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Abstract. A generalized version of the TKNN-equations computing Hall conductances for generalized Dirac-like Harper operators is derived. Geometrically these equations relate Chern numbers of suitable (dual) bundles naturally associated to spectral projections of the operators.

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GENERALIZED HARPER OPERATORS

The *integer quantum Hall effect* (IQHE) reveals a variety of surprising and attractive physical features, and has been the subject of several investigations (see [17, 13] and references therein). In fact, a complete spectral analysis of the Schrödinger operator for a single particle moving in a plane in a periodic potential and subject to an uniform orthogonal magnetic field of strength B (*magnetic Bloch electron*) is extremely difficult. Thus the need for simpler effective models which hopefully capture (some of) the main physical features in suitable physical regimes.

In the limit of a strong magnetic field, $B \gg 1$, the IQHE is well described by an effective Harper operator (cf. [23, 2, 15, 12]). For this model the quantization (in units of e^2/h) of the Hall conductance has a geometric meaning being related to Chern numbers of suitable naturally bundles associated to spectral projections of the operator. A family of Diophantine equations, the TKNN-equations of [22], provides a recipe for computing such integers. The aim of the present paper is to derive a generalized version of the TKNN-equations yielding the Hall conductances for more general Dirac-like Harper operators. Interest in such generalizations comes also from these Dirac-like operators appearing naturally in important physical models, notably models for the graphene.

With $\theta = 1/B$, the effective Harper operator is

$$(H^{\theta}_{1,0}\psi)(x) = \psi(x-\theta) + \psi(x+\theta) + 2\cos(2\pi x)\psi(x), \tag{1}$$

acting on the Hilbert space $\mathscr{H}_1 = L^2(\mathbb{R})$. That this operator is the simplest representative of a large family of generalized Harper operators, sharing similar mathematical properties, is our starting point. On the Hilbert space $\mathscr{H}_1$ consider the unitary operators

$$(T_1\psi)(x) = \mathrm{e}^{\mathrm{i}2\pi x}\psi(x)\,, \qquad (T_2^{\theta}\psi)(x) = \psi(x-\theta)\,, \tag{2}$$

with $\theta \in \mathbb{R}$. They are readily seen to obey the relation

$$T_1 T_2^\theta = \mathrm{e}^{\mathrm{i}2\pi\theta}\ T_2^\theta T_1, \tag{3}$$

yielding for the Harper operator the expression $H_{1,0}^\theta = T_1 + (T_1)^\dagger + T_2^\theta + (T_2^\theta)^\dagger$.

For any positive integer $q = 1,2,\dots$, on the vector space $\mathbb{C}^q$ consider two unitary $q\times q$ matrices $\mathbb{U}_q$ and $\mathbb{V}_q$ defined as follows. Let $\{e_0,\dots,e_{q-1}\}$ be the canonical basis of $\mathbb{C}^q$, then $\mathbb{U}_q$ is a diagonal matrix and $\mathbb{V}_q$ is a shift matrix acting as

$$\mathbb{U}_q : e_j \mapsto \mathrm{e}^{\mathrm{i}2\pi\frac{j}{q}}\ e_j, \quad \text{and} \quad \mathbb{V}_q : e_j \mapsto e_{[j+1]_q},$$

where $[\,\cdot\,]_q$ stays for modulo q. They obey

$$\mathbb{U}_q\mathbb{V}_q = \mathrm{e}^{\mathrm{i}2\pi\frac{1}{q}}\mathbb{V}_q\mathbb{U}_q \qquad \text{and} \qquad (\mathbb{U}_\mathrm{q})^\mathrm{q} = \mathbb{I}_\mathrm{q} = (\mathbb{V}_\mathrm{q})^\mathrm{q}.$$

Then, on the Hilbert space $\mathscr{H}_q = L^2(\mathbb{R})\otimes\mathbb{C}^q$ one defines a pair of unitary operators:

$$U_q = T_1 \otimes \mathbb{U}_q, \qquad V_{q,r}^\theta = T_2^\varepsilon \otimes (\mathbb{V}_q)^r, \tag{4}$$

with $\varepsilon(\theta,q,r) = \theta - \frac{r}{q}$ and T_1 and T_2^ε given by (2). The integer $r \in \{0,\pm1,\dots,\pm(q-1)\}$ is chosen coprime with respect to q. As for the case before (when $q=1, r=0$), the operators (4) also obey the relation (3):

$$U_q V_{q,r}^\theta = \mathrm{e}^{\mathrm{i}2\pi\theta}\ V_{q,r}^\theta U_q.$$

Following the definition (1) we can introduce the *generalized* (q,r)*-Harper operator*

$$H_{q,r}^\theta = U_q + (U_q)^\dagger + V_{q,r}^\theta + (V_{q,r}^\theta)^\dagger. \tag{5}$$

More generally one considers the collection $\mathscr{A}_{q,r}^\theta$ of bounded operators on the Hilbert space $\mathscr{H}_q$ generated by the unitaries U_q and $V_{q,r}^\theta$. Technically $\mathscr{A}_{q,r}^\theta$ is a C^*-algebra, i.e. an involutive algebra closed with respect to the operator norm, and it is named the *(ir)rational rotation algebra* or the *noncommutative torus algebra* [19, 8].

By writing the operator in (1) as $H_{1,0}^\theta = D_\theta + C$ with

$$(D_\theta\psi)(x) = \psi(x-\theta) + \psi(x+\theta), \qquad (C\psi)(x) = 2\cos(2\pi x)\psi(x), \tag{6}$$

in particular, the generalized $(2,1)$-Harper operator $H_{2,1}^\theta$ is just

$$H_{2,1}^\theta = \begin{pmatrix} C & D_{\theta-\frac{1}{2}} \\ D_{\theta-\frac{1}{2}} & -C \end{pmatrix} \tag{7}$$

acting on $\mathscr{H}_2 = L^2(\mathbb{R})\otimes\mathbb{C}^2$. This operator provides an interesting effective model for the IQHE on graphene [3, 14, 21]. Moreover, Dirac-like operators like $H_{2,1}^\theta$ can be used to describe effective models for electrons interacting with the periodic structure of a crystal through a periodic (internal) magnetic field and subjected to the action of an external strong magnetic field [9, 12].

BLOCH-FLOQUET TRANSFORM

For rational deformation parameter, $\theta = M/N$ (M and N taken to be coprime here and after), any family of operators $\mathscr{A}^{\theta}_{q,r}$ can be decomposed in a continuous way according to a generalized version of the Bloch theorem. More explicitly we have the following.

Proposition A. *Let $\theta = M/N$. For any (admissible) pair (q,r) the bounded operator algebra $\mathscr{A}^{\theta}_{q,r}$ on $\mathscr{H}_q$ admits a* bundle representation $\Pi_{q,r}$ *over the ordinary two-torus $\mathbb{T}^2$. That is to say, there is a Hermitian vector bundle $E_{N,q} \to \mathbb{T}^2$ together with a unitary transform $\mathscr{F}_{q,r} : \mathscr{H}_q \to L^2(E_{N,q})$ such that*

$$\Pi_{q,r}(\mathscr{A}^{\theta}_{q,r}) = \mathscr{F}_{q,r}\ \mathscr{A}^{\theta}_{q,r}\ \mathscr{F}_{q,r}{}^{-1} \subset \Gamma\big(\mathrm{End}(E_{N,q})\big). \tag{8}$$

The vector bundle $E_{N,q}$ has rank N and (first) Chern number $C_1(E_{N,q}) = q$.

Here $L^2(E_{N,q})$ denotes the Hilbert space of square integrable sections of the vector bundle $E_{N,q}$ and $\Gamma(\mathrm{End}\big(E_{N,q})\big)$ denotes the collection of continuous sections of the endomorphism bundle $\mathrm{End}(E_{N,q}) \to \mathbb{T}^2$, i.e. the vector bundle with fibers $\mathrm{End}(\mathbb{C}^q)$ associated with the vector bundle $E_{N,q}$. The unitary map $\mathscr{F}_{q,r}$ implementing the bundle representation of $\mathscr{A}^{\theta}_{q,r}$ is called *(generalized) Bloch-Floquet transform* [9, 11]. For the details of the proof of Prop. A, that we briefly sketch, we refer to [10] (see also [20]). Denote with $\alpha \in \mathbb{Z}$, $|\alpha| < q$ the unique solution of $\beta q - \alpha r = 1$ (due to q and r being coprime) and be $M_0 = qM - rN$. Then, a simple check shows that the unitary operators

$$A^{\theta}_{q,r} = (T_1)^{\frac{1}{q\varepsilon}} \otimes (\mathbb{U}_q)^{\alpha}, \qquad B^{\theta}_{q,r} = T_2^{\frac{M_0}{q}} \otimes (\mathbb{V}_q)^{rN}$$

commute, $[A^{\theta}_{q,r}, B^{\theta}_{q,r}] = 0$, while commuting with any element in $\mathscr{A}^{\theta}_{q,r}$. They generate a (indeed maximally) commutative sub-algebra of the commutant of $\mathscr{A}^{\theta}_{q,r}$, and in particular, of symmetries for the operator (5). Were N a multiple of q this commutative sub-algebra would reduce to a direct sum of q copies of a commutative algebra on $L^2(\mathbb{R})$. Thus to avoid this degeneracy, we take q and N to be coprime as well. This entails there exist two integers d_r and n_r such that $qd_r + Nn_r = 1$, a fact we shall exploit momentarily. Moreover, the commutative algebra generated by $A^{\theta}_{q,r}$ and $B^{\theta}_{q,r}$ is isomorphic to the algebra of continuos functions over the ordinary 2-torus $\mathbb{T}^2$.

A generalized simultaneous eigenvectors of $A^{\theta}_{q,r}$ and $B^{\theta}_{q,r}$ is a $\Xi_k \in S'(\mathbb{R}) \otimes \mathbb{C}^q$ ($S'(\mathbb{R})$ is the space of tempered distributions) such that

$$A^{\theta}_{q,r}\ \Xi_k = \mathrm{e}^{\mathrm{i}2\pi k_1}\ \Xi_k, \qquad B^{\theta}_{q,r}\ \Xi_k = \mathrm{e}^{\mathrm{i}2\pi k_2}\ \Xi_k.$$

For any $k = (k_1,k_2) \in [0,1]^2 \simeq \mathbb{T}^2$, the generalized eigenvectors make up a N-dimensional space, a basis of which being given by a fundamental family of distribution $\Upsilon^{(j)}(k) = (\zeta^{(j)}_0(k), \dots, \zeta^{(j)}_{q-1}(k)) \in S'(\mathbb{R}) \otimes \mathbb{C}^q$, for indices $j = 0, \dots, N-1$, with elements $\zeta^{(j)}_\ell(k)$, $\ell = 0, \dots, q-1$, defined by

$$\zeta^{(j)}_\ell(k) = \sqrt{\frac{|M_0|}{N}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi k_1(\tau_\ell + mq)}\ \delta\left[\ \cdot\ - \frac{M_0}{N}(k_2 + j) - mM_0 - \tau_\ell \frac{M_0}{q}\right]. \tag{9}$$

Here the permutation $\tau : \ell \mapsto \tau_\ell$ of the set $\{0,\dots,q-1\}$ is defined by $\ell = [\tau_\ell r N]_q$ and, as usual, the Dirac delta function $\delta(\cdot - x_0)$ acts on functions $f : \mathbb{R} \to \mathbb{C}$ as the evaluation at the point x_0, i.e. $\langle \delta(\cdot - x_0); f \rangle = f(x_0)$.

We let $\mathscr{H}_{q,r}(k) \subset S'(\mathbb{R}) \otimes \mathbb{C}^q$ denote the N-dimensional vector space spanned by the distributions $\Upsilon^{(0)}(k),\dots,\Upsilon^{(N-1)}(k)$. The total space of the vector bundle $E_{N,q}$ is just the disjoint union of the spaces $\mathscr{H}_{q,r}(k)$ glued together with transitionf functions coming from pseudo-periodic conditions satisfied by the $\Upsilon^{(j)}$'s. Indeed, from (9) one deduces that $\Upsilon^{(j)}(k_1+1,k_2) = \Upsilon^{(j)}(k_1,k_2)$ while $\Upsilon^{(j)}(k_1,k_2+1) = \Upsilon^{(j+1)}(k_1,k_2)$ for $j = 0,\dots,N-2$ and $\Upsilon^{(N-1)}(k_1,k_2+1) = \mathrm{e}^{\mathrm{i}2\pi q}\Upsilon^{(0)}(k_1,k_2)$. Also, there is an identification

$$\mathscr{F}_{q,r} : \mathscr{H}_q \to L^2(E_{N,q}) \simeq \int_{\mathbb{T}^2}^{\oplus} \mathscr{H}_{q,r}(k)\ \mathrm{d}z(k)$$

which is very reminiscent of the usual direct integral decomposition of the Bloch theory. We stress that Prop. A does not only states that any $H \in \mathscr{A}^\theta_{q,r}$ can be decomposed as a direct integral operator $H = \int_{\mathbb{T}^2}^{\oplus} h(k)\ \mathrm{d}z(k)$ with $h(k)$ an $N \times N$ matrix acting on $\mathscr{H}_{q,r}(k)$, but also that such a decomposition is continuous with respect to the topology of the vector bundle $E_{N,q}$, thus amounting to a bundle representation. For $H \in \mathscr{A}^\theta_{q,r}$, we denote $\widetilde{H} = \Pi_{q,r}(H)$. For the generators, when acting on the basis $\{\Upsilon^{(j)}(k)\}$ one finds

$$\widetilde{U}_q(k_1,k_2) = \mathrm{e}^{\mathrm{i}2\pi \frac{M_0}{N} k_2}\ (\mathbb{U}_N)^{qM}, \qquad \widetilde{V}^\theta_{q,r}(k_1,k_2) = \mathrm{e}^{\mathrm{i}2\pi n_r k_1}\ (\mathbb{V}_{N;k_1})^{d_r}. \tag{10}$$

Here $\mathbb{U}_N$ is the diagonal matrix $\mathbb{U}_N : e_j \mapsto \mathrm{e}^{\mathrm{i}2\pi \frac{j}{N}}\ e_j$; $\mathbb{V}_{N;k_1}$ is the twisted shift matrix sending e_j to e_{j+1} for $j = 0,\dots,N-2$ while e_{N-1} to $\mathrm{e}^{\mathrm{i}2\pi q k_1} e_0$. The matrices in (10) commute up to $\mathrm{e}^{\mathrm{i}2\pi \frac{M}{N} q d_r} = \mathrm{e}^{\mathrm{i}2\pi \frac{M}{N}}$ being $q d_r = 1 - n_r N$ as before, $\widetilde{U}_q \widetilde{V}^\theta_{q,r} = \mathrm{e}^{\mathrm{i}2\pi \frac{M}{N}} \widetilde{V}^\theta_{q,r} \widetilde{U}_q$, thus providing a representation of $\mathscr{A}^\theta_{q,r}$. Moreover, their pseudo-periodic conditions

$$\widetilde{U}_q(k_1+1,k_2+1) = \mathrm{e}^{\mathrm{i}2\pi \frac{M}{N}} \widetilde{U}_q(k_1,k_2), \qquad \widetilde{V}^\theta_{q,r}(k_1+1,k_2+1) = \widetilde{V}^\theta_{q,r}(k_1,k_2),$$

match those of the basis $\{\Upsilon^{(j)}(k)\}$ thus making $\Pi_{q,r}$ a representation of $\mathscr{A}^\theta_{q,r}$ as bundle endomorphisms, as expressed in (8).

The bundle $E_{N,q}$ comes equipped with the *Berry connection*

$$\omega_{i,j}(k) = \langle \Upsilon^i(k) | \, \mathrm{d}\Upsilon^j(k) \rangle\ , \quad i,j = 0,\dots,N-1, \tag{11}$$

Its curvature $K = \mathrm{d}\omega$ is constant, $K(k) = \left(\frac{2\pi q}{\mathrm{i}N}\ \mathbb{I}_N \right) \mathrm{d}k_1 \wedge \mathrm{d}k_2$ (up to an exact form) and when integrated it results in the first Chern number of the bundle being

$$C_1(E_{N,q}) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{T}^2} \mathrm{Tr}_N(K) = q.$$

For a rational $\theta = M/N$, the spectrum of $H^{\theta}_{q,r}$ in (5) has $N+1$ energy bands if N is odd or N energy bands if N is even [16, 6, 10]. These include the *inf-gap* (from $-\infty$ to the minimum of the spectrum) and the *sup-gap* (from the maximum of the spectrum to $+\infty$).

To each gap g one associates a spectral projection P_g with the convention that $P_0 = 0$ for the inf-gap $g = 0$ and $P_{\max} = \mathbb{I}$ for the sup-gap $g = N_{\max}$ with $N_{\max} = N-1$ or $N_{\max} = N$ according to whether N is odd or even. As usual, the projection P_g is defined via the Riesz formula for the operator $H^{\theta}_{q,r}$,

$$P_g = \frac{1}{\mathrm{i}2\pi} \oint_{\Lambda} (\lambda \mathbb{I} - H^{\theta}_{q,r})^{-1} \, \mathrm{d}\lambda. \tag{12}$$

The closed rectifiable path $\Lambda \subset \mathbb{C}$ encloses the spectral subset $I_g = [\varepsilon_0, \varepsilon_g] \cap \sigma(H^{\theta}_{q,r})$ (intersecting the real axis in ε_0 and ε_g) with the real numbers $\varepsilon_0, \varepsilon_g \in \mathbb{R} \setminus \sigma(H^{\theta}_{q,r})$ being such that $-\infty < \varepsilon_0 < \min \sigma(\mathrm{H}^{\theta}_{\mathrm{q,r}})$ and ε_g in the gap g.

The Hall conductance associated with the energy spectrum up to the gap g is related to the projection P_g via the Kubo formula (linear response theory). Its value is an integer number t_g; it is by now well known that t_g is to be thought of as the Chern number of a bundle determined by the projection P_g [22, 4, 1].

Any such a spectral projection P_g yields a projection $\Pi_{q,r}(P_g) \in \Gamma(\mathrm{End}(E_{N,q}))$, via the representation $\Pi_{q,r}$ in (8), and thus a vector subbundle $L_{q,r}(P_g) \subset E_{N,q}$. The related (first) Chern number $C_1(L_{q,r}(P_g))$ measures the degree of non triviality of the bundle $L_{q,r}(P_g)$. The geometric interpretation of the Hall conductance is none other than the equality $t_g = C_1(L_{q,r}(P_g))$. On the other hand, the Chern number $C_1(L_{q,r}(P_g))$ obeys a Diophantine equation which then provides a TKNN-type equation for the conductance t_g. We have the following.

Proposition B. *For any projection P in the algebra $\mathscr{A}^{\theta}_{q,r}$ there exists a "dual" vector bundle $L_{\mathrm{ref}}(P) \to \mathbb{T}^2$ s.t. the following duality between Chern numbers holds:*

$$C_1(L_{q,r}(P)) = q \left[\frac{1}{N} \, \mathrm{Rk}(L_{\mathrm{ref}}(P)) + \left(\frac{M}{N} - \frac{r}{q} \right) C_1(L_{\mathrm{ref}}(P)) \right]. \tag{13}$$

Before we sketch the proof of this result we turn to its interpretation in terms of conductances of the generalized Harper operators in (5). As mentioned, if P_g is its spectral projection up to the gap g, the associated Hall conductance t_g is the number $C_1(L_{q,r}(P_g))$. For the dual number we have $C_1(L_{\mathrm{ref}}(P_g)) = -s_g$, with s_g identified with the Hall conductance of the energy spectrum up to the gap g but in the opposite limit of a weak magnetic field ($B \ll 1$) [22, 1, 9]. Writing $d_g = \mathrm{Rk}(L_{\mathrm{ref}}(P_g))$, relation (13) translates to the *generalized* TKNN-*equations*

$$N \, t_g + (qM - rN) \, s_g = q \, d_g, \qquad g = 0, \dots, N_{\max}. \tag{14}$$

When $q = 1$ and $r = 0$, the above reduces to

$$N \, t_g + M \, s_g = d_g, \qquad g = 0, \dots, N_{\max}, \tag{15}$$

which is the original TKNN-equation derived in [22] for the Harper operator (1). In its spirit then, the integer d_g in the right-hand side coincides with the labeling of the gap when N is odd, i.e. $d_g = g$ for N odd. When N is even $d_g = g$ if $0 \leqslant g \leqslant N/2 - 1$ and $d_g = g + 1$ if $N/2 \leqslant g \leqslant N_{\max} = N - 1$. We remark that the bound

$$2\,|s_g| < N \tag{16}$$

(already present in [22]) still holds, owing to the bound $2\,|C_1(L_{\mathrm{ref}}(P_g))| < N$ for the spectral projections into the gaps of the Hofstadter operator [6].

Now, the subbundle $L_{q,r}(P) \subset E_{N,q}$ determined by the projection valued section $\Pi_{q,r}(P) = P(\cdot)$, for a projection $P \in \mathscr{A}^{\theta}_{q,r}$, will have as fiber over $k \in \mathbb{T}^2$ the space

$$L_{q,r}(P)|_k = \mathrm{Range}(P(k)) \subset \mathscr{H}_{q,r}(k). \tag{17}$$

We need a dual bundle representation, $\Pi^{\mathrm{ref}}_{q,r}$ of $\mathscr{A}^{\theta}_{q,r}$, s.t.

$$P(k_1, Nk_2) = P^{\mathrm{ref}}(k_1, M_0 k_2), \tag{18}$$

and $P^{\mathrm{ref}}(\cdot) = \Pi^{\mathrm{ref}}_{q,r}(P)$. We are lead to the representation

$$U^{\mathrm{ref}}_q(k) = \mathrm{e}^{\,\mathrm{i}\,2\pi k_2}\,(\mathbb{U}_N)^{qM}, \qquad V^{\mathrm{ref}}_{q,r}(k) = V^{\theta}_{q,r}(k) = \mathrm{e}^{\,\mathrm{i}\,2\pi n_r k_1}\,(\mathbb{V}_{N;k_1})^{d_r}\,. \tag{19}$$

which obey $U^{\mathrm{ref}}_q(\cdot)V^{\mathrm{ref}}_{q,r}(\cdot) = \mathrm{e}^{\,\mathrm{i}\,2\pi\frac{M}{N}}V^{\mathrm{ref}}_{q,r}(\cdot)U^{\mathrm{ref}}_q(\cdot)$. As elements in $C(\mathbb{T}^2) \otimes \mathrm{Mat_N}(\mathbb{C}) \simeq \mathrm{C}(\mathbb{T}^2; \mathrm{Mat_N}(\mathbb{C}))$ they yield a representation of $\mathscr{A}^{\theta}_{q,r}$ as endomorphisms of the the trivial bundle $\mathbb{T}^2 \times \mathbb{C}^N \to \mathbb{T}^2$. Then, any projection P in $\mathscr{A}^{\theta}_{q,r}$ is mapped to a projection-valued section $P^{\mathrm{ref}}(\cdot) = \Pi^{\mathrm{ref}}_{q,r}(p)$ which defines a vector subbundle $L_{\mathrm{ref}}(p) \to \mathbb{T}^2$ of the trivial vector bundle $\mathbb{T}^2 \times \mathbb{C}^N$. It will have as fiber over $k \in \mathbb{T}^2$ the space

$$L_{\mathrm{ref}}(P)|_k = \mathrm{Range}(P^{\mathrm{ref}}(k)) \subset \mathbb{C}^N. \tag{20}$$

Then, equation (18) say that the vector bundle $L_{q,r}(P)$ “winded” around N times in the second direction is (locally) isomorphic to the vector bundle $L_{\mathrm{ref}}(P)$ “winded” around M_0 times in the same direction. There is however an extra twist, due to the bundle $E_{N,q}$, of which $L_{q,r}(P)$ is a subbundle, being not trivial. Indeed, an analysis of the transition functions lead to the bundle isomorphism

$$\varphi^*_{(1,N)} L_{q,r}(P) \simeq \varphi^*_{(1,M_0)} L_{\mathrm{ref}}(P) \otimes \det(E_{N,q}). \tag{21}$$

Here $\det(E_{N,q}) \to \mathbb{T}^2$ is the determinant line bundle and the extra operation $\varphi^*_{(1,N)}$ (the pullback) stays for the extra winding by N (for the bundle $L_{q,r}(P)$) and the same for $\varphi^*_{(1,M_0)}$ (for the bundle $L_{\mathrm{ref}}(P)$). Formula (13) is the relation among corresponding first Chern numbers. Using the fact that $C_1(\varphi^*_{(1,N)} L_{q,r}(P)) = NC_1(L_{q,r}(P))$ and $C_1(\varphi^*_{(1,M_0)} L_{\mathrm{ref}}(P)) = M_0 C_1(L_{\mathrm{ref}}(P))$, as well as the identity $C_1(\det(E_{N,q})) = C_1(E_{N,q}) = q$, the relation (13) follows from (21) by standard arguments.

THE IRRATIONAL CASE

On the algebra $\mathscr{A}^{\theta}_{q,r}$ there is a faithful trace defined by

$$\tau\left((U_q)^n (V^{\theta}_{q,r})^m\right) = \delta_{n,0}\ \delta_{m,0}$$

on monomials, and extended by linearity. Derivations $\partial_j : \mathscr{A}^{\theta}_{q,r} \to \mathscr{A}^{\theta}_{q,r}$, for $j = 1,2$, defined on monomials by

$$\partial_1((U_q)^n (V^{\theta}_{q,r})^m) = \mathrm{i}\,2\pi\, n\, (U_q)^n (V^{\theta}_{q,r})^m, \qquad \partial_2((U_q)^n (V^{\theta}_{q,r})^m) = \mathrm{i}\,2\pi\, m\, (U_q)^n (V^{\theta}_{q,r})^m,$$

are extended by linearity and Leibniz rule. Lastly, we need the first Connes-Chern number which, for a projection $P \in \mathscr{A}^{\theta}_{q,r}$ (in the domain of the derivations) computes the integer (an index of a Fredholm operator)

$$\mathcal{C}_1(P) = \frac{1}{\mathrm{i}\,2\pi}\tau\big(P(\partial_1(P)\partial_2(P) - \partial_2(P)\partial_1(P))\big).$$

Let $H^{\theta}_{q,r} \in \mathscr{A}^{\theta}_{q,r}$ be the Hofstadter operator (5) with associated spectral projection P^{θ}_g for the gap g as in (12). For $\theta \in I \subset \mathbb{R}$, the functional expression of $H^{\theta}_{q,r} \in \mathscr{A}^{\theta}_{q,r}$ is fixed and $H^{\theta}_{q,r}$ depends on the parameter θ only through the fundamental commutation relation which defines $\mathscr{A}^{\theta}_{q,r}$. Now, if the gap g is open for all $\theta \in I$ (with I sufficiently small), the functions $\theta \mapsto \mathcal{C}_1(P^{\theta}_g)$ is constant in the interval I [5]. On the other hand, from the structure of the group $K_0(\mathscr{A}^{\theta}_{q,r})$, one deduces [18, 7] that

$$\tau(P^{\theta}_g) = m\, P^{\theta}_g - \theta\, \mathcal{C}_1(P^{\theta}_g), \tag{22}$$

with the integer $m(\cdot) \in \mathbb{Z}$ uniquely determined by the condition $0 \leqslant \tau(\cdot) \leqslant 1$. From (22), the integer $m(\cdot)$ is constant for $\theta \in I$. Hence, formula

$$C_{q,r}(P_g) = q\left[m(P_g) - \frac{r}{q}\,\mathcal{C}_1(P_g)\right] = q\left[\tau(P_g) + \left(\theta - \frac{r}{q}\right)\mathcal{C}_1(P_g)\right] \in \mathbb{Z} \tag{23}$$

is well defined and extends (13) for irrational values $\theta \in I$ (for which the gap g remains open). Indeed, for a rational $\theta = M/N$ one has natural identifications $\mathrm{Rk}(\mathrm{L}_{\mathrm{ref}}(\mathrm{P})) = \tau(\mathrm{P})$ and $C_1(L_{\mathrm{ref}}(P)) = \mathcal{C}_1(p)$ [10] and for the rational torus, formula (23) is the same as (13).

We think of (23) as relating conductances for the Harper operator $H^{\theta}_{q,r}$ in (5), thus generalizing (14) to

$$t_g + (q\theta - r)s_g = q\, d_g, \tag{24}$$

with P_g once again the spectral projections of the Harper operator $H^{\theta}_{q,r}$, and now identifying $t_g = C_{q,r}(P_g)$ and $s_g = -\,\mathcal{C}_1(P_g)$ as before, whereas $d_g = \ \tau(P_g)$.

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