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STABILITY OF THE DETERMINATION OF A TIME-DEPENDENT COEFFICIENT IN PARABOLIC EQUATIONS

MOURAD CHOULLI AND YAVAR KIAN

ABSTRACT. We establish a Lipschitz stability estimate for the inverse problem consisting in the determination of the coefficient $\sigma(t)$, appearing in a Dirichlet initial-boundary value problem for the parabolic equation $\partial_t u - \Delta_x u + \sigma(t)f(x)u = 0$, from Neumann boundary data. We extend this result to the same inverse problem when the previous linear parabolic equation is changed to the semi-linear parabolic equation $\partial_t u - \Delta_x u = F(t, x, \sigma(t), u(x, t))$.

Key words : parabolic equation, semi-linear parabolic equation, inverse problem, determination of time-depend coefficient, stability estimate.

AMS subject classifications : 35R30.

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1. INTRODUCTION

Throughout this paper, we assume that Ω is a \mathcal{C}^3 bounded domain of \mathbb{R}^n with $n \geq 2$. Let $T > 0$ and set

$$Q = \Omega \times (0, T), \quad \Gamma = \partial\Omega, \quad \Sigma = \Gamma \times (0, T).$$

We consider the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u + \sigma(t)f(x)u = 0, & (x, t) \in Q, \\ u(x, 0) = h(x), & x \in \Omega, \\ u(x, t) = g(x, t), & (x, t) \in \Sigma. \end{cases} \quad (1.1)$$

We introduce the following assumptions :

(H1) $f \in \mathcal{C}^2(\overline{\Omega})$, $h \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, $g \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Sigma})$, for some $0 < \alpha < 1$, and satisfy the compatibility condition

$$\partial_t g(x, 0) - \Delta_x h(x) + \sigma(0)f(x)h(x) = 0, \quad x \in \Gamma.$$

(H2) There exists $x_0 \in \Gamma$ such that

$$\inf_{t \in [0, T]} |g(x_0, t)f(x_0)| > 0.$$

Under assumption (H1), it is well known that, for $\sigma \in \mathcal{C}^1[0, T]$, the initial-boundary value problem (1.1) admits a unique solution $u = u(\sigma) \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})$ (see Theorem 5.2 of [LSU]). Moreover, given $M > 0$, there exists a constant $C > 0$ depending only on data (that is Ω , T , f , g and h) such that $\|\sigma\|_{W^{1,\infty}(0, T)} \leq M$ implies

$$\|u(\sigma)\|_{\mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})} \leq C. \quad (1.2)$$

In the present paper we are concerned with the inverse problem consisting in the determination of the time dependent coefficient $\sigma(t)$ from Neumann boundary data $\partial_\nu u(\sigma)$ on Σ , where ∂_ν is the derivative in the direction of the unit outward normal vector to Γ .

We prove the following theorem, where $B(M)$ is the ball of $C^1[0, T]$ centered at 0 and with radius $M > 0$.

Theorem 1. *Assume that (H1) and (H2) are fulfilled. For $i = 1, 2$, let $\sigma_i \in B(M)$ and $u_i = u(\sigma_i)$. Then there exists a constant $C > 0$, depending only on data, such that*

$$\|\sigma_2 - \sigma_1\|_{L^\infty(0, T)} \leq C \|\partial_t \partial_\nu u_2 - \partial_t \partial_\nu u_1\|_{L^\infty(\Sigma)}. \quad (1.3)$$

Following [COY], it is quite natural to extend Theorem 1 when the linear parabolic equation is changed to a semi-linear parabolic equation. To this end, introduce the following semi-linear initial-boundary value problem :

$$\begin{cases} \partial_t u - \Delta_x u = F(x, t, \sigma(t), u(x, t)), & (x, t) \in Q, \\ u(x, 0) = h(x), & x \in \Omega, \\ u(x, t) = g(x, t), & (x, t) \in \Sigma \end{cases} \quad (1.4)$$

and consider the following assumptions

(H3) $h \in \mathcal{C}^{2, \alpha}(\bar{\Omega})$, $g \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Sigma})$, for some $0 < \alpha < 1$, and satisfy the compatibility condition

$$\partial_t g(x, 0) - \Delta_x h(x) = F(0, x, \sigma(0), h(x)), \quad x \in \Gamma.$$

(H4) $F \in \mathcal{C}^1(\bar{\Omega}_x \times \mathbb{R}_t \times \mathbb{R}_\sigma \times \mathbb{R}_u)$ is such that $\partial_u F$ and $\partial_\sigma F$ are \mathcal{C}^1 , F and $\partial_\sigma F$ are \mathcal{C}^2 with respect to x and u .

(H5) There exist $M > 0$ and $x_0 \in \Gamma$ such that

$$\inf_{t \in [0, T], \sigma \in [-M, M]} |\partial_\sigma F(x_0, t, \sigma, g(x_0, t))| > 0.$$

(H6) There exist two non negative constants c and d such that

$$uF(x, t, \sigma(t), u) \leq cu^2 + d, \quad t \in [0, T], \quad x \in \bar{\Omega}, \quad u \in \mathbb{R}.$$

Under the above mentioned conditions, for any $\sigma \in \mathcal{C}^1[0, T]$, the initial-boundary value problem (1.4) admits a unique solution $u = u(\sigma) \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q})$ (see Theorem 6.1 in [LSU]) and, given $M > 0$, there exists a constant $C > 0$ depending only on data (that is Ω , T , F , g and h) such that $\|\sigma\|_{W^{1, \infty}(0, T)} \leq M$ implies

$$\|u(\sigma)\|_{\mathcal{C}^{2+\alpha, 1+\alpha/2}(\bar{Q})} \leq C. \quad (1.5)$$

We have the following extension of Theorem 1.

Theorem 2. *Assume that (H3), (H4), (H5) and (H6) are fulfilled. For $i = 1, 2$, let $\sigma_i \in B(M)$ and $u_i = u(\sigma_i)$. Then there exists a constant $C > 0$, depending only on data, such that*

$$\|\sigma_2 - \sigma_1\|_{L^\infty(0, T)} \leq C \|\partial_t \partial_\nu u_2 - \partial_t \partial_\nu u_1\|_{L^\infty(\Sigma)}. \quad (1.6)$$

Remark 1. *Let us observe that we can generalize the results in Theorems 1 and 2 as follows:*

i) In (1.1), we can replace $\sigma(t)f(x)$ by $\sum_{k=1}^p \sigma_k(t)f_k(x)$, where f_k , $1 \leq k \leq p$, are known. Assume that (H1) is satisfied, with $f = f_k$ for each k , where the compatibility condition is changed to

$$\partial_t g(x, 0) - \Delta_x h(x) + \sum_{k=1}^p \sigma_k(0)f_k(x)h(x) = 0, \quad x \in \Gamma.$$

Therefore, to each $(\sigma_1, \dots, \sigma_p) \in \mathcal{C}[0, T]^p$ corresponds a unique solution $u = u(\sigma_1, \dots, \sigma_p) \in \mathcal{C}^{2+\alpha, 1+\alpha/2}(\bar{Q})$ and $\max\{\|\sigma_k\|_{W^{1, \infty}(0, T)}; 1 \leq k \leq p\} \leq M$ implies

$$\|u(\sigma_1, \dots, \sigma_p)\|_{\mathcal{C}^{2+\alpha, 1+\alpha/2}(\bar{Q})} \leq C,$$

for some positive constant C depending only on data.

Following the proof of Theorem 1, we prove that, under the following conditions : there exist $x_1, \dots, x_p \in \Gamma$ such that the matrix $M(t) = (f_k(x_l)g(x_l, t))$ is invertible for any $t \in [0, T]$,

$$\max_{1 \leq k \leq p} \|\sigma_k^1 - \sigma_k^2\|_{L^\infty(0, T)} \leq C \|\partial_t \partial_\nu u_2 - \partial_t \partial_\nu u_1\|_{L^\infty(\Sigma)},$$

if $\sigma_k^j \in B(M)$, $1 \leq k \leq p$ and $j = 1, 2$. Here C is a constant that can depend only on data and $u_j = u(\sigma_1^j, \dots, \sigma_p^j)$, $j = 1, 2$.

ii) We can replace the semi-linear parabolic equation in (1.4) by a semi-linear integro-differential equation. In other words, F can be changed to

$$F_1(x, t, \sigma(t), u(x, t)) + \int_0^t F_2(x, s, \sigma(t-s), u(x, s)) ds.$$

Under appropriate assumptions on F_1 and F_2 , one can establish that Theorem 2 is still valid in the present case.

ii) Both in (1.1) and (1.4), the Laplace operator can be replaced by a second order elliptic operator in divergence form :

$$E = \nabla \cdot A(x) \nabla + B(x) \cdot \nabla,$$

where $A(x) = (a_{ij}(x))$ is a symmetric matrix with coefficients in $C^{1+\alpha}(\overline{\Omega})$, $B(x) = (b_i(x))$ is a vector with components in $C^\alpha(\overline{\Omega})$ and the following ellipticity condition holds

$$A(x)\xi \cdot \xi \geq \lambda |\xi|^2, \quad \xi \in \mathbb{R}^n, x \in \overline{\Omega}.$$

Actually, the normal derivative associated to E is the boundary operator $\partial_{\nu_E} = \nu(x) \cdot A(x) \nabla$.

To our knowledge, there are only few results concerning the determination of a time-dependent coefficient in an initial-boundary value problem for a parabolic equation from a single measurement. The determination of a source term of the form $f(t)\chi_D(x)$, where χ_D the characteristic function of the known subdomain D , was considered by J. R. Canon and S. P. Esteva. They established in [CE86-1] a logarithmic stability estimate in 1D case in a half line when the overdetermined data is the trace at the end point. A similar inverse problem in 3D case was studied by these authors in [CE86-2], where they obtained a Lipschitz stability estimate in weighted spaces of continuous functions. The case of a non local measurement was considered by J. R. Canon and Y. Lin in [CL88] and [CL90], where they proved existence and uniqueness for both quasilinear and semi-linear parabolic equations. The determination of a time dependent coefficient in an abstract integrodifferential equation was studied by the first author in [Ch91-1]. He proved existence, uniqueness and Lipschitz stability estimate, extending earlier results by [Ch91-2], [LS87], [LS88], [PO85-1] and [PO85-2]. In [CY06], the first author and M. Yamamoto obtained a stability result, in a restricted class, for the inverse problem of determining a source term $f(x, t)$, appearing in a Dirichlet initial-boundary value problem for the heat equation, from Neumann boundary data. In a recent work, the first author and M. Yamamoto [CY11] considered the inverse problem of finding a control parameter $p(t)$ that reach a desired temperature $h(t)$ along a curve $\gamma(t)$ for a parabolic semi-linear equation with homogeneous Neumann boundary data and they established existence, uniqueness as well as Lipschitz stability. Using geometric optic solutions, the first author [Ch09] proved uniqueness as well as stability for the inverse problem of determining a general time dependent coefficient of order zero for parabolic equations from Dirichlet to Neumann map. In [E07] and [E08], G. Eskin considered the same inverse problem for hyperbolic and the Schrödinger equations with time-dependent electric and magnetic potential and he established uniqueness by gauge invariance. Recently, R. Salazar [Sa] extended the result of [E07] and obtained a stability result for compactly supported coefficients.

We would like to mention that the determination of space dependent coefficient $f(x)$, in the source term $\sigma(t)f(x)$, from Neumann boundary data was already considered by the first author and M. Yamamoto [CY06]. But, it seems that our paper is the first work where one treats the determination of a time dependent coefficient, appearing in a parabolic initial-boundary value problem, from Neumann boundary data.

This paper is organized as follows. In section 2 we come back to the construction of the Neumann fundamental solution by [It] and establish time-differentiability of some potential-type functions, necessary for proving Theorems 1 and 2. Section 3 is devoted to the proof of Theorems 1 and 2.

2. TIME-DIFFERENTIABILITY OF POTENTIAL-TYPE FUNCTIONS

In this section, we establish time-differentiability of some potential-type functions, needed in the proof of our stability estimates. In our analysis we follow the construction of the fundamental solution by S. Itô [It].

First of all, we recall the definition of fundamental solution associate to the heat equation plus a time-dependent coefficient of order zero, in the case of Neumann boundary condition. Consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta_x u + q(x, t)u, & (x, t) \in \Omega \times (s, t_0), \\ \lim_{t \rightarrow s} u(x, t) = u_0(x), & x \in \Omega, \\ \partial_\nu u(x, t) = 0, & (x, t) \in \Gamma \times (s, t_0). \end{cases} \quad (2.1)$$

Here $s_0 < t_0$ are fixed, $s \in (s_0, t_0)$, u_0 and $q(x, t)$ are continuous respectively in $\bar{\Omega}$ and in $\bar{\Omega} \times [s, t_0]$. Let $U(x, t; y, s)$ be a continuous function in the domain $s_0 < s < t < t_0$, $x \in \bar{\Omega}$, $y \in \bar{\Omega}$. We recall that U is the fundamental solution of (2.1) if for any $u_0 \in \mathcal{C}(\bar{\Omega})$,

$$u(x, t) = \int_{\Omega} U(x, t; y, s) u_0(y) dy$$

is the solution of (2.1). We refer to [It] for the existence and uniqueness of this fundamental solution.

We start with time-differentiability of volume potential-type functions¹.

Lemma 1. *Fix $s \in (s_0, t_0)$. Let $f \in \mathcal{C}(\bar{\Omega} \times [s, t_0])$ be \mathcal{C}^2 with respect to x , $q \in \mathcal{C}^1(\bar{\Omega} \times [s, t_0])$ and define, for $(x, t) \in \bar{\Omega} \times (s, t_0)$,*

$$f^1(x, t; \tau) = \int_{\Omega} U(x, t; y, \tau) f(y, \tau) dy, \quad t > \tau > s.$$

Then, f^1 admits a derivative with respect to t and

$$\begin{aligned} \frac{\partial f^1}{\partial t}(x, t; \tau) &= \int_{\Omega} U(x, t; y, \tau) (\Delta_y + q(x, \tau)) f(y, \tau) dy \\ &+ \int_{\tau}^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau') \partial_t q(z, \tau') U(z, \tau'; y, \tau) f(y, \tau) dz dy d\tau'. \end{aligned} \quad (2.2)$$

Moreover, F given by

$$F(x, t) = \int_s^t f^1(x, t; \tau) d\tau, \quad (x, t) \in \Gamma \times (s_0, t_0),$$

possesses a derivative with respect to t ,

$$\frac{\partial F}{\partial t}(x, t) = f(x, t) + \int_s^t \frac{\partial f^1}{\partial t}(x, t; \tau) d\tau \quad (2.3)$$

and

$$\left| \int_s^t \frac{\partial f^1}{\partial t}(x, t; \tau) d\tau \right| \leq C \int_s^t \|f(\cdot, \tau)\|_{C_x^2(\bar{\Omega})} d\tau. \quad (2.4)$$

¹Recall that if $\varphi = \varphi(x, t)$ is a continuous function then the corresponding volume potential is given by

$$\psi(x, t) = \int_s^t \int_{\Omega} U(x, t; y, \tau) \varphi(y, \tau) dy d\tau.$$

Proof. We have only to prove (2.2) and (2.4), because (2.3) follows immediately from (2.2).

Let then $u_0 \in C^2(\bar{\Omega})$ and consider the function

$$u(x, t) = \int_{\Omega} U(x, t; y, s) u_0(y) dy, \quad x \in \bar{\Omega}, \quad s < t < t_0.$$

We show that u admits a derivative with respect to t and

$$\begin{aligned} \partial_t u(x, t) &= \partial_t \left(\int_{\Omega} U(x, t; y, s) u_0(y) dy \right) \\ &= \int_{\Omega} U(x, t; y, s) (\Delta_y + q(x, s)) u_0(y) dy \\ &\quad - \int_s^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau) q_t(z, \tau) U(z, \tau; y, s) u_0(y) dz dy d\tau. \end{aligned} \quad (2.5)$$

We need to consider first the case $u_0 = w_0 \in C^\infty(\bar{\Omega})$. Set

$$w(x, t) = \int_{\Omega} U(x, t; z, s) w_0(y) dy, \quad x \in \bar{\Omega}, \quad s < t < t_0.$$

Clearly, $w(x, t)$ is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t w - \Delta_x w - q(x, t)w = 0, & (x, t) \in \Omega \times (s, t_0), \\ \lim_{t \rightarrow s} w(x, t) = w_0(x), & x \in \Omega, \\ \partial_\nu w(x, t) = 0, & (x, t) \in \Gamma \times (s, t_0) \end{cases}$$

and $w_1 = \partial_t w$ satisfies

$$\begin{cases} \partial_t w_1 - \Delta_x w_1 - q(x, t)w_1 = -\partial_t q w, & (x, t) \in \Omega \times (s, t_0), \\ \lim_{t \rightarrow s} w_1(x, t) = (\Delta_x + q(x, s))w_0(x), & x \in \Omega, \\ \partial_\nu w_1(x, t) = 0, & (x, t) \in \Gamma \times (s, t_0). \end{cases}$$

Therefore, (2.5), with w in place of u , is a consequence of Theorem 9.1 of [It].

Next, let $(w_0^n)_n$ be a sequence in $C^\infty(\bar{\Omega})$ converging to u_0 in $C^2(\bar{\Omega})$ and $v(x, t)$ given by

$$\begin{aligned} v(x, t) &= \int_{\Omega} U(x, t; y, s) (\Delta_x + q(x, s)) u_0(y) dy \\ &\quad - \int_s^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau) \partial_t q(z, \tau) U(z, \tau; y, s) u_0(y) dz dy d\tau. \end{aligned}$$

Consider $(w_n)_n$, the sequence of functions, defined by

$$w_n(x, t) = \int_{\Omega} U(x, t; z, s) w_0^n(y) dy.$$

We proved that, for any $n \in \mathbb{N}$,

$$\begin{aligned} \partial_t w_n(x, t) &= \int_{\Omega} U(x, t; y, s) (\Delta_y + q(x, s)) w_0^n(y) dy \\ &\quad - \int_s^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau) \partial_t q(z, \tau) U(z, \tau; y, s) w_0^n(y) dz dy d\tau. \end{aligned} \quad (2.6)$$

From the proof of Theorem 7.1 of [It],

$$\int_{\Omega} |U(x, t; y, s)| dy \leq C e^{C(t-s)}, \quad (x, t) \in \bar{\Omega} \times (s, t_0). \quad (2.7)$$

Therefore, we can pass to the limit, as n goes to infinity, in (2.6). We deduce that $\partial_t w_n$ converges to v in $\mathcal{C}(\bar{\Omega} \times [s, t_0])$. But, w_n converges to u in $\mathcal{C}(\bar{\Omega} \times [s, t_0])$. Hence u admits a derivative with respect to t and

$\partial_t u = v$. That is we proved (2.5) and consequently (2.2) holds true. Finally, we note that (2.4) is deduced easily from (2.7). \square

Next, we consider time-differentiability a single layer potential-type function².

Lemma 2. *Fix $s \in (s_0, t_0)$. Let $f \in \mathcal{C}(\Gamma \times [s, t_0])$ be \mathcal{C}^1 with respect to $t \in [s, t_0]$ with $f(x, s) = 0$. Define, for $(x, t) \in \Gamma \times (s, t_0)$,*

$$f^1(x, t; \tau) = \int_{\Gamma} U(x, t; y, \tau) f(y, \tau) d\sigma(y), \quad t > \tau > s.$$

Then

$$F(x, t) = \int_s^t f^1(x, t; \tau) d\tau$$

is well defined, admits a derivative with respect to t and we have

$$\left\| \frac{\partial F}{\partial t} \right\|_{L^\infty(\Gamma \times (s, t_0))} \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (s, t_0))}. \quad (2.8)$$

Contrary to Lemma 1, for Lemma 2 we cannot use directly the general properties of the fundamental solutions developed in [It]. We need to come back to the construction of the fundamental solution of (2.1) introduced by [It]. First, consider the heat equation $\partial_t u = \Delta_x u$ in the half space $\Omega_1 = \{x = (x_1, \dots, x_n); x_1 > 0\}$ in \mathbb{R}^n with the boundary condition $\partial_{x_1} u = 0$ on $\Gamma_1 = \{x = (0, x_2, \dots, x_n); (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}$. For any $y = (y_1, y_2, \dots, y_n)$, we define \bar{y} by $\bar{y} = (-y_1, y_2, \dots, y_n)$. Let

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

denotes the Gaussian kernel and set

$$G_1(x, t; y) = G(x - y, t) + G(x - \bar{y}, t).$$

Then, the fundamental solution $U_0(x, t; y, s)$ of

$$\begin{cases} \partial_t u = \Delta_x u, & (x, t) \in \Omega_1 \times (s, t_0), \\ \lim_{t \rightarrow s} u(x, t) = u_0(x), & x \in \Omega_1, \\ \partial_\nu u(x, t) = 0, & (x, t) \in \Gamma_1 \times (s, t_0) \end{cases} \quad (2.9)$$

is given by

$$U_0(x, t; y, s) = G_1(x, t - s; y).$$

In order to construct the fundamental solution in the case of an arbitrary domain Ω , Itô introduced the following local coordinate system around each point $z \in \Gamma$.

Lemma 3. (Lemma 6.1 and its corollary, Chapter 6 of [It]) *For every point $z \in \Gamma$, there exist a coordinate neighborhood W_z of z and a coordinate system (x_1^*, \dots, x_n^*) satisfying the following conditions:*

- 1) *the coordinate transformation between the coordinate system (x_1^*, \dots, x_n^*) and the original coordinate system in W_z is of class C^2 and the partial derivatives of the second order of the transformation functions are Hölder continuous ;*
- 2) *$\Gamma \cap W_z$ is represented by the equation $x_1^* = 0$ and $\Omega \cap W_z$ is represented by $x_1^* > 0$;*
- 3) *let \mathcal{L} be the diffeomorphism from W_z to $\mathcal{L}(W_z)$ defined by*

$$\mathcal{L} : W_z \rightarrow \mathcal{L}(W_z)$$

$$x \mapsto (x_1^*(x), \dots, x_n^*(x)).$$

²The single-layer potential corresponding to a continuous function $\varphi = \varphi(x, t)$ is given by

$$\psi(x, t) = \int_s^t \int_{\Gamma} U(x, t; y, \tau) \varphi(y, \tau) d\sigma(y) d\tau.$$

Then, for any $u \in C^1(\overline{\Omega})$ we have

$$\partial_\nu u(\xi) = -\partial_{x_1}(u \circ \mathcal{L}^{-1})(x), \quad \xi \in \Gamma \cap W_z \text{ and } x = \mathcal{L}(\xi).$$

From now on, for any $z \in \Gamma$, we view coordinate system (x_1^*, \dots, x_n^*) as a rectangular coordinate system. Moreover, using the local coordinate system of Lemma 3, for any $y = (y_1, y_2, \dots, y_n) \in \mathcal{L}(W_z)$, we define $\overline{y} = (-y_1, y_2, \dots, y_n)$ and, without loss of generality, we assume that, for any $y \in \mathcal{L}(W_z)$, we have $\overline{y} \in \mathcal{L}(W_z)$. For any interior point z of Ω , we fix an arbitrary local coordinate system and a coordinate neighborhood W_z contained in Ω . For any $z \in \overline{\Omega}$ and $\delta > 0$, we set $W(z, \delta) = \{x : |x - z|^2 < \delta\}$ and $\delta_z > 0$ such that, for any $z \in \overline{\Omega}$ we have $\overline{W(z, \delta_z)} \subset W_z$.

Recall the following partition of unity lemma.

Lemma 4. (Lemma 7.1, Chapter 7 of [It]) *There exist a finite subset $\{z_1, \dots, z_m\}$ of $\overline{\Omega}$ and a finite sequence of functions $\{\omega_1, \dots, \omega_m\}$ with the following properties:*

- 1) *supp $\omega_l \subset W(z_l, \delta_{z_l})$, $l = 1, \dots, m$, and each ω_l is of class C^3 with respect to the local coordinates in W_{z_l} ;*
- 2) *$\{\omega_l(x)^2; l = 1, \dots, m\}$ forms a partition of unity in Ω ;*
- 3) *$\partial_\nu \omega_l(\xi) = 0$, $l = 1, \dots, m$, $\xi \in \Gamma$.*

Let $\{z_1, \dots, z_m\}$ be the finite subset of $\overline{\Omega}$, introduced in the previous lemma. For any $k \in \{1, \dots, m\}$, let \mathcal{L}_k denotes the diffeomorphism from W_{z_k} to $\mathcal{L}_k(W_{z_k})$ defined by

$$\begin{aligned} \mathcal{L}_k : W_{z_k} &\rightarrow \mathcal{L}_k(W_{z_k}) \\ x &\mapsto (x_1^*(x), \dots, x_n^*(x)), \end{aligned}$$

where (x_1^*, \dots, x_n^*) is the local coordinate system of Lemma 3 defined in W_{z_k} . For any $k \in \{1, \dots, m\}$, the differential operator

$$\partial_t - \Delta_x - q(x, t)$$

becomes, in terms of local coordinate system $x^* = (x_1^*, \dots, x_n^*)$,

$$L_{t, x^*}^k = \partial_t - \frac{1}{\sqrt{a_k(x^*)}} \sum_{i, j=1}^n \partial_{x_i^*} \left(\sqrt{a_k(x^*)} a_k^{ij}(x^*) \partial_{x_j^*} \right) - q_k(x^*, t)$$

in $\mathcal{L}_k(W_{z_k}) \times (s_0, t_0)$. Here $q_k(x^*, t)$ is Hölder continuous on $\mathcal{L}_k(W_{z_k}) \times (s_0, t_0)$ and $(a^{ij}(x^*))$ is the contravariant tensor of degree 2 defined by

$$(a_k^{ij}(x^*)) = (J_{\mathcal{L}_k}(\mathcal{L}_k^{-1}(x^*)))^T (J_{\mathcal{L}_k}(\mathcal{L}_k^{-1}(x^*))),$$

with

$$J_{\mathcal{L}_k}(x) = \left(\frac{\partial x_j^*(x)}{\partial x_i} \right).$$

According to the construction of [It] given in Chapter 6 (see pages 42 to 45 of [It]), $(a_k^{ij}(x^*))$ is of class C^2 in $\mathcal{L}_k(\overline{\Omega} \cap W_{z_k})$ and it is a positive definite symmetric matrix at every point $x^* \in \mathcal{L}_k(W_{z_k})$. We set $(a_{ij}^k(x^*)) = (a_k^{ij}(x^*))^{-1}$ and $a_k(x^*) = \det(a_{ij}^k(x^*))$. Consider the volume element $dx^* = \sqrt{a_k(x^*)} dx_1^* \dots dx_n^*$ on $\mathcal{L}_k(W_{z_k})$ and $dx' = \sqrt{a(0, x')} dx_2^* \dots dx_n^*$ on $\mathcal{L}_k(W_{z_k} \cap \Gamma)$ with $x' = (x_2^*, \dots, x_n^*)$. Note that, by the construction of S. Itô [It] (see page 45), for any $k = 1, \dots, m$, we have

$$a_k^{ij}(\mathcal{L}_k(x)) = a_k^{ij}(\overline{\mathcal{L}_k(x)}), \quad x \in \overline{\Omega} \cap W_{z_k}, \quad \text{for } i = j = 1 \text{ or } i, j = 2, \dots, n, \quad (2.10)$$

$$a_k^{1j}(\mathcal{L}_k(x)) = a_k^{j1}(\mathcal{L}_k(x)) = -a_k^{1j}(\overline{\mathcal{L}_k(x)}), \quad x \in \overline{\Omega} \cap W_{z_k}, \quad \text{for } j = 1, \dots, n \quad (2.11)$$

and

$$a_k^{1j}(\mathcal{L}_k(\xi)) = a_k^{j1}(\mathcal{L}_k(\xi)) = \delta_{j1}, \quad \xi \in \Gamma \cap W_{z_k}, \quad j = 1, \dots, n, \quad (2.12)$$

where δ_{j1} denotes the kronecker's symbol. For any $k \in \{1, \dots, m\}$, let $G_k(x, t; y)$ be defined, in the region

$$D_k = \{(x, t, y); x, y \in \mathcal{L}_k(W_{z_k}), 0 < t < t_0 - s\},$$

by

$$G_k(x, t; y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\sum_{i,j=1}^n \frac{a_{ij}^k(y)(x_i - y_i)(x_j - y_j)}{4t}}.$$

Next, define $H_{z_k}(x, t; y) = \overline{G_k(\mathcal{L}_k(x), t; \mathcal{L}_k(y))}$, for $k \in \{1, \dots, m\}$ and $z_k \in \Omega$; $H_{z_k}(x, t; y) = G_k(\mathcal{L}_k(x), t; \mathcal{L}_k(y)) + \overline{G_k(\mathcal{L}_k(x), t; \mathcal{L}_k(y))}$, for $k \in \{1, \dots, m\}$, $z_k \in \Gamma$, $x \in W_{z_k}$ and $y \in W_{z_k}$; $H_{z_k}(x, t; y) = 0$ if $x \notin W_{z_k}$ or $y \notin W_{z_k}$. Consider also $H(x, t; y)$, defined in the region

$$D = \{(x, t, y); x \in \overline{\Omega}, y \in \overline{\Omega}, 0 < t < t_0 - s\},$$

as follows

$$H(x, t; y) = \sum_{l=1}^m \omega_l(x) H_{z_l}(x, t; y) \omega_l(y).$$

As in Lemma 7.2 of [It], we define successively:

$$J_0(x, t; y, s) = (\partial_t - \Delta_x - q(x, t))(H(x, t - s; y)),$$

$$J_k(x, t; y, s) = \int_s^t \int_{\Omega} J_0(x, t; z, \tau) J_{k-1}(z, \tau; y, s) dz d\tau,$$

$$K(x, t; y, s) = \sum_{k=0}^{+\infty} J_k(x, t; y, s).$$

Then, following [It] (see page 53), the fundamental solution of (2.1) is given by

$$U(x, t; s, y) = H(x, t - s; y) + \int_s^t \int_{\Omega} H(x, t - \tau; z) K(z, \tau; y, s) dz d\tau. \quad (2.13)$$

We are now able to prove Lemma 2 with the help of representation (2.13), the properties of $H(x, t; y)$ and $K(x, t; y, s)$.

Proof of Lemma 2. Without loss of generality, we assume that $s = 0$. Set

$$F_1(x, t) = \int_0^t \int_{\Gamma} H(x, t - s; y) f(y, s) d\sigma(y) ds,$$

$$F_2(x, t) = \int_0^t \int_{\Gamma} \int_s^t \int_{\Omega} H(x, t - \tau; z) K(z, \tau; y, s) f(y, s) dz d\tau d\sigma(y) ds.$$

According to representation (2.13), one needs to show that F_1 and F_2 admit a derivative with respect to t and

$$|\partial_t F_1(x, t)| + |\partial_t F_2(x, t)| \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))} \quad (2.14)$$

for $(x, t) \in \Gamma \times (0, t_0)$. We start by considering F_1 . Applying a simple substitution, we obtain

$$F_1(x, t) = \int_0^t \int_{\Gamma} H(x, s; y) f(y, t - s) d\sigma(y) ds. \quad (2.15)$$

Next, for $x \in \Gamma$, there exist $l_1, \dots, l_r \subset \{1, \dots, m\}$ such that $x \in \text{supp } \omega_l$ for $l \in \{l_1, \dots, l_r\}$ and $x \notin \text{supp } \omega_l$ for $l \notin \{l_1, \dots, l_r\}$. Moreover, since $x \in \Gamma$, we have $z_{l_1}, \dots, z_{l_r} \in \Gamma$. Then, from the construction of $H(x, t; y)$, we obtain

$$\begin{aligned} \int_{\Gamma} H(x, s; y) d\sigma(y) &= \int_{\Gamma} \sum_{k=1}^r \omega_{l_k}(x) H_{z_{l_k}}(x, s; y) \omega_{l_k}(y) d\sigma(y) \\ &= 2 \sum_{k=1}^r \int_{\mathbb{R}^{n-1}} \chi_{l_k}(0, x') \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-\sum_{i,j=1}^n \frac{a_{ij}^{l_k}(0, y')(x'_i - y'_i)(x'_j - y'_j)}{4s}} \chi_{l_k}(0, y') \sqrt{a_{l_k}(0, y')} dy' \end{aligned}$$

with, for $l \in \{1, \dots, m\}$, $\chi_l \in C_0^3(\mathcal{L}_l(\text{supp } \omega_l))$ such that $\chi_l(x) = \omega_l(\mathcal{L}_l^{-1}(x))$ and with $(x'_1, \dots, x'_n) = (0, x')$, $(y'_1, \dots, y'_n) = (0, y')$. Using the substitution $y' \rightarrow z' = \frac{x' - y'}{\sqrt{s}}$, we derive

$$\begin{aligned} & \int_{\Gamma} H(x, s; y) d\sigma(y) \\ & \leq C \sum_{k=1}^r \int_{\mathbb{R}^{n-1}} \chi_{l_k}(0, x') \frac{1}{\sqrt{s}} e^{-\sum_{i,j=1}^n a_{ij}^{l_k}(0, x' - \sqrt{s}z') z'_i z'_j} \chi_{l_k}(0, x' - \sqrt{s}z') \sqrt{a_{l_k}(0, x' - \sqrt{s}z')} dz'. \end{aligned} \quad (2.16)$$

Therefore,

$$\int_{\Gamma} |H(x, s; y)| d\sigma(y) \leq \frac{C}{\sqrt{s}} \int_{\mathbb{R}^{n-1}} e^{-a_0 |z'|^2} dz' \leq \frac{C}{\sqrt{s}},$$

where $a_0 > 0$ is a constant. From this estimate, we deduce that

$$\int_{\Gamma} |H(x, s; y) f(y, t - s)| d\sigma(y) \leq C \frac{\|f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{s}}$$

and

$$\left| \partial_t \left(\int_{\Gamma} H(x, s; y) f(y, t - s) d\sigma(y) \right) \right| \leq C \frac{\|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{s}}.$$

Thus, F_1 admits a derivative with respect to t ,

$$\partial_t F_1(x, t) = \int_{\Gamma} H(x, t; y) f(y, 0) d\sigma(y) + \int_0^t \int_{\Gamma} H(x, s; y) \partial_t f(y, t - s) d\sigma(y) ds$$

and, since $f(y, 0) = 0$ for $y \in \Gamma$, we obtain

$$|\partial_t F_1(x, t)| \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}, \quad (x, t) \in \Gamma \times (0, t_0). \quad (2.17)$$

Let us now consider F_2 . We want to show that $\partial_t F_2$ exists and the following estimate holds:

$$|\partial_t F_2(x, t)| \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}, \quad (x, t) \in \Gamma \times (0, t_0). \quad (2.18)$$

For this purpose, using the local coordinate system, it suffices to prove

$$|\partial_t F_2(\mathcal{L}_l^{-1}(0, x'), t)| \leq C_l \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}, \quad ((0, x'), t) \in \mathcal{L}_l(\Gamma \cap W_{z_l}) \times (0, t_0), \quad l \in \{1, \dots, m\}. \quad (2.19)$$

From now on we set $x = \mathcal{L}_l^{-1}(0, x')$ with $(0, x') \in \mathcal{L}_l(\Gamma \cap W_{z_l}) \subset \{0\} \times \mathbb{R}^{n-1}$ and we will show (2.19). First, note that

$$\begin{aligned} J_0(z, \tau; s, y) &= (\partial_\tau - \Delta_z - q(z, t)) H(z, \tau - s; y) \\ &= \sum_{l=1}^m \omega_l(\mathcal{L}_l^{-1}(z^*)) L_{\tau, z^*}^l H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; y) \omega_l(y) \\ &\quad + \sum_{l=1}^m [L_{\tau, z^*}^l, \omega_l(\mathcal{L}_l^{-1}(z^*))] H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; y) \omega_l(y). \end{aligned}$$

According to the results in Chapter 4 of [It] (pages 26 and 27), using the local coordinate system, we obtain

$$\begin{aligned} L_{\tau, z^*}^l H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; \mathcal{L}_l^{-1}(y^*)) &= \sum_{i,j=1}^n (a_{ij}^{l^*}(z^*) - a_{ij}^{l^*}(y^*)) \frac{\partial^2 H_{z_l}}{\partial z_i^* \partial z_j^*}(\mathcal{L}_l^{-1}(z^*), \tau - s; \mathcal{L}_l^{-1}(y^*)) \\ &\quad + [B_l(z^*, y^*, \partial_{z^*}) + q_l(z^*, t)] H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; \mathcal{L}_l^{-1}(y^*)), \end{aligned}$$

where $B_l(z^*, y^*, \partial_{z^*})$ is a differential operator of order ≤ 1 in z^* with continuous coefficients in $z^*, y^* \in \mathcal{L}_l(\text{supp } \omega_l)$. In view of the results in Chapter 4 of [It] (see pages 26 and 27), combining (2.10), (2.11), (2.12)

and (2.16), applying the substitution $y'' = \frac{z' - y'}{\sqrt{\tau - s}}$, with $z^* = (z_1^*, z')$ and $y^* = (0, y')$, we obtain

$$\int_{\Gamma} J_0(\mathcal{L}_l^{-1}(z^*), \tau; y, s) d\sigma(y) = \sum_{j=0}^2 P_j \left(\frac{z_1^*}{\sqrt{\tau - s}} \right) e^{-\frac{(z_1^*)^2}{\tau - s}} \left[\int_{\mathbb{R}^{n-1}} \frac{J_0^j(z^*, \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right],$$

for $0 < s < \tau < t_0$ and $z^* \in \mathcal{L}_l(\Omega \cap W_{z_k})$, where, for $j = 0, 1, 2$, P_j are polynomials and J_0^j are continuous functions, \mathcal{C}^1 with respect to τ , $s \in (0, t_0)$ and satisfy

$$\max_{\substack{i=0,1,2 \\ \alpha_1 + \alpha_2 \leq 1}} \int_{\mathbb{R}^{n-1}} \left| \partial_{\tau}^{\alpha_1} \partial_s^{\alpha_2} J_0^j(z^*, \tau; y'', s; v_1) \right| dy'' \leq C_l, \quad 0 < s < \tau < t_0, \quad z_1^* > 0, \quad 0 < v_1 < t_0,$$

for some constant $C_l > 0$. We note that $\partial_{v_1} J_0^j((z_1', z''), \tau; y'', s; v_1)$ is not necessarily bounded. Indeed, we show

$$\left| \partial_{v_1} J_0^j((z_1', z''), \tau; y'', s; v_1) \right| \leq \frac{C_l}{\sqrt{v_1}}, \quad 0 < v_1 < t_0, \quad j = 0, 1, 2.$$

This representation and the construction of $K(z, \tau; y, s)$ in Chapter 5 of [It] (see pages 31 to 32 for the construction in \mathbb{R}^n and page 53 for the construction in a bounded domain) lead

$$\int_{\Gamma} K(\mathcal{L}_l^{-1}(z^*), \tau; y, s) d\sigma(y) = \sum_{j=0}^2 Q_j \left(\frac{z_1^*}{\sqrt{\tau - s}} \right) e^{-\frac{(z_1^*)^2}{\tau - s}} \left[\int_{\mathbb{R}^{n-1}} \frac{K_j(z^*, \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right], \quad (2.20)$$

for $0 < s < \tau < t_0$ and $z^* \in \mathcal{L}_l(\Omega \cap W_{z_k})$, where, for $j = 0, 1, 2$, Q_j are polynomials and K_j are continuous functions, \mathcal{C}^1 with respect to τ , $s \in (0, t_0)$ and satisfy

$$\max_{\substack{i=0,1,2 \\ \alpha_1 + \alpha_2 \leq 1}} \int_{\mathbb{R}^{n-1}} \left| \partial_{\tau}^{\alpha_1} \partial_s^{\alpha_2} K_j(z^*, \tau; y'', s; v_1) \right| dy'' \leq C_l, \quad 0 < s < \tau < t_0, \quad z_1^* > 0, \quad 0 < v_1 < t_0,$$

where $C_l > 0$ is a constant. Furthermore, using representation (2.20), we have, for $s < \tau < t < t_0$,

$$\begin{aligned} & \int_{\Omega} H(x, t - \tau; z) \int_{\Gamma} K(\tau, z; y, s) f(y, s) d\sigma(y) dz \\ &= \sum_{j=0}^2 \sum_{l=1}^m \int_{\mathbb{R}_+^n} \omega_l(x) H_{z_l}(x, t - \tau; \mathcal{L}_l^{-1}(z^*)) \chi_l(z^*) Q_j \left(\frac{z_1^*}{\sqrt{\tau - s}} \right) e^{-\frac{(z_1^*)^2}{\tau - s}} \left[\int_{\mathbb{R}^{n-1}} \frac{K_j(z^*, \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right] dz^* \end{aligned}$$

with $\mathbb{R}_+^n = \{(z_1^*, \dots, z_n^*) \in \mathbb{R}^n; z_1^* > 0\}$. Then, applying the substitutions $z'' = \frac{x' - z'}{\sqrt{t - \tau}}$ and $z_1' = \frac{z_1^*}{\sqrt{\tau - s}}$, we deduce, in view of the form of the functions K_j , the following

$$\begin{aligned} & \int_{\Omega} H(x, t - \tau; z) \int_{\Gamma} K(z, \tau; y, s) d\sigma(y) dz \\ &= \sum_{j=0}^2 \int_{\mathbb{R}_+^n} \frac{H_l'(x', t - \tau; (z_1', z''), \tau - s)}{\sqrt{t - \tau}} \left[\int_{\mathbb{R}^{n-1}} \frac{K_j'((z_1', z''), \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right] dz'' dz_1', \quad s < \tau < t < t_0, \end{aligned} \quad (2.21)$$

for some continuous functions K_0', K_1' and H_l' such that K_0', K_1' are \mathcal{C}^1 , with respect to s and τ , and the following estimates hold:

$$\int_{\mathbb{R}_+^n} |H_l'(x', t - \tau; (z_1', z''), \tau - s)| dz'' \leq C_l, \quad 0 < s < \tau < t < t_0, \quad (2.22)$$

$$\max_{\substack{j=0,1 \\ \alpha_1 + \alpha_2 \leq 1}} \int_{\mathbb{R}^{n-1}} \left| \partial_{\tau}^{\alpha_1} \partial_s^{\alpha_2} K_j'((z_1', z''), \tau; y'', s; v_1) \right| dy'' \leq C_l, \quad 0 < s < \tau < t_0, \quad 0 < v_1 < t_0, \quad (2.23)$$

for some constant $C_l > 0$. Repeating the arguments used for (2.21) and applying some results of page 31 of [It], we obtain, for $0 < t < t_0$,

$$\begin{aligned} \int_0^t \int_s^t \int_{\Omega} |H(x, t - \tau; z)| \int_{\Gamma} |K(z, \tau; y, s)| d\sigma(y) dz d\tau ds &\leq C_l \int_0^t \int_s^t \left[\sum_{j=0}^1 \frac{1}{\sqrt{t-\tau}} \cdot \frac{1}{(\tau-s)^{\frac{j}{2}}} \right] d\tau ds \\ &\leq C_l \sum_{j=0}^1 \int_0^t (t-s)^{1-\frac{j}{2}} ds \leq C_l. \end{aligned} \quad (2.24)$$

This estimate and Fubini's theorem imply

$$F_2(x, t) = \int_0^t \int_s^t \int_{\Omega} H(x, t - \tau; z) \int_{\Gamma} K(z, \tau; y, s) f(y, s) d\sigma(y) dz d\tau ds.$$

Then, in view of representation (2.21), for all $0 < t < t_0$,

$$\begin{aligned} F_2(x, t) &= \int_0^t \int_s^t \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', t - \tau; (z'_1, z''), \tau - s)}{\sqrt{t-\tau}} \\ &\quad \times \left[\int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} f_1(x', s; y'', z'') dy'' \right] dz'' dz'_1 d\tau ds, \end{aligned}$$

where $f_1(x', s; y'', z'') = f(\mathcal{L}_l^{-1}(0, x' - (\sqrt{t-s})z'' - (\sqrt{\tau-s})y''), s)$. Making the substitution $\tau' = t - \tau$, we obtain

$$\begin{aligned} F_2(t, x) &= \int_0^t \int_0^{t-s} \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), t - s - \tau')}{\sqrt{\tau'}} \\ &\quad \times \left[\int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', s; t - s - \tau')}{(t - s - \tau')^{\frac{j}{2}}} f_1(x', s; y'', z'') dy'' \right] dz'' dz'_1 d\tau' ds. \end{aligned}$$

Then, the substitution $s' = t - s$ yields

$$\begin{aligned} F_2(x, t) &= \int_0^t \int_0^{s'} \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \\ &\quad \times \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} f_1(x', t - s'; y'', z'') dy'' dz'_1 dz'' d\tau' ds'. \end{aligned}$$

But, for $0 < \tau' < s' < t < t_0$, estimates (2.22), (2.23) and $f(y, 0) = 0, y \in \Gamma$, imply

$$\begin{aligned} \left| \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} \right. \\ \left. \times f_1(x', t - s'; y'', z'') dy'' dz'_1 dz'' \right| \leq C_l \sum_{j=0}^1 \frac{\|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{\tau'}} \frac{1}{(s' - \tau')^{\frac{j}{2}}} \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} \left| \partial_t \left(\sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} \right. \right. \\ \left. \left. \times f_1(x', t - s'; y'', z'') dy'' dz'_1 dz'' \right) \right| \leq C_l \sum_{j=0}^1 \frac{\|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{\tau'}} \frac{1}{(s' - \tau')^{\frac{j}{2}}}. \end{aligned} \quad (2.26)$$

From estimates (2.25), (2.26) and $f(y, 0) = 0$, $y \in \Gamma$, we conclude that F_2 admits a derivative with respect to t and

$$\begin{aligned} \partial_t F_2(x, t) &= \int_0^t \int_0^{s'} \partial_t \left(\sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{1}{2}}} \right. \\ &\quad \left. \times f_1(x', t - s'; y''; z'') dy'' dz'_1 dz'' \right) d\tau' ds'. \end{aligned}$$

Moreover, (2.24) and (2.26) imply (2.19) and (2.18). Finally, we obtain (2.14) from (2.17) and (2.18). This completes the proof. \square

3. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1. Let $u = u_1 - u_2$ and $\sigma = \sigma_2 - \sigma_1$. Then u is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u + \sigma_2(t)f(x)u = \sigma(t)f(x)u_1(x, t), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & (x, t) \in \Sigma. \end{cases} \quad (3.1)$$

Let $U(x, t; y, s)$ be the fundamental solution of (2.1) with $q(x, t) = -\sigma_2(t)f(x)$. Applying Theorem 9.1 of [It], we obtain

$$u(x, t) = \int_0^t \int_{\Omega} U(x, t; y, s) \sigma(s) f(y) u_1(y, s) dy ds + \int_0^t \int_{\Gamma} U(x, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds. \quad (3.2)$$

Now, since $u(x, t) = 0$, $(t, x) \in \Sigma$ and $x \in \Gamma$,

$$\int_0^t \int_{\Omega} U(x, t; y, s) \sigma(s) f(y) u_1(y, s) dy ds = - \int_0^t \int_{\Gamma} U(x, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds. \quad (3.3)$$

In view of differentiability properties in Lemma 1 and 2, we can take the t -derivative of both sides of identity (3.2). We find

$$\begin{aligned} f(x)g(x, t)\sigma(t) &= - \int_0^t \partial_t \left(\int_{\Omega} U(x, t; y, s) \sigma(s) f(y) u_1(y, s) dy \right) ds \\ &\quad - \partial_t \left(\int_0^t \int_{\Gamma} U(x, t; y, s) (\partial_{\nu} u_2(y, s) - \partial_{\nu} u_1(y, s)) d\sigma(y) ds \right) \end{aligned}$$

and, for $x = x_0$, condition (H2) implies

$$\begin{aligned} \sigma(t) &= h(t) \int_0^t \partial_t \left(\int_{\Omega} U(x_0, t; y, s) \sigma(s) f(y) u_1(y, s) dy \right) ds \\ &\quad + h(t) \partial_t \left(\int_0^t \int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds \right), \end{aligned} \quad (3.4)$$

where $h(t) = -1/(g(t, x_0)f(x_0))$.

Since $u(x, 0) = 0$, $x \in \Omega$, we have $\partial_{\nu} u(x, 0) = 0$, $x \in \Gamma$. Thus, the estimates in Lemma 1 and 2 lead

$$\left| \int_0^t \partial_t \left(\int_{\Omega} U(x_0, t; y, s) \sigma(s) f(y) u_1(y, s) dy \right) ds \right| \leq C \int_0^t |\sigma(s)| ds, \quad (3.5)$$

$$\left| \partial_t \left(\int_0^t \int_{\Gamma} U(x, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds \right) \right| \leq C \|\partial_t \partial_{\nu} u\|_{L^{\infty}(\Sigma)}. \quad (3.6)$$

Therefore, representation (3.4) and estimates (1.5), (3.5), (3.6) imply

$$|\sigma(t)| \leq \int_0^t C |\sigma(s)| ds + C \|\partial_t \partial_\nu u\|_{L^\infty(\Sigma)}.$$

Here and henceforth, $C > 0$ is a generic constant depending only on data. Hence, Gronwall's lemma yields

$$|\sigma(t)| \leq C \|\partial_t \partial_\nu u\|_{L^\infty(\Sigma)} e^{Ct} \leq C e^{CT} \|\partial_t \partial_\nu u\|_{L^\infty(\Sigma)}, \quad t \in (0, T).$$

Then (1.3) follows and the proof is complete. \square

Proof of Theorem 2. Set $u = u_1 - u_2 = u(\sigma_1) - u(\sigma_2)$. Then, according to (H4) and (H6), u is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u - q(x, t)u = F(t, x, \sigma_1(t), u_2(x, t)) - F(t, x, \sigma_2(t), u_2(x, t)), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(t, x) = 0, & (t, x) \in \Sigma, \end{cases} \quad (3.7)$$

with

$$q(x, t) = \int_0^1 \partial_u F[t, x, \sigma_1(t), u_2(x, t) + \tau(u_1(x, t) - u_2(x, t))] d\tau. \quad (3.8)$$

Note that assumptions (H4) and (H6) imply that $q \in C^1(\overline{Q})$.

On the other hand, in view of (H4),

$$F(t, x, \sigma_1(t), u_2(x, t)) - F(t, x, \sigma_2(t), u_2(x, t)) = (\sigma_1(t) - \sigma_2(t))G(x, t),$$

with

$$G(x, t) = \int_0^1 \partial_\sigma F(t, x, \sigma_2(t) + s(\sigma_1(t) - \sigma_2(t)), u_2(x, t)) ds.$$

Using this representation, we deduce that u is the solution of

$$\begin{cases} \partial_t u - \Delta_x u - q(x, t)u = (\sigma_1(t) - \sigma_2(t))G(x, t), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & (t, x) \in \Sigma. \end{cases} \quad (3.9)$$

Let us remark that (H4) and (H6) imply that $G \in C^{2,1}(\overline{Q})$. Let $U(t, x; s, y)$ be the fundamental solution of (2.1) with $q(x, t)$ defined by (3.8). Then, according to Theorem 9.1 of [It], for $\sigma(t) = \sigma_1(t) - \sigma_2(t)$, we have the representation

$$u(x, t) = \int_0^t \int_\Omega U(x, t; y, s) \sigma(s) G(y, s) dy ds + \int_0^t \int_\Gamma U(x, t; y, s) \partial_\nu u(y, s) d\sigma(y) ds.$$

Since $u(x, t) = 0$, $(t, x) \in \Sigma$, we obtain

$$\int_0^t \int_\Omega U(x_0, t; y, s) \sigma(s) G(y, s) dy ds = - \int_0^t \int_\Gamma U(x_0, t; y, s) \partial_\nu u(y, s) d\sigma(y) ds, \quad (3.10)$$

with x_0 defined in assumption (H5). Combining Lemma 1 and Lemma 2 with some arguments used in the proof of Theorem 1, we prove that f_1 and f_2 defined respectively by

$$\begin{aligned} f_1(t) &= \int_0^t \int_\Omega U(x_0, t; y, s) \sigma(s) G(y, s) dy ds, \\ f_2(t) &= \int_0^t \int_\Gamma U(x_0, t; y, s) \partial_\nu u(y, s) d\sigma(y) ds, \end{aligned}$$

admit a derivative with respect to t and

$$f_1'(t) = \sigma(t)G(x_0, t) + \int_0^t \partial_t \left(\int_\Omega U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds,$$

$$\begin{aligned}
f_2'(t) &= \int_0^t \partial_t \left(\int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds, \\
\left| \int_0^t \partial_t \left(\int_{\Omega} U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds \right| \\
&\leq C \int_0^t |\sigma(s)| \|G(\cdot, s)\|_{C_x^2(\bar{\Omega})} ds \leq C \int_0^t |\sigma(s)| ds
\end{aligned} \tag{3.11}$$

and

$$\left| \int_0^t \partial_t \left(\int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds \right| \leq C \|\partial_t \partial_{\nu} u\|_{L^{\infty}(\Sigma)}. \tag{3.12}$$

Here and in the sequel $C > 0$ is a generic constant that can depend only on data.

Taking the t -derivative of both sides of identity (3.10), we obtain

$$\begin{aligned}
\sigma(t)G(x_0, t) &= - \int_0^t \partial_t \left(\int_{\Omega} U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds \\
&\quad - \int_0^t \partial_t \left(\int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds.
\end{aligned}$$

Let us observe that (H5) and $\max(\|\sigma_1\|_{\infty}, \|\sigma_2\|_{\infty}) \leq M$ imply

$$\begin{aligned}
|G(x_0, t)| &= \int_0^1 |\partial_{\sigma} F(t, x_0, \sigma_2(t) + s(\sigma_1(t) - \sigma_2(t)), g(x_0, t))| ds \\
&\geq \inf_{t \in [0, T], \sigma \in [-M, M]} |\partial_{\sigma} F(t, x_0, \sigma, g(x_0, t))| > 0.
\end{aligned} \tag{3.13}$$

Then,

$$\begin{aligned}
\sigma(t) &= H(t) \int_0^t \partial_t \left(\int_{\Omega} U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds \\
&\quad + H(t) \int_0^t \partial_t \left(\int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds,
\end{aligned}$$

where $H(t) = -1/G(x_0, t)$. Hence, (3.11), (3.12) and (3.13) imply

$$|\sigma(t)| \leq \int_0^t C |\sigma(s)| ds + C \|\partial_t \partial_{\nu} u\|_{L^{\infty}(\Sigma)}.$$

We complete the proof of Theorem 2 by applying Gronwall's lemma. \square

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