



HAL
open science

On the convergence to equilibrium for degenerate transport problems

Etienne Bernard, Francesco Salvarani

► **To cite this version:**

Etienne Bernard, Francesco Salvarani. On the convergence to equilibrium for degenerate transport problems. 2012. hal-00674093v2

HAL Id: hal-00674093

<https://hal.science/hal-00674093v2>

Preprint submitted on 9 Mar 2012 (v2), last revised 18 May 2012 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE CONVERGENCE TO EQUILIBRIUM FOR DEGENERATE TRANSPORT PROBLEMS

ÉTIENNE BERNARD AND FRANCESCO SALVARANI

ABSTRACT. We give a counterexample which shows that the asymptotic rate of convergence to the equilibrium state for the transport equation, with a degenerate cross section and in the periodic setting, cannot be better than $t^{-1/2}$ in the general case. We suggest moreover that the geometrical properties of the cross section are the key feature of the problem and impose, through the distribution of the forward exit time, the speed of convergence to the stationary state.

1. INTRODUCTION

The long-time behaviour of kinetic transport equations – on periodic domains or on bounded domains with specular reflection on the boundary – is well known when the cross sections are bounded from below by a strictly positive constant.

In this case, the exponential decay in time of the solutions to the unique equilibrium state of the system can be obtained, with explicit rates, by the method of hypocoercivity as in [7, 11].

This result has, however, no obvious extension in the case of cross sections vanishing in a portion of the domain. Such a transport problem is said to be degenerate, and the characterization of the long-time asymptotic in the general case is still an open problem.

Indeed, in the regions where the cross section is zero, the problem is reduced to the free transport equation, which has no equilibrium state in a periodic setting or when the problem is defined on a bounded domain with specular reflection.

A partial answer to this question has been obtained by Desvillettes and Salvarani in [3] in a special situation, namely when the cross section vanishes at a finite number of points.

The key point of the proof in [3] is the use of the Desvillettes-Villani lemma, based on a pair of differential inequalities that allows to prove a polynomial (in time) speed of convergence towards equilibrium for the solution of the transport problem [4].

We note that there exist other phenomena that can lead to the convergence to equilibrium of the solutions of free transport equations: we cite, for example, the interaction with the boundary of the domain that induces diffuse reflection [1, 10] or the presence of a dissipating obstacle (see [9] and the references therein).

Date: March 9, 2012.

Key words and phrases. Transport equation, degenerate cross section, asymptotic behaviour.

In this note, we consider a situation when the cross section vanishes on a set of strictly positive measure, and give a counterexample which shows that the L^2 -convergence cannot decay better than $t^{-1/2}$.

2. THE PROBLEM

The problem we are looking for is the long-time asymptotics of the non-homogeneous (in space) transport equation

$$(2.1) \quad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = \sigma(x) (\bar{f} - f),$$

where $f := f(t, x, v)$ represents the density of particles which at time $t \in \mathbb{R}^+$ and point $x \in \mathbb{T}^d$ move with velocity $v \in S^{d-1}$.

Here S^{d-1} denotes the hypersphere of radius 1 in the space of velocities \mathbb{R}^d ($d \in \mathbb{N}$, $d \geq 2$) and

$$\bar{f}(t, x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(t, x, v) dv,$$

where $|S^{d-1}|$ is the total surface of the hypersphere.

The equation is set in a periodic domain with respect to all components of x , that is $x \in \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ and satisfies the initial condition

$$(2.2) \quad f(0, x, v) = f^{\text{in}}(x, v) \in L^\infty(\mathbb{T}^d \times S^{d-1}),$$

where $f^{\text{in}} \geq 0$ for a.e. $(x, v) \in \mathbb{T}^d \times S^{d-1}$.

The nonnegative function $\sigma(x)$ designates the cross-section of the problem. We suppose here that

- (1) $\sigma(x) \in L^\infty(\mathbb{T}^d)$ and is nonnegative (that is $\sigma(x) \geq 0$ for all $x \in \mathbb{T}^d$);
- (2) $\|\sigma\|_{L^1(\mathbb{T}^d)} > 0$.

Since problem (2.1)-(2.2) is a Lipschitzian perturbation of the free transport equation, there exists a unique mild solution f of the problem (see, for example, [8]).

It is also straightforward to prove that the constants are stationary solutions of Equation (2.1), and that $f_\infty = \|f^{\text{in}}\|_{L^1(\mathbb{T}^d \times S^{d-1})} / |S^{d-1}|$ is the unique constant solution with the same mass of the initial data.

3. A COUNTEREXAMPLE

We give here a case of degenerate transport problem where the convergence towards equilibrium is algebraic, which therefore excludes the exponential convergence under the assumptions above and without supplementary requirements.

By following [5], for all $r \in (0, 1/2)$ we consider the domain

$$Z_r = \{x \in \mathbb{R}^d : \text{dist}(x, \mathbb{Z}^d) > r\}$$

and, subsequently, we define the domain $Y_r = Z_r / \mathbb{Z}^d$.

A crucial tool in studying Equation (2.1) is the *forward exit time* for a particle starting from $x \in Z_r$ in the direction $v \in S^{d-1}$, defined as

$$\tau_r(x, v) = \inf\{t > 0 : x + tv \in \partial Z_r\}.$$

The forward exit time can be extended on the space $Y_r \times S^{d-1}$ thanks to the invariance under \mathbb{Z}^d -translations of Z_r , namely

$$\tau_r(x+k, v) = \tau_r(x, v) \text{ for all } (x, v) \in Z_r \times S^{d-1} \text{ and } k \in \mathbb{Z}^d.$$

On the measurable space $Y_r \times S^{d-1}$, equipped with its Borel σ -algebra, we define μ_r as the probability measure proportional to the Lebesgue measure on $Y_r \times S^{d-1}$, that is

$$d\mu_r(y, v) = \frac{dydv}{|Y_r| |S^{d-1}|}.$$

We finally define the distribution of τ_r under μ_r by

$$\Phi_r(t) := \mu_r \left(\{(x, v) \in Y_r \times S^{d-1} : \tau_r(y, v) \geq t\} \right).$$

The distribution of forward exit time satisfies, in the periodic case, both a lower and an upper bound. This property, proved by Bourgain, Golse and Wennberg in [2] and by Golse and Wennberg in [6], is summarized in the following theorem:

Theorem 3.1. *Let $d \geq 2$. Then there exist two positive constants C_1 and C_2 such that, for all $r \in (0, 1/2)$ and each $t > 1/r^{d-1}$*

$$\frac{C_1}{r^{d-1}} t^{-1} \leq \Phi_r(t) \leq \frac{C_2}{r^{d-1}} t^{-1}.$$

We are now ready to describe our counterexample, which uses only the lower bound of Theorem 3.1. We underline that the lower bound is based on the fact that a particle can never meet the scattering region because of the existence of the so-called *infinite channels* [2].

We choose $\sigma(x) = \mathbb{1}_{\mathbb{T}^d \setminus Y_{r'}}$, $0 < r' < r$, and $f^{\text{in}}(x, v) = f^{\text{in}}(x) = \mathbb{1}_{Y_r}$ in (2.1)-(2.2).

It is easy to show that the only stationary solution of Equation (2.1) with the same mass of f^{in} is the constant function $f_\infty = |Y_r|$.

Let us now consider

(3.1)

$$\begin{aligned} \int_{\mathbb{T}^d \times S^{d-1}} (f - f_\infty)^2 dx dv &\geq \int_{Y_r \times S^{d-1}} (f - f_\infty)^2 dx dv = \\ &\int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x, -v) \geq t} (f - f_\infty)^2 dx dv + \int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x, -v) < t} (f - f_\infty)^2 dx dv. \end{aligned}$$

The first integral in the right-hand side of the previous equation being defined only when $\tau_r(x, -v) \geq t$, we have obviously that, in the aforementioned region of the space-time, $f(t, x, v) \geq f^{\text{in}}(x - vt, t) = 1$. Hence

$$\int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x, -v) \geq t} (f - f_\infty)^2 dx dv \geq (1 - |Y_r| |S^{d-1}|)^2 \int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x, -v) \geq t} dx dv.$$

We have by Theorem 3.1

$$\begin{aligned} \int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x, -v) \geq t} dx dv &= \text{meas} \left(\{(x, v) \in Y_r \times S^{d-1} \mid \tau_r(x, -v) \geq t\} \right) \\ &\geq \frac{C_1}{r^{d-1}} |Y_r| |S^{d-1}| t^{-1}. \end{aligned}$$

Therefore

$$\int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x,-v) \geq t} (f - f_\infty)^2 dx dv \geq (1 - |Y_r| |S^{d-1}|)^2 \frac{C_1}{r^{d-1}} |Y_r| |S^{d-1}| t^{-1}.$$

Since

$$\int_{Y_r \times S^{d-1}} \mathbb{1}_{\tau_r(x,-v) < t} (f - f_\infty)^2 dx dv \geq 0,$$

(3.1) implies

$$\int_{\mathbb{T}^d \times S^{d-1}} (f - f_\infty)^2 dx dv \geq \frac{C_1}{r^{d-1}} (1 - |Y_r| |S^{d-1}|)^2 |Y_r| |S^{d-1}| t^{-1}.$$

or equivalently

$$\|f - f_\infty\|_{L^2(\mathbb{T}^d \times S^{d-1})} \geq \frac{C}{\sqrt{t}}.$$

This particular example shows that the convergence cannot be better than polynomial in the general case. Here the L^2 -norm of the difference between the time-dependent solution and the corresponding stationary state evolves at best like $t^{-1/2}$.

Remark 3.2. *The counterexample has been produced by considering an initial condition $f^{\text{in}}(x, v) \in L^\infty(\mathbb{T}^d \times S^{d-1})$ and a cross section $\sigma(x)$ of class $L^\infty(\mathbb{T}^d)$.*

The same computations can be performed also for regular initial data and cross section and lead to the same result. For example, we can suppose that $\sigma(x) = 0$ on $Y_{r'}$, $\sigma(x) = 1$ on $\mathbb{T}^d \setminus Y_{r'-\varepsilon}$, $0 < \varepsilon < r'$, and with a C^∞ -connection between the two regions, and that $f^{\text{in}}(x, v) = 1$ on Y_r , $f^{\text{in}}(x, v) = 0$ on $\mathbb{T}^d \setminus Y_{r-\varepsilon}$, and with a C^∞ -connection between the two regions.

Hence, the regularity properties of the cross section or the initial data do not play here any role in governing the convergence speed towards equilibrium.

On the contrary, the distribution of the forward exit time, induced by the geometrical properties of the scattering region, is the key ingredient in the computations.

Hence, further hypotheses on the cross-section are necessary in order to improve the convergence rate.

Acknowledgements. The authors thank François Golse for useful discussions on the subject and Kazuo Aoki for providing us some references.

REFERENCES

- [1] Kazuo Aoki and François Golse. On the speed of approach to equilibrium for a collisionless gas. *Kinet. Relat. Models*, 4(1):87–107, 2011.
- [2] Jean Bourgain, François Golse, and Bernt Wennberg. On the distribution of free path lengths for the periodic Lorentz gas. *Comm. Math. Phys.*, 190(3):491–508, 1998.
- [3] Laurent Desvillettes and Francesco Salvarani. Asymptotic behavior of degenerate linear transport equations. *Bull. Sci. Math.*, 133(8):848–858, 2009.
- [4] Laurent Desvillettes and Cédric Villani. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math.*, 54(1):1–42, 2001.
- [5] François Golse. On the periodic Lorentz gas and the Lorentz kinetic equation. *Ann. Fac. Sci. Toulouse Math. (6)*, 17(4):735–749, 2008.

- [6] François Golse and Bernt Wennberg. On the distribution of free path lengths for the periodic Lorentz gas. II. *M2AN Math. Model. Numer. Anal.*, 34(6):1151–1163, 2000.
- [7] Clément Mouhot and Lukas Neumann. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity*, 19(4):969–998, 2006.
- [8] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [9] Tetsuro Tsuji and Kazuo Aoki. On the periodic Lorentz gas and the Lorentz kinetic equation. *J. Stat. Phys.*, 146:620–645, 2012.
- [10] Tetsuro Tsuji, Kazuo Aoki, and François Golse. Relaxation of a free-molecular gas to equilibrium caused by interaction with vessel wall. *J. Stat. Phys.*, 140(3):518–543, 2010.
- [11] Cédric Villani. Hypocoercive diffusion operators. In *International Congress of Mathematicians. Vol. III*, pages 473–498. Eur. Math. Soc., Zürich, 2006.

(É. B.) ECOLE POLYTECHNIQUE, CENTRE DE MATHÉMATIQUES L. SCHWARTZ, F91128 PALAISEAU CEDEX, & INSTITUT GÉOGRAPHIQUE NATIONAL, LABORATOIRE DE RECHERCHE EN GÉODÉSIE, 6-8 AVENUE BLAISE PASCAL, CITÉ DESCARTES-CHAMPS SUR MARNE, 77455 MARNE LA VALLE CEDEX 2

E-mail address: `bernard@math.polytechnique.fr`

(F. S.) DIPARTIMENTO DI MATEMATICA F. CASORATI, UNIVERSITÀ DEGLI STUDI DI PAVIA, VIA FERRATA 1, 27100 PAVIA, ITALIA

E-mail address: `francesco.salvarani@unipv.it`