonal coinvariant spaces in three sets of n variables. The representation of the symmetric group $\mathfrak{S}_n$ on these spaces is conjectured to be closely related to the natural representation of $\mathfrak{S}_n$ on (labelled) intervals of the m-Tamari lattice, which we study in this paper.

An interval $[P,Q]$ of $\mathcal{T}_n^{(m)}$ is *labelled* if the north steps of Q are labelled from 1 to n in such a way the labels increase along any sequence of consecutive north steps. The symmetric group $\mathfrak{S}_n$ acts on labelled intervals of $\mathcal{T}_n^{(m)}$ by permutation of the labels. We prove an explicit formula, conjectured by F. Bergeron and the third author, for the character of the associated representation of $\mathfrak{S}_n$. In particular, the dimension of the representation, that is, the number of labelled m-Tamari intervals of size n, is found to be

$$(m+1)^n(mn+1)^{n-2}.$$

These results are new, even when $m=1$.

The form of these numbers suggests a connection with parking functions, but our proof is not bijective. The starting point is a recursive description of m-Tamari intervals. It yields an equation for an associated generating function, which is a refined version of the Frobenius series of the representation. This equation involves two additional variables x and y, a derivative with respect to y and iterated divided differences with respect to x. The hardest part of the proof consists in solving it, and we develop original techniques to do so, partly inspired by previous work on polynomial equations with "catalytic" variables.

1. Introduction and main result

An *m-ballot path* of size n is a path on the square grid consisting of north and east unit steps, starting at $(0,0)$, ending at (mn,n), and never going below the line $\{x=my\}$. It is well-known that there are

$$\frac{1}{mn+1}\binom{(m+1)n}{n}$$

such paths [11], and that they are in bijection with $(m+1)$-ary trees with n inner nodes.

François Bergeron recently defined on the set $\mathcal{T}_n^{(m)}$ of m-ballot paths of size n an order relation. It is convenient to describe it via the associated covering relation, exemplified in Figure 1.

Definition 1. *Let P and Q be two m-ballot paths of size n. Then Q covers P if there exists in P an east step a, followed by a north step b, such that Q is obtained from P by swapping a and S, where S is the shortest factor of P that begins with b and is a (translated) m-ballot path.*

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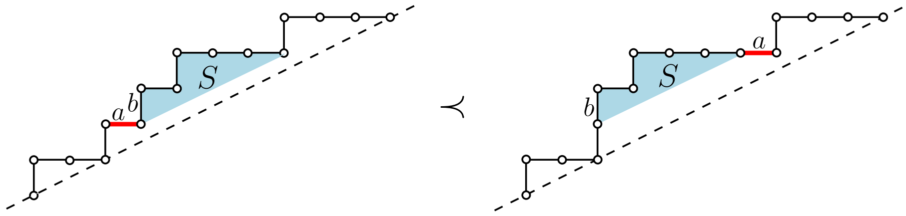

FIGURE 1. The covering relation between m-ballot paths ($m = 2$).

It was shown in [8] that this order endows $\mathcal{T}_n^{(m)}$ with a lattice structure, which is called the *m-Tamari lattice of size n*. When $m = 1$, it coincides with the classical Tamari lattice $\mathcal{T}_n$ [6, 12, 26, 27]. Figure 2 shows two of the lattices $\mathcal{T}_n^{(m)}$. The main result of [8] gives the number of intervals in $\mathcal{T}_n^{(m)}$ as

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}. \tag{1}$$

The lattices $\mathcal{T}_n^{(m)}$ are also known to be EL-shellable [30].

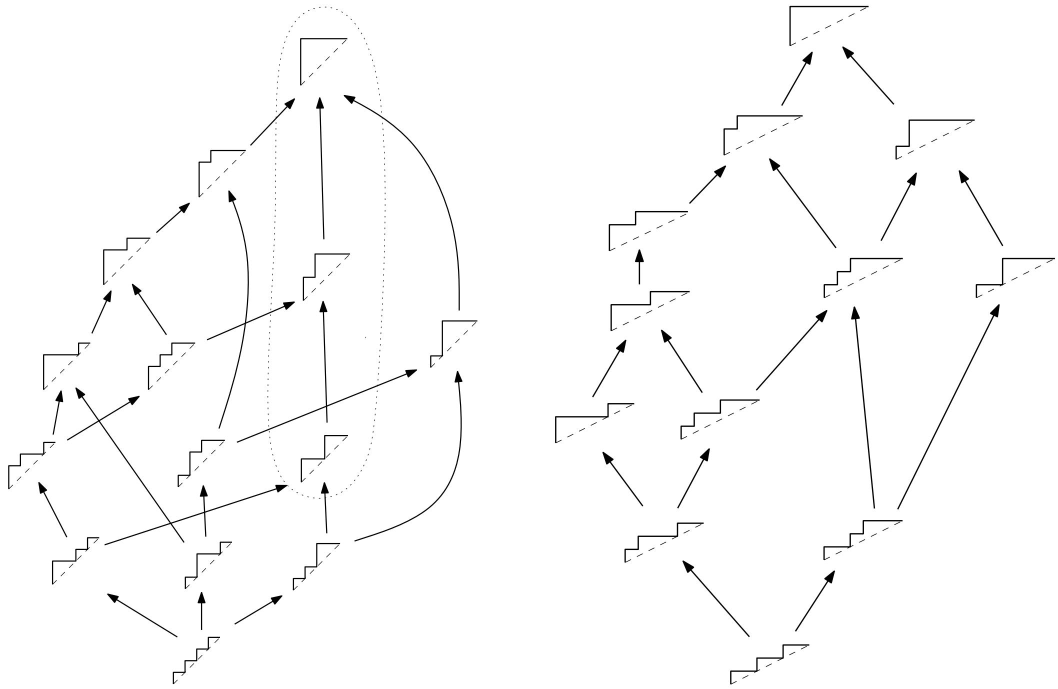

FIGURE 2. The m-Tamari lattice $\mathcal{T}_n^{(m)}$ for $m = 1$ and $n = 4$ (left) and for $m = 2$ and $n = 3$ (right). The three walks surrounded by a line in $\mathcal{T}_4^{(1)}$ form a lattice that is isomorphic to $\mathcal{T}_2^{(2)}$ (see Proposition 6).

The interest in these lattices is motivated by their — still conjectural — connections with trivariate diagonal coinvariant spaces [5, 8]. Some of these connections are detailed at the end of this introduction. In particular, it is believed that the representation of the symmetric group on these spaces is closely related to the representation of the symmetric group on *labelled* m-Tamari intervals. The aim of this paper is to characterize the latter representation, by describing explicitly its character.

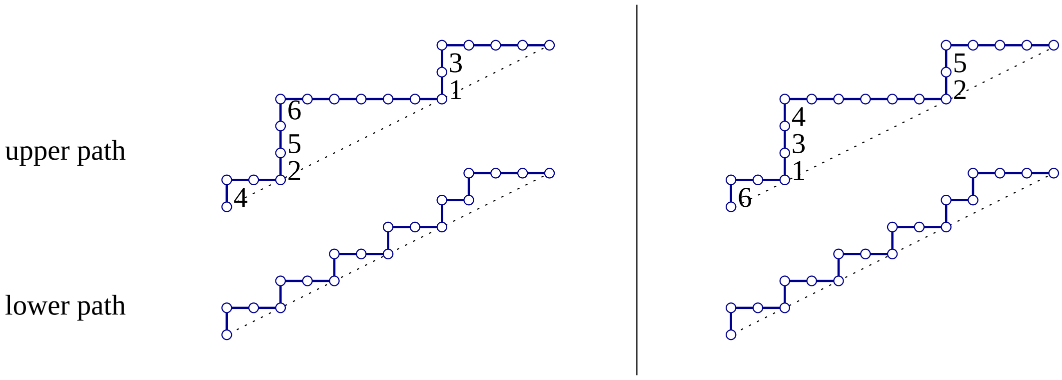

FIGURE 3. A labelled 2-Tamari interval, and its image under the action of $\sigma = 2\,3\,5\,6\,1\,4$.

So let us define this representation and state our main result. Let us call *ascent* of a path a maximal sequence of consecutive north steps. An m-ballot path of size n is *labelled* if the north steps are labelled from 1 to n, in such a way the labels increase along ascents (see the upper paths in Figure 3). These paths are in bijection with $(1, m, \dots, m)$-*parking functions* of size n, in the sense of [34, 35]: the function f associated with a path Q satisfies $f(i) = k$ if the north step of Q labelled i lies at abscissa $k-1$. The symmetric group $\mathfrak{S}_n$ acts on labelled m-ballot paths of size n by permuting labels, and then reordering them in each ascent (Figure 3, top paths). The character of this representation, evaluated at a permutation of cycle type $\lambda = (\lambda_1, \dots, \lambda_\ell)$, is

$$(mn+1)^{\ell-1}.$$

This formula is easily proved using the cycle lemma [31]. As recalled further down, this representation is closely related to the representation of $\mathfrak{S}_n$ on diagonal coinvariant spaces in two sets of variables.

Now an m-Tamari interval $[P, Q]$ is *labelled* if the upper path Q is labelled. The symmetric group $\mathfrak{S}_n$ acts on labelled intervals of $\mathcal{T}_n^{(m)}$ by rearranging the labels of Q as described above (Figure 3). We call this representation the m-*Tamari* representation of $\mathfrak{S}_n$. Our main result is an explicit expression for its character χ_m, which was conjectured by Bergeron and the third author [5].

Theorem 2. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of n and σ a permutation of $\mathfrak{S}_n$ having cycle type λ. Then for the m-Tamari representation of $\mathfrak{S}_n$,*

$$\chi_m(\sigma) = (mn+1)^{\ell-2} \prod_{1\le i\le \ell} \binom{(m+1)\lambda_i}{\lambda_i}. \tag{2}$$

Since $\mathfrak{S}_n$ acts by permuting labelled intervals, this is also the number of labelled m-Tamari intervals left unchanged under the action of σ. The value of the character only depends on the cycle type λ, and will sometimes be denoted $\chi_m(\lambda)$.

In particular, the dimension of the representation, that is, the number of labelled m-Tamari intervals of size n, is

$$\chi_m(\mathrm{id}) = (mn+1)^{n-2}(m+1)^n. \tag{3}$$

We were unable to find a bijective proof of these amazingly simple formulas. Instead, our proof uses generating functions and a recursive construction of intervals. Our main generating function records the numbers $\chi_m(\sigma)$, that is, the number of pairs (I, σ) where I is a labelled interval fixed by the permutation σ. This generating function involves variables $p_1, p_2, \dots$ (keeping track of the cycle type of σ) and t (for the size of I). The recursive construction of intervals that we use is borrowed from [8]. In order to translate it into an equation defining our generating function, we need to keep track of one more parameter defined on (I, σ), using an additional variable x

(Proposition 5, Section 3). The resulting equation involves discrete derivatives (a.k.a. divided differences) with respect to x, of unbounded order. The solution of equations with discrete derivatives of *bounded* order is now well-understood [9] (such equations are for instance involved in the enumeration (1) of unlabelled m-Tamari intervals). But this is the first time we meet an equation of unbounded order, and its solution is the most difficult and original part of the paper. Our approach requires to introduce one more variable y, and a derivative with respect to it. Its principles are explained in Section 4, and exemplified with the case $m = 1$. The general case is solved in Section 5. This approach was already used in a preprint by the same authors [7], where the special case (3) was proved. Since going from (3) to (2) implies a further complexification, this preprint may serve as an introduction to our techniques. The present paper is however self-contained. Section 6 gathers a few final comments. In particular, we reprove the main result of [8] giving the number of *unlabelled* intervals of $\mathcal{T}_n^{(m)}$.

In the remainder of this section, we recall some of the conjectured connections between Tamari intervals and trivariate diagonal coinvariant spaces. They seem to parallel the (now largely proved) connections between ballot paths and *bivariate* diagonal coinvariant spaces, which have attracted considerable attention in the past 20 years [14, 17, 18, 21, 24, 23, 29] and are still a very active area of research today [1, 2, 13, 16, 22, 19, 25, 28].

Let $X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ be a matrix of variables. The diagonal coinvariant space $\mathcal{DR}_{k,n}$ is defined as the quotient of the ring $\mathbb{C}[X]$ of polynomials in the coefficients of X by the ideal $\mathcal{J}$ generated by constant-term free polynomials that are invariant under permuting the columns of X. For example, when $k = 2$, denoting $x_{1,j} = x_j$ and $x_{2,j} = y_j$, the ideal $\mathcal{J}$ is generated by constant-term free polynomials f such that for all $\sigma \in \mathfrak{S}_n$,

$$f(X) = \sigma(f(X)), \quad \text{where} \quad \sigma(f(X)) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)}).$$

An *m-extension* of the spaces $\mathcal{DR}_{k,n}$ is of great importance here [15, p. 230]. Let $\mathcal{A}$ be the ideal of $\mathbb{C}[X]$ generated by *alternants* under the diagonal action described above; that is, by polynomials f such that $\sigma(f(X)) = \varepsilon(\sigma)f(X)$. There is a natural action of $\mathfrak{S}_n$ on the quotient space $\mathcal{A}^{m-1}\big/\mathcal{J}\mathcal{A}^{m-1}$. Let us twist this action by the $(m-1)^{\text{st}}$ power of the sign representation ε: this gives rise to spaces

$$\mathcal{DR}_{k,n}^m := \varepsilon^{m-1} \otimes \mathcal{A}^{m-1}\big/\mathcal{J}\mathcal{A}^{m-1},$$

so that $\mathcal{DR}_{k,n}^1 = \mathcal{DR}_{k,n}$. It is now a famous theorem of Haiman [23, 20] that, as representations of $\mathfrak{S}_n$,

$$\mathcal{DR}_{2,n}^m \cong \varepsilon \otimes \ \mathrm{Park}_m(n)$$

where $\mathrm{Park}_m(n)$ is the m-parking representation of $\mathfrak{S}_n$, that is, the representation on m-ballot paths of size n defined above.

In the case of three sets of variables, Bergeron and Préville-Ratelle [5] conjecture that, as representations of $\mathfrak{S}_n$,

$$\mathcal{DR}_{3,n}^m \cong \varepsilon \otimes \mathrm{Tam}_m(n),$$

where $\mathrm{Tam}_m(n)$ is the m-Tamari representation of $\mathfrak{S}_n$. The fact that the dimension of this space seems to be given by (3) is an earlier conjecture due to F. Bergeron. This was also observed earlier for small values of n by Haiman [24] in the case $m = 1$.

2. The refined Frobenius series

2.1. Definitions and notation

Let $\mathbb{L}$ be a commutative ring and t an indeterminate. We denote by $\mathbb{L}[t]$ (resp. $\mathbb{L}[[t]]$) the ring of polynomials (resp. formal power series) in t with coefficients in $\mathbb{L}$. If $\mathbb{L}$ is a field, then $\mathbb{L}(t)$ denotes the field of rational functions in t. This notation is generalized to polynomials, fractions and series in several indeterminates. We denote by bars the reciprocals of variables: for instance, $\bar{u} = 1/u$, so that $\mathbb{L}[u, \bar{u}]$ is the ring of Laurent polynomials in u with coefficients in $\mathbb{L}$. The coefficient of u^n in a Laurent polynomial $P(u)$ is denoted by $[u^n]P(u)$.

We use classical notation relative to integer partitions, which we recall briefly. A *partition* λ of n is a non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ summing to n. We write $\lambda \vdash n$ to mean that λ is a partition of n. Each component λ_i is called a *part*. The number of parts or *length* of the partition is denoted by $\ell(\lambda)$. The *cycle type* of a permutation $\sigma \in \mathfrak{S}_n$ is the partition of n whose parts are the lengths of the cycles of σ. This partition is denoted by $\lambda(\sigma)$. The number of permutations $\sigma \in \mathfrak{S}_n$ having cycle type $\lambda \vdash n$ equals $\frac{n!}{z_\lambda}$ where $z_\lambda := \prod_{i\geq 1} i^{\alpha_i}\alpha_i!$, where α_i is the number of parts equal to i in λ.

We let $p = (p_1, p_2, \dots)$ be an infinite list of independent variables, and for λ a partition of n, we let $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}$. The reader may view the p_λ's as power sums in some ground set of variables (see *e*.g. [32]). This point of view is not really needed in this paper, but it explains why we call our main generating function a *refined Frobenius series*. Throughout the paper, we denote by $\mathbb{K} = \mathbb{Q}(p_1, p_2, \dots)$ the field of rational fractions in the p_i's with rational coefficients.

Given a Laurent polynomial $P(u)$ in a variable u, we denote by $[u^{\geq}]P(u)$ the *non-negative part of* $P(u)$ *in* u, defined by

$$[u^{\geq}]P(u) = \sum_{i\geq 0} u^i\,[u^i]P(u).$$

The definition is then extended by linearity to power series whose coefficients are Laurent polynomials in u. We define similarly the positive part of $P(u)$, denoted by $[u^{>}]P(u)$.

We now introduce several series and polynomials which play an important role in this paper. They depend on two independent variables u and z. First, we let $v \equiv v(u)$ be the following Laurent polynomial in u:

$$v = (1+u)^{m+1}u^{-m}.$$

We now consider the following series:

$$V(v) = \sum_{k\geq 1} \frac{p_k}{k} v^k z^k. \tag{4}$$

It is is a formal power series in z whose coefficients are Laurent polynomials in u over the field $\mathbb{K}$. Finally we define the two following formal power series in z:

$$L \equiv L(z,p) \quad := \quad [u^0]V(v) = \sum_{k\geq 1} \frac{p_k}{k}\binom{(m+1)k}{k} z^k, \tag{5}$$

$$K(u) \equiv K(z,p;u) \quad := \quad [u^{>}]V(v) = \sum_{k\geq 1} \frac{p_k}{k} z^k \sum_{i=1}^{k} \binom{(m+1)k}{k-i} u^i. \tag{6}$$

As shown with these series, we often do not denote the dependence of our series in certain variables (like z and p above). This is indicated by the symbol $\equiv$.

2.2. A refined theorem

As stated in Theorem 2, the value of the character $\chi_m(\sigma)$ is the number of labelled intervals fixed under the action of σ, and one may see (2) as an enumerative result. Our main result is a refinement of (2) where we take into account two more parameters, which we now define. The first parameter is the number of *contacts* of the interval: A *contact* of a ballot path P is a vertex of P lying on the line $\{x = my\}$, and a *contact* of a Tamari interval $[P, Q]$ is a contact of the *lower* path P. We denote by $c(P)$ the number of contacts of P.

By definition of the action of $\mathfrak{S}_n$ on m-Tamari intervals, a labelled interval $I = [P, Q]$ is fixed by a permutation $\sigma \in \mathfrak{S}_n$ if and only if σ stabilizes the set of labels of each ascent of Q. Equivalently, each cycle of σ is contained in the label set of an ascent of Q. If this holds, we let $a_\sigma(Q)$ be the number of cycles of σ occurring in the *first* ascent of Q: this is our second parameter.

The main object we handle in this paper is a generating function for pairs (σ, I), where σ is a permutation and $I = [P, Q]$ is a labelled m-Tamari interval fixed by σ. In this series

$F^{(m)}(t,p;x,y)$, pairs (σ, I) are counted by the size $|I|$ of I (variable t), the number $c(P)$ of contacts (variable x), the parameter $a_\sigma(Q)$ (variable y), and the cycle type of σ (one variable p_i for each cycle of size i in σ). Moreover, $F^{(m)}(t,p;x,y)$ is an exponential series in t. That is,

$$F^{(m)}(t,p;x,y) = \sum_{I=[P,Q],\text{ labelled}} \frac{t^{|I|}}{|I|!} x^{c(P)} \sum_{\sigma\in\mathrm{Stab}(I)} y^{a_\sigma(Q)} p_{\lambda(\sigma)}, \tag{7}$$

where the first and second sums are taken respectively over all labelled m-Tamari intervals I, and over all permutations σ fixing I.

Note that when $(x,y)=(1,1)$, we have:

$$F^{(m)}(t,p;1,1) = \sum_{I=[P,Q]} \frac{t^{|I|}}{|I|!} \sum_{\sigma\in\mathrm{Stab}(I)} p_{\lambda(\sigma)} = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} \chi_m(\sigma) p_{\lambda(\sigma)} = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \chi_m(\lambda) \frac{p_\lambda}{z_\lambda},$$

since the number of intervals fixed by a permutation depends only on its cycle type, and since $\frac{n!}{z_\lambda}$ is the number of permutations of cycle type λ. Hence, in representation theoretic terms, $[t^n]F^{(m)}(t,p;1,1)$ is the *Frobenius characteristic* of the m-Tamari representation of $\mathfrak{S}_n$, also equal to

$$\sum_{\lambda\vdash n} c(\lambda) s_\lambda,$$

where s_λ is the Schur function of shape λ and $c(\lambda)$ is the multiplicity of the irreducible representation associated with λ in the m-Tamari representation [32, Chap. 4]. For this reason, we call $F^{(m)}(t,p;x,y)$ a *refined Frobenius series.*

The most general result of this paper is a (complicated) parametric expression of $F^{(m)}(t,p;x,y)$, which takes the following simpler form when $y=1$.

Theorem 3. *Let $F^{(m)}(t,p;x,y) \equiv F(t,p;x,y)$ be the refined Frobenius series of the m-Tamari representation, defined by* (7)*. Let z and u be two indeterminates, and write*

$$t = ze^{-mL} \quad \text{and} \quad x = (1+u)e^{-mK(u)}, \tag{8}$$

where $L \equiv L(z,p)$ and $K(u) \equiv K(z,p;u)$ are defined by (5) *and* (6)*. Then $F(t,p;x,1)$ becomes a series in z with polynomial coefficients in u and the p_i, and this series has a simple expression:*

$$F(t,p;x,1) = (1+\bar{u})e^{K(u)+L}\left((1+u)e^{-mK(u)} - 1\right) \tag{9}$$

with $\bar{u} = 1/u$. In particular, in the limit $u \to 0$, we obtain

$$F(t,p;1,1) = e^L \left(1 - m\sum_{k\geq 1} \frac{p_k}{k} z^k \binom{(m+1)k}{k-1}\right). \tag{10}$$

The form of this theorem is reminiscent of the enumeration of unlabelled m-Tamari intervals [8, Thm. 10], for which finding the "right" parametrization of the variables t, x and y was an important step in the solution. This will also be the case here.

Theorem 2 will follow from Theorem 3 by extracting the coefficient of p_λ/z_λ in $F(t,p;1,1)$ (via Lagrange's inversion). Our expression of $F^{(m)}(t,p;x,y)$ is given in Theorem 21. When $m=1$, it takes a reasonably simple form, which we now present. The case $m=2$ is also detailed at the end of Section 5 (Corollary 22).

Theorem 4. *Let $F^{(1)}(t,p;x,y) \equiv F(t,p;x,y)$ be the refined Frobenius series of the 1-Tamari representation, defined by* (7)*. Define the series $V(v), L$ and $K(u)$ by* (4–6)*, with $m=1$, and perform the change of variables* (8)*, still with $m=1$. Then $F(t,p;x,y)$ becomes a formal power series in z with polynomial coefficients in u and y, which is given by*

$$F(t,p;x,y) = (1+u)\ [u^{\geq}]\Big(e^{yV(v)-K(u)} - \bar{u}e^{yV(v)-K(\bar{u})}\Big), \tag{11}$$

with $\bar{u} = 1/u$.

Remarks
1. It is easily seen that the case $y = 1$ of (11) reduces to the case $m = 1$ of (9) (the proof relies on the fact that L and $K(u)$ are respectively the constant term and the positive part of $V(v)$ in u, and that $v = (1 + u)(1 + \bar{u})$ is symmetric in u and $\bar{u}$).

2. When $p_1 = 1$ and $p_i = 0$ for $i > 1$, the only permutation that contributes in (7) is the identity. We are thus simply counting labelled 1-Tamari intervals, by their size (variable t), the number of contacts (variable x) and the size of the first ascent (variable y). Still taking $m = 1$, we have $V(v) = zv = z(1+u)(1+\bar{u})$, $K(u) = zu$ and the extraction of the positive part in u in (11) can be performed explicitly:

$$\begin{aligned} F(t,p;x,y) &= (1+u)[u^{\geq}]\left(e^{yzv-zu} - \bar{u}e^{yzv-z\bar{u}}\right) \\ &= (1+u)e^{2yz}\left(\sum_{0\leq i\leq j} u^{j-i}\frac{z^{i+j}y^i(y-1)^j}{i!j!} - \sum_{0\leq j<i} u^{i-j-1}\frac{z^{i+j}y^i(y-1)^j}{i!j!}\right). \end{aligned}$$

When $x = 1$, that is, $u = 0$, the double sums in this expression reduce to simple sums, and the generating function of labelled Tamari intervals, counted by the size and the height of the first ascent, is expressed in terms of Bessel functions:

$$\frac{F(t,p;1,y)}{e^{2yz}} = \sum_{i\geq 0}\frac{z^{2i}y^i(y-1)^i}{i!^2} - \sum_{j\geq 0}\frac{z^{2j+1}y^{j+1}(y-1)^j}{(j+1)!j!}.$$

3. A functional equation

The aim of this section is to establish a functional equation satisfied by the series $F^{(m)}(t,p;x,y)$.

Proposition 5. *For $m \geq 1$, let $F^{(m)}(t,p;x,y) \equiv F(x,y)$ be the refined Frobenius series of the m-Tamari representation, defined by* (7)*. Then*

$$\begin{aligned} F(x,y) &= \sum_{k\geq 0}\tilde{h}_k(y)\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x) \\ &= \exp\left(y\sum_{k\geq 1}\frac{p_k}{k}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}\right)(x), \end{aligned}$$

where

$$\tilde{h}_k(y) = \sum_{\lambda\vdash k}\frac{p_\lambda}{z_\lambda}y^{\ell(\lambda)}, \tag{12}$$

Δ is the following divided difference operator

$$\Delta S(x) = \frac{S(x) - S(1)}{x-1},$$

and the powers (m) and (k) mean that the operators are applied respectively m times and k times.
Equivalently, $F(x,0) = x$ and

$$\frac{\partial F}{\partial y}(x,y) = \sum_{k\geq 1}\frac{p_k}{k}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(F(x,y)). \tag{13}$$

The above equations rely on a recursive construction of labelled m-Tamari intervals. Our description of the construction is self-contained, but we refer to [8] for several proofs and details.

3.1. Recursive construction of Tamari intervals

We start by modifying the appearance of 1-ballot paths. We apply a 45 degree rotation to transform them into *Dyck paths*. A Dyck path of size n consists of steps $u = (1, 1)$ (up steps) and steps $d = (1, -1)$ (down steps), starts at $(0, 0)$, ends at $(2n, 0)$ and never goes below the x-axis. We say that an up step has *rank* i if it is the i^{th} up step of the path. We often represent Dyck paths by words on the alphabet $\{u, d\}$. An ascent is thus now a maximal sequence of u steps.

Consider an m-ballot path of size n, and replace each north step by a sequence of m north steps. This gives a 1-ballot path of size mn, and thus, after a rotation, a Dyck path. In this path, for each $i \in [\![0, n-1]\!]$, the up steps of ranks $mi + 1, \ldots, m(i + 1)$ are consecutive. We call the Dyck paths satisfying this property *m-Dyck paths*, and say that the up steps of ranks $mi + 1, \ldots, m(i + 1)$ form a *block*. Clearly, m-Dyck paths of size mn (*i.e.*, having n blocks) are in one-to-one correspondence with m-ballot paths of size n.

We often denote by $\mathcal{T}_n$, rather than $\mathcal{T}_n^{(1)}$, the usual Tamari lattice of size n. Similarly, the intervals of this lattice are called Tamari intervals rather than 1-Tamari intervals. As proved in [8], the transformation of m-ballot paths into m-Dyck paths maps $\mathcal{T}_n^{(m)}$ on a sublattice of $\mathcal{T}_{mn}$.

Proposition 6 ([8, Prop. 4])**.** *The set of m-Dyck paths with n blocks is the sublattice of $\mathcal{T}_{nm}$ consisting of the paths that are larger than or equal to $u^m d^m \ldots u^m d^m$. It is order isomorphic to $\mathcal{T}_n^{(m)}$.*

We now describe a recursive construction of (unlabelled) Tamari intervals, again borrowed from [8]. Thanks to the embedding of $\mathcal{T}_n^{(m)}$ into $\mathcal{T}_{nm}$, it will also enable us to describe recursively m-Tamari intervals, for any value of m, in the next subsection.

A Tamari interval $I = [P, Q]$ is *pointed* if its lower path P has a distinguished contact. Such a contact splits P into two Dyck paths P^ℓ and P^r, respectively located to the left and to the right of the contact. The pointed interval I is *proper* if P^ℓ is not empty, *i.e.*, if the distinguished contact is not $(0, 0)$. We often use the notation $I = [P^\ell P^r, Q]$ to denote a pointed Tamari interval. The contact $(0, 0)$ is called the *initial* contact.

Proposition 7. *Let $I_1 = [P_1^\ell P_1^r, Q_1]$ be a pointed Tamari interval, and let $I_2 = [P_2, Q_2]$ be a Tamari interval. Construct the Dyck paths*

$$P = uP_1^\ell dP_1^r P_2 \quad \textit{and} \quad Q = uQ_1 dQ_2$$

as shown in Figure 4. Then $I = [P, Q]$ is a Tamari interval. Moreover, the mapping $(I_1, I_2) \mapsto I$ is a bijection between pairs (I_1, I_2) formed of a pointed Tamari interval and a Tamari interval, and Tamari intervals I of positive size. Note that I_1 is proper if and only if the first ascent of P has height larger than 1.

Remarks
1. To recover P_1^ℓ, P_1^r, Q_1, P_2 and Q_2 from P and Q, one proceeds as follows: Q_2 is the part of Q that follows the first return of Q to the x-axis; this also defines Q_1 unambiguously. The path P_2 is the suffix of P having the same size as Q_2. This also defines $P_1 := uP_1^\ell dP_1^r$ unambiguously. Finally, P_1^r is the part of P_1 that follows the first return of P_1 to the x-axis, and this also defines P_1^ℓ unambiguously.
2. This proposition is obtained by combining Proposition 5 in [8] and the case $m = 1$ of Lemma 9 in [8]. With the notation $(P'; p_1)$ and (Q', q_1) used therein, the above paths P_2 and Q_2 are respectively the parts of P' and Q' that lie to the right of q_1, while $P_1^\ell P_1^r$ and Q_1 are the parts of P' and Q' that lie to the left of q_1. The pointed vertex p_1 is the endpoint of P_1^ℓ. Proposition 5 in [8] guarantees that, if $P \preceq Q$ in the Tamari order, then $P_1^\ell P_1^r \preceq Q_1$ and $P_2 \preceq Q_2$.
3. One can keep track of several parameters in the construction of Proposition 7. For instance, the number of non-initial contacts of P is

$$c(P) - 1 = (c(P_1^r) - 1) + c(P_2). \tag{14}$$

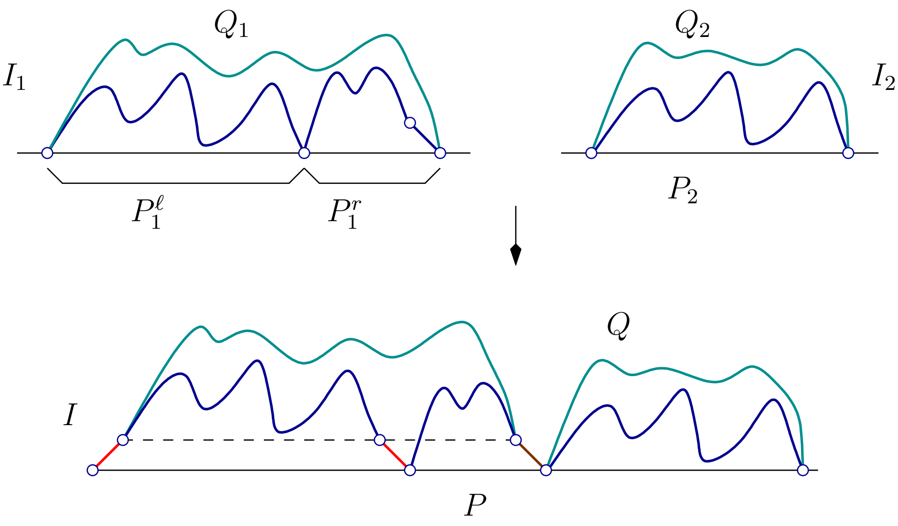

FIGURE 4. The recursive construction of Tamari intervals.

3.2. From the construction to the functional equation

We now prove Proposition 5 through a sequence of lemmas. The first one describes $F^{(m)}(t,p;x,y)$ in terms of homogeneous symmetric functions rather than power sums.

Lemma 8. *Let $\tilde{h}_k(y)$ be defined by (12), and set*

$$h_k = \tilde{h}_k(1) = \sum_{\lambda \vdash k} \frac{p_\lambda}{z_\lambda}.$$

Hence h_k is the k^{th} homogenous symmetric function if p_k is the k^{th} power sum. Then the refined Frobenius series $F^{(m)}(t,p;x,y)$, defined by (7), can also be written as the following ordinary *generating function:*

$$F^{(m)}(t,p;x,y) = \sum_{I=[P,Q],\ \text{unlabelled}} t^{|I|} x^{c(P)} \tilde{h}_{a_1}(y) \prod_{i \geq 2} h_{a_i}, \tag{15}$$

where the sum runs over unlabelled *m-Tamari intervals I, and a_i is the height of the i^{th} ascent of the upper path Q. In particular, $F^{(m)}(t,p;x,1) \equiv F^{(m)}(x,1)$ is the ordinary generating function of m-Tamari intervals, counted by the size (t), the number of contacts (x), and the distribution of ascents (h_i for each ascent of height i in the upper path).*

Proof. Let $I = [P,Q]$ be an unlabelled Tamari interval, and let a_i be the height of the i^{th} ascent of Q. Denote $n = |I|$. An *I-partitioned permutation* is a permutation $\sigma \in \mathfrak{S}_n$, together with a partition of the set of cycles of σ into labelled subsets $A_1, A_2, \ldots$, such that the sum of the lengths of the cycles of A_i is a_i. In the expression (7) of $F^{(m)}$, the contribution of labelled intervals $\bar{I} = [P, \bar{Q}]$ obtained by labelling Q in all possible ways is $x^{c(P)}\phi(I)$, where

$$\phi(I) = \frac{t^{|I|}}{|I|!} \sum_{\bar{I}=[P,\bar{Q}]} \sum_{\sigma \in \mathrm{Stab}(\bar{I})} y^{a_\sigma(\bar{Q})} p_{\lambda(\sigma)}.$$

In other words, $\phi(I)$ is the exponential generating function of I-partitioned permutations, counted by the size (variable t), the number of cycles in the block A_1 (variable y), and the number of cycles of length j (variable p_j), for all j. By elementary results on exponential generating functions, this series factors over ascents of Q. The contribution of the i^{th} ascent is the exponential generating function of permutations of $\mathfrak{S}_{a_i}$, counted by the size, the number of cycles of length j for all j, and also by the number of cycles if $i = 1$. But this is exactly $t^{a_i} h_{a_i}$ (or $t^{a_1} \tilde{h}_{a_1}(y)$ if $i = 1$), since

$$t^a \tilde{h}_a(y) = t^a \sum_{\lambda \vdash a} \frac{p_\lambda}{z_\lambda} y^{\ell(\lambda)} = \frac{t^a}{a!} \sum_{\sigma \in \mathfrak{S}_a} p_\lambda(\sigma)\, y^{\ell(\lambda(\sigma))}.$$

Hence

$$\phi(I) = t^{|I|} \tilde{h}_{a_1}(y) \prod_{i \geq 2} h_{a_i},$$

and the proof is complete[1]. ∎

Lemma 9. *In the expression* (15) *of* $F^{(m)}(t, p; x, y) \equiv F(x, y)$*, the contribution of intervals* $I = [P, Q]$ *such that the first ascent of* Q *has height* k *is*

$$\tilde{h}_k(y) \left(tx (F(x,1)\Delta)^{(m)} \right)^{(k)} (x).$$

This proves the first equation satisfied by $F^{(m)}(x, y)$ *in Proposition* 5.

Proof. We constantly use in this proof the inclusion $\mathcal{T}_n^{(m)} \subset \mathcal{T}_{nm}$ given by Proposition 6. That is, we identify elements of $\mathcal{T}_n^{(m)}$ with m-Dyck paths having n blocks. The size of an interval is thus now the number of blocks, and the height of the first ascent becomes the number of blocks in the first ascent.

Lemma 9 relies on the recursive description of Tamari intervals given in Proposition 7. We actually apply this construction to a slight generalization of m-Tamari intervals. For $\ell \geq 0$, an *ℓ-augmented m-Dyck path* is a Dyck path Q of size $\ell + mn$ for some integer n, where the first ℓ steps are up steps, and all the other up steps can be partitioned into *blocks* of m consecutive up steps. The ℓ first steps of Q are not considered to be part of a block, even if ℓ is a multiple of m. We denote by $a(Q)$ the number of blocks contained in the first ascent[2] of Q. A Tamari interval $I = [P, Q]$ is an *ℓ-augmented m-Tamari interval* if both P and Q are ℓ-augmented m-Dyck paths.

For $k, \ell \geq 0$ let $F_{k,\ell}(x)^{(m)} \equiv F_{k,\ell}(x)$ be the generating function of ℓ-augmented m-Tamari intervals $I = [P, Q]$ such that $a(Q) = k$, counted by the number of blocks (variable t), the number of *non-initial* contacts (variable x) and the number of non-initial ascents of Q having i blocks (one variable h_i for each $i \geq 1$, as before). We are going to prove that for all $k, \ell \geq 0$,

$$F_{k,\ell}(x) = \begin{cases} \frac{1}{x} \left(tx (F(x,1)\Delta)^{(m)} \right)^{(k)} (x) & \text{if } \ell = 0, \\ (F(x,1)\Delta)^{(\ell)} \left(tx (F(x,1)\Delta)^{(m)} \right)^{(k)} (x) & \text{if } \ell > 0. \end{cases} \tag{16}$$

We claim that (16) implies Lemma 9. Indeed, m-Tamari intervals coincide with 0-augmented m-Tamari intervals. Since the initial contact and the first ascent are not counted in $F_{k,0}(x)$, but are counted in $F^{(m)}(x, y)$, the contribution in $F^{(m)}(x, y)$ of intervals such that $a(Q) = k$ is $x \tilde{h}_k(y) F_{k,0}(x)$, as stated in the lemma.

We now prove (16), by lexicographic induction on (k, ℓ). For $(k, \ell) = (0, 0)$, the unique interval involved in $F_{k,\ell}(x)$ is $\{\bullet\}$, where $\bullet$ is the path of length 0. Its contribution is 1 (since the initial and only contact is not counted). Therefore $F_{0,0}(x) = 1$ and (16) holds. Let $(k, \ell) \neq (0, 0)$ and assume that (16) holds for all lexicographically smaller pairs $(k', \ell') < (k, \ell)$. We are going to show that (16) holds at rank (k, ℓ).

If $k > 0$ and $\ell = 0$, then we are considering 0-augmented m-Tamari intervals, that is, usual m-Tamari intervals. But an m-Tamari interval $I = [P, Q]$ having n blocks and k blocks in the first ascent can be seen as an m-augmented m-Tamari interval having $n - 1$ blocks and $k - 1$

[1] An analogous result was used without proof in the study of the parking representation of the symmetric group [24, p. 28].

[2] Since the number of blocks does not depend on Q only, but also on ℓ, it should in principle be denoted $a^{(\ell)}(Q)$. We hope that our choice of a lighter notation will not cause any confusion.

blocks in the first ascent, by considering that the first m up steps of P and Q are not part of a block. This implies that:

$$F_{k,0}(x) = tF_{k-1,m}(x) = \frac{1}{x}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x)$$

by the induction hypothesis (16) applied at rank $(k-1, m)$. This is exactly (16) at rank $(k, \ell = 0)$.

Now assume $\ell \neq 0$. The series $F_{k,\ell}(x)$ counts ℓ-augmented m-Tamari intervals $I = [P, Q]$ such that $a(Q) = k$. By Proposition 7, such an interval can be decomposed into a pointed interval $I_1 = [P_1^\ell P_1^r, Q_1]$ and an interval $I_2 = [P_2, Q_2]$ (the "ℓ" in the notation P_1^ℓ is a bit unfortunate here; we hope it will not raise any difficulty). Note that I_2 is an m-Tamari interval, while I_1 is an $(\ell-1)$-augmented pointed m-Tamari interval. Conversely, starting from such a pair (I_1, I_2), the construction of Proposition 7 produces an ℓ-augmented m-Tamari interval, unless I_1 is not proper and $\ell > 1$. Moreover, $a(Q_1) = a(Q)$. Using (14), we obtain:

$$F_{k,\ell}(x) = F(x,1)\left(F^\bullet_{k,\ell-1}(x) + 1_{\ell=1}F^\circ_{k,\ell-1}(x)\right) \tag{17}$$

where $F^\bullet_{k,\ell-1}(x)$ (resp. $F^\circ_{k,\ell-1}(x)$) is the generating function of proper (resp. non-proper) pointed $(\ell-1)$-augmented m-Tamari intervals $I_1 = [P_1^\ell P_1^r, Q_1]$ such that $a(Q_1) = k$, counted by the number of blocks (variable t), the number of non-initial ascents of Q_1 of height i (variable h_i) for each $i \geq 1$, and the number of non-initial contacts of P_1^r (variable x). The factor $F(x,1)$ in (17) is the contribution of the interval I_2.

To determine the series $F^\bullet_{k,\ell-1}(x)$, expand the series $F_{k,\ell-1}(x)$ as

$$F_{k,\ell-1}(x) = \sum_{i\geq 1} F_{k,\ell-1,i}\, x^i,$$

where $F_{k,\ell-1,i} = [x^i]F_{k,\ell-1}(x)$ is the generating function of $(\ell-1)$-augmented m-Tamari intervals $[P_1, Q_1]$ such that $r(Q_1) = k$, and having i non-initial contacts. Each such interval can be pointed in i different ways to give rise to i different proper pointed intervals $[P_1^\ell P_1^r, Q_1]$, having respectively $0, 1, \ldots, i-1$ non-initial contacts. Therefore,

$$\begin{aligned} F^\bullet_{k,\ell-1}(x) &= \sum_i F_{k,\ell-1,i}(1 + x + \cdots + x^{i-1}) \\ &= \sum_i F_{k,\ell-1,i}\frac{x^i - 1}{x - 1} \\ &= \frac{1}{x-1}\left(F_{k,\ell-1}(x) - F_{k,\ell-1}(1)\right) \\ &= \Delta F_{k,\ell-1}(x). \end{aligned} \tag{18}$$

This, together with (17), already allows us to prove (16) when $\ell > 1$. Indeed, one then has:

$$F_{k,\ell}(x) = F(x,1)\Delta F_{k,\ell-1}(x) = (F(x,1)\Delta)^{(\ell)}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x),$$

by the induction hypothesis. This is (16) at rank (k, ℓ).

It remains to treat the case $\ell = 1$. To this end we need to determine the series $F^\circ_{k,0}(x)$. By definition, a pointed interval $I_1 = [P_1^\ell P_1^r, Q_1]$ is non-proper if P_1^ℓ is empty, in which case I_1 can be identified with the (non-pointed) interval $[P_1^r, Q_1]$. This implies that $F^\circ_{k,0}(x) = F_{k,0}(x)$. Therefore (17) and (18) give:

$$\begin{aligned} F_{k,1}(x) &= F(x,1)\left(\Delta F_{k,0}(x) + F_{k,0}(x)\right) \\ &= F(x,1)\Delta\big(xF_{k,0}(x)\big). \end{aligned}$$

By the induction hypothesis, $F_{k,0}(x) = \frac{1}{x}\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x)$, so that

$$F_{k,1}(x) = F(x,1)\Delta\left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}.$$

We recognise (16) at rank $(k, \ell = 1)$, and this settles the last case of the induction. ∎

Proof of Proposition 5. By Lemmas 8 and 9, and the definition (12) of $\tilde{h}_k(y)$, we have:

$$F(x,y) = \sum_{k\geq 0} \tilde{h}_k(y) \left(tx(F(x,1)\Delta)^{(m)}\right)^{(k)}(x) = \sum_\lambda y^{\ell(\lambda)} \frac{p_\lambda}{z_\lambda} \left(tx(F(x,1)\Delta)^{(m)}\right)^{(|\lambda|)}(x).$$

Letting α_i be the number of parts equal to i in the partition λ, and summing on the α_i's rather than on λ, we can rewrite this sum as:

$$\begin{aligned} F(x,y) &= \sum_{\alpha_1,\alpha_2,\ldots} \prod_i \left(\frac{y^{\alpha_i}}{\alpha_i!} \left(\frac{p_i}{i}\right)^{\alpha_i} \left(tx(F(x,1)\Delta)^{(m)}\right)^{(i\alpha_i)} \right)(x) \\ &= \prod_{i\geq 1} \exp\left(y\,\frac{p_i}{i} \left(tx(F(x,1)\Delta)^{(m)}\right)^{(i)} \right)(x) \\ &= \exp\left(y \sum_i \frac{p_i}{i} \left(tx(F(x,1)\Delta)^{(m)}\right)^{(i)} \right)(x). \end{aligned}$$

We have used the fact that the operator Δ commutes with the multiplication by y and by p_i. This is the second functional equation satisfied by $F(x,y)$ given in Proposition 5. The third one, (13), follows by differentiating with respect to y. ■

4. Principle of the proof, and the case $m = 1$

4.1. Principle of the proof

Let us consider the functional equation (13), together with the initial condition $F(t,p;x,0) = x$. Perform the change of variables (8), and denote $G(z,p;u,y) \equiv G(u,y) = F(t,p;x,y)$. Then $G(u,y)$ is a series in z with coefficients in $\mathbb{K}[u,y]$ (where $\mathbb{K} = \mathbb{Q}(p_1,p_2,\ldots)$) satisfying

$$\frac{\partial G}{\partial y}(u,y) = \sum_{k\geq 1} \frac{p_k}{k} \left(z(1+u)e^{-m(K(u)+L)} \left(\frac{uG(u,1)}{(1+u)e^{-mK(u)}-1}\ \Delta_u \right)^{(m)} \right)^{(k)} G(u,y), \tag{19}$$

with $\Delta_u H(u) = \frac{H(u)-H(0)}{u}$, and the initial condition

$$G(u,0) = (1+u)e^{-mK(u)}. \tag{20}$$

Observe that this pair of equations defines $G(u,y) \equiv G(z,p;u,y)$ uniquely as a formal power series in z. Indeed, the coefficient of z^n in G can be computed inductively from these equations: one first determines the coefficient of z^n in $\frac{\partial G}{\partial y}$, which can be expressed, thanks to (19), in terms of the coefficients of z^i in G for $i < n$; then the coefficient of z^n in G is obtained by integration with respect to y, using the initial condition (20). Hence, if we exhibit a series $\tilde{G}(z,p;u,y)$ that satisfies both equations, then $\tilde{G}(z,p;u,y) = G(z,p;u,y)$. We are going to construct such a series.

Let

$$G_1(z,p;u) \equiv G_1(u) = (1+\bar{u})e^{K(u)+L} \left((1+u)e^{-mK(u)} - 1\right). \tag{21}$$

Then $G_1(u)$ is a series in z with coefficients in $\mathbb{K}[u]$, which, as we will see, coincides with $G(u,1)$. Consider now the following equation, obtained from (19) by replacing $G(u,1)$ by its conjectured value $G_1(u)$:

$$\begin{aligned} \frac{\partial \tilde{G}}{\partial y}(z,p;u,y) &= \sum_{k\geq 1} \frac{p_k}{k} \left(z(1+u)e^{-m(L+K(u))} \left(\frac{uG_1(u)}{(1+u)e^{-mK(u)}-1}\ \Delta_u \right)^{(m)} \right)^{(k)} \tilde{G}(z,p;u,y) \\ &= \sum_{k\geq 1} \frac{p_k}{k} \left(z(1+u)e^{-m(L+K(u))} \left((1+u)e^{K(u)+L}\ \Delta_u \right)^{(m)} \right)^{(k)} \tilde{G}(z,p;u,y), \end{aligned} \tag{22}$$

with the initial condition

$$\tilde{G}(z,p;u,0) = (1+u)e^{-mK(u)}. \tag{23}$$

Eq. (22) can be rewritten as

$$\frac{\partial \tilde{G}}{\partial y}(z,p;u,y) = \sum_{k\geq 1} \frac{p_k}{k}\left(zvA(u)^m\Lambda^{(m)}\right)^{(k)} \tilde{G}(z,p;u,y), \tag{24}$$

where

$$A(u) = \frac{u}{1+u}e^{-K(u)}, \tag{25}$$

Λ is the operator defined by

$$\Lambda(H)(u) = \frac{H(u)-H(0)}{A(u)}, \tag{26}$$

and $v=(1+u)^{m+1}u^{-m}$ as before. Again, it is not hard to see that (24) and the initial condition (23) define a unique series in z, denoted $\tilde{G}(z,p;u,y)\equiv\tilde{G}(u,y)$. The coefficients of this series lie in $\mathbb{K}[u,y]$. The principle of our proof can be described as follows.

> *If we prove that $\tilde{G}(u,1)=G_1(u)$, then the equation (22) satisfied by $\tilde{G}$ coincides with the equation (19) that defines G, and thus $\tilde{G}(u,y)=G(u,y)$. In particular, $G_1(z,p;u)=\tilde{G}(z,p;u,1)=G(z,p;u,1)=F(t,p;x,1)$, and Theorem 3 is proved.*

Remark. Our proof relies on the fact that we have been able to *guess* the value of $G(u,1)$, given by (21). This was a difficult task, which we discuss in greater details in Section 6.1.

4.2. The case $m=1$

Take $m=1$. In this subsection, we describe the three steps that, starting from (24), prove that $\tilde{G}(u,1)=G_1(u)$. In passing, we establish the expression (11) of $F(t,p;x,1)$ (equivalently, of $\tilde{G}(z,p;u,1)$) given in Theorem 4. The case of general m is difficult, and we hope that studying in details the case $m=1$ will make the ideas of the proof more transparent. Should this specialization not suffice, we invite the reader to set further $p_i=1_{i=1}$, in which case we are simply counting labelled Tamari intervals (see also [7]).

4.2.1. ***A homogeneous differential equation and its solution.*** When $m=1$, the equation (24) defining $\tilde{G}(z;u,y)\equiv\tilde{G}(u,y)$ reads

$$\frac{\partial \tilde{G}}{\partial y}(u,y) = \sum_{k\geq 1} \frac{p_k}{k}z^k\left((1+u)(1+\bar{u})\,\Omega\right)^{(k)} \tilde{G}(u,y), \tag{27}$$

where $\bar{u}=1/u$ and the operator Ω is defined by $\Omega H(u)=H(u)-H(0)$, with the initial condition

$$\tilde{G}(u,0)=(1+u)e^{-K(u)}. \tag{28}$$

These equations imply that $\tilde{G}(-1,y)=0$.

Observe that the differential equation (27) is not homogeneous: the term obtained for $k=1$ involves the (unknown) series $\tilde{G}(0,y)$, and more and more unknown series independent of u occur as k grows. By exploiting the symmetry of the term $(1+u)(1+\bar{u})$, we are going to obtain an equation that does not involve these series. This idea has already been used for other equations with divided differences [10].

Lemma 10. *For $k\geq 0$ one has:*

$$\left((1+u)(1+\bar{u})\Omega\right)^{(k)}\tilde{G}(u,y) = (1+u)^k(1+\bar{u})^k\tilde{G}(u,y) - P_k(v),$$

where $P_k\in\mathbb{K}[y][[z]][v]$ and $v=(1+u)(1+\bar{u})$.

Proof. This is easily proved by induction on k. Let us also give a combinatorial argument.

Clearly, it suffices to prove that for any $i\geq 0$, the above property holds with $\tilde{G}(u,y)$ replaced by u^i. Consider walks on the line $\mathbb{N}$, starting from i and consisting of k steps taken in $-1,0^1,0^2,1$ (the steps 0 are thus bicolored). The term $(1+u)^k(1+\bar{u})^k u^i$ in right-hand side of the above identity counts these walks by their final position. The left-hand side counts those walks that never reach 0, except possibly at their final point.

Hence the difference P_k between the two terms counts walks that reach 0 before their final point. Such walks consist of a walk visiting 0 only at its end, of length, say, ℓ, followed by an arbitrary walk of length $k-\ell$. This shows that P_k has the following form:

$$P_k = \sum_{\ell=0}^{k-1} a_\ell ((1+u)(1+\bar u))^{k-\ell}$$

where a_ℓ is the number of ℓ-step walks going from i to 0 and visiting 0 only once. It is now clear that P_k is a polynomial in v. ■

Observe that the quantity $P_k(v)$, being a function of $v=(1+u)(1+\bar u)$, is left invariant by the substitution $u \mapsto \bar u$. This symmetry is the keystone of our approach, as it enables us to eliminate some *a priori* intractable terms in (27). Replacing u by $\bar u$ in (27) gives

$$\frac{\partial \tilde G}{\partial y}(\bar u, y) = \sum_k \frac{p_k}{k} z^k \Big((1+u)(1+\bar u)\Omega\Big)^{(k)} \tilde G(\bar u, y),$$

so that, applying Lemma 10 and using $v(u)=v(\bar u)$ we obtain:

$$\frac{\partial}{\partial y}\left(\tilde G(u,y)-\tilde G(\bar u,y)\right) = \sum_{k\ge 1}\frac{p_k}{k} z^k (1+u)^k(1+\bar u)^k \left(\tilde G(u,y)-\tilde G(\bar u,y)\right) = V(v)\left(\tilde G(u,y)-\tilde G(\bar u,y)\right),$$

where $V(v)$ is given by (4). This is a *homogeneous* linear differential equation satisfied by $\tilde G(u,y)-\tilde G(\bar u,y)$. It is readily solved, and the initial condition (28) yields

$$\tilde G(u,y)-\tilde G(\bar u,y) = (1+u)\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right) e^{yV(v)}. \tag{29}$$

4.2.2. ***Reconstruction of*** $\tilde G(u,y)$. Recall that $\tilde G(u,y) \equiv \tilde G(z,p;u,y)$ is a series in z with coefficients in $\mathbb{K}[u,y]$. Hence, by extracting from the above equation the positive part in u (as defined in Section 2.1), we obtain

$$\tilde G(u,y)-\tilde G(0,y) = [u^>]\left((1+u)\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right) e^{yV(v)}\right).$$

For any Laurent polynomial P, we have

$$[u^>]\left((1+u)P(u)\right) = (1+u)[u^>]P(u) + u[u^0]P(u). \tag{30}$$

Hence

$$\begin{aligned}\tilde G(u,y)-\tilde G(0,y) = (1+u)[u^>]\left(e^{yV(v)}\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right)\right)& \\ + u[u^0]\left(e^{yV(v)}\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right)\right).&\end{aligned}$$

Since $\tilde G(-1,y)=0$, setting $u=-1$ in this equation gives the value of $\tilde G(0,y)$ (this is an instance of the *kernel method*, see *e.g.* [3]):

$$-\tilde G(0,y) = -[u^0]\left(e^{yV(v)}\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right)\right),$$

so that finally,

$$\begin{aligned}\tilde G(u,y) &= (1+u)[u^>]\left(e^{yV(v)}\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right)\right) \\ &\qquad + (1+u)[u^0]\left(e^{yV(v)}\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right)\right) \\ &= (1+u)[u^\ge]\left(e^{yV(v)}\left(e^{-K(u)} - \bar u e^{-K(\bar u)}\right)\right). \end{aligned} \tag{31}$$

As explained in Section 4.1, $\tilde G(u,y) = G(u,y)$ will be proved if we establish that $\tilde G(u,1)$ coincides with the series $G_1(u)$ given by (21). This is the final step of our proof.

4.2.3. ***The case*** $y = 1$**.** Equation (31) completely describes the solution of (27). It remains to check that $\tilde{G}(u,1) = G_1(u)$, that is

$$\tilde{G}(u,1) = (1+\bar{u})e^{K(u)+L}\left((1+u)e^{-K(u)} - 1\right). \tag{32}$$

Let us set $y = 1$ in (31). We find, using $V(v) = K(\bar{u}) + L + K(u)$:

$$\begin{aligned}\tilde{G}(u,1) &= (1+u)[u^{\geq}]\left(e^{L+K(\bar{u})} - \bar{u}e^{L+K(u)}\right)\\ &= (1+u)e^{L}\left(1 - \bar{u}e^{K(u)} + \bar{u}\right),\end{aligned}$$

which coincides with (32). Hence $\tilde{G}(z,p;u,y) = G(z,p;u,y) = F(t,p;x,y)$ (with the change of variables (8)), and Theorem 4 is proved, using (31).

5. Solution of the functional equation: the general case

We now adapt to the general case the solution described for $m = 1$ in Section 4.2. Recall from Section 4.1 that we start from (24), and want to prove that $\tilde{G}(u,1) = G_1(u)$. We first obtain in Section 5.1 the counterpart of (29), that is, an explicit expression for a linear combination of the series $\tilde{G}(u_i,y)$, where $u_0 = u, u_1, \ldots, u_m$ are the $m+1$ roots of the equation $v(u) = v(u_i)$, with $v(u) = (1+u)^{m+1}\bar{u}^m$. In Section 5.2, we reconstruct from this expression the series $\tilde{G}(u,y)$, by taking iterated positive parts. This generalizes (31). The third part of the proof differs from Section 4.2.3, because we are not able to derive from our expression of $\tilde{G}(u,y)$ that $\tilde{G}(u,1) = G_1(u)$. Instead, the arguments of Section 5.2 imply that the counterpart of (29) also has a unique solution when $y = 1$, and we check that $G_1(u)$ is a solution.

5.1. A homogeneous differential equation and its solution

Let us return to the equation (24) satisfied by $\tilde{G}(u,y)$. This equations involves the quantity

$$v \equiv v(u) = (1+u)^{m+1}\bar{u}^m$$

In the case $m = 1$, this (Laurent) polynomial was $(1+u)(1+\bar{u})$, and took the same value for u and $\bar{u}$. We are again interested in the series u_i such that $v(u_i) = v(u)$.

Lemma 11. *Denote $v = (1+u)^{m+1}u^{-m}$, and consider the following polynomial equation in U:*

$$(1+U)^{m+1} = U^m v.$$

This equation has no double root. We denote its $m+1$ roots by $u_0 = u, u_1, \ldots, u_m$.

Proof. A double root would also satisfy

$$(m+1)(1+U)^m = mU^{m-1}v,$$

and this is easily shown to be impossible. ∎

Remark. One can express the u_i's as Puiseux series in u (see [33, Ch. 6]), but this will not be needed here, and we will think of them as abstract elements of an algebraic extension of $\mathbb{Q}(u)$. In fact, in this paper, the u_i's always occur in *symmetric rational functions* of the u_i's, which are thus rational functions of v. At some point, we will have to prove that a symmetric polynomial in the u_i's (and thus a polynomial in v) vanishes at $v = 0$, that is, at $u = -1$, and we will then consider series expansions of the u_i's around $u = -1$.

The following proposition generalizes (29).

Proposition 12. *Denote $v = (1+u)^{m+1}u^{-m}$, and let the series u_i be defined as above. Denote $A_i = A(u_i)$, where $A(u)$ is given by (25). Then*

$$\sum_{i=0}^{m} \frac{\tilde{G}(u_i,y)}{\prod_{j\neq i}(A_i - A_j)} = ve^{yV(v)}. \tag{33}$$

By $\prod_{j\neq i}(A_i - A_j)$ we mean $\prod_{0\leq j\leq m, j\neq i}(A_i - A_j)$ but we prefer the shorter notation when the bounds on j are clear. Observe that the A_i's are distinct since the u_i's are distinct (the coefficient of z^0 in $A(u)$ is $u/(1+u)$). Note also that when $m = 1$, then $u_0 = u$, $u_1 = \bar{u}$, and (33) coincides with (29). In order to prove the proposition, we need the following two lemmas.

Lemma 13. *Let $x_0, x_1, \ldots, x_m$ be $m+1$ variables. Then*

$$\sum_{i=0}^{m} \frac{{x_i}^m}{\prod_{j\neq i}(x_i - x_j)} = 1 \tag{34}$$

and

$$\sum_{i=0}^{m} \frac{1/x_i}{\prod_{j\neq i}(x_i - x_j)} = (-1)^m \prod_{i=0}^{m} \frac{1}{x_i}. \tag{35}$$

Moreover, for any polynomial Q of degree less than m,

$$\sum_{i=0}^{m} \frac{Q(x_i)}{\prod_{j\neq i}(x_i - x_j)} = 0. \tag{36}$$

Proof. By Lagrange interpolation, any polynomial R of degree at most m satisfies:

$$R(X) = \sum_{i=0}^{m} R(x_i) \prod_{j\neq i} \frac{X - x_j}{x_i - x_j}.$$

Equations (35) and (36) follow by evaluating this equation at $X = 0$, respectively with $R(X) = 1$ and $R(X) = XQ(X)$. Equation (34) is obtained by taking $R(X) = X^m$ and extracting the leading coefficient. ∎

Our second lemma replaces Lemma 10 for general values of m.

Lemma 14. *Let $k \geq 0$, and let Λ be the operator defined by (26). Let $H(z; u) \equiv H(u)$ be a formal power series in z, having coefficients in $\mathbb{L}(u)$, with $\mathbb{L} = \mathbb{K}(y)$. Assume that these coefficients have no pole at $u = 0$. Then there exists a polynomial $P_k(X, Y) \in \mathbb{L}[[z]][X, Y]$ of degree less than m in X, such that*

$$\left(zvA(u)^m\Lambda^{(m)}\right)^{(k)} H(u) = (zv)^k H(u) - P_k(A(u), v). \tag{37}$$

Proof. We give here a power series argument, but an analogue of the combinatorial argument proving Lemma 10 (carefully justified) also exists.

We denote by $\mathcal{L}$ the subring of $\mathbb{L}(u)[[z]]$ formed by formal power series whose coefficients have no pole at $u = 0$. By assumption, $H(u) \in \mathcal{L}$. We use the notation $O(u^k)$ to denote an element of $\mathbb{L}(u)[[z]]$ of the form $u^k J(z; u)$ with $J(z; u) \in \mathcal{L}$.

First, note that $A(u) = ue^{-K(u)}/(1 + u)$ belongs to $\mathcal{L}$. Moreover,

$$A(u) = u + O(u^2). \tag{38}$$

We first prove that for all series $I(u) \in \mathcal{L}$, there exists a sequence of formal power series $(g_j^I)_{j\geq 0} \in \mathbb{L}[[z]]^{\mathbb{N}}$ such that for all $\ell \geq 0$,

$$I(u) = \sum_{j=0}^{\ell-1} g_j^I A(u)^j + O(u^\ell). \tag{39}$$

We prove (39) by induction on $\ell \geq 0$. The identity holds for $\ell = 0$ since $I(u) \in \mathcal{L}$. Assume it holds for some $\ell \geq 0$: there exists series $g_0^I, \ldots, g_{\ell-1}^I$ in $\mathbb{L}[[z]]$ and $J(u) \in \mathcal{L}$ such that

$$I(u) = \sum_{j=0}^{\ell-1} g_j^I A(u)^j + u^\ell J(u).$$

By (38) and by induction on r, we have $u^r = A(u)^r + O(u^{r+1})$ for all $r \geq 0$. Using this identity with $r = \ell$, and rewriting $J(u) = J(0) + O(u)$, we obtain $u^\ell J(u) = J(0)A(u)^\ell + O(u^{\ell+1})$, so that:

$$I(u) = \sum_{j=0}^{\ell} g_j^I A(u)^j + O(u^{\ell+1}),$$

with $g_\ell^I := J(0) \in \mathbb{L}[[z]]$. Thus (39) holds for $\ell + 1$.

We now prove that for all $q \geq 0$, one has:

$$\Lambda^{(q)} I(u) = \frac{1}{A(u)^q} \left(I(u) - \sum_{j=0}^{q-1} g_j^I A(u)^j \right), \tag{40}$$

where the series g_j^I are those that satisfy (39). Again, we proceed by induction on $q \geq 0$. The identity clearly holds for $q = 0$. Assume it holds for some $q \geq 0$. In (40), replace $I(u)$ by its expression (39) obtained with $\ell = q+1$, and let u tend to 0: this shows that g_q^I is in fact $\Lambda^{(q)} I(0)$. From the definition of Λ one then obtains

$$\Lambda^{(q+1)} I(u) = \frac{\Lambda^{(q)} I(u) - g_q^I}{A(u)} = \frac{1}{A(u)^{q+1}} \left(I(u) - \sum_{j=0}^{q} g_j^I A(u)^j \right).$$

Thus (40) holds for $q + 1$.

We finally prove, by induction on $k \geq 0$, that (37) holds and that the left-hand side of (37) is an element of $\mathcal{L}$. For $k = 0$, these results are clear, with $P_0 = 0$. Assume they hold for some $k \geq 0$, for any $H(u) \in \mathcal{L}$. Let $H(u) \in \mathcal{L}$ and let $M(u)$ be the left-hand side of (37). By the induction hypothesis, $M(u) \in \mathcal{L}$, so that applying (40) with $I(u) = M(u)$ and $q = m$ gives:

$$\begin{aligned} zvA(u)^m \Lambda^{(m)} M(u) &= zv \left(M(u) - \sum_{j=0}^{m-1} g_j^M A(u)^y \right). \qquad (41) \\ &= \left(zvA(u)^m \Lambda^{(m)} \right)^{(k+1)} H(u) \qquad \text{by definition of } M. \end{aligned}$$

By the induction hypothesis (37), we have $M(u) = (zv)^k H(u) - P_k(A(u), v)$ with $P_k(X, Y)$ of degree less than m in X, so that the above equation gives:

$$\left(zvA(u)^m \Lambda^{(m)} \right)^{(k+1)} H(u) = (zv)^{k+1} H(u) - P_{k+1}(A(u), v),$$

with

$$P_{k+1}(X, Y) := zY \left(P_k(X, Y) + \sum_{j=0}^{m-1} g_j^M X^j \right).$$

Note that $P_{k+1}(X, Y)$ still has degree less than m in X.

It remains to prove that $\left((zvA(u)^m \Lambda^{(m)}\right)^{(k+1)} H(u) \in \mathcal{L}$. Applying (39) with $I(u) = M(u)$ and $\ell = m + 1$, and substituting in (41), we obtain:

$$\begin{aligned} \left(zvA(u)^m \Lambda^{(m)} \right)^{(k+1)} H(u) &= zv \left(g_m^M A(u)^m + O(u^{m+1}) \right) \\ &= zvu^m \left(g_m^M + O(u) \right), \end{aligned}$$

since $A(u)^m = u^m + O(u^{m+1})$. Since $v = (1+u)^{m+1} u^{-m}$, this shows that $\left(zvA(u)^m \Lambda^{(m)} \right)^{(k+1)} H(u)$ belongs to $\mathcal{L}$, which completes the proof. ∎

Proof of Proposition 12. Thanks to Lemma 14, we can rewrite (24) as

$$\frac{\partial \tilde{G}}{\partial y}(u,y) = \sum_{k\geq 1} \frac{p_k}{k}\Big((zv)^k \tilde{G}(u,y) - P_k(A(u),v)\Big), \tag{42}$$

where $v \equiv v(u) = (1+u)^{m+1}\bar{u}^m$, and for all $k \geq 1$, $P_k(X,Y)$ is a polynomial of degree less than m in X with coefficients in $\mathbb{K}(y)[[z]]$.

As was done in Section 4.2.1, we are going to use the fact that $v(u_i) = v$ for all $i \in [\![0,m]\!]$ to eliminate the (infinitely many) unknown polynomials $P_k(A(u),v)$. For $0 \leq i \leq m$, the substitution $u \mapsto u_i$ in (42) gives:

$$\frac{\partial \tilde{G}}{\partial y}(u_i,y) = \sum_{k\geq 1} \frac{p_k}{k}\Big((zv)^k \tilde{G}(u_i,y) - P_k(A_i,v)\Big), \tag{43}$$

with $A_i = A(u_i)$. Consider the linear combination

$$R(u,y) := \sum_{i=0}^{m} \frac{\tilde{G}(u_i,y)}{\prod_{j\neq i}(A_i - A_j)}. \tag{44}$$

Recall that A_i is independent of y. Thus by (43),

$$\begin{aligned}
\frac{\partial R}{\partial y}(u,y) &= \sum_{k\geq 1}\frac{p_k}{k}\left((zv)^k R(u,y) - \sum_{i=0}^{m}\frac{P_k(A_i,v)}{\prod_{j\neq i}(A_i-A_j)}\right),\\
&= \sum_{k\geq 1}\frac{p_k}{k}(zv)^k R(u,y), \qquad \text{by (36)},\\
&= V(v)R(u,y),
\end{aligned}$$

where $V(v)$ is defined by (4). This homogeneous linear differential equation is readily solved:

$$R(u,y) = R(u,0)e^{yV(v)}.$$

Recall the expression (44) of R in terms of $\tilde{G}$. The initial condition (23) can be rewritten $\tilde{G}(u,0) = vA(u)^m$, which yields

$$\begin{aligned}
R(u,0) &= v\sum_{i=0}^{m}\frac{{A_i}^m}{\prod_{j\neq i}(A_i-A_j)}\\
&= v
\end{aligned}$$

by (34). Hence $R(u,y) = ve^{yV(v)}$, and the proposition is proved. ∎

5.2. Reconstruction of $\tilde{G}(u,y)$

We are now going to prove that (33), together with the condition $\tilde{G}(-1,y) = 0$ derived from (22–23), characterizes the series $\tilde{G}(u,y)$. We will actually obtain a (complicated) expression for this series, generalizing (31).

We first introduce some notation. Consider a formal power series in z, denoted $H(z;u) \equiv H(u)$, having coefficients in $\mathbb{L}[u]$, where $\mathbb{L} = \mathbb{K}(y)$. We define a series H_k in z whose coefficients are rational symmetric functions of $k+1$ variables $x_0,\ldots,x_k$:

$$H_k(x_0,\ldots,x_k) = \sum_{i=0}^{k}\frac{H(x_i)}{\displaystyle\prod_{0\leq j\leq k, j\neq i}(A(x_i)-A(x_j))}, \tag{45}$$

where, as above, A is defined by (25).

Lemma 15. *The series $H_k(x_0,\ldots,x_k)$ has coefficients in $\mathbb{L}[x_0,\ldots,x_k]$. If, moreover, $H(-1) = 0$, then the coefficients of H_k are multiples of $(1+x_0)\cdots(1+x_k)$.*

Proof. Using the fact that $e^{-K(u)} = 1 + O(z)$, it is not hard to prove that

$$\frac{1}{A(x_i) - A(x_j)} = \frac{1}{x_i - x_j} B(x_i, x_j), \tag{46}$$

where $B(x_i, x_j)$ is a series in z with *polynomial* coefficients in x_i and x_j. Hence

$$H_k(x_0, \dots, x_k) \prod_{0 \le i < j \le k} (x_i - x_j)$$

has polynomial coefficients in the x_i's. But these polynomials are anti-symmetric in the x_i's (since H_k is symmetric), hence they must be multiples of the Vandermonde $\prod_{i<j}(x_i - x_j)$. Hence $H_k(x_0, \dots, x_k)$ has polynomial coefficients.

A stronger property than (46) actually holds, namely:

$$\frac{1}{A(x_i) - A(x_j)} = \frac{(1 + x_i)(1 + x_j)}{x_i - x_j} C(x_i, x_j),$$

where $C(x_i, x_j)$ is a series in z with polynomial coefficients in x_i and x_j. Hence, if $H(-1) = 0$, that is, if $H(x) = (1 + x)K(x)$ where $K(x)$ has polynomial coefficients in x,

$$H_k(x_0, \dots, x_k) = \sum_{i=0}^{k} K(x_i)(1 + x_i)^{k+1} \prod_{j \ne i} \frac{(1 + x_j)C(x_i, x_j)}{x_i - x_j}.$$

Setting $x_0 = -1$ shows that $H_k(-1, x_1, \dots, x_k) = 0$, so that $H_k(x_0, \dots, x_k)$ is a multiple of $(1 + x_0)$. By symmetry, it is also a multiple of all $(1 + x_i)$, for $1 \le i \le k$. ∎

Our treatment of (33) actually applies to equations with an arbitrary right-hand side. We consider a formal power series $H(z; u) \equiv H(u)$ with coefficients in $\mathbb{L}[u]$, satisfying $H(-1) = 0$ and

$$\sum_{i=0}^{m} \frac{H(u_i)}{\prod_{j \ne i}(A_i - A_j)} = \Phi_m(v),$$

for some series $\Phi_m(v) \equiv \Phi_m(z; v)$ with coefficients in $v\mathbb{L}[v]$, where $v = (1 + u)^{m+1}\bar{u}^m$. Using the notation (45), this equation can be rewritten as

$$H_m(u_0, \dots, u_m) = \Phi_m(v).$$

We will give an explicit expression of $H(u)$ involving two standard families of symmetric functions, namely the homogeneous functions h_λ and the monomial functions m_λ.

> **Caveat.** *These symmetric functions will be evaluated at $(u_0, u_1, \dots u_m)$ or $(A(u_0), \dots, A(u_m))$. They have nothing to do with the variables p_k involved in the generating function $F^{(m)}(t, p; x, y)$.*

We also use the following notation: For any subset $V = \{i_1, \dots, i_k\}$ of $[\![0, m]\!]$, of cardinality k, and any sequence $(x_0, \dots, x_m)$, we denote $x_V = \{x_{i_1}, \dots, x_{i_k}\}$.

Proposition 16. *Let $H(z; u) \equiv H(u)$ be a power series in z with coefficients in $\mathbb{L}[u]$, satisfying $H(-1) = 0$ and*

$$H_m(u_0, \dots, u_m) = \Phi_m(v), \tag{47}$$

where $\Phi_m(v) \equiv \Phi_m(z; v)$ is a series in z with coefficients in $v\mathbb{L}[v]$.

There exists a sequence $\Phi_0, \dots, \Phi_m$ of series in z with coefficients in $v\mathbb{L}[v]$, which depend only on Φ_m, such that for $0 \le k \le m$, and for all subset V of $[\![0, m]\!]$ of cardinality $k + 1$,

$$H_k(u_V) = \sum_{j=k}^{m} \Phi_j(v) h_{j-k}(A_V). \tag{48}$$

In particular, $H(u) \equiv H_0(u)$ is completely determined if Φ_m is known:

$$H(u) = \sum_{j=0}^{m} \Phi_j(v) A(u)^j.$$

The series $\Phi_k(v) \equiv \Phi_k(z; v)$ can be computed by a descending induction on k as follows. Let us denote by $\Phi_{k-1}^{>}(u)$ the positive part in u of $\Phi_{k-1}(v)$, that is

$$\Phi_{k-1}^{>}(u) := [u^{>}]\Phi_{k-1}(\bar{u}^m(1+u)^{m+1}).$$

Then for $1 \le k \le m$, this series can be expressed in terms of $\Phi_k, \dots, \Phi_m$:

$$\Phi_{k-1}^{>}(u) \quad = \quad -\frac{1}{\binom{m}{k}}[u^{>}]\left(\sum_{j=k}^{m} \Phi_j(v) \sum_{\lambda \vdash j-k+1} \binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(A_1, \dots, A_m)\right). \tag{49}$$

The extraction makes sense since, as will be seen, $v m_\lambda(A_1, \dots, A_m)$ belongs to $\mathbb{K}[u, \bar{u}][[z]]$. Finally, $\Phi_{k-1}(v)$ can be expressed in terms of $\Phi_{k-1}^{>}$:

$$\Phi_{k-1}(v) = \sum_{i=0}^{m} \left(\Phi_{k-1}^{>}(u_i) - \Phi_{k-1}^{>}(-1)\right). \tag{50}$$

We first establish three lemmas dealing with symmetric functions of the series u_i defined in Lemma 11.

Lemma 17. *The elementary symmetric functions of $u_0 = u, u_1, \dots, u_m$ are*

$$e_j(u_0, u_1, \dots, u_m) = (-1)^j \binom{m+1}{j} + v 1_{j=1}$$

with $v = u^{-m}(1+u)^m$.

The elementary symmetric functions of $u_1, \dots, u_m$ are

$$e_{m-j}(u_1, \dots, u_m) = \begin{cases} 1 & \text{if } j = m, \\ (-1)^{m-j-1} \sum_{p=0}^{j} \binom{m+1}{p} u^{p-j-1} & \text{otherwise.} \end{cases}$$

In particular, they are polynomials in $1/u$, and so is any symmetric polynomial in $u_1, \dots, u_m$.

Finally,

$$\prod_{i=0}^{m}(1+u_i) = v.$$

Proof. The symmetric functions of the roots of a polynomial can be read from the coefficients of this polynomial. Hence the first result follows directly from the equation satisfied by the u_i's, for $0 \le i \le m$, namely

$$(1+u_i)^{m+1} = v u_i^m.$$

For the second one, we need to find the equation satisfied by $u_1, \dots, u_m$, which is

$$0 = \frac{(1+u_i)^{m+1}u^m - (1+u)^{m+1}u_i^m}{u_i - u} = u^m u_i^m - \sum_{j=0}^{m-1} u_i^j u^{m-j-1} \sum_{p=0}^{j} \binom{m+1}{p} u^p.$$

The second result follows.

The third one is obtained by evaluating at $U = -1$ the identity

$$\prod_{i=0}^{m}(U - u_i) = (1+U)^{m+1} - vU^m.$$

∎

Lemma 18. *Denote $v = \bar{u}^m(1+u)^{m+1}$. Let P be a polynomial. Then $P(v)$ is a Laurent polynomial in u. Let $P^>(u) := [u^>]P(v)$ denote its positive part. Then*

$$P(v) = P(0) + \sum_{i=0}^{m}(P^>(u_i) - P^>(-1)). \tag{51}$$

Proof. The right-hand side of (51) is a symmetric polynomial of $u_0, \dots, u_m$, and thus, by the first part of Lemma 17, a polynomial in v. Denote it by $\tilde{P}(v)$. The second part of Lemma 17 implies that the positive part of $\tilde{P}(v)$ in u is $P^>(u_0) = P^>(u)$. That is, $P(v)$ and $\tilde{P}(v)$ have the same positive part in u. In other words, the polynomial $Q := P - \tilde{P}$ is such that $Q(v)$ is a Laurent polynomial in u of non-positive degree. But since $v = (1+u)^{m+1}\bar{u}^m$, the degree in u of $Q(v)$ coincides with the degree of Q, and so Q must be a constant. Finally, by setting $u = -1$ in $\tilde{P}(v)$, we see that $\tilde{P}(0) = P(0)$ (because $u_i = -1$ for all i when $u = -1$, as follows for instance from Lemma 17). Hence $Q = 0$ and the lemma is proved. ∎

Lemma 19. *Let $0 \leq k \leq m$, and let $R(x_0, \dots, x_k)$ be a symmetric rational function of $k+1$ variables $x_0, \dots, x_k$, such that for any subset V of $[\![0, k]\!]$ of cardinality $k+1$,*

$$R(u_V) = R(u_0, \dots, u_k).$$

Then there exists a rational fraction in v equal to $R(u_0, \dots, u_k)$.

Proof. Let $\tilde{R}$ be the following rational function in $x_0, \dots, x_m$:

$$\tilde{R}(x_0, \dots, x_m) = \frac{1}{\binom{m+1}{k+1}} \sum_{V \subset [\![0,m]\!],\ |V|=k+1} R(x_V).$$

Then $\tilde{R}$ is a symmetric function of $x_0, \dots, x_m$, and hence a rational function in the elementary symmetric functions $e_j(x_0, \dots, x_m)$, say $S(e_1(x_0, \dots, x_m), \dots, e_{m+1}(x_0, \dots, x_m))$. By assumption,

$$\tilde{R}(u_0, \dots, u_m) = S(e_1(u_0, \dots, u_m), \dots, e_{m+1}(u_0, \dots, u_m)) = R(u_0, \dots, u_k).$$

Since S is a rational function, it follows from the first part of Lemma 17 that $R(u_0, \dots, u_k)$ can be written as a rational function in v. ∎

Proof of Proposition 16. We prove (48) by descending induction on k. For $k = m$, it holds by assumption. Let us assume that (48) holds for some $k > 0$, and prove it for $k-1$.

Observe that

$$(A(x_{k-1}) - A(x_k))H_k(x_0, \dots, x_k) = H_{k-1}(x_0, \dots, x_{k-2}, x_{k-1}) - H_{k-1}(x_0, \dots, x_{k-2}, x_k).$$

This is easily proved by collecting the coefficient of $H(x_i)$, for all $i \in [\![0, k]\!]$, in both sides of the equation. We also have, for any indeterminates $a_0, \dots, a_m$,

$$(a_{k-1} - a_k)h_{j-k}(a_0, \dots, a_k) = h_{j-k+1}(a_0, \dots, a_{k-2}, a_{k-1}) - h_{j-k+1}(a_0, \dots, a_{k-2}, a_k).$$

Let V be a subset of $[\![0, m]\!]$ of cardinality $k-1$, and let p and q be two elements of $[\![0, m]\!] \setminus V$. Multiplying (48) by $A_p - A_q$, and using the two equations above gives

$$H_{k-1}(u_V, u_p) - \sum_{j=k}^{m} \Phi_j(v)h_{j-k+1}(A_V, A_p) = H_{k-1}(u_V, u_q) - \sum_{j=k}^{m} \Phi_j(v)h_{j-k+1}(A_V, A_q).$$

This implies that the series

$$H_{k-1}(x_0, \dots, x_{k-1}) - \sum_{j=k}^{m} \Phi_j(v)h_{j-k+1}(A(x_0), \dots, A(x_{k-1}))$$

takes the same value at all points u_V, for $V \subset [\![0,m]\!]$ of cardinality k. Hence Lemma 19, applied to the coefficients of this series, implies that there exists a series in z with *rational* coefficients in v, denoted $\Phi_{k-1}(v)$, such that for all $V \subset [\![0,m]\!]$ with $|V| = k$:

$$H_{k-1}(u_V) - \sum_{j=k}^{m} \Phi_j(v) h_{j-k+1}(A_V) = \Phi_{k-1}(v). \tag{52}$$

This is exactly (48) with k replaced by $k-1$.

The next point we will prove is that the coefficients of Φ_{k-1} belong to $v\mathbb{L}[v]$. In order to do so, we symmetrize (52) over $u_0, \ldots, u_m$. By (52),

$$\binom{m+1}{k} \Phi_{k-1}(v) = \sum_{V \subset [\![0,m]\!], |V|=k} H_{k-1}(u_V) - \sum_{j=k}^{m} \left(\Phi_j(v) \sum_{V \subset [\![0,m]\!], |V|=k} h_{j-k+1}(A_V) \right). \tag{53}$$

We will prove that both sums in the right-hand side of this equation are series in z with coefficients in $v\mathbb{L}[v]$.

By Lemma 15,

$$\sum_{V \subset [\![0,m]\!], |V|=k} H_{k-1}(x_V)$$

is a series in z with polynomial coefficients in $x_0, \ldots, x_m$, which is symmetric in these variables. By Lemma 17, the first sum in (53) is thus a series in z with *polynomial* coefficients in v. We still need to prove that this series vanishes at $v = 0$, that is, at $u = -1$. But this follows from the second part of Lemma 15, since $u_i = -1$ for all i when $u = -1$.

Let us now consider the second sum in (53), and more specifically the term

$$\Phi_j(v) \sum_{V \subset [\![0,m]\!], |V|=k} h_{j-k+1}(A_V). \tag{54}$$

Recall that

$$A_i = \frac{u_i}{1+u_i} e^{-K(u_i)}.$$

But by Lemma 17,

$$\frac{1}{1+u_i} = \frac{1}{v} \prod_{0 \le j \ne i \le m} (1+u_j).$$

Hence (54) can be written as a series in z with coefficients in $\mathbb{L}[1/v, u_0, \ldots, u_m]$, symmetric in $u_0, \ldots, u_m$. By the first part of Lemma 17, these coefficients belong to $\mathbb{L}[v, 1/v]$. We want to prove that they actually belong to $v\mathbb{L}[v]$, that is, that they are not singular at $v = 0$ (equivalently, at $u = -1$) and even vanish at this point. From the equation $(1+u_i)^{m+1} = v u_i^m$, it follows that we can label $u_1, \ldots, u_m$ in such a way that

$$1 + u_i = \xi^i (1+u) + o(1+u),$$

where ξ is a primitive $(m+1)^{\text{st}}$ root of unity. Since $\Phi_j(v)$ is a multiple of $v = \bar{u}^m (1+u)^{m+1}$, and the symmetric function h_{j-k+1} has degree $j - k + 1 \le m$, it follows that the series (54) is not singular at $u = -1$, and even vanishes at this point. Hence its coefficients belong to $v\mathbb{L}[v]$.

So far, $\Phi_{k-1}(v)$ has been expressed in terms of H (and the series Φ_j), and we now want to obtain an expression in terms of the Φ_j only. Lemma 18, together with $\Phi_{k-1}(0) = 0$, establishes (50). To express $\Phi_{k-1}^{>}(u)$, we now symmetrize (52) over $u_1, \ldots, u_m$. With the above notation,

$$\binom{m}{k} \Phi_{k-1}(v) = \sum_{V \subset [\![1,m]\!], |V|=k} H_{k-1}(u_V) - \sum_{j=k}^{m} \left(\Phi_j(v) \sum_{V \subset [\![1,m]\!], |V|=k} h_{j-k+1}(A_V) \right). \tag{55}$$

As above,

$$\sum_{V\subset [\![1,m]\!],|V|=k} H_{k-1}(x_V)$$

is a series in z with polynomial coefficients in $x_1,\dots,x_m$, which is symmetric in these variables. By the second part of Lemma 17, the first sum in (55) is thus a series in z with polynomial coefficients in $1/u$. Since $\Phi_{k-1}(v)$ has coefficients in $\mathbb{L}[v]$, and hence in $\mathbb{L}[u,1/u]$, the second sum in (55) is also a zeries in z with coefficients in $\mathbb{L}[u,1/u]$. We can now extract from (55) the positive part in u, and this gives

$$\binom{m}{k}\Phi^{>}_{k-1}(u) = -[u^>]\left(\sum_{j=k}^{m}\left(\Phi_j(v)\sum_{V\subset [\![1,m]\!],|V|=k} h_{j-k+1}(A_V)\right)\right).$$

One easily checks that, for indeterminates $a_1,\dots,a_m$,

$$\sum_{V\subset [\![1,m]\!],|V|=k} h_{j-k+1}(a_V) = \sum_{\lambda\vdash j-k+1}\binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(a_1,\dots,a_m),$$

so that the above expression of $\Phi^{>}_{k-1}(u)$ coincides with (49). ∎

5.3. The case $y=1$

As explained in Section 4.1, Theorem 3 will be proved if we establish $\tilde G(u,1)=G_1(u)$, where

$$G_1(u) = (1+\bar u)e^{K(u)+L}\left((1+u)e^{-mK(u)}-1\right).$$

A natural attempt would be to set $y=1$ in the expression of $\tilde G(u,y)$ that can be derived from Proposition 16, as we did when $m=1$ in Section 4.2.3. However, we have not been able to do so, and will proceed differently.

We have proved in Proposition 12 that the series $\tilde G(u,y)$ satisfies (47) with $\Phi_m(v)=ve^{yV(v)}$. In particular, $\tilde G(u,1)$ satisfies (47) with $\Phi_m(v)=ve^{V(v)}$. By Proposition 16, this equation, together with the initial condition $\tilde G(-1,1)=0$, characterizes $\tilde G(u,1)$. It is clear that $G_1(-1)=0$. Hence it suffices to prove the following proposition.

Proposition 20. *The series $G_1(u)$ satisfies (47) with $\Phi_m(v)=ve^{V(v)}$.*

Proof. First observe that

$$G_1(u) = e^L\left(vA(u)^{m-1}-\frac{1}{A(u)}\right).$$

Using Lemma 13 with $x_i=A_i$, it follows that

$$\begin{aligned}\sum_{i=0}^{m}\frac{G_1(u_i)}{\prod_{j\neq i}(A_i-A_j)} &= 0+(-1)^{m+1}e^L\prod_{i=0}^{m}\frac{1}{A_i} \qquad \text{(by (35) and (36))}\\ &= (-1)^{m+1}e^{L+\sum_i K(u_i)}\prod_{i=0}^{m}\frac{(1+u_i)}{u_i}.\end{aligned}$$

By Lemma 17 one has $\prod_i(1+u_i)=v$ and $\prod_i u_i=(-1)^{m+1}$, so it only remains to show that $L+\sum_{i=0}^m K(u_i)=V(v)$.

Recall that $V(v)$ belongs to $v\mathbb{K}[v][[z]]$ and that $K(u)=[u^>]V(v)$. Therefore Lemma 18 gives:

$$V(v) = 0+\sum_{i=0}^{m}\big(K(u_i)-K(-1)\big).$$

But it follows from (6) that

$$K(-1)=\sum_{k\geq 1}\frac{p_k}{k}z^k\sum_{i=1}^{k}\binom{(m+1)k}{k-i}(-1)^i = -\sum_{k\geq1}\frac{p_k}{k}z^k\frac{k}{(m+1)k}\binom{(m+1)k}{k} = \frac{-L}{m+1},$$

where we have used the identity

$$\sum_{i=1}^{a}\binom{b}{a-i}(-1)^i=-\binom{b-1}{a-1}=-\frac{a}{b}\binom{b}{a},$$

valid for $b\geq a\geq 0$, which is easily proved by induction on a. Therefore $V(v)=L+\sum_{i=0}^m K(u_i)$, and the proof is complete. ∎

We have finally proved that $\tilde G(u,1)=G_1(u)$. As explained in Section 4.1, this implies that $F^{(m)}(x,y)=\tilde G(u,y)$ after the change of variables (8). In particular, $F^{(m)}(x,1)=G_1(u)$, and (9) is proved. One then obtains (10) in the limit $u\to 0$, using

$$[u]K(u)=\sum_{k\geq 1}\frac{p_k}{k}\binom{(m+1)k}{k-1}z^k.$$

5.4. From series to numbers

We now derive from (10) the expression of the character given in Theorem 2. We will extract from $F^{(m)}(t,p;1,1)$ the coefficient of t^n. We find convenient to rewrite the factor e^L occurring in this series as $\tilde z/s$, where $s^m=t$ and $\tilde z=se^L$ (so that $\tilde z^m=z$).

Hence

$$\begin{aligned}
[t^n]F(t,p;1,1) &= [s^{mn+1}]\left(\tilde z-m\sum_{k\geq 1}\frac{p_k}{k}\tilde z^{km+1}\binom{(m+1)k}{k-1}\right)\\
&= \frac{1}{mn+1}[\tilde z^{mn}]\left(1-m\sum_{k\geq 1}\frac{p_k}{k}(km+1)\tilde z^{km}\binom{(m+1)k}{k-1}\right)e^{(mn+1)L}
\end{aligned}$$

by the Lagrange inversion formula. This can be rewritten in terms of $z=\tilde z^m$:

$$[t^n]F(t,p;1,1) = \frac{1}{mn+1}[z^n]\left(1-m\sum_{k\geq 1}p_k z^k\binom{(m+1)k}{k}\right)e^{(mn+1)L}.$$

The sum inside the brackets is closely related to the derivative of L with respect to z:

$$\begin{aligned}
[t^n]F(t,p;1,1) &= \frac{1}{mn+1}[z^n]\left(1-mz\frac{\partial L}{\partial z}\right)e^{(mn+1)L}\\
&= \frac{1}{mn+1}[z^n]\left(1-\frac{mz}{mn+1}\frac{\partial}{\partial z}\right)e^{(mn+1)L}\\
&= \frac{1}{mn+1}\left(1-\frac{mn}{mn+1}\right)[z^n]e^{(mn+1)L}\\
&= \frac{1}{(mn+1)^2}[z^n]\prod_{k\geq 1}\exp\left((mn+1)\frac{p_k}{k}z^k\binom{(m+1)k}{k}\right)\\
&= \frac{1}{(mn+1)^2}\sum_{\alpha_1+2\alpha_2+\cdots=n}(mn+1)^{\sum\alpha_k}\prod_k\frac{p_k^{\alpha_k}}{k^{\alpha_k}\alpha_k!}\binom{(m+1)k}{k}^{\alpha_k}\\
&= \frac{1}{(mn+1)^2}\sum_{\lambda=(\lambda_1,\ldots)\vdash n}(mn+1)^{\ell(\lambda)}\frac{p_\lambda}{z_\lambda}\prod_{i\geq 1}\binom{(m+1)\lambda_i}{\lambda_i}.
\end{aligned}$$

The final equation is precisely Theorem 2.

5.5. **The complete series** $F(t,p;x,y)$

We finally give an explicit expression of the complete series $F(x,y) \equiv F^{(m)}(t,p;x,y)$. Recall that $F(x,y) = \tilde{G}(u,y)$ after the change of variables (8), and that the series $\tilde{G}(u,y)$ satisfies (47) with $\Phi_m(v) = ve^{yV(v)}$ (Proposition 12). Hence Proposition 16 gives an explicit, although complicated, expression of the complete series $F(t,p;x,y)$.

Theorem 21. *Let $F^{(m)}(t,p;x,y) \equiv F(t,p;x,y)$ be the refined Frobenius series of the m-Tamari representation, defined by (7). Let z and u be two indeterminates, and write*

$$t = ze^{-mL} \quad \text{and} \quad x = (1+u)e^{-mK(u)}.$$

where L and $K(u)$ are defined by (5–6). Then $F(t,p;x,y)$ becomes a series in z with polynomial coefficients in u, y and the p_i, and this series can be computed by an iterative extraction of positive parts. More precisely,

$$F(t,p;x,y) = \sum_{k=0}^{m} \Phi_k(v)A(u)^k, \tag{56}$$

where $v = u^{-m}(1+u)^{m+1}$, $A(u)$ is defined by (25), and $\Phi_k(v) \equiv \Phi_k(z;v)$ is a series in z with polynomial coefficients in v, y and the p_i's. This series can be computed by a descending induction on k as follows. First, $\Phi_m(v) = ve^{yV(v)}$ where $V(v)$ is defined by (4). Then for $1 \le k \le m$,

$$\Phi_{k-1}(v) = \sum_{i=0}^{m} \left(\Phi_{k-1}^{>}(u_i) - \Phi_{k-1}^{>}(-1)\right)$$

where

$$\begin{aligned}\Phi_{k-1}^{>}(u) &= [u^{>}]\Phi_{k-1}(v) \\ &= -\frac{1}{\binom{m}{k}}[u^{>}]\left(\sum_{j=k}^{m} \Phi_j(v) \sum_{\lambda \vdash j-k+1} \binom{m-\ell(\lambda)}{k-\ell(\lambda)} m_\lambda(A(u_1),\dots,A(u_m))\right),\end{aligned} \tag{57}$$

and $u_0 = u, u_1, \dots, u_m$ are the $m+1$ roots of the equation $(1+u_i)^{m+1} = u_i^m v$.

We can rewrite (56) in a slightly different form, which gives directly (11) when $m = 1$. This rewriting combines (56) with the expression of $[u^{>}]\Phi_0(v)$ derived from (57). The case $k = 1$ of (57) reads

$$[u^{>}]\Phi_0(v) = -\frac{1}{m}[u^{>}]\left(\sum_{j=1}^{m} \Phi_j(v) \sum_{i=1}^{m} A(u_i)^j\right). \tag{58}$$

Recall that $F(t,p;x,y) = \tilde{G}(z,p;u,y)$ has polynomial coefficients in u, and that $x = 1$ when $u = 0$. Hence, returning to (56):

$$\begin{aligned}F(t,p;x,y) &= F(t,p;1,y) + [u^{>}]\left(\sum_{k=0}^{m} \Phi_k(v)A(u)^k\right) \\ &= F(t,p;1,y) + [u^{>}]\left(\sum_{k=1}^{m} \Phi_k(v)\left(A(u)^k - \frac{1}{m}\sum_{i=1}^{m} A(u_i)^k\right)\right) \qquad \text{(by (58))} \\ &= (1+u)[u^{\ge}]\left(\sum_{k=1}^{m} \frac{\Phi_k(v)}{1+u}\left(A(u)^k - \frac{1}{m}\sum_{i=1}^{m} A(u_i)^k\right)\right)\end{aligned} \tag{59}$$

by (30), and given that $F(t,p;x,y) = 0$ when $u = -1$. The proof that $\Phi_k(v)\sum_{i=1}^{m} A(u_i)^k$ has coefficients in $(1+u)\mathbb{K}[u,\bar{u}]$ (which is needed to apply (30)) is similar to the proof that (54) has coefficients in $v\mathbb{K}[v]$.

Examples. We now specialize (59) to $m = 1$ and $m = 2$. When $m = 1$, (59) coincides with (11) (recall that $\Phi_m = ve^{yV(v)}$). When $m = 2$, we obtain the following expression for $F^{(2)}$.

Corollary 22. *Let $V(v), L$ and $K(u)$ be the series given by (4–6), with $m = 2$. Perform the change of variables (8), still with $m = 2$. Then the weighted Frobenius series of the 2-Tamari representation satisfies*

$$\frac{F^{(2)}(t,p;x,y)}{1+u} =$$
$$[u^{\geq}]\left(\frac{\Phi_1(v)}{1+u}\left(A(u)-\frac{A(u_1)}{2}-\frac{A(u_2)}{2}\right)+(1+\bar u)^2 e^{yV(v)}\left(A(u)^2-\frac{A(u_1)^2}{2}-\frac{A(u_2)^2}{2}\right)\right),$$

where

$$u_{1,2} = \frac{1+3u\pm(1+u)\sqrt{1+4u}}{2u^2}, \qquad A(u) = \frac{u}{1+u}e^{-K(u)},$$

and

$$\Phi_1(v) = \Phi_1^{>}(u) + \Phi_1^{>}(u_1) + \Phi_1^{>}(u_2) - 3\Phi_1^{>}(-1),$$

with

$$\Phi_1^{>}(u) = -[u^{>}]\left((1+u)^3\bar u^2 e^{yV(v)}\left(A(u_1)+A(u_2)\right)\right).$$

This expression has been checked with Maple, after computing the first coefficients of $F^{(2)}(t,p;x,y)$ from the functional equation (13).

6. Final comments

6.1. A constructive proof?

Our proof would not have been possible without a preliminary task consisting in *guessing* the expression (9) of $F(t,p;x,1)$. This turned out to be difficult, in particular because the standard guessing tools, like the Maple package Gfun, can only guess D-finite generating functions, while the generating function of the numbers (2), or even (3), is not D-finite. The expression of $F(t,p;x,1)$ actually *becomes* D-finite in z (at least when only finitely many p_i's are non-zero) after the change of variables (8). The correct parametrization of the variable t by z was not hard to obtain using the (former) conjecture (2) and the Lagrange inversion formula, but we had no indication on the correct parametrization of x. Our discovery of it only came after a long study of special cases (for instance $m = 1$ and $p_i = 1_{i=1}$), and an analogy with the enumeration of unlabelled Tamari intervals [8]. Obviously, a constructive proof of our result would be most welcome, not to mention a bijective one.

6.2. The action of $\mathfrak{S}_n$ on prime m-Tamari intervals

Other remarkable formulas, as simple as (2) and (3), can be derived from our expression (9) of the series $F^{(m)}(t,p;x,1)$. Let us for instance focus on the action of $\mathfrak{S}_n$ on *prime* intervals, that is, intervals $[P,Q]$ such that P has only two contacts with the line $\{x = my\}$. The character $\tilde\chi_m$ of this representation is obtained by extracting the coefficient of x^2 from $F^{(m)}(t,p;x,1)$, and the Lagrange inversion formula gives, for a partition λ of length ℓ:

$$\tilde\chi_m(\lambda) = ((m+1)n-1)^{\ell-2}\prod_{1\leq i\leq \ell}\binom{(m+1)\lambda_i-1}{\lambda_i}.$$

In particular, the number of prime labelled m-Tamari intervals of size n is

$$((m+1)n-1)^{n-2}m^n.$$

For *unlabelled* intervals, it follows from [8, Coro. 11] that the corresponding numbers are

$$\frac{m}{n((m+1)n-1)}\binom{(m+1)^2n-m-1}{n-1}.$$

6.3. The number of unlabelled m-Tamari intervals

Recall from Lemma 8 that the series $F^{(m)}(t,p;x,y)$ can also be understood as the generating function of (weighted) unlabelled m-Tamari intervals. In particular, when $p=\mathbf{1}=(1,1,\ldots)$ and $y=1$, we have

$$h_k=\sum_{\lambda\vdash k}\frac{1}{z_\lambda}=1,$$

(because $k!/z_\lambda$ counts permutations of cycle type λ), so that

$$F^{(m)}(t,\mathbf{1};x,1)=\sum_{I=[P,Q]\text{ unlabelled}} t^{|I|}x^{c(P)}.$$

By specializing Theorem 3 to the case $y=1, p=\mathbf{1}$, we recover the following result, already proved in [8]. The result of [8] also keeps track of the size of the first ascent, but we have not been able to recover it in this generality.

Proposition 23 ([8]). *Let z' and u' be two indeterminates, and write*

$$t=z'(1-z')^{m^2+2m} \qquad \text{and} \qquad x=\frac{1+u'}{(1+z'u')^{m+1}}. \tag{60}$$

Then the ordinary generating function of unlabelled m-Tamari intervals, counted by the size and the number of contacts, becomes a series in z' with polynomial coefficients in u', and admits the following closed form expression:

$$F^{(m)}(t,\mathbf{1};x,1)=\frac{(1+u')(1+z'u')}{u'(1-z')^{m+2}}\left(\frac{1+u'}{(1+z'u')^{m+1}}-1\right). \tag{61}$$

As shown in [8], this implies that the number of unlabelled m-Tamari intervals of size n is

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

Proof. We need to relate the parametrizations (8) and (60), and then the expressions (9) and (61). Let $M\equiv M(z)$ be the unique formal power series in z satisfying

$$M=1+zM^{m+1}. \tag{62}$$

We claim that, when $p=\mathbf{1}$,

$$e^L=M^{m+1} \quad \text{and} \quad e^{K(u)}=\frac{1}{1-zuM^{m+1}}=\frac{1}{1-u(M-1)}. \tag{63}$$

This establishes the equivalence between the parametrizations (8) and (60), with

$$M=\frac{1}{1-z'} \quad \text{and} \quad u=\frac{u'(1-z')}{1+u'z'}.$$

The equivalence between the two expressions of $F^{(m)}$, namely (9) and (61), also follows.

We will prove (63) using combinatorial interpretations of the series K, L, M in terms of lattice paths on the square grid, starting at $(0,0)$ and formed of north and east steps. First, note that M counts m-ballot paths (defined in the introduction) by the size. Also,

$$B(z):=z\frac{d}{dz}L(z,\mathbf{1})=\sum_{k\geq 1}\binom{(m+1)k}{k}z^k$$

counts, by the number of north steps, non-empty paths ending on the line $\{x=my\}$ (often called *bridges*, hence the notation B). We have $M=1/(1-P)$, where P counts *prime* ballot paths (those that only have two contacts). By a variant of the cycle lemma [4, Section 4.1], there exists a size preserving bijection between non-empty bridges and pairs formed of a prime excursion with a marked step, and an excursion. Since a bridge having n north steps has $(m+1)n$ steps in total, this gives:

$$z\frac{d}{dz}L(z,\mathbf{1}) = B(z) = \frac{z(m+1)P'(z)}{1-P(z)} = z\frac{d}{dz}\left(\ln M(z)^{m+1}\right). \tag{64}$$

Integrating over z and then exponentiating gives the first part of (63).

Let us now consider the series $K(z,\mathbf{1};u)$. We will interpret it in terms of paths of length $k(m+1)$ for some k (to generalize the terminology used for ballot paths, we say that such paths have *size* k). The *depth* of path ending at (x,y) is $x-my$. Observe that

$$z\frac{d}{dz}K(z,\mathbf{1};u) = \sum_{k\geq 1} z^k \sum_{i=1}^{k} \binom{(m+1)k}{k-i} u^i,$$

counts paths of length multiple of $(m+1)$ having a positive depth (z accounts for the size, divided by $(m+1)$, and u for the depth, also divided by $(m+1)$). Let w be such a path, and look at the shortest prefixes of w of depth 1, then depth 2, and so on up to depth $(m+1)i$. This factors w into a sequence $(M_1, e, M_2, e, \ldots, M_{(m+1)i}, e, B)$, where the M_i are ballot paths, e stands for an east step and B is a bridge. Accordingly,

$$z\frac{d}{dz}K(z,\mathbf{1};u) = (1+B(z))\left(\frac{1}{1-zuM(z)^{m+1}}-1\right) = z\frac{d}{dz}\left(\ln\frac{1}{1-zuM(z)^{m+1}}\right),$$

by (64). Integrating and exponentiating gives the second part of (63). ■

6.4. A q-analogue of the functional equation

As described in the introduction, the numbers (3) are conjectured to give the dimension of certain polynomial rings generalizing $\mathrm{DR}_{3,n}$. These rings are tri-graded (with respect to the sets of variables $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$), and it is conjectured [5] that the dimension of the homogeneous component in the x_i's of degree k is the number of labelled intervals $[P,Q]$ in $\mathcal{T}_n^{(m)}$ such that the longest chain from P to Q, in the Tamari order, has length k. One can recycle the recursive description of intervals described in Section 3 to generalize the functional equation of Proposition 5 (taken when $p_i = 1_{i=1}$), by taking into account (with a new variable q) this distance. Eq. (13) becomes

$$\frac{\partial F}{\partial y}(x,y) = tx(F(x,1)\Delta)^{(m)}(F(x,y)),$$

where now

$$\Delta S(x) = \frac{S(qx)-S(1)}{qx-1}.$$

Here $F(1,1)$ counts labelled m-Tamari intervals by the size and the above defined distance. But we have not been able to conjecture any simple q-analogue of (3).

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MBM: CNRS, LaBRI, UMR 5800, Université de Bordeaux, 351 cours de la Libération, 33405 Talence Cedex, France

E-mail address: `mireille.bousquet@labri.fr`

GC: CNRS, LIAFA, UMR 7089, Université Paris Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, France

E-mail address: `guillaume.chapuy@liafa.jussieu.fr`

LFPR: LACIM, UQAM, C.P. 8888 Succ. Centre-Ville, Montréal H3C 3P8, Canada

(Current address: Instituto de Matemática y Física, Universidad de Talca, 2 norte 685, Talca, Chile)

E-mail address: `preville-ratelle@inst-mat.utalca.cl`