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# The topological Bloch-Floquet transform and some applications

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## Abstract

We investigate the relation between the symmetries of a Schrödinger operator and the related topological quantum numbers. We show that, under suitable assumptions on the symmetry algebra, a generalization of the Bloch-Floquet transform induces a direct integral decomposition of the algebra of observables. More relevantly, we prove that the generalized transform selects uniquely the set of “continuous sections” in the direct integral decomposition, thus yielding a Hilbert bundle. The proof is constructive and provides an explicit description of the fibers. The emerging geometric structure is a rigorous framework for a subsequent analysis of some topological invariants of the operator, to be developed elsewhere [DFP11]. Two running examples provide an Ariadne’s thread through the paper. For the sake of completeness, we begin by reviewing two related classical theorems by von Neumann and Maurin.

**Key words:** Topological quantum numbers, spectral decomposition, Bloch-Floquet transform, Hilbert bundle.

## 1 Introduction

In view of the competition between different space-scales, magnetic Schrödinger operators with a periodic background exhibit striking features, as fractal spectrum and anomalous density of states. Beyond the spectrum and the density of states, other properties of these operators attracted the interest of physicists and, more recently, mathematicians: the so-called *Topological Quantum Numbers* (TQN), related to observable effects whose origin is geometric.

The prototypical example is the *Hall conductance* of a 2-dimensional gas of non-interacting electrons in a periodic background potential and a uniform orthogonal magnetic field. The dynamics of the single-electron wavefunction  $\psi \in L^2(\mathbb{R}^2, dx dy)$  is governed by the Hamiltonian operator

$$H_\beta = \frac{1}{2} \left[ \left( -i \frac{\partial}{\partial x} - \frac{\beta}{2} y \right)^2 + \left( -i \frac{\partial}{\partial y} + \frac{\beta}{2} x \right)^2 \right] + V_\Gamma(x, y) \quad (1)$$

where  $V_\Gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$  is periodic with respect to the lattice  $\Gamma \cong \mathbb{Z}^2$  and  $\beta$  is proportional to the modulus of a uniform magnetic field in the direction orthogonal to the plane. More precisely, the magnetic flux through the unit cell of  $\Gamma$  is divided by the fundamental unit of flux to obtain the dimensionless parameter  $\beta$ .

If a current of intensity  $I$  is forced in the  $x$ -direction the charge carriers experience the Lorentz force, resulting in motion of the carriers and a non-zero equilibrium voltage  $V_H$  along the  $y$ -direction. The *Hall conductance*  $\sigma_H$  is experimentally defined as  $\sigma_H = V_H/I$ , its value depending on both the magnetic flux  $\beta$  and the density of carriers, which depends on the Fermi energy  $\mu$ . While at room temperature the measured values of  $\sigma_H$  are in accordance with the predictions of classical electrodynamics, the same measurement performed at zero temperature show striking quantum features [vDP80], whose discovery deserved the Nobel Prize. The value of  $\sigma_H$ , when varying either  $\beta$  or  $\mu$ , exhibit extremely accurate “plateaux” (which are flat up to an accuracy of one part over  $10^7$ ) corresponding to integer multiples of the constant  $e^2/h$  ( $= 1/2\pi$  in the natural units used in (1)), where  $e$  is the charge of the electron and  $h$  the Planck constant.

By replacing (1) with a simplified Hamiltonian  $K_\beta$  (the Hofstadter operator [DP10]) a numerical simulation becomes feasible. For  $\beta \in [0, 1]$  and  $\mu \in [0, 4]$  the integer corresponding to  $\sigma_H(\beta, \mu)$  is coded by a color (warm colors for positive integers, cold colors for negative integers and white for zero) yielding a beautiful picture known as the “colored Hofstadter butterfly” [OA01, Avr04]. It is assumed that  $\mu$  is not in the spectrum of  $K_\beta$ , which appears for reader’s convenience on the left-hand side of the Figure 1.

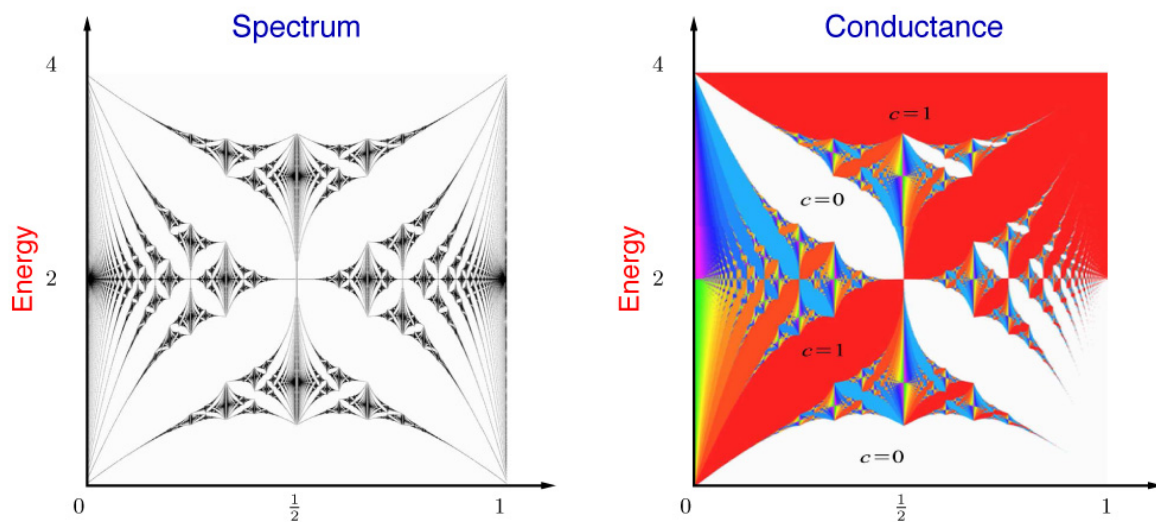


Figure 1: The black and white Hofstadter butterfly, showing the spectrum of the Hofstadter operator as a function of the parameter  $\beta$ , *versus* the colored Hofstadter butterfly, labeling the points of the resolvent set with a color corresponding to a suitable integer (Topological Quantum Number). The colored butterfly was originally obtained in [OA01]

How are these integers (colors) related to the proprieties of the corresponding Schrödinger operators? How can one read these Topological Quantum Numbers from the Hamiltonian? The goal of this contribution is to provide a tool to investigate the simplest framework,

namely the case of TQNs related to an abelian algebra of symmetries  $\mathfrak{S}$  generated by a finite family of unitary operators. This framework includes the case of the Hall conductance for the Hamiltonian (1) which, as we shall see, is related to the algebra generated by the magnetic translations. It also includes the relevant examples of Hofstadter and Harper operators, which are discussed in [DFP11]. In particular, the latter paper exploits the methods here developed to prove properties of the Hall conductance corresponding to the operator  $K_\beta$ .

We firstly recall the standard construction. To simplify the notation, we assume that the lattice  $\Gamma$  in the definition of  $H_\beta$  is simply  $\mathbb{Z}^2$ . The unitary operators  $T_{\beta,1}$  and  $T_{\beta,2}$ , acting on  $\psi \in L^2(\mathbb{R}^2, dx dy)$  by

$$(T_{\beta,1}\psi)(x, y) = e^{-i\frac{\beta}{2}y} \psi(x-1, y) \quad (T_{\beta,2}\psi)(x, y) = e^{i\frac{\beta}{2}x} \psi(x, y-1), \quad (2)$$

describe symmetries of the Hamiltonian (1) in the sense that

$$[T_{\beta,1}; H_\beta] = 0 = [T_{\beta,2}; H_\beta].$$

The operators (2) are known as *magnetic translations*. Unfortunately, in general  $T_{\beta,1}$  and  $T_{\beta,2}$  do not commute and thus do not correspond to simultaneously implementable symmetries (except for  $\beta \in 2\pi\mathbb{Z}$ ), indeed

$$T_{\beta,1} T_{\beta,2} = e^{-i\beta} T_{\beta,2} T_{\beta,1}.$$

In particular, the map  $(n_1, n_2) \mapsto T_{\beta,1}^{n_1} T_{\beta,2}^{n_2}$ , with  $(n_1, n_2) \in \mathbb{Z}^2$ , provides only a projective unitary representation of the group  $\mathbb{Z}^2$ . Nevertheless, under the so called *rational flux condition*  $\beta \in 2\pi\mathbb{Q}$  it is still possible to recover a  $\mathbb{Z}^2$ -symmetry for  $H_\beta$ . We are mainly interested in this case. Let  $\beta/2\pi = p/q$  with  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$  coprime. By replacing the standard lattice  $\Gamma = \mathbb{Z}^2$  with the *super-lattice*  $\Gamma' := (q\mathbb{Z})^2$ , one obtains a unitary representation  $(qn_1, qn_2) \mapsto T'_{\beta,1}{}^{n_1} T'_{\beta,2}{}^{n_2}$  of  $\Gamma'$  given by the commuting pair of unitary operators  $T'_{\beta,j} := T_{\beta,j}^q$ ,  $j = 1, 2$ . Thus, assuming the rational flux condition, one can still define a  $\mathbb{Z}^2$ -symmetry for  $H_\beta$  implemented by the ‘‘gauged’’ translations  $\{T'_{\beta,1}, T'_{\beta,2}\}$ . One defines the magnetic *Bloch-Floquet* (BF) transform, initially for  $\psi \in \mathfrak{S}(\mathbb{R}^2)$ , by posing

$$(\mathcal{U}_{\text{BF}}\psi)(k, \cdot) := \sum_{n \in \mathbb{Z}^2} e^{-ik \cdot n} (T'_{\beta,1}{}^{n_1} T'_{\beta,2}{}^{n_2} \psi)(\cdot), \quad k \in \mathbb{R}^d, \quad (3)$$

where  $n := (n_1, n_2)$ . Definition (3) extends to a unitary operator

$$\mathcal{U}_{\text{BF}} : L^2(\mathbb{R}^2) \longrightarrow \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}(k) dk \quad (4)$$

where  $\mathbb{T}^2 := \mathbb{R}^2/\Gamma^*$  corresponds to the first (magnetic) Brillouin zone in the physics literature, and

$$\mathcal{H}(k) := \{\varphi \in L^2_{\text{loc}}(\mathbb{R}^2) : T'_{\beta,1}{}^{n_1} T'_{\beta,2}{}^{n_2} \varphi = e^{ik \cdot n} \varphi \quad \forall n \in \mathbb{Z}^2\}.$$

While we focused on the bidimensional case in view of its relevance for the Hall conductance, the definition of the magnetic translations and the magnetic BF transform effortlessly extend to any dimension  $d \in \mathbb{N}$ .

In the magnetic BF representation, the Fermi projector  $P_\mu = \chi_{(-\infty, \mu)}(H_\beta)$ , with  $\chi_I$  the characteristic function of the set  $I$ , is a decomposable operator, in the sense that

$$\mathcal{U}_{\text{BF}} P_\mu \mathcal{U}_{\text{BF}}^{-1} = \int_{\mathbb{T}^2}^{\oplus} P_\mu(k) dk.$$

If  $\mu$  lies in a spectral gap, the dimension of the range of  $P_\mu(k)$  is constant. Thus it would be tempting to consider the *measurable* collection of vector spaces  $\{\text{Ran } P_\mu(k)\}_{k \in \mathbb{T}^2}$  as a vector bundle  $\mathcal{E}$  over  $\mathbb{T}^2$ , and to consider its first Chern number  $C_1(\mathcal{E}) \in \mathbb{Z}$  as a topological quantum number (analogously, for  $d \geq 3$  one considers the first and the higher Chern numbers). However, as already emphasized, the decomposition (4) is a measure-theoretic object, yielding only a measurable collection of vector spaces, thus the Chern number might be undefined; even if one circumvents this obstacle, its value might not be invariant under unitary equivalence. In this paper we develop a construction that yields a *topological* decomposition analogous to (4). Moreover, the vector bundle  $\mathcal{E}_{\mathfrak{S}} \rightarrow \mathbb{T}^d$  defined by this procedure is essentially unique, in the sense that it is invariant under any unitary equivalence commuting with the elements of  $\mathfrak{S}$ .

We formulate the result in a general framework:  $\mathcal{H}$  is a separable Hilbert space which corresponds to the physical states;  $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra of bounded operators which contains the relevant physical models (the self-adjoint elements of  $\mathfrak{A}$  can be interpreted as Hamiltonians); the commutant  $\mathfrak{A}'$  (the set of all the elements in  $\mathcal{B}(\mathcal{H})$  which commute with  $\mathfrak{A}$ ) can be seen as the set of all the physical symmetries; any commutative unital  $C^*$ -algebra  $\mathfrak{S} \subset \mathfrak{A}'$  describes a set of simultaneously implementable physical symmetries.

**DEFINITION 1.1** (Physical frame). *A physical frame is a triple  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  where  $\mathcal{H}$  is a separable Hilbert space,  $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra and  $\mathfrak{S} \subset \mathfrak{A}'$  is a commutative unital  $C^*$ -algebra. The physical frame  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  is called irreducible if  $\mathfrak{S}$  is maximal commutative. Two physical frames  $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$  and  $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$  are equivalent if there exists a unitary map  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $\mathfrak{A}_2 = U \mathfrak{A}_1 U^{-1}$  and  $\mathfrak{S}_2 = U \mathfrak{S}_1 U^{-1}$ .*

We focus on triples  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  whose  $C^*$ -algebra  $\mathfrak{S}$  describes symmetries with an intrinsic group structure. In particular, in this contribution, we focus on the case of the group  $\mathbb{Z}^N$ .

**DEFINITION 1.2** ( $\mathbb{Z}^N$ -algebra). *Let  $\mathbb{Z}^N \ni n \mapsto U(n) \in \mathcal{U}(\mathcal{H})$  be a unitary representation of  $\mathbb{Z}^N$  in the group  $\mathcal{U}(\mathcal{H})$  of the unitary operators on  $\mathcal{H}$ . The representation is faithful if  $U(n) = \mathbb{1}$  implies  $n = 0$  and is algebraically compatible if the operators  $\{U(n) : n \in \mathbb{Z}^N\}$  are linearly independent in  $\mathcal{B}(\mathcal{H})$ . Let  $\mathfrak{S}(\mathbb{Z}^N)$  be the unital  $C^*$ -algebra generated by  $\{U(n) : n \in \mathbb{Z}^N\}$ . When the representation is faithful and algebraically compatible we say that  $\mathfrak{S}(\mathbb{Z}^N)$  is a  $\mathbb{Z}^N$ -algebra in  $\mathcal{H}$ .*

In a nutshell, our main result is the following. Let  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  be a physical frame with  $\mathfrak{S}$  a  $\mathbb{Z}^N$ -algebra satisfying the wandering property (see Definition 5.1). Then there exist (and one can explicitly construct):

- a Hermitian vector bundle  $\mathcal{E}_{\mathfrak{S}} \rightarrow \mathbb{T}^N$ , whose rank is equal to the cardinality of the wandering system;

- a unitary operator  $\mathcal{F}_{\mathfrak{E}} : \mathcal{H} \rightarrow \Gamma_{L^2}(\mathcal{E}_{\mathfrak{E}} \rightarrow \mathbb{T}^N)$ , the latter being the Hilbert space consisting of the  $L^2$ -sections of the Hermitian vector bundle  $\mathcal{E}_{\mathfrak{E}} \rightarrow \mathbb{T}^N$ ;

such that the  $*$ -subalgebra  $\mathfrak{A}^0 \subset \mathfrak{A}$ , consisting of some adjointable operators in  $\mathfrak{A}$  (see Proposition 7.13), satisfies

$$\mathcal{F}_{\mathfrak{E}} \mathfrak{A}^0 \mathcal{F}_{\mathfrak{E}}^{-1} \subset \Gamma(\text{End}(\mathcal{E}_{\mathfrak{E}}) \rightarrow \mathbb{T}^N)$$

where  $\Gamma(\cdot)$  is the space of the *continuous* sections of the vector bundle appearing as the argument. Moreover, if  $\{\mathcal{H}, \mathfrak{A}_1, \mathfrak{S}_1\}$  and  $\{\mathcal{H}, \mathfrak{A}_2, \mathfrak{S}_2\}$  are equivalent physical frames, then  $\mathcal{E}_{\mathfrak{S}_1}$  and  $\mathcal{E}_{\mathfrak{S}_2}$  are isomorphic Hermitian vector bundles. In particular, their topological invariants are the same, and can thus be considered a fingerprint of the physical frame.

Last but not least, it is physically interesting to consider the case of perturbed abelian symmetries, e.g. to take into account the effects of disorder and impurities in the physical system, or, in the case of the Hamiltonian (1), an irrational value of the parameter  $\beta/2\pi$ . In this case the algebra of symmetries is replaced by a non-abelian  $C^*$ -algebra, and the related TQNs should be investigated either with the tools of Non Commutative Geometry [BSE94], or with other abstract methods [Gru01]. On the other hand, the aim of this paper is not to pursue the greatest generality, but rather to provide an explicit and manageable tool to define and to compute the TQNs hidden in the symmetries of Schrödinger operators.

The paper is organized as follows:

- **Section 2** introduces two relevant examples which will reappear later, as an Ariadne's thread through the paper.

- in **Section 3 and 4** we review two cornerstones in the classical literature concerning the measurable decomposition of a physical frame: the *von Neumann's complete spectral theorem* (Theorem 3.1) and the *Maurin's nuclear spectral theorem* (Theorem 4.1).

- **Section 5** concerns the notion of *wandering property* for a commutative  $C^*$ -algebra generated by a finite family of operators. This notion is of particular relevance when the generators of the  $C^*$ -algebra are a finite set of unitary operators. In this case we prove that the wandering property assures that the  $C^*$ -algebra is a  $\mathbb{Z}^N$ -algebra with  $N$  the number of generators. Moreover the Gel'fand spectrum of the  $C^*$ -algebra is forced to be the dual group of  $\mathbb{Z}^N$ , i.e. the  $N$ -dimensional torus  $\mathbb{T}^N$ .

- in **Section 6** we extend the decomposition (3) to the case of a  $\mathbb{Z}^N$ -algebra which satisfies the wandering property. This *generalized Bloch-Floquet transform* provides a concrete recipe to decompose the Hilbert space and the algebra  $\mathfrak{A}$  according to the von Neumann and Maurin theorems.

- **Section 7** contains our novel results: we show that a *topological* decomposition of the algebra  $\mathfrak{A}$  emerges in a canonical way from the generalized Bloch-Floquet transform, and we prove two decomposition theorems (Theorems 7.8 and 7.14). The topological structure is essentially unique, so the emerging information is a fingerprint of the given physical frame.

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## 2 Some guiding examples

We elucidate the theory with two explicit examples, while other relevant applications are covered elsewhere [DFP11] [DL11].

**EXAMPLE 2.1** (*Periodic systems, part one*). Let  $H_\Gamma$  be a  $\Gamma$ -periodic operator defined on  $L^2(\mathbb{R}^d)$ . With the word  $\Gamma$ -periodic we mean that there exists a linear basis  $\{\gamma_1, \dots, \gamma_d\}$  of  $\mathbb{R}^d$  which spans the lattice  $\Gamma \simeq \mathbb{Z}^d$  and  $d$  gauged translations  $\{T_1, \dots, T_d\}$  defined by  $(T_j\psi)(x) = g_j(x) \psi(x - \gamma_j)$ , where  $g_j(\cdot - \gamma) = g_j(\cdot)$  for all  $\gamma \in \Gamma$ , such that  $[H_\Gamma; T_j] = 0$  for all  $j = 1, \dots, d$ . From the definition it follows that  $[T_i; T_j] = 0$  for any  $i, j$ . Both the cases of the magnetic translations with rational flux condition (cf. Section 1) and the “genuine” translations (i.e.  $g_j \equiv 1$ ) fit in this scheme.

The Gel’fand-Naïmark Theorem shows that there exists an isomorphism between the commutative  $C^*$ -algebra  $C_0(\sigma(H_\Gamma))$  and a commutative non-unital  $C^*$ -algebra  $\mathfrak{A}_0(H_\Gamma)$  of bounded operators in  $\mathcal{H}$ . The elements of  $\mathfrak{A}_0(H_\Gamma)$  are the operators  $f(H_\Gamma) \in \mathcal{B}(\mathcal{H})$ , for  $f \in C_0(\sigma(H_\Gamma))$ , obtained via the spectral theorem. Let  $\mathfrak{A}(H_\Gamma)$  be the multiplier algebra of  $\mathfrak{A}_0(H_\Gamma)$  in  $\mathcal{B}(\mathcal{H})$ . This is a unital commutative  $C^*$ -algebra which contains  $\mathfrak{A}_0(H_\Gamma)$  (as an essential ideal), its Gel’fand spectrum is a (Stone-Ćech) compactification of  $\sigma(H_\Gamma)$  and the Gel’fand isomorphism maps  $\mathfrak{A}(H_\Gamma)$  into the unital  $C^*$ -algebra of the continuous and bounded functions on  $\sigma(H_\Gamma)$  denoted by  $C_b(\sigma(H_\Gamma))$  (see Appendix A for details). We assume that  $\mathfrak{A}(H_\Gamma)$  is the  $C^*$ -algebra of physical models.

The  $C^*$ -algebra  $\mathfrak{S}_T$  generated by the gauged translations  $T_j$  is clearly commutative. Since  $[H_\Gamma; T_j] = 0$  it follows that  $\mathfrak{S}_T \subset \mathfrak{A}(H_\Gamma)'$ . Thus  $\{L^2(\mathbb{R}^d), \mathfrak{A}(H_\Gamma), \mathfrak{S}_T\}$  is a physical frame. It is a convenient model to study the properties of an electron in a periodic medium.

◀▶

**EXAMPLE 2.2** (*Mathieu-like Hamiltonians, part one*). Let  $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$  be the one-dimensional torus. In the Hilbert space  $L^2(\mathbb{T})$  consider the Fourier orthonormal basis  $\{e_n\}_{n \in \mathbb{Z}}$  defined by  $e_n(\theta) := (2\pi)^{-\frac{1}{2}} e^{in\theta}$ . Let  $\mathbf{u}$  and  $\mathbf{v}$  be the unitary operators defined, for  $g \in L^2(\mathbb{T})$ , by

$$(\mathbf{u}g)(\theta) := e^{i\theta} g(\theta), \quad (\mathbf{v}g)(\theta) := g(\theta - 2\pi\alpha), \quad \mathbf{u}\mathbf{v} = e^{i2\pi\alpha} \mathbf{v}\mathbf{u} \quad (5)$$

with  $\alpha \in \mathbb{R}$ . The last equation in (5) shows that the unitaries  $\mathbf{u}$  and  $\mathbf{v}$  satisfy the commutation relation of a *noncommutative torus* with deformation parameter  $\alpha$  (see [Boc01] Chapter 1 or [GVF01] Chapter 12 for more details). We denote by  $\mathfrak{A}_M^\alpha \subset \mathcal{B}(L^2(\mathbb{T}))$  the unital  $C^*$ -algebra generated by  $\mathbf{u}, \mathbf{v}$ . We call  $\mathfrak{A}_M^\alpha$  the *Mathieu  $C^*$ -algebra*<sup>(1)</sup> and we refer to its elements as Mathieu-like operators. This name is due to the fact that the Hamiltonian  $\mathfrak{h} := \mathbf{u} + \mathbf{u}^\dagger + \mathbf{v} + \mathbf{v}^\dagger \in \mathfrak{A}_M^\alpha$  appears in the well know (almost-)Mathieu eigenvalue equation

$$(\mathfrak{h}g)(\theta) \equiv g(\theta - 2\pi\alpha) + g(\theta + 2\pi\alpha) + 2 \cos(\theta)g(\theta) = \varepsilon g(\theta). \quad (6)$$

The action of  $\mathbf{u}$  and  $\mathbf{v}$  on the Fourier basis is given explicitly by  $\mathbf{u}e_n = e_{n+1}$  and  $\mathbf{v}e_n = e^{-i2\pi n\alpha} e_n$  for all  $n \in \mathbb{Z}$ .

<sup>(1)</sup>Such an algebra is a representation of the rotation  $C^*$ -algebra and in particular it is a faithful representations when  $\alpha \notin \mathbb{Q}$  [Boc01]. Since in this paper we focus on properties which *do* depend on the representation, we will adopt different names for images of the same abstract  $C^*$ -algebra under unitarily inequivalent representations.

We focus now on the commutant  $\mathfrak{A}_M^{\alpha'}$  of the Mathieu  $C^*$ -algebra. Let  $\mathfrak{s} \in \mathcal{B}(L^2(\mathbb{T}))$  be a bounded operator such that  $[\mathfrak{s}; \mathbf{u}] = 0 = [\mathfrak{s}; \mathbf{v}_\alpha]$  and let  $\mathfrak{s}e_n = \sum_{m \in \mathbb{Z}} s_{n,m} e_m$  be the action of  $\mathfrak{s}$  on the basis vectors. The relation  $[\mathfrak{s}; \mathbf{u}] = 0$  implies  $s_{n+1,m+1} = s_{n,m}$  and the relation  $[\mathfrak{s}; \mathbf{v}_\alpha] = 0$  implies  $e^{-i2\pi(m-n)\alpha} s_{n,m} = s_{n,m}$  for all  $n, m \in \mathbb{Z}$ . If  $\alpha \notin \mathbb{Q}$  then  $e^{-i2\pi(m-n)\alpha} \neq 1$  unless  $n = m$ , hence  $s_{n,m} = 0$  if  $n \neq m$  and the condition  $s_{n+1,n+1} = s_{n,n}$  implies that  $\mathfrak{s} = s\mathbb{1}$  with  $s \in \mathbb{C}$ . This shows that in the irrational case  $\alpha \notin \mathbb{Q}$  the commutant of the Mathieu  $C^*$ -algebra is trivial.

To have a non trivial commutant we need to assume that  $\alpha := p/q$  with  $p, q$  non zero integers such that  $\gcd(q, p) = 1$ . In this case the condition  $\mathfrak{s} \in (\mathfrak{A}_M^{p/q})'$  implies that  $s_{n,m} \neq 0$  if and only if  $m - n = kq$  for some  $k \in \mathbb{Z}$ , moreover  $s_{n,n+kq} = s_{0,kq} =: s'_k$  for all  $n \in \mathbb{Z}$ . Let  $\mathfrak{w}$  be the unitary operator defined on the orthonormal basis by  $\mathfrak{w}e_n := e_{n+q}$ , namely  $\mathfrak{w} = (\mathbf{u})^q$ . The relations for the commutant imply that  $\mathfrak{s} \in (\mathfrak{A}_M^{p/q})'$  if and only if  $\mathfrak{s} = \sum_{k \in \mathbb{Z}} s'_k \mathfrak{w}^k$ . Then in the rational case the commutant of the Mathieu  $C^*$ -algebra is the von Neumann algebra generated in  $\mathcal{B}(L^2(\mathbb{T}))$  as the strong closure of the family of finite polynomials in  $\mathfrak{w}$ . We will denote by  $\mathfrak{S}_M^q$  the unital commutative  $C^*$ -algebra generated by  $\mathfrak{w}$ . Observe that it does not depend on  $p$ . The triple  $\{L^2(\mathbb{T}), \mathfrak{A}_M^{q/p}, \mathfrak{S}_M^q\}$  is an example of a physical frame.  $\blacktriangleleft \blacktriangleright$

Finally, we introduce some notation which will be useful in the following.

**REMARK 2.3** (*Notation*). The  $N$ -dimensional torus  $\mathbb{T}^N := \mathbb{R}^N / (2\pi\mathbb{Z})^N$  is parametrized by the cube  $[0, 2\pi)^N$ : for every  $t = (t_1, \dots, t_N)$  in the cube,  $z(t) := (z_1(t), \dots, z_N(t))$ , with  $z_j(t) := e^{it_j}$ , is a point of  $\mathbb{T}^N$ . The normalized Haar measure is  $dz(t) = dt_1 \dots dt_N / (2\pi)^N$ .  $\blacklozenge$

### 3 The complete spectral theorem by von Neumann

The complete spectral theorem is a useful generalization of the usual spectral decomposition of a normal operator on a Hilbert space. It shows that the symmetries reduce the description of the full algebra  $\mathfrak{A}$  to a family of simpler representations. The main tool used in the theorem is the notion of the direct integral of Hilbert spaces (Appendix B). The ‘‘spectral’’ content of the theorem amounts to the characterization of the base space for the decomposition (the ‘‘set of labels’’) and of the measure which glues together the spaces so that the Hilbert space structure is preserved. This information emerges essentially from the Gel’fand theory (Appendix A). The definitions of decomposable and continuously diagonal operator are reviewed in Appendix B.

**THEOREM 3.1** (von Neumann’s complete spectral theorem). *Let  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  be a physical frame and  $\mu$  the basic measure carried by the spectrum  $X$  of  $\mathfrak{S}$  (see Appendix A). Then there exist*

- a) a direct integral  $\mathfrak{H} := \int_X^\oplus \mathcal{H}(x) d\mu(x)$  with  $\mathcal{H}(x) \neq \{0\}$  for all  $x \in X$ ,
- b) a unitary map  $\mathcal{F}_\mathfrak{S} : \mathcal{H} \rightarrow \mathfrak{H}$ , called  $\mathfrak{S}$ -Fourier transform <sup>(2)</sup>,

such that:

<sup>(2)</sup> According to the terminology used in [Mau68].



- (i) the unitary map  $\mathcal{F}_{\mathfrak{S}}$  intertwines the Gel'fand isomorphism  $C(X) \ni f \xrightarrow{\mathcal{G}} A_f \in \mathfrak{S}$  and the canonical isomorphism of  $C(X)$  onto the continuously diagonal operators  $C(\mathfrak{H})$ , i.e. the following diagram commutes

$$\begin{array}{ccc} & f \in C(X) & \\ \mathcal{G} \swarrow & & \searrow \\ \mathfrak{S} \ni A_f & \xrightarrow{\mathcal{F}_{\mathfrak{S}} \dots \mathcal{F}_{\mathfrak{S}}^{-1}} & M_f(\cdot) \in C(\mathfrak{H}) \end{array}$$

- (ii) the unitary conjugation  $\mathcal{F}_{\mathfrak{S}} \dots \mathcal{F}_{\mathfrak{S}}^{-1}$  maps the elements of  $\mathfrak{A}$  to decomposable operators on  $\mathfrak{H}$ ; more precisely, there is a family  $\{\pi_x\}_{x \in X}$  such that  $\pi_x$  is a representation of  $\mathfrak{A}$  on  $\mathcal{H}(x)$  and, for every  $A \in \mathfrak{A}$ , the map  $x \mapsto \pi_x(A)$  is measurable and

$$\mathcal{F}_{\mathfrak{S}} A \mathcal{F}_{\mathfrak{S}}^{-1} = \int_X^{\oplus} \pi_x(A) d\mu(x);$$

- (iii) the representations  $\pi_x$  are irreducible if and only if the physical frame  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  is irreducible.

**REMARK 3.2.** For a complete proof of the above theorem one can see [Mau68] (Theorem 25 in Chapter I and Theorem 2 in Chapter V) or [Dix81] (Theorem 1 in Part II, Chapter 6). For our purposes it is interesting to recall how the fiber Hilbert spaces  $\mathcal{H}(x)$  are constructed. For  $\psi, \varphi \in \mathcal{H}$  let  $\mu_{\psi, \varphi} = h_{\psi, \varphi} \mu$  be the relation between the spectral measure  $\mu_{\psi, \varphi}$  with the basic measure  $\mu$ . For  $\mu$ -almost every  $x \in X$  the value of the Radon-Nikodym derivative  $h_{\psi, \varphi}$  in  $x$  defines a semi-definite sesquilinear form on  $\mathcal{H}$ , i.e.  $(\psi; \varphi)_x := h_{\psi, \varphi}(x)$ . Let  $\mathcal{I}_x := \{\psi \in \mathcal{H} : h_{\psi, \psi}(x) = 0\}$ . Then the quotient space  $\mathcal{H}/\mathcal{I}_x$  is a pre-Hilbert space and  $\mathcal{H}'(x)$  is defined to be its completion. By construction  $\mathcal{H}'(x) \neq \{0\}$  for  $\mu$ -almost every  $x \in X$ . Let  $N \subset X$  be the  $\mu$ -negligible set on which  $\mathcal{H}'(x)$  is trivial or not well defined. Then  $\mathfrak{H} := \int_X^{\oplus} \mathcal{H}(x) d\mu(x)$  with  $\mathcal{H}(x) := \mathcal{H}'(x)$  if  $x \in X \setminus N$  and  $\mathcal{H}(x) := H$  if  $x \in N$  where  $H$  is an arbitrary non trivial Hilbert space.  $\blacklozenge$

Given the triple  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ , the direct integral decomposition invoked in the statement of Theorem 3.1 is essentially unique in measure-theoretic sense. The space  $X$  is unique up to homeomorphism: it agrees with the spectrum of  $C(\mathfrak{H})$  in such a way that the canonical isomorphism of  $C(X)$  onto  $C(\mathfrak{H})$  may be identified with the Gel'fand isomorphism. As for the uniqueness of the direct integral decomposition, the following result holds true (see [Dix81] Theorem 3 in Part II Chapter 6).

**THEOREM 3.3** (Uniqueness). *With the notation of Theorem 3.1, let  $\nu$  be a positive measure with support  $X$ ,  $\prod_{x \in X} \mathcal{K}(x)$  a field of non-zero Hilbert spaces over  $X$  endowed with a measurable structure,  $\mathfrak{K} := \int_X^{\oplus} \mathcal{K}(x) d\nu(x)$ ,  $C(\mathfrak{K})$  the commutative unital  $C^*$ -algebra of continuously diagonal operators on  $\mathfrak{K}$  and  $C(X) \rightarrow C(\mathfrak{K})$  the canonical isomorphism. Let  $\mathcal{W}$  be a unitary (antiunitary) map from  $\mathcal{H}$  onto  $\mathfrak{K}$  transforming by conjugation  $A_f \in \mathfrak{S}$  into  $M'_f(\cdot) \in C(\mathfrak{K})$  for all  $f \in C(X)$ , in such a way that the diagram on the right hand side*

commutes.

$$\begin{array}{ccc}
 & M_f(\cdot) \in C(\mathfrak{H}) & \\
 \mathcal{F}_{\mathfrak{S}} \dots \mathcal{F}_{\mathfrak{S}}^{-1} \nearrow & & \nwarrow \\
 \mathfrak{S} \ni A_f & \xleftrightarrow{\mathcal{G}} & f \in C(X) \\
 \searrow \mathcal{W} \dots \mathcal{W}^{-1} & & \nearrow \\
 & M'_f(\cdot) \in C(\mathfrak{K}) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathfrak{H} & \\
 \mathcal{F}_{\mathfrak{S}} \nearrow & \vdots & \nwarrow W(\cdot) \\
 \mathcal{H} & & \mathfrak{K} \\
 \searrow \mathcal{W} & & \downarrow
 \end{array}$$

Then,  $\mu$  and  $\nu$  are equivalent measures (so one can assume that  $\mu = \nu$  up to a rescaling isomorphism). Moreover there exists a decomposable unitary (antiunitary)  $W(\cdot)$  from  $\mathfrak{H}$  onto  $\mathfrak{K}$  such that  $W(x) : \mathcal{H}(x) \rightarrow \mathcal{K}(x)$  is a unitary (antiunitary) operator  $\mu$ -almost everywhere and  $\mathcal{W} = W(\cdot) \circ \mathcal{F}_{\mathfrak{S}}$ , i.e. the diagram on the left hand side commutes.

**COROLLARY 3.4** (Unitary equivalent triples). *Let  $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$  and  $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$  be two equivalent physical frames and  $U$  the unitary map which intertwines them. Let  $\mathfrak{H}_1$  and  $\mathfrak{H}_2$  denote the direct integral decomposition of the two triples and let  $\mathcal{F}_{\mathfrak{S}_1}$  and  $\mathcal{F}_{\mathfrak{S}_2}$  be the two  $\mathfrak{S}$ -Fourier transforms. Then  $W(\cdot) := \mathcal{F}_{\mathfrak{S}_2} \circ U \circ \mathcal{F}_{\mathfrak{S}_1}^{-1}$  is a decomposable unitary operator from  $\mathfrak{H}_1$  to  $\mathfrak{H}_2$ , so that  $W(x) : \mathcal{H}_1(x) \rightarrow \mathcal{H}_2(x)$  is a unitary map for  $\mu$ -almost every  $x \in X$ .*

## 4 The nuclear spectral theorem by Maurin

The complete spectral theorem by von Neumann shows that any physical frame  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  admits a representation in which the Hilbert space is decomposed (in a measure-theoretically unique way) into a direct integral  $\int_X^{\oplus} \mathcal{H}(x) d\mu(x)$ , the elements of  $\mathfrak{S}$  are simultaneously diagonalized and the  $C^*$ -algebra  $\mathfrak{A}$  is decomposed on the fibers. The contribution of Maurin is a characterization of the fiber spaces  $\mathcal{H}(x)$  as common generalized eigenspaces for  $\mathfrak{S}$ .

A key ingredient of Maurin's theorem is the notion of (*nuclear*) *Gel'fand triple*. The latter is a triple  $\{\Phi, \mathcal{H}, \Phi^*\}$  with  $\mathcal{H}$  a separable Hilbert space,  $\Phi \subset \mathcal{H}$  a norm-dense subspace such that  $\Phi$  has a topology for which it is a nuclear space and the inclusion map  $\iota : \Phi \hookrightarrow \mathcal{H}$  is continuous, and  $\Phi^*$  is topological dual of  $\Phi$ . By identifying  $\mathcal{H}$  with its dual space  $\mathcal{H}^*$  one gets an *antilinear* injection  $\iota^* : \mathcal{H} \hookrightarrow \Phi^*$ . Since the duality pairing between  $\Phi$  and  $\Phi^*$  is compatible with the scalar product on  $\mathcal{H}$ , namely  $\langle \iota^*(\psi_1); \psi_2 \rangle = (\psi_1; \psi_2)_{\mathcal{H}}$  whenever  $\psi_1 \in \mathcal{H}$  and  $\psi_2 \in \Phi$ , we write  $\langle \psi_1; \psi_2 \rangle$  for  $\langle \iota^*(\psi_1); \psi_2 \rangle$ .

If  $A$  is a bounded operator on  $\mathcal{H}$  such that  $A^\dagger$  leaves invariant  $\Phi$  and  $A^\dagger : \Phi \rightarrow \Phi$  is continuous with respect to the nuclear topology of  $\Phi$ , one defines  $\hat{A} : \Phi^* \rightarrow \Phi^*$  by posing  $\langle \hat{A}\eta; \varphi \rangle := \langle \eta; A^\dagger \varphi \rangle$  for all  $\eta \in \Phi^*$  and  $\varphi \in \Phi$ . Then  $\hat{A}$  is continuous and is an extension of  $A$ , defined on  $\mathcal{H}$ , to  $\Phi^*$ .

Assume the notation of Theorem 3.1. Let  $\{\xi_k(\cdot) : k \in I\}$  be a *fundamental family* of orthonormal measurable vector fields (see Appendix B) for the direct integral  $\mathfrak{H}$  defined by the  $\mathfrak{S}$ -Fourier transform  $\mathcal{F}_{\mathfrak{S}}$ . Any square integrable vector field  $\varphi(\cdot)$  can be written in a unique way as  $\varphi(\cdot) = \sum_{k \in I} \hat{\varphi}_k(\cdot) \xi_k(\cdot)$  where  $\hat{\varphi}_k \in L^2(X, d\mu)$  for all  $k \in I$ . Equipped with

this notation, the scalar product in  $\mathfrak{H}$  reads

$$\langle \varphi(\cdot); \psi(\cdot) \rangle_{\mathfrak{H}} = \int_X \sum_{k=1}^{\dim \mathcal{H}(x)} \overline{\widehat{\varphi}_k(x)} \widehat{\psi}_k(x) d\mu(x).$$

For any  $\varphi \in \mathcal{H}$  let  $\varphi(\cdot) := \mathcal{F}_{\mathfrak{S}}\varphi$  be the square integrable vector field obtained from  $\varphi$  by the  $\mathfrak{S}$ -Fourier transform. Denote by  $A_f \in \mathfrak{S}$  the operator associated with  $f \in C(X)$  through the Gel'fand isomorphism. One checks that

$$(\widehat{\mathcal{F}_{\mathfrak{S}}A_f\varphi})_k(x) = (\xi_k(x); f(x)\varphi(x))_x = f(x) \widehat{\varphi}_k(x) \quad k = 1, 2, \dots, \dim \mathcal{H}(x). \quad (7)$$

Suppose that  $\{\Phi, \mathcal{H}, \Phi^*\}$  is a Gel'fand triple for the space  $\mathcal{H}$ . If  $\varphi \in \Phi$  then the map  $\Phi \ni \varphi \mapsto \widehat{\varphi}_k(x) := (\xi_k(x); \varphi(x))_x \in \mathbb{C}$  is linear; moreover it is possible to show that it is continuous with respect to the nuclear topology of  $\Phi$ , for an appropriate choice of  $\Phi$ . This means that there exists  $\eta_k(x) \in \Phi^*$  such that

$$\langle \eta_k(x); \varphi \rangle := \widehat{\varphi}_k(x) = (\xi_k(x); \varphi(x))_x \quad k = 1, 2, \dots, \dim \mathcal{H}(x). \quad (8)$$

Suppose that  $A_f : \Phi \rightarrow \Phi$  is continuous with respect to the nuclear topology for every  $f \in C(X)$ . Then from equations (7) and (8) one has that the extended operator  $\hat{A}_f : \Phi^* \rightarrow \Phi^*$ , namely  $\langle \hat{A}_f \eta; \varphi \rangle := \langle \eta; A_{\bar{f}} \varphi \rangle$  for all  $\eta \in \Phi^*$  and  $\varphi \in \Phi$ , satisfies

$$\langle \hat{A}_f \eta_k(x); \varphi \rangle = \langle \eta_k(x); A_{\bar{f}} \varphi \rangle = \bar{f}(x) \widehat{\varphi}_k(x) = \langle f(x) \eta_k(x); \varphi \rangle \quad k = 1, 2, \dots, \dim \mathcal{H}(x) \quad (9)$$

for all  $\varphi \in \Phi$ . Hence,

$$\hat{A}_f \eta_k(x) = f(x) \eta_k(x) \quad \text{in } \Phi^*.$$

In this sense  $\eta_k(x)$  is a *generalized eigenvector* for  $A_f$ . These claims are made precise in the following statement.

**THEOREM 4.1** (Maurin's nuclear spectral theorem). *With the notation and the assumptions of Theorem 3.1, let  $\{\Phi, \mathcal{H}, \Phi^*\}$  be a nuclear Gel'fand triple for the space  $\mathcal{H}$  such that  $\Phi$  is  $\mathfrak{S}$ -invariant, i.e. each  $A \in \mathfrak{S}$  is a continuous linear map  $A : \Phi \rightarrow \Phi$ . Then:*

- (i) *for all  $x \in X$  the  $\mathfrak{S}$ -Fourier transform  $\mathcal{F}_{\mathfrak{S}}|_x : \Phi \rightarrow \mathcal{H}(x)$ ,  $\varphi \mapsto \varphi(x) \in \mathcal{H}(x)$  is continuous with respect to the nuclear topology for  $\mu$ -almost every  $x \in X$ ;*
- (ii) *there is a family of linear functionals  $\{\eta_k(x) : k = 1, 2, \dots, \dim \mathcal{H}(x)\} \subset \Phi^*$  such that equations (8) and (9) hold true for  $\mu$ -almost all  $x \in X$ ;*
- (iii) *with the identification  $\eta_k(x) \leftrightarrow \xi_k(x)$  the Hilbert space  $\mathcal{H}(x)$  is (isomorphic to) a vector subspace of  $\Phi^*$ ; with this identification the  $\mathcal{F}_{\mathfrak{S}}$ -Fourier transform is defined on the dense set  $\Phi$  by*

$$\Phi \ni \varphi \xrightarrow{\mathcal{F}_{\mathfrak{S}}|_x} \sum_{k=1}^{\dim \mathcal{H}(x)} \langle \eta_k(x); \varphi \rangle \eta_k(x) \in \Phi^* \quad (10)$$

*and the scalar product in  $\mathcal{H}(x)$  is formally defined by posing  $(\eta_k(x); \eta_j(x))_x := \delta_{k,j}$ ;*

- (iv) under the identification in (iii) the spaces  $\mathcal{H}(x)$  become the common generalized eigenspaces of the operators in  $\mathfrak{S}$  in the sense that if  $A_f \in \mathfrak{S}$  then  $\hat{A}_f \eta_k(x) = f(x) \eta_k(x)$  for  $\mu$ -almost every  $x \in X$  and all  $k = 1, 2, \dots, \dim \mathcal{H}(x)$ .

For a proof we refer to [Mau68] (Chapter II). The identification at point (iii) of Theorem 4.1 depends on the choice of a fundamental family of orthonormal measurable vector fields  $\{\xi_k(\cdot) : k \in I\}$  for the direct integral  $\mathfrak{H}$ , which is clearly not unique. If  $\{\zeta_k(\cdot) : k \in I\}$  is a second fundamental family of orthonormal measurable vector fields for  $\mathfrak{H}$ , then there exists a decomposable unitary map  $W(\cdot)$  such that  $W(x)\xi_k(x) = \zeta_k(x)$  for  $\mu$ -almost every  $x \in X$  and every  $k \in I$ . The composition  $U := \mathcal{F}_{\mathfrak{S}}^{-1} \circ W(\cdot) \circ \mathcal{F}_{\mathfrak{S}}$  is a unitary isomorphism of the Hilbert space  $\mathcal{H}$  which induces a linear isomorphism between the Gel'fand triples  $\{\Phi, \mathcal{H}, \Phi^*\}$  and  $\{\Psi, \mathcal{H}, \Psi^*\}$  where  $\Psi := U\Phi$ . One checks that  $\Psi$  is a nuclear space in  $\mathcal{H}$  with respect to the topology induced from  $\Phi$  by the map  $U$  (i.e. defined by the family of seminorms  $p'_\alpha := p_\alpha \circ U^{-1}$ ).  $\Psi^*$ , the topological dual of  $\Psi$ , is  $\hat{U}\Phi^*$ , in view of the continuity of  $U^{-1} : \Psi \rightarrow \Phi$ . The isomorphism of the Gel'fand triples is compatible with the direct integral decomposition. Indeed if  $\vartheta_k(x) \leftrightarrow \zeta_k(x)$  is the identification between the new orthonormal basis  $\{\zeta_k(x) : k = 1, 2, \dots, \dim \mathcal{H}(x)\}$  of  $\mathcal{H}(x)$  and a family of linear functionals  $\{\vartheta_k(x) : k = 1, 2, \dots, \dim \mathcal{H}(x)\} \subset \Psi^*$  then equation (8) implies that for any  $\varphi \in \Psi$

$$\langle \vartheta_k(x); \varphi \rangle := (\zeta_k(x); \varphi(x))_x = (\xi_k(x); W(x)^{-1}\varphi(x))_x = \langle \eta_k(x); U^{-1}\varphi \rangle = \langle \hat{U} \eta_k(x); \varphi \rangle. \quad (11)$$

As a consequence, we get the following result.

**PROPOSITION 4.2.** *Up to a canonical identification of isomorphic Gel'fand triples the realization (10) of the fiber spaces  $\mathcal{H}(x)$  as common generalized eigenspaces is canonical in the sense that it does not depend on the choice of a fundamental family of orthonormal measurable fields.*

From Proposition 4.2 and Corollary 3.4 it follows that:

**COROLLARY 4.3.** *Up to a canonical identification of isomorphic Gel'fand triples, the realization (10) of the fiber spaces  $\mathcal{H}(x)$  as generalized common eigenspaces is preserved by a unitary transform of the triple  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ .*

Theorem 4.1 assumes the existence of a  $\mathfrak{S}$ -invariant nuclear space and the related Gel'fand triple. If  $\mathfrak{S}$  is generated by a countable family, such a nuclear space does exist and there is an algorithmic procedure to construct it.

**THEOREM 4.4** (Existence of the nuclear space [Mau68]). *Let  $\{A_1, A_2, \dots\}$  be a countable family of commuting bounded normal operators on the separable Hilbert space  $\mathcal{H}$ , generating the commutative  $C^*$ -algebra  $\mathfrak{S}$ . Then there exists a countable  $\mathfrak{S}$ -cyclic system  $\{\psi_1, \psi_2, \dots\}$  which generates a nuclear space  $\Phi \subset \mathcal{H}$  such that: a)  $\Phi$  is dense in  $\mathcal{H}$ ; b) the embedding  $\iota : \Phi \hookrightarrow \mathcal{H}$  is continuous; c) the maps  $A_j^m : \Phi \rightarrow \Phi$  are continuous for all  $j, m \in \mathbb{N}$ .*

**REMARK 4.5.** For the proof of Theorem 4.4 see [Mau68] (Chapter II, Theorem 6). We recall that a countable (or finite) family  $\{\psi_1, \psi_2, \dots\}$  of orthonormal vectors in  $\mathcal{H}$  is a  $\mathfrak{S}$ -cyclic system if the set  $\{A^{\dagger b} A^a \psi_k : k \in I, a, b \in \mathbb{N}_{\text{fin}}^\infty\}$  is total in  $\mathcal{H}$ , where  $\mathbb{N}_{\text{fin}}^\infty$  is the

space of  $\mathbb{N}$ -valued sequences which are definitely zero (i.e.  $a_n = 0$  for any  $n \in \mathbb{N} \setminus I$  with  $|I| < +\infty$ ) and  $A^a := A_1^{a_1} A_2^{a_2} \dots A_N^{a_N}$  for some integer  $N$ .

Any  $C^*$ -algebra  $\mathfrak{S}$  (not necessarily commutative) has many  $\mathfrak{S}$ -cyclic systems. Indeed one can start from any normalized vector  $\psi_1 \in \mathcal{H}$  to build the closed subspace  $\mathcal{H}_1$  spanned by the action of  $\mathfrak{S}$  on  $\psi_1$ . If  $\mathcal{H}_1 \neq \mathcal{H}$  one can choose a second normalized vector  $\psi_2$  in the orthogonal complement of  $\mathcal{H}_1$  to build the closed subspace  $\mathcal{H}_2$ . Since  $\mathcal{H}$  is separable, this procedure produces a countable (or finite) family  $\{\psi_1, \psi_2, \dots\}$  such that  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ . Obviously this construction is not unique. The nuclear space  $\Phi$  claimed in Theorem 4.4 depends on the choice of a  $\mathfrak{S}$ -cyclic system and generally many inequivalent choices are possible.  $\blacklozenge$

## 5 The wandering property

An interesting and generally unsolved problem is the construction of the invariant subspaces of an operator or of a family of operators. Let  $\mathfrak{S}$  be a  $C^*$ -algebra contained in  $\mathcal{B}(\mathcal{H})$ . If  $\psi \in \mathcal{H}$  then the subspace  $\mathfrak{S}[\psi]$  generated by the action of  $\mathfrak{S}$  on the vector  $\psi$  is an invariant subspace for the  $C^*$ -algebra. The existence of a particular decomposition of the Hilbert space into invariant subspaces depends on the nature of the  $C^*$ -algebra. The problem is reasonably simple to solve for the  $C^*$ -algebras which satisfy the wandering property.

**DEFINITION 5.1** (wandering property). *Let  $\mathfrak{S}$  be a commutative unital  $C^*$ -algebra generated by the countable family  $\{A_1, A_2, \dots\}$  of commuting bounded normal operators in a separable Hilbert space  $\mathcal{H}$ . We say that  $\mathfrak{S}$  has the wandering property if there exists a (at most) countable family  $\{\psi_1, \psi_2, \dots\} \subset \mathcal{H}$  of orthonormal vectors which is  $\mathfrak{S}$ -cyclic (according to Remark 4.5) and such that*

$$(\psi_k; A^{\dagger b} A^a \psi_h)_{\mathcal{H}} = \|A^a \psi_k\|_{\mathcal{H}}^2 \delta_{k,h} \delta_{a,b} \quad \forall h, k \in I, \quad \forall a, b \in \mathbb{N}_{\text{fin}}^{\infty}, \quad (12)$$

where  $A^a := A_1^{a_1} A_2^{a_2} \dots A_N^{a_N}$ ,  $\delta_{k,h}$  is the usual Kronecker delta and  $\delta_{a,b}$  is the Kronecker delta for the multiindices  $a$  and  $b$ .

Let  $\mathcal{H}_k := \mathfrak{S}[\psi_k]$  be the Hilbert subspace generated by the action of  $\mathfrak{S}$  on the vector  $\psi_k$ . If  $\mathfrak{S}$  has the wandering property then the Hilbert space decomposes as  $\mathcal{H} = \bigoplus_{k \in I} \mathcal{H}_k$  and each  $\mathcal{H}_k$  is an  $\mathfrak{S}$ -invariant subspace. We will refer to  $\mathcal{H}_k$  as a *wandering subspace* and to  $\{\psi_1, \psi_2, \dots\}$  as the *wandering system*. In these subspaces each operator  $A_j$  acts as a unilateral weighted shift and this justifies the use of the adjective “wandering” (see [NF70] Chapter 1, Sections 1 and 2). The wandering property implies many interesting consequences.

**PROPOSITION 5.2.** *Let  $\mathfrak{S}$  be a commutative unital  $C^*$ -algebra generated by the (at most) countable family  $\{A_1, A_2, \dots\}$  of commuting bounded normal operators on a separable Hilbert space  $\mathcal{H}$ . Suppose that  $\mathfrak{S}$  has the wandering property with respect to the family of vectors  $\{\psi_1, \psi_2, \dots\}$ , then:*

- (i) *the generators are not selfadjoint, and  $A_j^n \neq \mathbb{1}$  for every  $n \in \mathbb{N} \setminus \{0\}$ ;*

- (ii) every generator which is unitary has no eigenvectors;
- (iii) if  $\mathfrak{S}$  is generated by  $N$  unitary operators then  $\mathfrak{S}$  is a  $\mathbb{Z}^N$ -algebra.

**Proof.** To prove (i) observe that the condition  $A_j = A_j^\dagger$  implies that  $A_j \psi_k = 0$  for all  $\psi_k$  in the system and the  $\mathfrak{S}$ -cyclicity imposes  $A_j = 0$ . As for the second claim, by setting  $b = 0$  and  $h = k$  in equation (12) one sees that  $A^a = \mathbb{1}$  implies  $a = 0$ .

To prove (ii) observe that if  $\{U, A_1, A_2, \dots\}$  is a set of commuting generators for  $\mathfrak{S}$  with  $U$  unitary, then each vector  $\varphi \in \mathcal{H}$  can be written as  $\varphi = \sum_{n \in \mathbb{Z}} U^n \chi_n$  where  $\chi_n = \sum_{k \in I, a \in \mathbb{Z}^N} \alpha_{k,a} A^a \psi_k$ . Clearly  $U\varphi = \sum_{n \in \mathbb{Z}} U^n \chi_{n-1}$  and equation (12) implies that  $\|\varphi\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{Z}} \|\chi_n\|_{\mathcal{H}}^2$ . If  $U\varphi = \lambda\varphi$ , with  $\lambda \in \mathbb{S}^1$ , then a comparison between the components provides  $\chi_{n-1} = \lambda\chi_n$ , i.e.  $\chi_n = \lambda^{-n}\chi_0$  for all  $n \in \mathbb{Z}$ . This contradicts the convergence of the series expressing the norm of  $\varphi$ .

To prove (iii) observe that the map  $\mathbb{Z}^N \ni a := (a_1, \dots, a_N) \mapsto U^a = U_1^{a_1} \dots U_N^{a_N} \in \mathcal{U}(\mathcal{H})$  is a unitary representation of  $\mathbb{Z}^N$  on  $\mathcal{H}$ . To show that the representation is algebraically compatible, suppose that  $\sum_{a \in \mathbb{Z}^N} \alpha_a U^a = 0$ ; then from equation (12) it follows that  $0 = (U^b \psi_k; \sum_{a \in \mathbb{Z}^N} \alpha_a U^a \psi_k)_{\mathcal{H}} = \alpha_b$  for all  $b \in \mathbb{Z}^N$ , and this concludes the proof.  $\blacksquare$

Proposition 5.2 shows that the wandering property forces a commutative  $C^*$ -algebra generated by a finite number of unitary operators to be a  $\mathbb{Z}^N$ -algebra. This is exactly what happens in the cases in which we are mostly interested.

**EXAMPLE 5.3** (*Periodic systems, part two*). The commutative unital  $C^*$ -algebra  $\mathfrak{S}_T$  defined in Example 2.1 is generated by a unitary faithful representation of  $\mathbb{Z}^d$  on  $L^2(\mathbb{R}^d)$ , given by  $\mathbb{Z}^d \ni m \mapsto T^m \in \mathcal{U}(L^2(\mathbb{R}^d))$  where  $m := (m_1, \dots, m_d)$  and  $T^m := T_1^{m_1} \dots T_d^{m_d}$ . It is easy to show that the  $C^*$ -algebra  $\mathfrak{S}_T$  has the wandering property. Indeed let  $\mathcal{Q}_0 := \{x = \sum_{j=1}^d x_j \gamma_j : -1/2 \leq x_j \leq 1/2, j = 1, \dots, d\}$  be the *fundamental unit cell* of the lattice  $\Gamma$  and  $\mathcal{Q}_m := \mathcal{Q}_0 + m$  its translation by the lattice vector  $m := \sum_{j=1}^d m_j \gamma_j$ . Let  $\{\psi_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$  be a family of functions with support in  $\mathcal{Q}_0$  providing an orthonormal basis of  $L^2(\mathcal{Q}_0)$  up to the natural inclusion  $L^2(\mathcal{Q}_0) \hookrightarrow L^2(\mathbb{R}^d)$ . This system is  $\mathfrak{S}_T$ -cyclic since  $L^2(\mathbb{R}^d) = \bigoplus_{m \in \mathbb{Z}^d} L^2(\mathcal{Q}_m)$ . Moreover, it is wandering under the action of  $\mathfrak{S}_T$  since the intersection  $\mathcal{Q}_0 \cap \mathcal{Q}_m$  has zero measure for every  $m \neq 0$ . The cardinality of the wandering system is  $\aleph_0$ . Proposition 5.2 assures that  $\mathfrak{S}_T$  is a  $\mathbb{Z}^d$ -algebra. Moreover, as a consequence of Proposition 5.5, the Gel'fand spectrum of  $\mathfrak{S}_T$  is homeomorphic to the  $d$ -dimensional torus  $\mathbb{T}^d$  and the normalized basic measure is the Haar measure  $dz$  on  $\mathbb{T}^d$ .  $\blacktriangleleft$

**EXAMPLE 5.4** (*Mathieu-like Hamiltonians, part two*). The unital commutative  $C^*$ -algebra  $\mathfrak{S}_M^q \subset \mathcal{B}(L^2(\mathbb{T}))$  defined in Example 2.2 is generated by a unitary faithful representation of the group  $\mathbb{Z}$  on the Hilbert space  $L^2(\mathbb{T})$ . Indeed, the map  $\mathbb{Z} \ni k \mapsto \mathfrak{w}^k \in \mathcal{U}(L^2(\mathbb{T}))$  is an injective group homomorphism. The set of vectors  $\{e_0, \dots, e_{q-1}\} \subset L^2(\mathbb{T})$  shows that the  $C^*$ -algebra  $\mathfrak{S}_M^q$  has the wandering property. In this case the cardinality of the wandering system is  $q$ . Proposition 5.2 assures that  $\mathfrak{S}_M^q$  is a  $\mathbb{Z}$ -algebra. Moreover, Proposition 5.5 will show that the Gel'fand spectrum of  $\mathfrak{S}_M^q$  is homeomorphic to the 1-dimensional torus  $\mathbb{T}$  and the normalized basic measure on the spectrum coincides with the Haar measure  $dz$  on  $\mathbb{T}$ . The first claim agrees with the fact that the Gel'fand spectrum of  $\mathfrak{S}_M^q$  coincides with the (Hilbert space) spectrum of  $\mathfrak{w}$ , the generator of the  $C^*$ -algebra, and  $\sigma(\mathfrak{w}) = \mathbb{T}$ .

The claim about the basic measure agrees with the fact that the vector  $e_0$  is cyclic for the commutant of  $\mathfrak{S}_M^q$  (which is the von Neumann algebra generated by  $\mathfrak{A}_M^{p/q}$ ). Indeed, a general result (see Appendix A) assures that the spectral measure  $\mu_{e_0, e_0}$  provides the basic measure. To determine  $\mu_{e_0, e_0}$  let  $\mathcal{P}(\mathfrak{w}) := \sum_{k \in \mathbb{Z}} \alpha_k \mathfrak{w}^k$  be any element of  $\mathfrak{S}_M^q$ . From the definition of spectral measure it follows that

$$\alpha_0 = (e_0; \mathcal{P}(\mathfrak{w})e_0) = \int_{\mathbb{T}} \mathcal{P}(z) d\mu_{e_0, e_0}(z) = \sum_{k \in \mathbb{Z}} \alpha_k \int_0^{2\pi} e^{ikt} d\tilde{\mu}_{e_0, e_0}(t). \quad (13)$$

where the measure  $\tilde{\mu}_{e_0, e_0}$  is related to  $\mu_{e_0, e_0}$  by the change of variables  $\mathbb{T} \ni z \mapsto t \in [0, 2\pi)$  defined by  $z := e^{it}$  according to Remark 2.3. Equation (13) implies that  $\tilde{\mu}_{e_0, e_0}$  agrees with  $dt/2\pi$  on  $C(\mathbb{T})$ , namely the basic measure  $\mu_{e_0, e_0}$  is the normalized Haar measure.  $\blacktriangleleft \blacktriangleright$

In the relevant cases of commutative unital  $C^*$ -algebras generated by a finite set of unitary operators the wandering property provides a useful characterization of the Gel'fand spectrum and the basic measure. We firstly introduce some notation and terminology. Let  $\mathbb{G}$  be a discrete group and  $\ell^1(\mathbb{G})$  be the set of sequences  $c = \{c_g\}_{g \in \mathbb{G}}$  such that  $\|c\|_{\ell^1} = \sum_{g \in \mathbb{G}} |c_g| < +\infty$ . Equipped with the convolution product  $(c * d)_g := \sum_{h \in \mathbb{G}} c_h d_{g-h}$  and involution  $c^\dagger := \{\bar{c}_{-g}\}_{g \in \mathbb{G}}$ ,  $\ell^1(\mathbb{G})$  becomes a unital Banach  $*$ -algebra called the *group algebra*  $\mathbb{G}$ . The latter is not a  $C^*$ -algebra since the norm  $\|\cdot\|_{\ell^1}$  does not verify the  $C^*$ -condition  $\|c * c^*\|_{\ell^1} = \|c\|_{\ell^1}^2$ . In general there exist several inequivalent ways to complete  $\ell^1(\mathbb{G})$  to a  $C^*$ -algebra by introducing suitable  $C^*$ -norms. Two of these  $C^*$ -extensions are of particular interest. The first is obtained as the completion of  $\ell^1(\mathbb{G})$  with respect to the *universal enveloping norm*

$$\|c\|_u := \sup\{\|\pi(c)\|_{\mathcal{H}} : \pi : \ell^1(\mathbb{G}) \rightarrow \mathcal{B}(\mathcal{H}) \text{ is a } * \text{-representation}\}.$$

The resulting abstract  $C^*$ -algebra, denoted by  $C^*(\mathbb{G})$ , is called the *group  $C^*$ -algebra* of  $\mathbb{G}$  (or *enveloping  $C^*$ -algebra*).

The second relevant extension is obtained by means of the concrete representation of the elements  $\ell^1(\mathbb{G})$  as (convolution) multiplicative operators on the Hilbert space  $\ell^2(\mathbb{G})$ . In other words, for any  $\xi = \{\xi_g\}_{g \in \mathbb{G}} \in \ell^2(\mathbb{G})$  and  $c = \{c_g\}_{g \in \mathbb{G}} \in \ell^1(\mathbb{G})$  one defines the representation  $\pi_r : \ell^1(\mathbb{G}) \rightarrow \mathcal{B}(\ell^2(\mathbb{G}))$  as

$$\pi_r(c)\xi := c * \xi = \left\{ \sum_{h \in \mathbb{G}} c_h \xi_{g-h} \right\}_{g \in \mathbb{G}}.$$

The representation  $\pi_r$ , known as *left regular representation*, is injective. The norm  $\|c\|_r := \|\pi_r(c)\|_{\mathcal{B}(\ell^2(\mathbb{G}))}$  defines a new  $C^*$ -norm on  $\ell^1(\mathbb{G})$ , called *reduced norm*, and a new  $C^*$ -extension denoted by  $C_r^*(\mathbb{G})$  and called *reduced group  $C^*$ -algebra*. Since  $\|\cdot\|_r \leq \|\cdot\|_u$  it follows that  $C_r^*(\mathbb{G})$  is  $*$ -isomorphic to a quotient  $C^*$ -algebra of  $C^*(\mathbb{G})$ . Nevertheless, if the group  $\mathbb{G}$  is abelian, one has the relevant isomorphism  $C_r^*(\mathbb{G}) = C^*(\mathbb{G}) \simeq C(\widehat{\mathbb{G}})$  where  $\widehat{\mathbb{G}}$  denotes the dual (or character) group of  $\mathbb{G}$ . For more details the reader can refer to [Dix82] (Chapter 13) or [Dav96](Chapter VII).

**PROPOSITION 5.5.** *Let  $\mathcal{H}$  be a separable Hilbert space and  $\mathfrak{S} \subset \mathcal{B}(\mathcal{H})$  a unital commutative  $C^*$ -algebra generated by a finite family  $\{U_1, \dots, U_N\}$  of unitary operators. Assume the wandering property. Then:*

- (i) the Gel'fand spectrum of  $\mathfrak{S}$  is homeomorphic to the  $N$ -dimensional torus  $\mathbb{T}^N$ ;
- (ii) the basic measure of  $\mathfrak{S}$  is the normalized Haar measure  $dz$  on  $\mathbb{T}^N$ .

**Proof.** We use the short notation  $U^a = U_1^{a_1} \dots U_N^{a_N}$  for any  $a = (a_1, \dots, a_N) \in \mathbb{Z}^N$ .

To prove (i) one notices that the map  $F : \ell^1(\mathbb{Z}^N) \rightarrow \mathcal{B}(\mathcal{H})$ , defined by  $F(c) := \sum_{a \in \mathbb{Z}^N} c_a U^a$ , is a  $*$ -representation of  $\ell^1(\mathbb{Z}^N)$  in  $\mathcal{B}(\mathcal{H})$ . As in the proof of Proposition 5.2, one exploits the wandering property to see that for any  $c \in \ell^1(\mathbb{Z}^N)$ ,  $\sum_a c_a U^a = 0$  implies  $c = 0$ . Thus  $F$  is a faithful representation. Moreover  $\|F(c)\|_{\mathcal{B}(\mathcal{H})} \leq \|c\|_{\ell^1}$  for all  $c \in \ell^1(\mathbb{C})$ . Finally, the unital  $*$ -algebra  $\mathfrak{L}^1(\mathbb{Z}^N) := F(\ell^1(\mathbb{Z}^N)) \subset \mathcal{B}(\mathcal{H})$  is dense in  $\mathfrak{S}$  (with respect to the operator norm), since it does contain the polynomials in  $U_1, \dots, U_N$ , which are a dense subset of  $\mathfrak{S}$ .

In view of the fact that  $\mathbb{Z}^N$  is abelian, to prove (i) it is sufficient to show that  $\mathfrak{S} \simeq C_r^*(\mathbb{Z}^N)$ . Since  $\ell^1(\mathbb{Z}^N)$  and  $\mathfrak{L}^1(\mathbb{Z}^N)$  are isomorphic Banach  $*$ -algebras, and  $\mathfrak{L}^1(\mathbb{Z}^N)$  is dense in  $\mathfrak{S}$ , the latter claim follows if one proves that  $\|c\|_r = \|F(c)\|_{\mathcal{B}(\mathcal{H})}$  for any  $c \in \ell^1(\mathbb{Z}^N)$ . Let  $\{\psi_k\}_{k \in I}$  be the wandering system of vectors for  $\mathfrak{S}$ . The wandering property assures that the closed subspace  $\mathfrak{S}[\psi_k] =: \mathcal{H}_k \subset \mathcal{H}$  is isometrically isomorphic to  $\ell^2(\mathbb{Z}^N)$ , with unitary isomorphism given by  $\mathcal{H}_k \ni \sum_{a \in \mathbb{Z}^N} \xi_a U^a \psi_k \mapsto \{\xi_a\}_{a \in \mathbb{Z}^N} \in \ell^2(\mathbb{Z}^N)$ . Then, due to the mutual orthogonality of the spaces  $\mathcal{H}_k$ , there exists a unitary map  $\mathcal{R} : \mathcal{H} \rightarrow \bigoplus_{k \in I} \ell^2(\mathbb{Z}^N)$  which extends all the isomorphisms above. A simple computation shows that  $\mathcal{R}F(c)\mathcal{R}^{-1} = \bigoplus_{k \in I} \pi_r(c)$  for any  $c \in \ell^1(\mathbb{Z}^N)$ . Since  $\mathcal{R}$  is isometric, it follows that  $\|F(c)\|_{\mathcal{B}(\mathcal{H})} = \|\bigoplus_{k \in I} \pi_r(c)\|_{\bigoplus_{k \in I} \ell^2} = \|\pi_r(c)\|_{\ell^2}$ , which is exactly the definition of the norm  $\|c\|_r$ .

To prove (ii) let  $\mu_k := \mu_{\psi_k, \psi_k}$  be the spectral measure defined by the vector of the wandering system  $\psi_k$ . The Gel'fand isomorphism identifies the generator  $U_j \in \mathfrak{S}$  with  $z_j \in C(\mathbb{T}^N)$ . It follows that for every  $a \in \mathbb{Z}^N$  one has

$$\delta_{a,0} = (\psi_k; U^a \psi_k) = \int_{\mathbb{T}^N} z^a d\mu_k(z) := \int_0^{2\pi} \dots \int_0^{2\pi} z_1^{a_1}(t) \dots z_N^{a_N}(t) d\tilde{\mu}_k(t), \quad (14)$$

where the measure  $\tilde{\mu}_k$  is related to  $\mu_k$  by the change of variables  $\mathbb{T}^N \ni z \mapsto t \in [0, 2\pi)^N$  defined by  $z := e^{it}$  according to Remark 2.3. Equation (14) shows that for all  $k$  the spectral measure  $\tilde{\mu}_k$  agrees with  $dz(t) := dt_1 \dots dt_N / (2\pi)^N$ .

Let  $A_f$  be the element of  $\mathfrak{S}$  whose image via the Gel'fand isomorphism is the function  $f \in C(\mathbb{T}^N)$ . Then

$$(U^b \psi_j; A_f U^a \psi_k)_{\mathcal{H}} = \delta_{j,k} (\psi_k; A_f U^{a-b} \psi_k)_{\mathcal{H}} = \int_{\mathbb{T}^N} f(z) \delta_{j,k} z^{a-b} dz.$$

So the spectral measure  $\mu_{U^b \psi_j, U^a \psi_k}$  is related to the Haar measure  $dz$  by the function  $\delta_{j,k} z^{a-b}$ . Let  $\varphi := \sum_{k \in I, a \in \mathbb{N}^N} \alpha_{a,k} U^a \psi_k$  be any vector in  $\mathcal{H}$ . Notice that, in view of the wandering property, one has  $\alpha_{k,a} \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}^N)$ . Then a direct computation shows that  $\mu_{\varphi, \varphi}(z) = h_{\varphi, \varphi}(z) dz$ , where  $h_{\varphi, \varphi}(z) = \sum_{k \in I} |F_{\varphi}^{(k)}(z)|^2$  with  $F_{\varphi}^{(k)}(z) := \sum_{a \in \mathbb{N}^N} \alpha_{k,a} z^a$ . Since  $F_{\varphi}^{(k)} \in L^2(\mathbb{T}^N)$ , one has  $|F_{\varphi}^{(k)}|^2 \in L^1(\mathbb{T}^N)$ . Let  $h_{\varphi, \varphi}^{(M)}(z) = \sum_{k=0}^M |F_{\varphi}^{(k)}(z)|^2$ . Since  $h_{\varphi, \varphi}^{(M+1)} \geq h_{\varphi, \varphi}^{(M)} \geq 0$  and  $\int_{\mathbb{T}^N} h_{\varphi, \varphi}^{(M)}(z) dz = \sum_{k=0}^M \sum_{a \in \mathbb{N}^N} |\alpha_{k,a}|^2 \leq \|\varphi\|_{\mathcal{H}}^2$  for all  $M$ , one concludes by the monotone convergence theorem that  $h_{\varphi, \varphi} \in L^1(\mathbb{T}^N)$ .  $\blacksquare$



Not every commutative  $C^*$ -algebra generated by a faithful unitary representation of  $\mathbb{Z}^N$  has a wandering system. In this situation, even if the spectrum is still a torus, the basic measure can be inequivalent to the Haar measure, as illustrated by the following example.

**EXAMPLE 5.6.** Let  $R_\alpha$  be the unitary operator on  $L^2(\mathbb{R}^2)$  which implements a rotation around the origin of the angle  $\alpha$ , with  $\alpha \notin 2\pi\mathbb{Q}$ . Since  $R_\alpha^N = R_{N\alpha} \neq 1$  for every integer  $N$ , it follows that the commutative unital  $C^*$ -algebra  $\mathfrak{R}_\alpha$  generated by  $R_\alpha$  is a  $\mathbb{Z}$ -algebra. The Gel'fand spectrum of  $\mathfrak{R}_\alpha$ , which coincides with the spectrum of  $R_\alpha$ , is  $\mathbb{T}$ . Indeed, the vector  $\psi_N(\rho, \phi) := e^{iN\phi} f(\rho)$  (in polar coordinates) is an eigenvector corresponding to the eigenvalue  $e^{iN\alpha}$ . The spectrum of  $R_\alpha$  is the closure of  $\{e^{iN\alpha} : N \in \mathbb{Z}\}$ , which is  $\mathbb{T}$  in view of the irrationality of  $\alpha$ . The existence of eigenvectors excludes the existence of a wandering system (see Proposition 5.2). Moreover, since  $R_\alpha$  has point spectrum it follows that the basic measure is not the Haar measure. Indeed, the spectral measure  $\mu_{\psi_N, \psi_N}$  corresponding to the eigenvector  $\psi_N$  is the *Dirac measure* concentrated in  $\{e^{iN\alpha}\} \subset \mathbb{C}$ . ◀▶

## 6 The generalized Bloch-Floquet transform

The aim of this section is to provide a general algorithm for constructing the direct integral decomposition of a commutative  $C^*$ -algebra which appears in the von Neumann's complete spectral theorem. In this approach a relevant role will be played by the wandering property. We consider a commutative unital  $C^*$ -algebra  $\mathfrak{S}$  on a separable Hilbert space  $\mathcal{H}$  generated by the finite family  $\{U_1, U_2, \dots, U_N\}$  of unitary operators admitting a wandering system  $\{\psi_k\}_{k \in I} \subset \mathcal{H}$ . According to the results of Section 5,  $\mathfrak{S}$  is a  $\mathbb{Z}^N$ -algebra with Gel'fand spectrum  $\mathbb{T}^N$  and with the Haar measure  $dz$  as basic measure.

### Construction of the wandering nuclear space

The existence of a wandering system makes possible the explicit construction of a  $\mathfrak{S}$ -invariant nuclear space, which we call the *wandering nuclear space*.

Consider the orthonormal basis  $\{U^a \psi_k\}_{k \in I, a \in \mathbb{Z}^N}$ , where  $\{\psi_k\}_{k \in I}$  is the wandering system, and denote by  $\mathcal{L} \subset \mathcal{H}$  the family of all finite linear combinations of the vectors of this basis. For every integer  $m \geq 0$  denote by  $\mathcal{H}_m$  the finite dimensional Hilbert space generated by the finite set of vectors  $\{U^a \psi_k : 0 \leq k \leq m, 0 \leq |a| \leq m\}$ , where  $|a| := |a_1| + \dots + |a_N|$ . Obviously  $\mathcal{H}_m \subset \mathcal{L}$ . Let  $D_m$  denote the dimension of the space  $\mathcal{H}_m$ . If  $\varphi = \sum_{k \in I, a \in \mathbb{Z}^N} \alpha_{k,a} U^a \psi_k$  is any element of  $\mathcal{H}$  then the formula

$$p_m^2(\varphi) := D_m \sum_{\substack{0 \leq k \leq m \\ 0 \leq |a| \leq m}} |(U^a \psi_k; \varphi)_{\mathcal{H}}|^2 = D_m \sum_{\substack{0 \leq k \leq m \\ 0 \leq |a| \leq m}} |\alpha_{k,a}|^2, \quad (15)$$

defines a seminorm for every  $m \geq 0$ . From (15) it follows that  $p_m \leq p_{m+1}$  for all  $m$ . The countable family of seminorms  $\{p_m\}_{m \in \mathbb{N}}$  provides a locally convex topology for the vector space  $\mathcal{L}$ . Let  $\Sigma$  denote the pair  $\{\mathcal{L}, \{p_m\}_{m \in \mathbb{N}}\}$ , i.e. the vector space  $\mathcal{L}$  endowed with the locally convex topology induced by the seminorms (15).  $\Sigma$  is a complete and metrizable

(i.e. Fréchet) space. However, for our purposes, we need a topology on  $\mathcal{L}$  which is strictly stronger than the metrizable topology induced by the seminorms (15).

The quotient space  $\Phi_m := \mathcal{L}/\mathcal{N}_m$ , with  $\mathcal{N}_m := \{\varphi \in \mathcal{L} : p_m(\varphi) = 0\}$ , is isomorphic to the finite dimensional vector space  $\mathcal{H}_m$ , hence it is nuclear and Fréchet. This follows immediately observing that the norm  $p_m$  on  $\Phi_m$  coincides, up to the positive constant  $\sqrt{D_m}$ , with the usual Hilbert norm. Obviously  $\Phi_m \subset \Phi_{m+1}$  for all  $m \geq 0$  and the topology of  $\Phi_m$  agrees with the topology inherited from  $\Phi_{m+1}$ , indeed  $p_{m+1}|_{\Phi_m} = \sqrt{\frac{D_{m+1}}{D_m}} p_m$ . We define  $\Phi$  to be  $\bigcup_{m \in \mathbb{N}} \Phi_m$  (which is  $\mathcal{L}$  as a set) endowed with the *strict inductive limit topology* which is the strongest topology which makes all injections  $\iota_m : \Phi_m \hookrightarrow \Phi$  continuous. The space  $\Phi$  is called a *LF-space* (according to the definition of [Tre67] Chapter 13) and it is a nuclear space since it is the strict inductive limit of nuclear spaces (see [Tre67] Proposition 50.1). We will say that  $\Phi$  is the *wandering nuclear space* defined by the  $\mathbb{Z}^N$ -algebra  $\mathfrak{S}$  on the wandering system  $\{\psi_k\}_{k \in I}$ .

**PROPOSITION 6.1.** *The wandering nuclear space  $\Phi$  defined by the previous construction satisfies all the properties stated in Theorem 4.4.*

**Proof.** A linear map  $j : \Phi \rightarrow \Psi$ , with  $\Psi$  is an arbitrary locally convex topological vector space, is continuous if and only if the restriction  $j|_{\Phi_m}$  of  $j$  to  $\Phi_m$  is continuous for each  $m \geq 0$  (see [Tre67] Proposition 13.1). This implies that the canonical embedding  $\iota : \Phi \hookrightarrow \mathcal{H}$  is continuous, since its restrictions are linear operators defined on finite dimensional spaces. The linear maps  $U^a : \Phi \rightarrow \Phi$  for all  $a \in \mathbb{N}^N$  are also continuous for the same reason. Finally  $\Phi$  is norm-dense in  $\mathcal{H}$  since as a set it is the dense domain  $\mathcal{L}$ .  $\blacksquare$

## The transform

We are now in position to define the *generalized Bloch-Floquet transform*  $\mathcal{U}_{\mathfrak{S}}$  for the  $C^*$ -algebra  $\mathfrak{S}$ . The Gel'fand spectrum of  $\mathfrak{S}$  is  $\mathbb{T}^N$  and the Gel'fand isomorphism associates to the generator  $U_j$  the function  $z_j \in C(\mathbb{T}^N)$ , according to the notation of Remark 2.3. For all  $t \in [0, 2\pi)^N$  and for all  $\varphi \in \Phi$  we define (formally for the moment) the Bloch-Floquet transform of  $\varphi$  at point  $t$  as

$$\Phi \ni \varphi \xrightarrow{\mathcal{U}_{\mathfrak{S}}|_t} (\mathcal{U}_{\mathfrak{S}}\varphi)(t) := \sum_{a \in \mathbb{Z}^N} z^{-a}(t) U^a \varphi \tag{16}$$

where  $z^a(t) := e^{ia_1 t_1} \dots e^{ia_N t_N}$  and  $U^a := U_1^{a_1} \dots U_N^{a_N}$ . The structure of equation (16) suggests that  $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)$  is a common generalized eigenvector for the elements of  $\mathfrak{S}$ , indeed a formal computation shows that

$$U_j(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = z_j(t) \sum_{a \in \mathbb{Z}^N} z_j^{-1}(t) z^{-a}(t) U_j U^a \varphi = e^{it_j} (\mathcal{U}_{\mathfrak{S}}\varphi)(t). \tag{17}$$

This guess is clarified by the following result.

**THEOREM 6.2** (Generalized Bloch-Floquet transform). *Let  $\mathfrak{S}$  be a  $\mathbb{Z}^N$ -algebra in the separable Hilbert space  $\mathcal{H}$  with generators  $\{U_1, \dots, U_N\}$  and wandering system  $\{\psi_k\}_{k \in I}$ , and let  $\Phi$  be the corresponding wandering nuclear space. Then the generalized Bloch-Floquet transform (16) defines an injective linear map from the nuclear space  $\Phi$  into its topological dual  $\Phi^*$  for every  $t \in [0, 2\pi)^N$ . More precisely, the transform  $\mathcal{U}_{\mathfrak{S}}|_t$  maps  $\Phi$  onto a subspace  $\Phi^*(t) \subset \Phi^*$  which is a common generalized eigenspace for the commutative  $C^*$ -algebra  $\mathfrak{S}$ , i. e.*

$$\hat{U}_j(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = e^{it_j}(\mathcal{U}_{\mathfrak{S}}\varphi)(t) \quad \text{in } \Phi^*.$$

The map  $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \rightarrow \Phi^*(t) \subset \Phi^*$  is a continuous linear isomorphism, provided  $\Phi^*$  is endowed with the weak-\* topology.

**Proof.** We need to verify that the right-hand side of (16) is well defined as a linear functional on  $\Phi$ . Any vector  $\varphi \in \Phi$  is a finite linear combination  $\varphi = \sum_{k \in I}^{\text{fin}} \sum_{b \in \mathbb{Z}^N}^{\text{fin}} \alpha_{k,b} U^b \psi_k$  (the complex numbers  $\alpha_{k,b}$  are different from zero only for a finite set of the values of the index  $k$  and the multiindex  $b$ ). Let  $\phi = \sum_{h \in \mathbb{N}}^{\text{fin}} \sum_{c \in \mathbb{N}^N}^{\text{fin}} \beta_{h,c} U^c \psi_h$  be another element in  $\Phi$ . The linearity of the dual pairing between  $\Phi^*$  and  $\Phi$  and the compatibility of the pairing with the Hermitian structure of  $\mathcal{H}$  imply

$$\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi \rangle := \sum_{k \in I}^{\text{fin}} \sum_{b,c \in \mathbb{Z}^N}^{\text{fin}} \bar{\alpha}_{k,b} \beta_{k,c} \left( \sum_{a \in \mathbb{Z}^N} z^a(t) (U^{a+b} \psi_k; U^c \psi_k)_{\mathcal{H}} \right) \quad (18)$$

where in the right-hand side we used the orthogonality between the spaces generated by  $\psi_k$  and  $\psi_h$  if  $k \neq h$ . Without further conditions equation (18) is a finite sum in  $k, b, c$  (this is simply a consequence of the fact that  $\varphi$  and  $\phi$  are “test functions”) but it is an infinite sum in  $a$  which generally does not converge. However, in view of the wandering property one has that  $(U^{a+b} \psi_k; U^c \psi_k)_{\mathcal{H}} = \delta_{a+b,c}$ , so that (18) reads

$$\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi \rangle = \sum_{k \in I}^{\text{fin}} \sum_{b,c \in \mathbb{Z}^N}^{\text{fin}} \bar{\alpha}_{k,b} \beta_{k,c} z^c(t) z^{-b}(t). \quad (19)$$

Let  $C_{\varphi;k} := \sum_{b \in \mathbb{Z}^N}^{\text{fin}} |\alpha_{k,b}|$  and  $C_{\varphi} := \max_{k \in I} \{C_{\varphi;k}\}$  (which is well defined since the set contains only a finite numbers of non-zero elements). An easy computation shows that

$$|\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi \rangle| \leq \sum_{k \in I}^{\text{fin}} C_{\varphi,k} \left( \sum_{c \in \mathbb{Z}^N}^{\text{fin}} |\beta_{k,c}| \right) \leq C_{\varphi} \sum_{k \in I}^{\text{fin}} \sum_{c \in \mathbb{Z}^N}^{\text{fin}} |\beta_{k,c}|.$$

Let  $m \geq 0$  be the smallest integer such that  $\phi \in \Phi_m$ . The number of the coefficients  $\beta_{k,c}$  different from zero is smaller than the dimension  $D_m$  of  $\Phi_m$ . Using the Cauchy-Schwarz inequality one has

$$|\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi \rangle| \leq C_{\varphi} \sqrt{D_m} \left( \sum_{k \in I}^{\text{fin}} \sum_{c \in \mathbb{N}^N}^{\text{fin}} |\beta_{k,c}|^2 \right)^{\frac{1}{2}} = C_{\varphi} p_m(\phi). \quad (20)$$

The inequality (20) shows that the linear map  $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) : \Phi \rightarrow \mathbb{C}$  is continuous when it is restricted to each finite dimensional space  $\Phi_m$ . Since  $\Phi$  is endowed with the strict inductive

limit topology, this is enough to assure that  $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)$  is a continuous linear functional on  $\Phi$ . So, in view of (20),  $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) \in \Phi^*$  for all  $t \in [0, 2\pi)^N$  and for all  $\varphi \in \Phi$ .

The linearity of the map  $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \rightarrow \Phi^*$  is immediate and from equation (19) it follows that  $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = 0$  (as functional) implies that  $\alpha_{k,b} = 0$  for all  $k$  and  $b$ , hence  $\varphi = 0$ . This proves the injectivity. To prove the continuity of the map  $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \rightarrow \Phi^*$ , in view of the strict inductive topology on  $\Phi$ , we only need to check the continuity of the maps  $\mathcal{U}_{\mathfrak{S}}|_t : \Phi_m \rightarrow \Phi^*$  for all  $m \geq 0$ . Since  $\Phi_m$  is a finite dimensional vector space with norm  $p_m$ , it is sufficient to prove that the norm-convergence of the sequence  $\varphi_n \rightarrow 0$  in  $\Phi_m$  implies the weak-\* convergence  $(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \rightarrow 0$  in  $\Phi^*$ , i.e.  $|\langle (\mathcal{U}_{\mathfrak{S}}\varphi_n)(t); \phi \rangle| \rightarrow 0$  for all  $\phi \in \Phi$ . As inequality (20) suggests, it is enough to show that  $C_{\varphi_n} \rightarrow 0$ . This is true since  $\varphi_n := \sum_{0 < k, |b| \leq m} \alpha_{k,b}^{(n)} U^b \psi_k \rightarrow 0$  in  $\Phi_m$  implies  $\alpha_{k,b}^{(n)} \rightarrow 0$ .

Finally, since the map  $U^{-a} = (U^a)^\dagger$  is continuous on  $\Phi$  for all  $a \in \mathbb{Z}^N$  then  $(\hat{U}^a) : \Phi^* \rightarrow \Phi^*$  defines a continuous map which extends the operator  $U^a$  originally defined on  $\mathcal{H}$ . In this context the equation (17) is meaningful and shows that  $\Phi^*(t) := \mathcal{U}_{\mathfrak{S}}|_t(\Phi) \subset \Phi^*$  is a space of common generalized eigenvectors for the elements of  $\mathfrak{S}$ .  $\blacksquare$

## The decomposition

The wandering system  $\{\psi_k\}_{k \in I}$  generates under the Bloch-Floquet transform a special family of elements of  $\Phi^*$ , denoted by

$$\zeta_k(t) := (\mathcal{U}_{\mathfrak{S}}\psi_k)(t) = \sum_{a \in \mathbb{Z}^N} z^{-a}(t) U^a \psi_k \quad \forall k \in I. \quad (21)$$

The injectivity of the map  $\mathcal{U}_{\mathfrak{S}}$  implies that the functionals  $\{\zeta_k(t)\}_{k \in I}$  are linearly independent for every  $t$ . If  $\varphi = \sum_{k \in I}^{\text{fin}} \sum_{b \in \mathbb{Z}^N}^{\text{fin}} \alpha_{k,b} U^b \psi_k$  is any element in  $\Phi$  then a simple computation shows that

$$(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = \sum_{k \in I}^{\text{fin}} \sum_{b \in \mathbb{Z}^N}^{\text{fin}} \alpha_{k,b} \sum_{a \in \mathbb{N}^N} z^{-a}(t) U^{a+b} \psi_k = \sum_{k \in I}^{\text{fin}} f_{\varphi;k}(t) \zeta_k(t) \quad (22)$$

where  $f_{\varphi;k}(t) := \sum_{b \in \mathbb{Z}^N}^{\text{fin}} \alpha_{k,b} z^b(t)$ . The equalities in (22) should be interpreted in the sense of “distributions”, i.e. elements of  $\Phi^*$ . The functions  $f_{\varphi;k} : \mathbb{T}^N \rightarrow \mathbb{C}$ , for all  $k \in I$ , are finite linear combinations of continuous functions, hence continuous. Equation (22) shows that any subspace  $\Phi^*(t)$  is generated by finite linear combinations of the functionals (21). For every  $t \in [0, 2\pi)^N$  we denote by  $\mathcal{K}(t)$  the space of the elements of the form  $\sum_{k \in I} \alpha_k \zeta_k(t)$  with  $\{\alpha_k\}_{k \in I} \in \ell^2(I)$ . This is a Hilbert space with the inner product induced by the isomorphism with  $\ell^2(I)$ . In other words the inner product is induced by the “formal” conditions  $(\zeta_k(t); \zeta_h(t))_t := \delta_{k,h}$ . All the Hilbert spaces  $\mathcal{K}(t)$  have the same dimension which is the cardinality of the system  $\{\psi_k\}_{k \in I}$ .

**PROPOSITION 6.3.** *For all  $t \in [0, 2\pi)^N$  the inclusions  $\Phi^*(t) \subset \mathcal{K}(t) \subset \Phi^*$  hold true. Moreover the generalized Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}}|_t$  extends to a unitary isomorphism*

between the Hilbert space  $\mathbb{H} \subset \mathcal{H}$  spanned by the orthonormal system  $\{\psi_k\}_{k \in I}$  and the Hilbert space  $\mathcal{K}(t) \subset \Phi^*$  spanned by  $\{\zeta_k(t)\}_{k \in I}$  (assumed as orthonormal basis).

**Proof.** The first inclusion  $\Phi^*(t) \subset \mathcal{K}(t)$  follows from the definition. For the second inclusion we need to prove that  $\omega(t) := \sum_{k \in I} \alpha_k \zeta_k(t)$  is a continuous functional if  $\{\alpha_k\}_{k \in I} \in \ell^2(\mathbb{N})$ . Let  $\phi = \sum_{0 \leq h, |c| \leq m} \beta_{h,c} U^c \psi_h$  be an element of  $\Phi_m \subset \Phi$  then, from the sesquilinearity of the dual pairing and the Cauchy-Schwarz inequality it follows that

$$|\langle \omega(t); \phi \rangle|^2 \leq \left( \sum_{k \in I} |\alpha_k| |\langle (\mathcal{U}_{\mathfrak{S}} \psi_k)(t); \phi \rangle| \right)^2 \leq \|\alpha\|_{\ell^2}^2 \sum_{k \in I} |\langle (\mathcal{U}_{\mathfrak{S}} \psi_k)(t); \phi \rangle|^2 \quad (23)$$

where  $\|\alpha\|_{\ell^2}^2 = \sum_{k \in I} |\alpha_k|^2 < \infty$ . From equation (18) it is clear that  $\langle (\mathcal{U}_{\mathfrak{S}} \psi_k)(t); \phi \rangle = 0$  if  $\psi_k \notin \Phi_m$ , then equation (20) and  $C_{\psi_k} = 1$  imply  $|\langle \omega(t); \phi \rangle| \leq \|\alpha\|_{\ell^2} \sqrt{m} p_m(\phi)$ . This inequality shows that  $\omega(t)$  is a continuous functional when it is restricted to each subspace  $\Phi_m$  and, because the strict inductive limit topology, this proves that  $\omega(t)$  lies in  $\Phi^*$ .

As for the second claim, consider  $\omega_n(t) := \sum_{0 \leq k \leq n} \alpha_k \zeta_k(t)$ . Obviously  $\omega_n(t) = (\mathcal{U}_{\mathfrak{S}} \varphi_n)(t) \in \Phi^*(t)$  since  $\varphi_n := \sum_{0 \leq k \leq n} \alpha_k \psi_k \in \Phi$ . Moreover the inequality (23) can be used to show that  $(\mathcal{U}_{\mathfrak{S}} \varphi_n)(t) \rightarrow \omega(t)$  when  $n \rightarrow \infty$  with respect to the weak-\* topology of  $\Phi^*$ . This enables us to define  $\omega(t) := (\mathcal{U}_{\mathfrak{S}} \varphi)(t)$  for all  $\varphi := \sum_{k \in I} \alpha_k \psi_k \in \mathbb{H}$ . The generalized Bloch-Floquet transform acts as a unitary isomorphism between  $\mathbb{H}$  and  $\mathcal{K}(t)$  with respect to the Hilbert structure induced in  $\mathcal{K}(t)$  by the orthonormal basis  $\{\zeta_k(t)\}_{k \in I}$ .  $\blacksquare$

**THEOREM 6.4** (Bloch-Floquet spectral decomposition). *Let  $\mathfrak{S}$  be a  $\mathbb{Z}^N$ -algebra in the separable Hilbert space  $\mathcal{H}$  with generators  $\{U_1, \dots, U_N\}$ , wandering system  $\{\psi_k\}_{k \in I}$  and wandering nuclear space  $\Phi$ . The generalized Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}}$ , defined on  $\Phi$  by equation (16), induces a direct integral decomposition of the Hilbert space  $\mathcal{H}$  which is equivalent (in the sense of Theorem 3.3) to the decomposition of von Neumann's theorem (Theorem 3.1). Moreover, the spaces  $\mathcal{K}(t)$  spanned in  $\Phi^*$  by the functionals (21) provide an explicit realization for the family of common eigenspaces of  $\mathfrak{S}$  appearing in Maurin's theorem (Theorem 4.1).*

**Proof.** Proposition 5.5 assures that the Gel'fand spectrum of  $\mathfrak{S}$  is the  $N$ -dimensional torus  $\mathbb{T}^N$  and the basic measure agrees with the normalized Haar measure  $dz$ . On the field of Hilbert spaces  $\prod_{t \in \mathbb{T}^N} \mathcal{K}(t)$  we can introduce a measurable structure by the fundamental family of orthonormal vector fields  $\{\zeta_k(\cdot)\}_{k \in I}$  defined by (21). For all  $\varphi \in \Phi$  the generalized Bloch-Floquet transform defines a square integrable vector field  $(\mathcal{U}_{\mathfrak{S}} \varphi)(\cdot) \in \mathfrak{K} := \int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t) dz(t)$ . Indeed equation (22) shows that  $(\mathcal{U}_{\mathfrak{S}} \varphi)(t) \in \mathcal{K}(t)$  for any  $t$  and  $\|(\mathcal{U}_{\mathfrak{S}} \varphi)(t)\|_t^2 = \sum_{k \in I}^{\text{fin}} |f_{\varphi;k}(t)|^2$  is a continuous function (finite sum of continuous functions) hence integrable on  $\mathbb{T}^N$ . In particular

$$\|(\mathcal{U}_{\mathfrak{S}} \varphi)(\cdot)\|_{\mathfrak{K}}^2 = \int_{\mathbb{T}^N} \|(\mathcal{U}_{\mathfrak{S}} \varphi)(t)\|_t^2 dz(t) = \sum_{k \in I} \int_{\mathbb{T}^N} \underbrace{\left( \sum_{b,c \in \mathbb{Z}^N}^{\text{fin}} \bar{\alpha}_{k,b} \alpha_{k,c} z^{c-b}(t) \right)}_{=|f_{\varphi;k}(x)|^2} dz(t) = \|\varphi\|_{\mathcal{H}}^2.$$

In view of the density of  $\Phi$ ,  $\mathcal{U}_{\mathfrak{S}}$  can be extended to an isometry from  $\mathcal{H}$  to  $\mathfrak{K}$ .

It remains to show that  $\mathcal{U}_{\mathfrak{S}}$  is surjective. Any square integrable vector field  $\varphi(\cdot) \in \mathfrak{K}$  is uniquely characterized by its expansion on the basis  $\{\zeta_k(\cdot)\}_{k \in I}$ , i.e.  $\varphi(\cdot) = \sum_{k \in I} \widehat{\varphi}_k(\cdot) \zeta_k(\cdot)$  where  $\{\widehat{\varphi}_k(t)\}_{k \in I} \in \ell^2(\mathbb{N})$  for all  $t \in [0, 2\pi)^N$ . The condition

$$\|\varphi(\cdot)\|_{\mathfrak{K}}^2 = \int_{\mathbb{T}^N} \sum_{k \in I} |\widehat{\varphi}_k(t)|^2 dz(t) < +\infty$$

shows that  $\widehat{\varphi}_k \in L^2(\mathbb{T}^N)$  for all  $k \in I$ . Let  $\widehat{\varphi}_k(t) = \sum_{b \in \mathbb{Z}^N} \alpha_{k,b} z^b(t)$  be the Fourier expansion of  $\widehat{\varphi}_k$ . Since

$$\sum_{k \in I} \sum_{b \in \mathbb{Z}^N} |\alpha_{k,b}|^2 = \sum_{k \in I} \|\widehat{\varphi}_k\|_{L^2(\mathbb{T}^N)}^2 = \|\varphi(\cdot)\|_{\mathfrak{K}}^2 < +\infty$$

it follows that  $\{\alpha_{k,b}\}_{k \in I, b \in \mathbb{Z}^N}$  is an  $\ell^2$ -sequences and the mapping

$$\varphi(\cdot) = \sum_{k \in I} \sum_{b \in \mathbb{Z}^N} \alpha_{k,b} z^b(\cdot) \zeta_k(\cdot) \xrightarrow{\mathcal{U}_{\mathfrak{S}}^{-1}} \varphi := \sum_{k \in I} \sum_{b \in \mathbb{Z}^N} \alpha_{k,b} U^b \psi_k \quad (24)$$

defines an element  $\varphi \in \mathcal{H}$  starting from the vector field  $\varphi(\cdot) \in \mathfrak{K}$ . It is immediate to check that  $\mathcal{U}_{\mathfrak{S}}$  maps  $\varphi$  in  $\varphi(\cdot)$ , hence  $\mathcal{U}_{\mathfrak{S}}$  is surjective.

If  $A_f \in \mathfrak{S}$  is an operator associated with the continuous function  $f \in C(\mathbb{T}^N)$  via the Gel'fand isomorphism, then  $\mathcal{U}_{\mathfrak{S}} A_f \mathcal{U}_{\mathfrak{S}}^{-1} \varphi(\cdot) = f(\cdot) \varphi(\cdot)$ , i.e.  $\mathcal{U}_{\mathfrak{S}}$  maps  $A_f \in \mathfrak{S}$  in  $M_f(\cdot) \in C(\mathfrak{K})$ . This allows us to apply the Theorem 3.3 which assures that the direct integral  $\mathfrak{K}$  coincides, up to a decomposable unitary transform, with the spectral decomposition of  $\mathfrak{S}$  established in Theorem 3.1.  $\blacksquare$

The generalized Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}}$  can be seen as a “computable” realization of the abstract  $\mathfrak{S}$ -Fourier transform  $\mathcal{F}_{\mathfrak{S}}$ . From Proposition 6.3 and from general results about direct integrals (see [Dix81] Part II, Chapter 1, Section 8) one obtains the following identifications:

$$\mathcal{H} \xrightarrow{\mathcal{U}_{\mathfrak{S}} \dots \mathcal{U}_{\mathfrak{S}}^{-1}} \int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t) dz(t) \simeq \int_{\mathbb{T}^N}^{\oplus} \mathbb{H} dz(t) \simeq L^2(\mathbb{T}^N, \mathbb{H}). \quad (25)$$

Since the dimension of  $\mathbb{H}$  is the cardinality of the wandering system chosen to define the Bloch-Floquet transform, and since Theorem 3.3 assures that the direct integral decomposition is essentially unique (in measure theoretic sense), one has the following:

**COROLLARY 6.5.** *Any two wandering systems associated with a  $\mathbb{Z}^N$ -algebra  $\mathfrak{S}$  have the same cardinality. Any two wandering systems for  $\mathfrak{S}$  are intertwined by a unitary operator which commutes with  $\mathfrak{S}$ .*

**EXAMPLE 6.6** (*Periodic systems, part three*). In the case of Example 2.1, the Bloch-Floquet transform is the usual one (see [Kuc93], [Pan07])

$$(\mathcal{U}_{\mathfrak{S}_T} \varphi)(t, y) := \sum_{m \in \Gamma} z^{-m}(t) T^m \varphi(y) = \sum_{m \in \Gamma} e^{-im_1 t_1} \dots e^{-im_d t_d} \varphi(y - m),$$

where  $m := \sum_{j=1}^d m_j \gamma_j$ , for all  $\varphi$  in the wandering nuclear space  $\Phi \subset L^2(\mathbb{R}^d)$ , built according to Proposition 6.1 from any orthonormal basis of  $L^2(\mathcal{Q}_0)$ . The fiber spaces in the

direct integral decomposition are all unitarily equivalent to  $L^2(\mathcal{Q}_0)$  hence the Hilbert space decomposition is

$$L^2(\mathbb{R}^d) \xrightarrow{\mathcal{U}_{\mathfrak{S}_T} \dots \mathcal{U}_{\mathfrak{S}_T}^{-1}} \int_{\mathbb{T}^d}^{\oplus} L^2(\mathcal{Q}_0) dz(t).$$

◀▷

**EXAMPLE 6.7** (*Mathieu-like Hamiltonians, part three*). In this case the wandering nuclear space  $\Phi$  is the set of the finite linear combinations of the Fourier basis  $\{e_n\}_{n \in \mathbb{Z}}$  and for all  $g(\theta) = \sum_{n \in \mathbb{Z}}^{\text{fin}} \alpha_n e^{in\theta}$  in  $\Phi$  the Bloch-Floquet transform is

$$(\mathcal{U}_{\mathfrak{S}_M^q} g)(\theta, t) := \sum_{m \in \mathbb{Z}} e^{-imt} \mathfrak{w}^m g(\theta) = \sum_{n \in \mathbb{Z}}^{\text{fin}} \alpha_n \left( \sum_{m \in \mathbb{Z}} e^{i[n\theta + m(q\theta - t)]} \right).$$

The collection  $\zeta_k(\cdot; t) \in \Phi^*$ , given by  $\zeta_k(\theta; t) := e^{ik\theta} \sum_{m \in \mathbb{Z}} e^{im(q\theta - t)}$  with  $k = 0, \dots, q-1$ , defines a fundamental family of orthonormal fields. The fiber spaces in the direct integral decomposition are all unitarily equivalent to  $\mathbb{C}^q$  hence the Hilbert space decomposition is

$$L^2(\mathbb{T}) \xrightarrow{\mathcal{U}_{\mathfrak{S}_M^q} \dots \mathcal{U}_{\mathfrak{S}_M^q}^{-1}} \int_{\mathbb{T}}^{\oplus} \mathbb{C}^q dz(t).$$

The images of the generators  $\mathfrak{u}$  and  $\mathfrak{v}$  under the map  $\mathcal{U}_{\mathfrak{S}_M^q} \dots \mathcal{U}_{\mathfrak{S}_M^q}^{-1}$  are the two  $t$  dependent  $q \times q$  matrices

$$\mathfrak{u}(t) := \begin{pmatrix} 0 & & & e^{it} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \quad \mathfrak{v}(t) := \begin{pmatrix} 1 & & & \\ & e^{-i2\pi \frac{t}{q}} & & \\ & & \ddots & \\ & & & e^{-i2\pi \frac{t}{q}(q-1)} \end{pmatrix}.$$

For every  $t \in \mathbb{T}$  the matrices  $\mathfrak{u}(t)$  and  $\mathfrak{v}(t)$  generate a faithful irreducible representation of the  $C^*$ -algebra  $\mathfrak{A}_M^{p/q}$  on the Hilbert space  $\mathbb{C}^q$  (see [Boc01] Theorem 1.9). ◀▷

## 7 Emergent geometry

From a geometric viewpoint, the field of Hilbert spaces  $\mathfrak{F} := \prod_{x \in X} \mathcal{H}(x)$  can be regarded as a pseudo vector-bundle  $\mathcal{E} \xrightarrow{\pi} X$ , where

$$\mathcal{E} := \bigsqcup_{x \in X} \mathcal{H}(x) \tag{26}$$

is the disjoint union of the Hilbert spaces  $\mathcal{H}(x)$ . The use of the prefix “pseudo” refers to the fact that more ingredients are needed to turn  $\mathcal{E} \xrightarrow{\pi} X$  into a vector bundle. First of all, the map  $\pi$  must be continuous, which requires a topology on  $\mathcal{E}$ . As a first attempt, assuming that  $\mathcal{H}(x) \subset \Phi^*$  for every  $x \in X$ , one might consider  $\mathcal{E} \xrightarrow{\pi} X$  as a sub-bundle of the trivial vector bundle  $X \times \Phi^* \xrightarrow{\pi} X$ , equipped with the topology induced by the inclusion, so that  $\mathcal{E} \xrightarrow{\pi} X$  becomes a topological bundle whose fibers are Hilbert spaces. However, nothing ensures that the Hilbert space topology defined fiberwise is compatible with the topology of  $\mathcal{E}$ , a necessary condition to have a meaningful topological theory.

## Geometric vs. analytic viewpoint

We begin our analysis with the definition of topological fibration of Hilbert spaces. Following [FD88] (Chapter II, Section 13) we pose the following

**DEFINITION 7.1** (Geometric viewpoint: Hilbert bundle). *A Hilbert bundle is the datum of a topological Hausdorff spaces  $\mathcal{E}$  (the total space), a compact Hausdorff space  $X$  (the base space) and a map  $\mathcal{E} \xrightarrow{\pi} X$  (the canonical projection) which is a continuous open surjection such that:*

- a) *for all  $x \in X$  the fiber  $\pi^{-1}(x) \subset \mathcal{E}$  is a Hilbert space;*
- b) *the map  $\mathcal{E} \ni p \mapsto \|p\| \in \mathbb{C}$  is continuous;*
- c) *the operation  $+$  is continuous as a function on  $\mathcal{S} := \{(p, s) \in \mathcal{E} \times \mathcal{E} : \pi(p) = \pi(s)\}$  to  $\mathcal{E}$ ;*
- d) *for each  $\lambda \in \mathbb{C}$  the map  $\mathcal{E} \ni p \mapsto \lambda p \in \mathcal{E}$  is continuous;*
- e) *let  $0_x$  be the null vector in the Hilbert space  $\pi^{-1}(x)$ ; for each  $x \in X$ , the collection of all subsets of  $\mathcal{E}$  of the form  $\mathcal{U}(O, x, \varepsilon) := \{p \in \mathcal{E} : \pi(p) \in O, \|p\| < \varepsilon\}$ , where  $O$  is a neighborhood of  $x$  and  $\varepsilon > 0$ , is a basis of neighborhoods of  $0_x \in \pi^{-1}(x)$  in  $\mathcal{E}$ .*

We denote by the short symbol  $\mathcal{E}_{\pi, X}$  the Hilbert bundle  $\mathcal{E} \xrightarrow{\pi} X$ . A *section* of  $\mathcal{E}_{\pi, X}$  is a function  $\psi : X \rightarrow \mathcal{E}$  such that  $\pi \circ \psi = \text{id}_X$ . We denote by  $\Gamma(\mathcal{E}_{\pi, X})$  the set of all *continuous sections* of  $\mathcal{E}_{\pi, X}$ . As shown in [FD88], from Definition 7.1 it follows that: (i) the scalar multiplication  $\mathbb{C} \times \mathcal{E} \ni (\lambda, p) \mapsto \lambda p \in \mathcal{E}$  is continuous; (ii) the open sets of  $\mathcal{E}$ , restricted to a fiber  $\pi^{-1}(x)$ , generate the Hilbert space topology of  $\pi^{-1}(x)$ ; (iii) the set  $\Gamma(\mathcal{E}_{\pi, X})$  has the structure of a (left)  $C(X)$ -module. In other words, the definition of Hilbert bundle includes all the conditions which a “formal” fibration such as (26) needs to fulfill to be a topological fibration with a topology compatible with the Hilbert structure of the fibers. In this sense the Hilbert bundle is the “geometric object” of our interest.

However, the structure that emerges in a natural way from the Bloch-Floquet decomposition (Theorem 6.4) is more easily understood from the analytic viewpoint. Switching the focus from the total space  $\mathcal{E}$  to the space of sections  $\mathfrak{F}$ , the relevant notion is that of *continuous field of Hilbert spaces*, according to [Dix82] (Section 10.1) or [DD63] (Section 1).

**DEFINITION 7.2** (Analytic viewpoint: continuous field of Hilbert spaces). *Let  $X$  be a compact Hausdorff space and  $\mathfrak{F} := \prod_{x \in X} \mathcal{H}(x)$  a field of Hilbert spaces. A continuous structure on  $\mathfrak{F}$  is the datum of a linear subspace  $\Gamma \subset \mathfrak{F}$  such that:*

- a) *for each  $x \in X$  the set  $\{\sigma(x) : \sigma(\cdot) \in \Gamma\}$  is dense in  $\mathcal{H}(x)$ ;*
- b) *for any  $\sigma(\cdot) \in \Gamma$  the map  $X \ni x \mapsto \|\sigma(x)\|_x \in \mathbb{R}$  is continuous;*
- c) *if  $\psi(\cdot) \in \mathfrak{F}$  and if for each  $\varepsilon > 0$  and each  $x_0 \in X$ , there is some  $\sigma(\cdot) \in \Gamma$  such that  $\|\sigma(x) - \psi(x)\|_x < \varepsilon$  on a neighborhood of  $x_0$ , then  $\psi(\cdot) \in \Gamma$ .*



We denote by the short symbol  $\mathfrak{F}_{\Gamma, X}$  the field of Hilbert spaces  $\mathfrak{F}$  endowed with the continuous structure  $\Gamma$ . The elements of  $\Gamma$  are called *continuous vector fields*. The condition b) may be replaced by the requirement that for any  $\sigma(\cdot), \varrho(\cdot) \in \Gamma$ , the function  $X \ni x \mapsto (\sigma(x); \varrho(x))_x \in \mathbb{C}$  is continuous. Condition c) is called *locally uniform closure* and guarantees that the linear space  $\Gamma$  is stable under multiplication by continuous functions on  $X$ . This condition implies that  $\Gamma$  is a (left)  $C(X)$ -module. A *total set* of continuous vector fields for  $\mathfrak{F}_{\Gamma, X}$  is a subset  $\Lambda \subset \Gamma$  such that  $\Lambda(x) := \{\sigma(x) : \sigma(\cdot) \in \Lambda\}$  is dense in  $\mathcal{H}(x)$  for all  $x \in X$ . The continuous field of Hilbert spaces is said to be *separable* if it has a countable total set of continuous vector fields.

The link between the notion of continuous field of Hilbert spaces and that of Hilbert bundle is clarified by the following result.

**PROPOSITION 7.3** (Equivalence between geometric and analytic viewpoint [DD63] [FD88]). *Let  $\mathfrak{F}_{\Gamma, X}$  be a continuous field of Hilbert spaces over the compact Hausdorff space  $X$ . Let  $\mathcal{E}(\mathfrak{F}_{\Gamma, X}) := \bigsqcup_{x \in X} \mathcal{H}(x)$  be the disjoint union of the Hilbert spaces  $\mathcal{H}(x)$  and  $\pi$  the canonical surjection of  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  onto  $X$ . Then there exists a unique topology  $\mathcal{T}$  on  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  making  $\mathcal{E}(\mathfrak{F}_{\Gamma, X}) \xrightarrow{\pi} X$  a Hilbert bundle over  $X$  such that all the continuous vector fields in  $\mathfrak{F}_{\Gamma, X}$  are continuous sections of  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$ . Moreover, every Hilbert bundle comes from a continuous field of Hilbert spaces.*

For the proof we refer to [DD63] (Section 2) or [FD88] (Chapter II, Theorem 13.18). We say that the set  $\mathcal{E}(\mathfrak{F})$  endowed with the topology  $\mathcal{T}$  and the canonical surjection  $\pi$  is the Hilbert bundle associated with the continuous structure  $\Gamma$  of  $\mathfrak{F}$ .

## Triviality, local triviality and vector bundle structure

A Hilbert bundle is a generalization of a (infinite dimensional) vector bundle, in the sense that some other extra conditions are needed in order to turn it into a genuine vector bundle. For the axioms of vector bundle we refer to [Lan85]. The most relevant missing condition is the *local triviality property*.

Two Hilbert bundles  $\mathcal{E}_{\pi, X}$  and  $\mathcal{F}_{\tau, X}$  over the same base space  $X$  are said to be (isometrically) isomorphic if there exists a homeomorphism  $\Theta : \mathcal{E} \rightarrow \mathcal{F}$  such that a)  $\tau \circ \Theta = \pi$ , b)  $\Theta_x := \Theta|_{\pi^{-1}(x)}$  is a unitary map from the Hilbert space  $\pi^{-1}(x)$  to the Hilbert space  $\tau^{-1}(x)$ . From the definition it follows that if the Hilbert bundles  $\mathcal{E}_{\pi, X}$  and  $\mathcal{F}_{\tau, X}$  are isomorphic then the map  $\Gamma(\mathcal{E}_{\pi, X}) \ni \sigma \mapsto \Theta \circ \sigma \in \Gamma(\mathcal{F}_{\tau, X})$  is one to one. A Hilbert bundle is said to be *trivial* if it is isomorphic to the *constant* Hilbert bundle  $X \times \mathbb{H} \rightarrow X$  where  $\mathbb{H}$  is a fixed Hilbert space. It is called *locally trivial* if for every  $x \in X$  there is a neighborhood  $O$  of  $x$  such that the *reduced Hilbert bundle*  $\mathcal{E}|_O := \{p \in \mathcal{E} : \pi(p) \in O\} = \pi^{-1}(O)$  is isomorphic to the constant Hilbert bundle  $O \times \mathbb{H} \rightarrow O$ . Two continuous fields of Hilbert spaces  $\mathfrak{F}_{\Gamma, X}$  and  $\mathfrak{G}_{\Delta, X}$  over the same space  $X$  are said to be (isometrically) isomorphic if the associated Hilbert bundles  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  and  $\mathcal{E}(\mathfrak{G}_{\Delta, X})$  are isomorphic. A continuous field of Hilbert spaces  $\mathfrak{F}_{\Gamma, X}$  is said to be *trivial* (resp. *locally trivial*) if  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  is *trivial* (resp. *locally trivial*).

**PROPOSITION 7.4** ([FD88] [DD63]). *Let  $\mathfrak{F}_{\Gamma, X}$  be a continuous field of Hilbert spaces over the compact Hausdorff space  $X$  and  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  the associated Hilbert bundle. Then:*

- (i) if  $\mathfrak{F}_{\Gamma, X}$  is separable and  $X$  is second-countable (or equivalently metrizable) then the topology defined on the total space  $\mathcal{E}(\mathfrak{F})$  is second-countable;
- (ii) if  $\dim \mathcal{H}(x) = \aleph_0$  for all  $x \in X$  and if  $X$  is a finite dimensional manifold then the Hilbert bundle  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  is trivial;
- (iii) if  $\dim \mathcal{H}(x) = q < +\infty$  for all  $x \in X$  then the Hilbert bundle  $\mathcal{E}(\mathfrak{F}_{\Gamma, X})$  is a Hermitian vector bundle with typical fiber  $\mathbb{C}^q$ .

For the proof of (i) one can see [FD88] (Chapter II, Proposition 13.21). The proof of (ii) is in [DD63] (Theorem 5). As for the proof of (iii), we recall that a Hilbert bundle has the (stronger) structure of a vector bundle whenever the local triviality and the continuity of the transition functions (see [Lan85] Chapter III) hold true. However, if the fibers are finite dimensional then the continuity of the transition functions follows from the existence of the local trivializations (see [Lan85] Chapter III, Proposition 1), hence one needs only to prove the local triviality. The latter follows from standard arguments, as in the final remark of Section 1 of [DD63].

## Algebraic viewpoint

Roughly speaking, a continuous field of Hilbert spaces is an “analytic object” while a Hilbert bundle is a “geometric object”. There is also a third point of view which is of algebraic nature. We introduce an “algebraic object” which encodes all the relevant properties of the set of continuous vector fields (or continuous sections).

**DEFINITION 7.5** (Algebraic viewpoint: Hilbert module). *A (left) pre- $C^*$ -module over a commutative unital  $C^*$ -algebra  $\mathcal{A}$  is a complex vector space  $\Omega_0$  that is also a (left)  $\mathcal{A}$ -module endowed with a pairing  $\{\cdot; \cdot\} : \Omega_0 \times \Omega_0 \rightarrow \mathcal{A}$  satisfying, for  $\sigma, \varrho, \varsigma \in \Omega_0$  and for  $a \in \mathcal{A}$  the following requirements:*

- a)  $\{\sigma; \varrho + \varsigma\} = \{\sigma; \varrho\} + \{\sigma; \varsigma\}$ ;
- b)  $\{\sigma; a\varrho\} = a\{\sigma; \varrho\}$ ;
- c)  $\{\sigma; \varrho\}^* = \{\varrho; \sigma\}$ ;
- d)  $\{\sigma; \sigma\} > 0$  if  $\sigma \neq 0$ .

The map  $||| \cdot ||| : \Omega_0 \rightarrow [0, +\infty)$  defined by  $|||\sigma||| := \sqrt{\|\{\sigma; \sigma\}\|_{\mathcal{A}}}$  is a norm in  $\Omega_0$ . The completion  $\Omega$  of  $\Omega_0$  with respect to the norm  $||| \cdot |||$  is called (left)  $C^*$ -module or Hilbert module over  $\mathcal{A}$ .

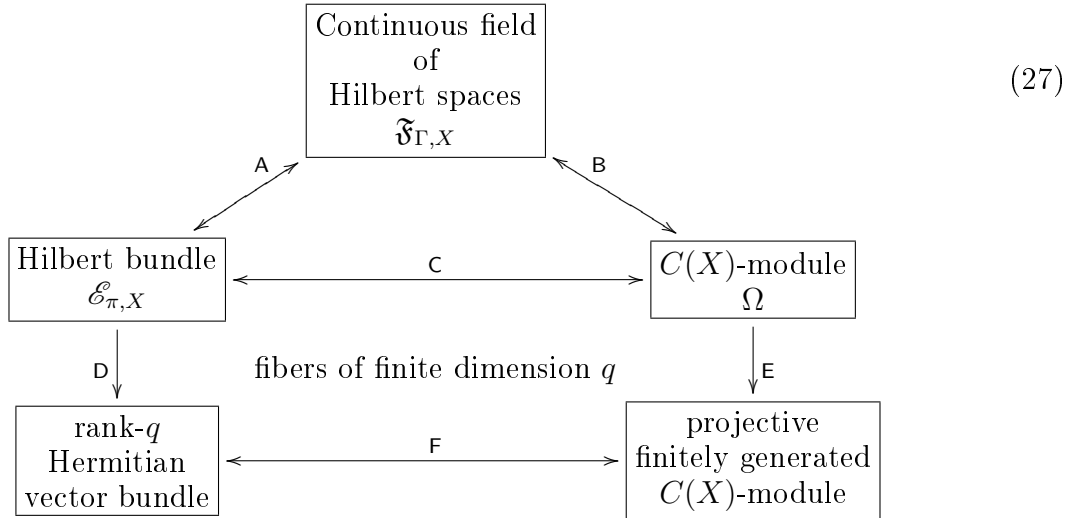
**PROPOSITION 7.6** (Equivalence between algebraic and analytic viewpoint [DD63]). *Let  $\mathfrak{F}_{\Gamma, X}$  be a continuous field of Hilbert spaces over the compact Hausdorff space  $X$ . The set of continuous vector fields  $\Gamma$  has the structure of a Hilbert module over  $C(X)$ . Conversely, any Hilbert module over  $C(X)$  defines a continuous field of Hilbert spaces. This correspondence is one-to-one.*

**Proof.** We shortly sketch the proof, see [DD63] (Section 3) for details. To prove the first part of the statement one observes that for all pairs of continuous vector fields  $\sigma(\cdot), \varrho(\cdot) \in \Gamma$  the pairing  $\{;\cdot\} : \Gamma \times \Gamma \rightarrow C(X)$  defined fiberwise by the inner product, i.e. by posing  $\{\sigma; \varrho\}(x) := (\sigma(x); \varrho(x))_x$ , satisfies Definition 7.5. The norm is defined by  $|||\sigma||| := \sup_{x \in X} \|\sigma(x)\|_x$  and  $\Gamma$  is closed with respect to this norm in view of the property of locally uniform closure.

Conversely let  $\Omega$  be a  $C^*$ -module over  $C(X)$ . For all  $x \in X$  define a pre-Hilbert structure on  $\Omega$  by  $(\sigma; \varrho)_x := \{\sigma; \varrho\}(x)$ . The set  $\mathcal{I}_x := \{\sigma \in \Omega : \{\sigma; \sigma\}(x) = 0\}$  is a linear subspace of  $\Omega$ . On the quotient space  $\Omega/\mathcal{I}_x$  the inner product  $(; )_x$  is a positive definite sesquilinear form and we denote by  $\mathcal{H}(x)$  the related Hilbert space. The collection  $\{\mathcal{H}(x) : x \in X\}$  defines a field of Hilbert spaces  $\mathfrak{F}(\Omega) = \prod_{x \in X} \mathcal{H}(x)$ . For all  $\sigma \in \Omega$  the canonical projection  $\Omega \ni \sigma \xrightarrow{j_x} \sigma(x) \ni \Omega/\mathcal{I}_x$  defines a vector field  $\sigma(\cdot) \in \mathfrak{F}(\Omega)$ . It is easy to check that the map  $\Omega \ni \sigma \xrightarrow{\Gamma} \sigma(\cdot) \ni \mathfrak{F}(\Omega)$  is injective. We denote by  $\Gamma(\Omega)$  the image of  $\Omega$  in  $\mathfrak{F}(\Omega)$ . The family  $\Gamma(\Omega)$  defines a continuous structure on  $\mathfrak{F}(\Omega)$ . Indeed  $\{\sigma(x) : \sigma(\cdot) \in \Gamma(\Omega)\} = \Omega/\mathcal{I}_x$  is dense in  $\mathcal{H}(x)$  and  $\|\sigma(x)\|_x^2 = \{\sigma; \sigma\}(x)$  is continuous. Finally locally uniform closure of  $\Gamma(\Omega)$  follows from the closure of  $\Omega$  with respect to the norm  $|||\sigma||| := \sup_{x \in X} \sqrt{\{\sigma; \sigma\}(x)}$  and the existence of a partition of the unit subordinate to a finite cover of  $X$  (since  $X$  is compact). ■

## The Hilbert bundle emerging from the Bloch-Floquet decomposition

Before proceeding with our analysis, it is useful to summarize in the following diagram the relations between the algebraic, the analytic and the geometric descriptions.



Arrows **A** and **B** summarize the content of Propositions 7.3 and 7.6 respectively, arrow **D** corresponds to point (iii) of Proposition 7.4, and arrow **E** follows by Proposition 53 in [Lan97]. Arrow **F** corresponds to the remarkable Serre-Swan Theorem (see [Lan97] Proposition 21), so arrow **C** can be interpreted as a generalization of the Serre-Swan Theorem.

Coming back to our original problem, let  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  be a physical frame with  $\mathcal{H}$  a separable Hilbert space and  $\mathfrak{S}$  a  $\mathbb{Z}^N$ -algebra with generators  $\{U_1, \dots, U_N\}$  and wandering system

$\{\psi_k\}_{k \in I}$ . The Bloch-Floquet decomposition (Theorem 6.4) ensures the existence of a unitary map  $\mathcal{U}_{\mathfrak{S}}$ , which maps  $\mathcal{H}$  into the direct integral  $\mathfrak{K} := \int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t) dz(t)$ . Let  $\mathfrak{F} := \prod_{t \in \mathbb{T}^N} \mathcal{K}(t)$  be the corresponding field of Hilbert spaces. The space  $\mathfrak{K}$  is a subset of  $\mathfrak{F}$  which has the structure of a Hilbert space and whose elements can be seen as  $L^2$ -sections of a “pseudo-Hilbert bundle”  $\mathcal{E}(\mathfrak{F}) := \bigsqcup_{t \in \mathbb{T}^N} \mathcal{K}(t)$ . This justifies the use of the notation  $\mathfrak{K} = \Gamma_{L^2}(\mathcal{E})$ .

To obtain a topological decomposition, we need to know *a priori* how to select a continuous structure  $\Gamma \subset \mathfrak{K}$  for the field of Hilbert spaces  $\mathfrak{F}$ . In view of Proposition 7.3, this procedure is equivalent to selecting *a priori* the family of the continuous section  $\Gamma(\mathcal{E})$  of the Hilbert bundle  $\mathcal{E}$  inside the Hilbert space of the  $L^2$ -sections  $\Gamma_{L^2}(\mathcal{E})$ . We can use the generalized Bloch-Floquet transform to push back this problem at the level of the original Hilbert space  $\mathcal{H}$  and to adopt the algebraic viewpoint. With this change of perspective the new, but equivalent, question which we need to answer is: does the physical frame  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  select a Hilbert module over  $C(\mathbb{T}^N)$  inside the Hilbert space  $\mathcal{H}$ ?

Generalizing an idea of [Gru01], we can use the transform  $\mathcal{U}_{\mathfrak{S}}$  and the notion of wandering nuclear space  $\Phi$  to provide a positive answer. The core of our analysis is the following result.

**PROPOSITION 7.7.** *Let  $\mathfrak{S}$  be a  $\mathbb{Z}^N$ -algebra in the separable Hilbert space  $\mathcal{H}$  with generators  $\{U_1, \dots, U_N\}$ , wandering system  $\{\psi_k\}_{k \in I}$  and wandering nuclear space  $\Phi$ . Let  $\mathfrak{K}$  be the direct integral defined by the Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}} : \mathcal{H} \rightarrow \mathfrak{K}$ . Then the Bloch-Floquet transform endows  $\Phi$  with the structure of a (left) pre- $C^*$ -module over  $C(\mathbb{T}^N)$ . Let  $\Omega_{\mathfrak{S}}$  be the completion of  $\Phi$  with respect to the  $C^*$ -module norm. Then  $\Omega_{\mathfrak{S}}$  is a Hilbert module over  $C(\mathbb{T}^N)$  such that  $\Omega_{\mathfrak{S}} \subset \mathcal{H}$ .*

**Proof.** The set  $\Phi$  is a complex vector space which can be endowed with the structure of a  $C(\mathbb{T}^N)$ -module by means of the Gel'fand isomorphism. For any  $f \in C(\mathbb{T}^N)$  and  $\varphi \in \Phi$  we define the (left) module product  $\star$  by

$$C(\mathbb{T}^N) \times \Phi \ni (f, \varphi) \longmapsto f \star \varphi := A_f \varphi \in \Phi \quad (28)$$

where  $A_f \in \mathfrak{S}$  is the operator associated with  $f \in C(\mathbb{T}^N)$ . The product is well defined since  $\Phi$  is  $\mathfrak{S}$ -invariant by construction. The Bloch-Floquet transform allows us also to endow  $\Phi$  with a pairing  $\{\cdot, \cdot\} : \Phi \times \Phi \rightarrow C(\mathbb{T}^N)$ . Indeed, for any pair  $\varphi, \phi \in \Phi$  and for all  $t \in \mathbb{T}^N$  we define a sesquilinear form

$$\Phi \times \Phi \ni (\varphi, \phi) \longmapsto \{\varphi, \phi\}(t) := ((\mathcal{U}_{\mathfrak{S}}\varphi)(t); (\mathcal{U}_{\mathfrak{S}}\phi)(t))_t \in \mathbb{C}. \quad (29)$$

Moreover  $\{\varphi, \phi\}(t)$  is a continuous function of  $t$ . Indeed  $\varphi, \phi \in \Phi$  means that  $\varphi$  and  $\phi$  are finite linear combinations of the vectors  $U^a \psi_k$  and from equation (24) and the orthonormality of the fundamental vector fields  $\zeta_k(\cdot)$  it follows that  $\{\varphi, \phi\}(t)$  consists of a finite linear combination of the exponentials  $e^{it_1}, \dots, e^{it_N}$ .

Endowed with the operations (28) and (29), the space  $\Phi$  becomes a (left) pre- $C^*$ -module over  $C(\mathbb{T}^N)$ . The Hilbert module  $\Omega_{\mathfrak{S}}$  is defined to be the completion of  $\Phi$  with respect to the norm

$$\|\varphi\|^2 := \sup_{t \in \mathbb{T}^N} \|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t^2 = \sup_{t \in \mathbb{T}^N} \left( \sum_{k \in I}^{\text{fin}} |f_{\varphi; k}(t)|^2 \right) \quad (30)$$

according to the notation in the proof of Theorem 6.4. Let  $\{\varphi_n\}_{n \in \mathbb{N}}$  be a sequence in  $\Phi$  which is Cauchy with respect to the norm  $||| \cdot |||$ . From (30), the unitarity of  $\mathcal{U}_{\mathfrak{S}}$  and the normalization of the Haar measure  $dz$  on  $\mathbb{T}^N$  it follows that  $\|\varphi_n - \varphi_m\|_{\mathcal{H}} \leq |||\varphi_n - \varphi_m|||$ , hence  $\{\varphi_n\}_{n \in \mathbb{N}}$  is also Cauchy with respect to the norm  $\|\cdot\|_{\mathcal{H}}$ , so the limit  $\varphi_n \rightarrow \varphi$  is an element of  $\mathcal{H}$ .  $\blacksquare$

Once the Hilbert module  $\Omega_{\mathfrak{S}}$  is selected, we can use it to define a continuous field of Hilbert spaces as explained in Proposition 7.6. It is easy to convince oneself that the abstract construction proposed in Proposition 7.6 is concretely implemented by the generalized Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}}$ . Then the set of vector fields  $\Gamma_{\mathfrak{S}} := \mathcal{U}_{\mathfrak{S}}(\Omega_{\mathfrak{S}})$  defines a continuous structure on the field of Hilbert spaces  $\mathfrak{F} := \prod_{t \in \mathbb{T}^N} \mathcal{K}(t)$  and, in view of Proposition 7.3, a Hilbert bundle over the base manifold  $\mathbb{T}^N$ . This Hilbert bundle, which we will denote by  $\mathcal{E}_{\mathfrak{S}}$ , is the set  $\bigsqcup_{t \in \mathbb{T}^N} \mathcal{K}(t)$  equipped by the topology prescribed by the set of the continuous sections  $\Gamma_{\mathfrak{S}}$ . The structure of  $\mathcal{E}_{\mathfrak{S}}$  depends only on the equivalence class of the physical frame  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  and we will refer to it as the *Bloch-Floquet Hilbert bundle*.

**THEOREM 7.8** (Emerging geometric structure). *Let  $\mathfrak{S}$  be a  $\mathbb{Z}^N$ -algebra in the separable Hilbert space  $\mathcal{H}$  with generators  $\{U_1, \dots, U_N\}$ , wandering system  $\{\psi_k\}_{k \in I}$  and wandering nuclear space  $\Phi$ . Let  $\mathfrak{K}$  be the direct integral defined by the Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}} : \mathcal{H} \rightarrow \mathfrak{K}$  and  $\Omega_{\mathfrak{S}} \subset \mathcal{H}$  the Hilbert module over  $C(\mathbb{T}^N)$  defined in Proposition 7.7. Then:*

- (i) *the family of vector fields  $\mathcal{U}_{\mathfrak{S}}(\Omega_{\mathfrak{S}}) =: \Gamma_{\mathfrak{S}}$  defines a continuous structure on  $\mathfrak{F} = \prod_{t \in \mathbb{T}^N} \mathcal{K}(t)$  which realizes the correspondence stated in Proposition 7.6;*
- (ii) *the Bloch-Floquet Hilbert bundle  $\mathcal{E}_{\mathfrak{S}}$ , defined by  $\Gamma_{\mathfrak{S}}$  according to Proposition 7.3, depends only on the equivalence class of the physical frame  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ .*

**Proof.** To prove (i) let  $\mathcal{I}_t := \{\varphi \in \Phi : ((\mathcal{U}_{\mathfrak{S}}\varphi)(t); (\mathcal{U}_{\mathfrak{S}}\varphi)(t))_t = 0\}$ . The space  $\Phi/\mathcal{I}_t$  is a pre-Hilbert space with respect to the scalar product induced by  $\mathcal{U}_{\mathfrak{S}}|_t$ . The map  $\mathcal{U}_{\mathfrak{S}}|_t : \Phi/\mathcal{I}_t \rightarrow \mathcal{K}(t)$  is obviously isometric and so can be extended to a linear isometry from the norm-closure of  $\Phi/\mathcal{I}_t$  into  $\mathcal{K}(t)$ . The map  $\mathcal{U}_{\mathfrak{S}}|_t$  is also surjective, indeed  $\mathcal{K}(t)$  is generated by the orthonormal basis  $\{\zeta_k(t)\}_{k \in I}$  and  $\mathcal{U}_{\mathfrak{S}}|_t^{-1}\zeta_k(t) = \psi_k \in \Phi/\mathcal{I}_t$ . Then the fiber Hilbert spaces appearing in the proof of Proposition 7.6 coincide, up to a unitary equivalence, with the fiber Hilbert spaces  $\mathcal{K}(t)$  obtained through the Bloch-Floquet decomposition. Moreover the Bloch-Floquet transform acts as the map defined in the proof of Proposition 7.6, which sends any element of the Hilbert module  $\Phi$  to a continuous section of  $\mathfrak{F}$ .

To prove (ii) let  $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$  and  $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$  be two physical frames related by a unitary map  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ . If  $\mathfrak{S}_1$  is a  $\mathbb{Z}^N$ -algebra in  $\mathcal{H}_1$  then also  $\mathfrak{S}_2 = U\mathfrak{S}_1U^{-1}$  is a  $\mathbb{Z}^N$ -algebra in  $\mathcal{H}_2$  and if  $\{\psi_k\}_{k \in I} \subset \mathcal{H}_1$  is a wandering system for  $\mathfrak{S}_1$  then  $\{\tilde{\psi}_k := U\psi_k\}_{k \in I} \subset \mathcal{H}_2$  is a wandering system for  $\mathfrak{S}_2$  (with the same cardinality). The two wandering nuclear spaces  $\Phi_1 \subset \mathcal{H}_1$  and  $\Phi_2 \subset \mathcal{H}_2$  are related by  $\Phi_2 = U\Phi_1$ . Let  $\mathcal{U}_{\mathfrak{S}_1} : \mathcal{H}_1 \rightarrow \mathfrak{H}_1$  and  $\mathcal{U}_{\mathfrak{S}_2} : \mathcal{H}_2 \rightarrow \mathfrak{H}_2$  be the two generalized Bloch-Floquet transforms defined by the two equivalent physical frames. From the explicit expression of  $\mathcal{U}_{\mathfrak{S}_2}$  and  $\mathcal{U}_{\mathfrak{S}_1}^{-1}$ , and in accordance with Corollary 3.4, one argues that  $\mathcal{U}_{\mathfrak{S}_2} \circ U \circ \mathcal{U}_{\mathfrak{S}_1}^{-1} =: W(\cdot)$  is a decomposable unitary which is well defined for all

t. Let  $\varphi, \phi \in \Phi_1$  then

$$\begin{aligned} \{\varphi; \phi\}_1(t) &:= ((\mathcal{U}_{\mathfrak{S}_1}\varphi)(t); (\mathcal{U}_{\mathfrak{S}_1}\phi)(t))_t = (W(t)(\mathcal{U}_{\mathfrak{S}_1}\varphi)(t); W(t)(\mathcal{U}_{\mathfrak{S}_1}\phi)(t))_t \\ &= ((\mathcal{U}_{\mathfrak{S}_2}U\varphi)(t); (\mathcal{U}_{\mathfrak{S}_2}U\phi)(t))_t = \left( (\mathcal{U}_{\mathfrak{S}_2}\tilde{\varphi})(t); (\mathcal{U}_{\mathfrak{S}_2}\tilde{\phi})(t) \right)_t =: \{\tilde{\varphi}; \tilde{\phi}\}_2(t) \end{aligned}$$

where  $\tilde{\varphi} := U\varphi$  and  $\tilde{\phi} := U\phi$  are elements of  $\Phi_2$ . This equation shows that  $\Phi_1$  and  $\Phi_2$  have the same  $C(\mathbb{T}^N)$ -module structure and so define the same abstract Hilbert module over  $C(\mathbb{T}^N)$ . The claim follows from the generalization of the Serre-Swan Theorem summarized by arrow C in (27).  $\blacksquare$

**REMARK 7.9.** With a proof similar to that of point (ii) of Theorem 7.8, one deduces also that the Bloch-Floquet-Hilbert bundle  $\mathcal{E}_{\mathfrak{S}}$  does not depend on the choice of two unitarily (or antiunitarily) equivalent commutative  $C^*$ -algebras  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$  inside  $\mathfrak{A}'$ . Indeed, also in this case the abstract Hilbert module structure induced by the two Bloch-Floquet transforms  $\mathcal{U}_{\mathfrak{S}_1}$  and  $\mathcal{U}_{\mathfrak{S}_2}$  is the same.  $\blacklozenge$

After defining the topology of the Bloch-Floquet Hilbert bundle, it is natural to deduce more information about its structure. An interesting property arises from the cardinality of the wandering system, which depends only on the physical frame (see Corollary 6.5).

**COROLLARY 7.10.** *The Hilbert bundle  $\mathcal{E}_{\mathfrak{S}}$  over the torus  $\mathbb{T}^N$  defined by the continuous structure  $\Gamma_{\mathfrak{S}}$  is trivial if the cardinality of the wandering system is  $\aleph_0$ , and is a rank- $q$  Hermitian vector bundle if the cardinality of the wandering system is  $q$ . In the latter case the transition functions of the vector bundle can be expressed in terms of the fundamental orthonormal frame  $\{\zeta_k(\cdot) := (\mathcal{U}_{\mathfrak{S}}\psi_k)\}_{k=1,\dots,q}$ .*

## Decomposition of the observables and endomorphism sections

According to Theorem 6.4, the Bloch-Floquet transform (16) provides a concrete realization for the unitary map ( $\mathfrak{S}$ -Fourier transform) whose existence is claimed by von Neumann's complete spectral theorem. Point (ii) of Theorem 3.1 implies that under the Bloch-Floquet transform any  $O \in \mathfrak{A}$  is mapped into a decomposable operator on the direct integral  $\int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t) dz(t)$ , i.e.  $\mathcal{U}_{\mathfrak{S}}O\mathcal{U}_{\mathfrak{S}}^{-1} =: O(\cdot) : t \mapsto O(t) \in \mathcal{B}(\mathcal{K}(t))$  with  $O(\cdot)$  weakly measurable.

The natural question which arises is the following: there exists any topological structure in the  $C^*$ -algebra  $\mathfrak{A}$  compatible with the Bloch-Floquet Hilbert bundle which emerges from the Bloch-Floquet transform? In order to answer this question, we first analyze the nature of the linear maps which preserve the (Hilbert module) structure of the set of the continuous sections.

**DEFINITION 7.11** (Hilbert module endomorphism). *Let  $\Omega$  be a (left) Hilbert module over the commutative unital  $C^*$ -algebra  $\mathcal{A}$ . An endomorphism of  $\Omega$  is an  $\mathcal{A}$ -linear map  $O : \Omega \rightarrow \Omega$  which is adjointable, i.e. there exists a map  $O^\dagger : \Omega \rightarrow \Omega$  such that  $\{\sigma; O\rho\} = \{O^\dagger\sigma; \rho\}$  for all  $\sigma, \rho \in \Omega$ . We denote by  $\text{End}_{\mathcal{A}}(\Omega)$  the set of all endomorphisms of  $\Omega$ .*

As proven in [GVF01] (Section 2.5) or [Lan97] (Appendix A), if  $O \in \text{End}_{\mathcal{A}}(\Omega)$ , then  $O^\dagger \in \text{End}_{\mathcal{A}}(\Omega)$  and  $\dagger$  is an involution over  $\text{End}_{\mathcal{A}}(\Omega)$ . Moreover,  $\text{End}_{\mathcal{A}}(\Omega)$  endowed with the *endomorphism norm*

$$\|O\|_{\text{End}(\Omega)} := \sup\{\|O(\sigma)\| : \|\sigma\| \leq 1\} \quad (31)$$

becomes a  $C^*$ -algebra (of bounded operators). For any  $\sigma, \rho \in \Omega$  one defines the *rank-1 endomorphism*  $|\sigma\rangle\langle\rho| \in \text{End}_{\mathcal{A}}(\Omega)$  by  $|\sigma\rangle\langle\rho|(\zeta) := \{\rho; \zeta\} \sigma$  for all  $\zeta \in \Omega$ . The adjoint of  $|\sigma\rangle\langle\rho|$  is given by  $|\rho\rangle\langle\sigma|$ . The linear span of the rank-1 endomorphisms is a selfadjoint two-sided ideal of  $\text{End}_{\mathcal{A}}(\Omega)$  (*finite rank endomorphisms*) and its (operator) norm closure is denoted by  $\text{End}_{\mathcal{A}}^0(\Omega)$ . The elements of the latter are called *compact endomorphisms* of  $\Omega$ . Since  $\text{End}_{\mathcal{A}}^0(\Omega)$  is an *essential ideal* of  $\text{End}_{\mathcal{A}}(\Omega)$ , it follows that  $\text{End}_{\mathcal{A}}^0(\Omega) = \text{End}_{\mathcal{A}}(\Omega)$  if and only if  $\mathbf{1}_\Omega \in \text{End}_{\mathcal{A}}^0(\Omega)$ .

A remarkable result which emerges from the above theory is the characterization of the compact endomorphisms of the  $C(X)$  Hilbert module  $\Gamma(\mathcal{E})$  of the continuous sections of a rank- $q$  Hermitian vector bundle.

**PROPOSITION 7.12.** *Let  $\mathcal{E} \rightarrow X$  be a rank- $q$  Hermitian vector bundle over the compact Hausdorff space  $X$  and let  $\Gamma(\mathcal{E})$  be the Hilbert module over  $C(X)$  of its continuous sections. Then*

$$\text{End}_{C(X)}^0(\Gamma(\mathcal{E})) = \text{End}_{C(X)}(\Gamma(\mathcal{E})) \simeq \Gamma(\text{End}(\mathcal{E})) \quad (32)$$

where  $\Gamma(\text{End}(\mathcal{E}))$  denotes the continuous sections of the vector bundle  $\text{End}(\mathcal{E}) \rightarrow X$ . The localization isomorphism appearing in right-hand side of (32) preserves the composition and the structure of  $C(X)$ -module.

The proof is a consequence of the Serre-Swan Theorem (see [GVF01], Theorems 2.10 and 3.8) and of Proposition 3.2 in [GVF01].

In Proposition 7.7 we proved that the Gel'fand isomorphism and the Bloch-Floquet transform equip the wandering nuclear space  $\Phi$  with the structure of a (left) pre- $C^*$ -module over  $C(\mathbb{T}^N)$  by means of the (left) product  $\star$  defined by (28) and the pairing  $\{ ; \}$  defined by (29). The closure of  $\Phi$  with respect to the module norm defines a Hilbert module over  $C(\mathbb{T}^N)$  denoted by  $\Omega_{\mathfrak{S}} \subset \mathcal{H}$ . In this description, what is the role played by  $\mathfrak{A}$ ? Is it possible, at least under some condition, to interpret the elements of  $\mathfrak{A}$  as endomorphism of the Hilbert module  $\Omega_{\mathfrak{S}}$ ? One could try to answer these questions by observing that for any  $O \in \mathfrak{A}$ , any  $A_f \in \mathfrak{S}$  and any  $\varphi \in \Omega_{\mathfrak{S}}$  one has that  $O(f \star \varphi) := OA_f\varphi = A_fO\varphi$ . The latter might be interpreted as  $f \star O(\varphi)$ , implying the  $C(\mathbb{T}^N)$ -linearity of  $O \in \mathfrak{A}$  as operator on  $\Omega_{\mathfrak{S}}$ . However it may happen that  $O\varphi \notin \Omega_{\mathfrak{S}}$  which implies that  $O$  can not define an endomorphism of  $\Omega_{\mathfrak{S}}$ . Everything works properly if one considers only elements in the subalgebra  $\mathfrak{A}^0 \subset \mathfrak{A}$  defined by

$$\mathfrak{A}^0 := \{O \in \mathfrak{A} : O : \Omega_{\mathfrak{S}} \rightarrow \Omega_{\mathfrak{S}}\}. \quad (33)$$

**PROPOSITION 7.13.** *Let  $\Omega_{\mathfrak{S}}$  be the Hilbert module over  $C(\mathbb{T}^N)$  defined by means of the Bloch-Floquet transform according to Proposition 7.7. Let  $\mathfrak{A}_{\text{s.a.}}^0$  be the  $C^*$ -subalgebra of  $\mathfrak{A}$  defined by  $\mathfrak{A}_{\text{s.a.}}^0 := \{O \in \mathfrak{A} : O, O^\dagger \in \mathfrak{A}^0\}$  (self-adjoint part of  $\mathfrak{A}^0$ ). Then  $\mathfrak{A}_{\text{s.a.}}^0 \subset \text{End}_{C(\mathbb{T}^N)}(\Omega_{\mathfrak{S}})$ .*

**Proof.** Let  $O \in \mathfrak{A}_{\text{s.a.}}^0$ . By definition  $O$  is a linear map from  $\Omega_{\mathfrak{S}}$  to itself; it is also  $C(\mathbb{T}^N)$ -linear since  $O(f \star \varphi) = OA_f\varphi = A_fO\varphi$  as mentioned. We need to prove that  $O$  is bounded with respect to the endomorphism norm (31). From the definition (30) of the module norm  $||| \cdot |||$  it follows that

$$|||O\varphi||| = \sup_{t \in \mathbb{T}^N} \|(\mathcal{U}_{\mathfrak{S}}O\varphi)(t)\|_t = \sup_{t \in \mathbb{T}^N} \|\pi_t(O)(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t \leq \|O\|_{\mathcal{B}(\mathcal{H})} |||\varphi|||$$

where  $\pi_t(O) := \mathcal{U}_{\mathfrak{S}}|_t O \mathcal{U}_{\mathfrak{S}}|_t^{-1}$  defines a representation of the  $C^*$ -algebra  $\mathfrak{A}$  on the fiber Hilbert space  $\mathcal{K}(t)$  and  $\|\pi_t(O)\|_{\mathcal{B}(\mathcal{K}(t))} \leq \|O\|_{\mathcal{B}(\mathcal{H})}$  since any  $C^*$  representation decreases the norm. Thus  $\|O\|_{\text{End}(\Omega_{\mathfrak{S}})} \leq \|O\|_{\mathcal{B}(\mathcal{H})}$ , therefore  $O$  defines a continuous  $C(\mathbb{T}^N)$ -linear map from  $\Omega_{\mathfrak{S}}$  to itself. To prove that  $O \in \text{End}_{C(\mathbb{T}^N)}(\Omega_{\mathfrak{S}})$  we must show that  $O$  is adjointable, which follows from the definition of  $\mathfrak{A}_{\text{s.a.}}^0$ .  $\blacksquare$

It is of particular interest to specialize the previous result to the case of a finite wandering system.

**THEOREM 7.14** (Bloch-Floquet endomorphism bundle). *Let  $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$  be a physical frame where  $\mathfrak{S}$  is a  $\mathbb{Z}^N$ -algebra with generators  $\{U_1, \dots, U_N\}$  and wandering system  $\{\psi_1, \dots, \psi_q\}$  of finite cardinality. Then:*

- (i)  $\mathfrak{A}_{\text{s.a.}}^0 = \mathfrak{A}^0$ ;
- (ii)  $\mathcal{U}_{\mathfrak{S}} \mathfrak{A}^0 \mathcal{U}_{\mathfrak{S}}^{-1} \subseteq \Gamma(\text{End}(\mathcal{E}_{\mathfrak{S}}))$  where  $\mathcal{E}_{\mathfrak{S}} \rightarrow \mathbb{T}^N$  is the rank  $q$  Bloch-Floquet vector bundle defined in Corollary 7.10.

**Proof.** To prove (i) let  $O \in \mathfrak{A}^0$  and observe that if  $O\psi_k = \sum_{h=1}^q \sum_{b \in \mathbb{Z}^N} \alpha_{h,b}^{(k)} U^b \psi_h$  then  $O^\dagger \psi_k = \sum_{h=1}^q \sum_{b \in \mathbb{Z}^N} \bar{\alpha}_{k,b}^{(h)} U^{-b} \psi_h$ . Since  $O\psi_k \in \Omega_{\mathfrak{S}}$ , then  $f_h^{(k)}(t) := \sum_{b \in \mathbb{Z}^N} \alpha_{h,b}^{(k)} z^b(t)$  is a continuous function on  $\mathbb{T}^N$  and

$$|||O^\dagger \psi_k|||^2 = \sup_{t \in \mathbb{T}^N} \left( \sum_{h=1}^q |f_h^{(k)}(t)|^2 \right) < +\infty.$$

Then  $O^\dagger \psi_k \in \Omega_{\mathfrak{S}}$  for all  $k = 1, \dots, q$ . Since  $O^\dagger(U^b \psi_k) = U^b(O^\dagger \psi_k) \in \Omega_{\mathfrak{S}}$  for all  $b \in \mathbb{Z}^N$  it follows that also  $O^\dagger \in \mathfrak{A}^0$ . Point (ii) is an immediate consequence of Proposition 7.13, Corollary 7.10 and Proposition 7.12.  $\blacksquare$

**EXAMPLE 7.15** (*Mathieu-like Hamiltonians, part four*). It is immediate to check that both  $\mathbf{u}$  and  $\mathbf{v}$  preserve the wandering nuclear space  $\Phi$ , so that the full  $C^*$ -algebra  $\mathfrak{A}_{\mathbb{M}}^{p/q}$  consists of endomorphisms for the Hilbert module realized by means of the Bloch-Floquet transform  $\mathcal{U}_{\mathfrak{S}_{\mathbb{M}}}^q$ . Theorem 7.14 claims that  $\mathcal{U}_{\mathfrak{S}_{\mathbb{M}}}^q$  maps  $\mathfrak{A}_{\mathbb{M}}^{p/q}$  into a subalgebra of the endomorphisms of the trivial bundle  $\mathbb{T} \times \mathbb{C}^q \rightarrow \mathbb{T}$ . The matrices  $\mathbf{u}(t)$  and  $\mathbf{v}(t)$  in Example 6.7 define the representation of the generators as elements of  $\Gamma(\text{End}(\mathbb{T} \times \mathbb{C}^q)) \simeq C(\mathbb{T}) \otimes \text{Mat}_q(\mathbb{C})$ .  $\blacktriangleleft$



## A Gel'fand theory, joint spectrum and basic measures

Let  $\mathfrak{A}$  be a unital (not necessarily commutative)  $C^*$ -algebra and  $\mathfrak{A}^\times$  the group of the invertible elements of  $\mathfrak{A}$ . The algebraic spectrum of  $A \in \mathfrak{A}$  is defined to be  $\sigma_{\mathfrak{A}}(A) := \{\lambda \in \mathbb{C} : (A - \lambda\mathbf{1}) \notin \mathfrak{A}^\times\}$ . If  $\mathfrak{A}_0$  is a non unital  $C^*$ -algebra and  $\iota : \mathfrak{A}_0 \hookrightarrow \mathfrak{A}$  is the canonical embedding of  $\mathfrak{A}_0$  in the unital  $C^*$ -algebra  $\mathfrak{A}$  (see [BR87] Proposition 2.1.5) then one defines  $\sigma_{\mathfrak{A}_0}(A) := \sigma_{\mathfrak{A}}(\iota(A))$  for all  $A \in \mathfrak{A}_0$ . If  $\mathfrak{A}$  is unital and  $C^*(A) \subset \mathfrak{A}$  is the unital  $C^*$ -subalgebra generated algebraically by  $A$ , its adjoint  $A^\dagger$  and  $\mathbf{1}$  ( $=: A^0$  for definition) then  $\sigma_{\mathfrak{A}}(A) = \sigma_{C^*(A)}(A)$  (see [BR87] Proposition 2.2.7). As a consequence we have that if  $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$  is a concrete  $C^*$ -algebra of operators on the Hilbert space  $\mathcal{H}$  and  $A \in \mathfrak{A}$  then the algebraic spectrum  $\sigma_{\mathfrak{A}}(A)$  agrees with the Hilbert space spectrum  $\sigma(A) := \{\lambda \in \mathbb{C} : (A - \lambda\mathbf{1}) \notin \text{GL}(\mathcal{H})\}$  where  $\text{GL}(\mathcal{H}) := \mathcal{B}(\mathcal{H})^\times$  is the group of the invertible bounded linear operators on the Hilbert space  $\mathcal{H}$ .

Let us denote by  $\mathfrak{S}$  a commutative  $C^*$ -algebra. A character of  $\mathfrak{S}$  is a nonzero homomorphism  $x : \mathfrak{S} \rightarrow \mathbb{C}$  (also called pure state). The Gel'fand spectrum of  $\mathfrak{S}$ , denoted by  $X(\mathfrak{S})$  or simply by  $X$ , is the set of all characters of  $\mathfrak{S}$ . The space  $X$ , endowed with the weak- $*$  topology (topology of the pointwise convergence on  $\mathfrak{S}$ ) becomes a topological Hausdorff space, which is compact if  $\mathfrak{S}$  is unital and only locally compact otherwise (see [BR87] Theorem 2.1.11A). If  $\mathfrak{S}$  is separable (namely it is generated algebraically by a countable family of commuting elements) then the weak- $*$  topology in  $X$  is metrizable (see [Br687] Theorem III.25) and if, in addition,  $\mathfrak{S}$  is also unital then  $X$  is compact and metrizable which implies (see [Cho66] Proposition 18.3 and Theorem 20.9) that  $X$  is second-countable (has a countable basis of open sets), separable (has a countable everywhere dense subset) and complete. Summarizing, the Gel'fand spectrum of a commutative separable unital  $C^*$ -algebra has the structure of a Polish space (separable complete metric space).

The Gel'fand-Naïmark Theorem (see [BR87] Section 2.3.5 or [GVF01] Section 1.2 or [Lan97] Section 2.2) states that there is a canonical isomorphism between any commutative unital  $C^*$ -algebra  $\mathfrak{S}$  and the commutative  $C^*$ -algebra  $C(X)$  of the continuous complex valued functions on its spectrum endowed with the norm of the uniform convergence. The Gel'fand isomorphism  $C(X) \ni f \xrightarrow{\mathcal{G}} A_f \in \mathfrak{S}$  maps any continuous  $f$  into the unique element  $A_f$  which satisfies the relation  $f(x) = x(A_f)$  for all  $x \in X$ . Then we can use the continuous functions on  $X$  to “label” the elements of  $\mathfrak{S}$ . If  $\mathfrak{S}_0$  is a non-unital commutative  $C^*$ -algebra then the Gel'fand-Naïmark Theorem proves the isomorphism between  $\mathfrak{S}_0$  and the commutative  $C^*$ -algebra  $C_0(X_0)$  of the continuous complex valued functions vanishing at infinity on the locally compact space  $X_0$  which is the spectrum of  $\mathfrak{S}_0$ . If  $\mathfrak{S}_0 \subset \mathcal{B}(\mathcal{H})$  we define the multiplier algebra (or idealizer) of  $\mathfrak{S}_0$  to be  $\mathfrak{S} := \{B \in \mathcal{B}(\mathcal{H}) : BA, AB \in \mathfrak{S}_0 \ \forall A \in \mathfrak{S}_0\}$  (see [GVF01] Definition 1.8 and Lemma 1.9). Obviously  $\mathfrak{S}$  is a unital  $C^*$ -algebra and the commutativity of  $\mathfrak{S}_0$  implies the commutativity of  $\mathfrak{S}$ . Moreover  $\mathfrak{S}$  contains  $\mathfrak{S}_0$  as an essential ideal. The Gel'fand spectrum  $X$  of  $\mathfrak{S}$  corresponds to the Stone-Ćech compactification of the spectrum  $X_0$ . Since  $C(X) \simeq C_b(X_0)$ , the Gel'fand isomorphism asserts that the multiplier algebra  $\mathfrak{S}$  can be described as the unital commutative  $C^*$ -algebra of the bounded continuous functions on the locally compact space  $X_0$  (see [GVF01] Proposition 1.10). For every  $A_f \in \mathfrak{S}$  one has that  $\sigma_{\mathfrak{S}}(A_f) = \{f(x) : x \in X\}$  (see [Hör90] Theorem 3.1.6) then  $A_f$  is invertible if and only if  $0 < |f(x)| \leq \|A_f\|_{\mathfrak{S}}$  for all  $x \in X$ .

We often consider the relevant case when the unital commutative  $C^*$ -algebra is *finitely generated*, i.e. when  $\mathfrak{S}$  is generated by a finite family  $\{A_1, \dots, A_N\}$  of commuting normal elements and the identity  $\mathbf{1}$  ( $=: A_j^0$  by definition). Let  $f_1, \dots, f_N$  be the continuous functions which label the elements of the generating system. The map  $X \ni x \xrightarrow{\varpi} (f_1(x), \dots, f_N(x)) \in \mathbb{C}^N$  is a homeomorphism from the Gel'fand spectrum  $X$  to a compact subset of  $\mathbb{C}^N$  called the *joint spectrum* of the generating system  $\{A_1, \dots, A_N\}$  (see [Hör90] Theorem 3.1.15). Then, when  $\mathfrak{S}$  is finitely generated, we can identify the Gel'fand spectrum  $X$  with its homeomorphic image  $\varpi(X)$  (the joint spectrum) which is a compact, generally proper, subset of  $\sigma_{\mathfrak{S}}(A_1) \times \dots \times \sigma_{\mathfrak{S}}(A_N)$ . When  $\{A_1, \dots, A_N\} \subset \mathcal{B}(\mathcal{H})$  a necessary and sufficient condition for  $\lambda := (\lambda_1, \dots, \lambda_N)$  to be in  $\varpi(X)$  is that there exists a sequence of normalized vectors  $\{\psi_n\}_{n \in \mathbb{N}}$  such that  $\|(A_j - \lambda_j)\psi_n\| \rightarrow 0$  if  $n \rightarrow \infty$  for all  $j = 1, \dots, N$  (see [Sam91] Proposition 2).

**REMARK A.1** (*Dual group*). The Gel'fand theory has an interesting application to abelian locally compact groups  $\mathbb{G}$ . Usually the *dual group* (or character group)  $\widehat{\mathbb{G}}$  is defined to be the set of all continuous characters of  $\mathbb{G}$ , namely the set of all continuous homomorphisms of  $\mathbb{G}$  into the group  $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ . However, to endow  $\widehat{\mathbb{G}}$  with a natural topology it is useful to give an equivalent definition of dual group. Since  $\mathbb{G}$  is locally compact and abelian there exists a unique (up to a multiplicative constant) invariant *Haar measure* on  $\mathbb{G}$  denoted by  $dg$ . The space  $L^1(\mathbb{G})$  becomes a commutative Banach  $*$ -algebra, if multiplication is defined by convolution; it is called the *group algebra* of  $\mathbb{G}$ . If  $\mathbb{G}$  is discrete then  $L^1(\mathbb{G})$  is unital otherwise  $L^1(\mathbb{G})$  has always an *approximate unit* (see [Rud62] Theorems 1.1.7 and 1.1.8). Every  $\chi \in \widehat{\mathbb{G}}$  defines a linear multiplicative functional  $\widehat{\chi}$  on  $L^1(\mathbb{G})$  by  $\widehat{\chi}(f) := \int_{\mathbb{G}} f(g)\chi(-g) d\mu(g)$  for all  $f \in L^1(\mathbb{G})$  (the Fourier transform). This map defines a one to one correspondence between  $\widehat{\mathbb{G}}$  and the Gel'fand spectrum of the algebra  $L^1(\mathbb{G})$  (see [Rud62] Theorem 1.2.2). This enables us to consider  $\widehat{\mathbb{G}}$  as the Gel'fand spectrum of  $L^1(\mathbb{G})$ . When  $\widehat{\mathbb{G}}$  is endowed with the weak- $*$  topology with respect to  $L^1(\mathbb{G})$  then it becomes a Hausdorff locally compact space. Moreover  $\widehat{\mathbb{G}}$  is compact if  $\mathbb{G}$  is discrete and it is discrete when  $\mathbb{G}$  is compact (see [Rud62] Theorem 1.2.5).  $\blacklozenge$

Let  $X$  be a compact Polish space and  $\mathcal{B}(X)$  the Borel  $\sigma$ -algebra generated by the topology of  $X$ . The pair  $\{X, \mathcal{B}(X)\}$  is called *standard Borel space*. A mapping  $\mu : \mathcal{B}(X) \rightarrow [0, +\infty]$  such that  $\mu(\emptyset) = 0$  and  $\mu(X) < \infty$ , which is additive with respect to the union of countable families of pairwise disjoint subsets of  $X$ , is called a *finite Borel measure*. If  $\mu(X) = 1$  then we will say that  $\mu$  is a *probability Borel measure*. Any Borel measure on a standard Borel space  $\{X, \mathcal{B}(X)\}$  is *regular*, i.e. for all  $Y \in \mathcal{B}(X)$  one has that  $\mu(Y) = \sup\{\mu(K) : K \subset Y, K \text{ compact}\} = \inf\{\mu(O) : Y \subset O, O \text{ open}\}$ .

Let  $N$  be the union of all the open sets  $O_\alpha \subset X$  such that  $\mu(O_\alpha) = 0$ . The closed set  $X \setminus N$  is called the *support* of  $\mu$ . If  $\mu$  is a regular Borel measure then  $\mu(N) = 0$  and  $\mu$  is concentrated on its support.

Let  $\mathfrak{S}$  be a unital commutative  $C^*$ -algebra acting on the separable Hilbert space  $\mathcal{H}$  with Gel'fand spectrum  $X$ . For all pairs  $\psi, \varphi \in \mathcal{H}$  the mapping  $C(X) \ni f \mapsto (\psi; A_f \varphi)_{\mathcal{H}} \in \mathbb{C}$  is a continuous linear functional on  $C(X)$ ; hence the *Riesz-Markov Theorem* (see [Rud87] Theorem 2.14) implies the existence of a unique regular (complex) Borel measure  $\mu_{\psi, \varphi}$ , with finite total variation, such that  $(\psi; A_f \varphi)_{\mathcal{H}} = \int_X f(x) d\mu_{\psi, \varphi}(x)$  for all  $f \in C(X)$ . We will

refer to  $\mu_{\psi,\varphi}$  as a *spectral measure*. The union of the supports of the (positive) spectral measures  $\mu_{\psi,\psi}$  is dense, namely for every non-void open set  $O \subset X$  there exists a  $\psi \in \mathcal{H}$  such that  $\mu_{\psi,\psi}(O) > 0$ . A positive measure  $\mu$  on  $X$  is said to be *basic* for the unital  $C^*$ -algebra  $\mathfrak{S}$  if: for every  $Y \subset X$ ,  $\mu(Y) = 0$  if and only if  $\mu_{\psi,\psi}(Y) = 0$  for every  $\psi \in \mathcal{H}$ . From the definition it follows that: (i) if there exists a basic measure  $\mu$  on  $X$ , then every other basic measure is *equivalent* (has the same null sets) to  $\mu$ ; (ii) for all  $\psi, \varphi \in \mathcal{H}$  the spectral measure  $\mu_{\psi,\varphi}$  is *absolutely continuous* with respect to  $\mu$ , and there exists a unique element  $h_{\psi,\varphi} \in L^1(X)$  (the *Radon-Nikodym derivative*) such that  $\mu_{\psi,\varphi} = h_{\psi,\varphi}\mu$ ; (iii) since the union of the supports of the measures  $\mu_{\psi,\psi}$  is dense in  $X$ , then the support of a basic measure  $\mu$  is all of  $X$  (see [Dix81] Part I, Chapter 7). The existence of a basic measure for a commutative  $C^*$ -algebra  $\mathfrak{S} \subset \mathcal{B}(\mathcal{H})$  follows from general arguments. Indeed the existence of a basic measure is equivalent to the existence of a cyclic vector  $\phi$  for the commutant  $\mathfrak{S}'$  and the basic measure can be chosen to be the spectral measure  $\mu_{\phi,\phi}$  (see [Dix81] Part I, Chapter 7, Proposition 3). Since a vector  $\phi$  is cyclic for  $\mathfrak{S}'$  if and only if it is separating for the commutative von Neumann algebra  $\mathfrak{S}'' \supset \mathfrak{S}$ , and since any commutative von Neumann algebra of operators on a separable Hilbert space has a separating vector, it follows that any commutative unital  $C^*$ -algebra  $\mathfrak{S}$  of operators which acts on a separable Hilbert space has a basic measure carried on its spectrum (see [Dix81] Part I, Chapter 7, Propositions 4).

## B Direct integral of Hilbert spaces

General references about the notion of a direct integral of Hilbert spaces can be found in [Dix81] (Part II, Chapters 1-5) or in [Mau68] (Chapter I, Section 6). In the following we assume that the pair  $\{X, \mathcal{B}(X)\}$  is a standard Borel space and  $\mu$  a (regular) Borel measure on  $X$ . For every  $x \in X$  let  $\mathcal{H}(x)$  be a Hilbert space with scalar product  $(\ ; \ )_x$ . The set  $\mathfrak{F} := \prod_{x \in X} \mathcal{H}(x)$  (Cartesian product) is called a *field of Hilbert spaces* over  $X$ . A *vector field*  $\varphi(\cdot)$  is an element of  $\mathfrak{F}$ , namely a map  $X \ni x \mapsto \varphi(x) \in \mathcal{H}(x)$ . A countable family  $\{\xi_j(\cdot) : j \in \mathbb{N}\}$  of vector fields is called a *fundamental family of measurable vector fields* if:

- a) for all  $i, j \in \mathbb{N}$  the functions  $X \ni x \mapsto (\xi_i(x); \xi_j(x))_x \in \mathbb{C}$  are measurable;
- b) for each  $x \in X$  the set  $\{\xi_j(x) : j \in \mathbb{N}\}$  spans the space  $\mathcal{H}(x)$ .

The field  $\mathfrak{F}$  has a *measurable structure* if it has a fundamental family of measurable vector fields. A vector field  $\varphi(\cdot) \in \mathfrak{F}$  is said to be *measurable* if all the functions  $X \ni x \mapsto (\xi_j(x); \varphi(x))_x \in \mathbb{C}$  are measurable for all  $j \in \mathbb{N}$ . The set of all measurable vector fields is a linear subspace of  $\mathfrak{F}$ . By the Gram-Schmidt orthonormalization we can always build a fundamental family of orthonormal measurable fields (see [Dix81] Part II, Chapter 1, Propositions 1 and 4). Such a family is called a *measurable field of orthonormal frames*. Two fields are said to be equivalent if they are equal  $\mu$ -almost everywhere on  $X$ . The *direct integral*  $\mathfrak{H}$  of the Hilbert spaces  $\mathcal{H}(x)$  (subordinate to the measurable structure of  $\mathfrak{F}$ ), is the Hilbert space of the equivalence classes of measurable vector fields  $\varphi(\cdot) \in \mathfrak{F}$  satisfying

$$\|\varphi(\cdot)\|_{\mathfrak{H}}^2 := \int_X \|\varphi(x)\|_x^2 d\mu(x) < \infty. \quad (34)$$

The scalar product on  $\mathfrak{H}$  is defined by

$$\langle \varphi_1(\cdot); \varphi_2(\cdot) \rangle_{\mathfrak{H}} := \int_X (\varphi_1(x); \varphi_2(x))_x d\mu(x) < \infty. \quad (35)$$

The Hilbert space  $\mathfrak{H}$  is often denoted by the symbol  $\int_X^{\oplus} \mathcal{H}(x) d\mu(x)$ . It is separable if  $X$  is separable.

Let  $\nu$  be a positive measure equivalent to  $\mu$ . The Radon-Nikodym theorem ensures the existence of a positive  $\rho \in L^1(X, \mu)$  with  $\frac{1}{\rho} \in L^1(X, \nu)$  such that  $\nu = \rho\mu$ . Let  $\mathfrak{H}$  be the direct integral with respect to  $\mu$ ,  $\mathfrak{K}$  the direct integral with respect to  $\nu$  and  $\varphi(\cdot) \in \mathfrak{H}$ . The mapping  $\mathfrak{H} \ni \varphi(\cdot) \mapsto \varphi'(\cdot) \in \mathfrak{K}$  defined by  $\varphi'(x) = \frac{1}{\sqrt{\rho(x)}}\varphi(x)$  for all  $x \in X$  is an unitary map of  $\mathfrak{H}$  onto  $\mathfrak{K}$  and for fixed  $\mu$  and  $\nu$ . This isomorphism does not depend on the choice of the representative for  $\rho$  and it is called the *canonical rescaling isomorphism*.

A (bounded) operator field  $A(\cdot)$  is a map  $X \ni x \mapsto A(x) \in \mathcal{B}(\mathcal{H}(x))$ . It is called measurable if the function  $X \ni x \mapsto (\xi_i(x); A(x)\xi_j(x))_x \in \mathbb{C}$  is measurable for all  $i, j \in \mathbb{N}$ . A measurable operator field is called a *decomposable operator* in the Hilbert space  $\mathfrak{H}$ . Let  $f \in L^\infty(X)$  (with respect to the measure  $\mu$ ); then the map  $X \ni x \mapsto M_f(x) := f(x)\mathbf{1}_x \in \mathcal{B}(\mathcal{H}(x))$  (with  $\mathbf{1}_x$  the identity in  $\mathcal{H}(x)$ ) defines a simple example of decomposable operator called *diagonal operator*. When  $f \in C(X)$ , the diagonal operator  $M_f(\cdot)$  is called a *continuously diagonal operator*. Denote by  $C(\mathfrak{H})$  (resp.  $L^\infty(\mathfrak{H})$ ) the set of the continuously diagonal operators (resp. the set of diagonal operators) on  $\mathfrak{H}$ . Suppose that  $\mathcal{H}(x) \neq 0$   $\mu$ -almost everywhere on  $X$ , then the following facts hold true (see [Dix81] Part II, Chapter 2, Section 4): (i)  $L^\infty(\mathfrak{H})$  is a commutative von Neumann algebra and the mapping  $L^\infty(X) \ni f \mapsto M_f(\cdot) \in L^\infty(\mathfrak{H})$  is a (canonical) isomorphism of von Neumann algebras; (ii) the commutant  $L^\infty(\mathfrak{H})'$  is the von Neumann algebra of decomposable operators on  $\mathfrak{H}$ ; (iii) the mapping  $C(X) \ni f \mapsto M_f(\cdot) \in C(\mathfrak{H})$  is a (canonical) homomorphism of  $C^*$ -algebras which becomes an isomorphism if the support of  $\mu$  is all  $X$ ; in this case  $X$  is the Gel'fand spectrum of  $C(\mathfrak{H})$  and  $\mu$  is a basic measure.

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