



# Stability of the determination of a time-dependent coefficient in parabolic equations

Mourad Choulli, Yavar Kian

## ► To cite this version:

Mourad Choulli, Yavar Kian. Stability of the determination of a time-dependent coefficient in parabolic equations. 2012. hal-00673690v1

**HAL Id: hal-00673690**

**<https://hal.science/hal-00673690v1>**

Preprint submitted on 24 Feb 2012 (v1), last revised 28 Feb 2012 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STABILITY OF THE DETERMINATION OF A TIME-DEPENDENT COEFFICIENT IN PARABOLIC EQUATIONS

MOURAD CHOULLI AND YAVAR KIAN

**ABSTRACT.** We establish a Lipschitz stability estimate for the inverse problem consisting in the determination of the coefficient  $\sigma(t)$ , appearing in a Dirichlet initial-boundary value problem for the parabolic equation  $\partial_t u - \Delta_x u + \sigma(t)f(x)u = 0$ , from Neumann boundary data. We extend this result to the same inverse problem when the previous linear parabolic equation is changed to the semi-linear parabolic equation  $\partial_t u - \Delta_x u = F(t, x, \sigma(t), u(x, t))$ .

**Key words :** parabolic equation, semi-linear parabolic equation, inverse problem, determination of time-dependent coefficient, stability estimate.

**AMS subject classifications :** 35R30.

## CONTENTS

1.	Introduction	1
2.	Time-differentiability of potential-type functions	4
3.	Proof of Theorems 1 and 2	12
	References	14

## 1. INTRODUCTION

Throughout this paper, we assume that  $\Omega$  is a  $\mathcal{C}^3$  bounded domain of  $\mathbb{R}^n$  with  $n \geq 2$ . Let  $T > 0$  and set

$$Q = \Omega \times (0, T), \quad \Gamma = \partial\Omega, \quad \Sigma = \Gamma \times (0, T).$$

We consider the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u + \sigma(t)f(x)u = 0, & (x, t) \in Q, \\ u(x, 0) = h(x), & x \in \Omega, \\ u(x, t) = g(x, t), & (x, t) \in \Sigma. \end{cases} \quad (1.1)$$

We introduce the following assumptions :

(H1)  $f \in \mathcal{C}^2(\overline{\Omega})$ ,  $h \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ ,  $g \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Sigma})$ , for some  $0 < \alpha < 1$ , and satisfy the compatibility condition

$$\partial_t g(x, 0) - \Delta_x h(x) + \sigma(0)f(x)h(x) = 0, \quad x \in \Gamma.$$

(H2) There exists  $x_0 \in \Gamma$  such that

$$\inf_{t \in [0, T]} |g(x_0, t)f(x_0)| > 0.$$

Under assumption (H1), It is well known that, for  $\sigma \in \mathcal{C}^1[0, T]$ , the initial-boundary value problem (1.1) admits a unique solution  $u = u(\sigma) \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})$  (see Theorem 5.2 of [LSU]). Moreover, given  $M > 0$ , there exists a constant  $C > 0$  depending only on data (that is  $\Omega$ ,  $T$ ,  $f$ ,  $g$  and  $h$ ) such that  $\|\sigma\|_{W^{1,\infty}(0, T)} \leq M$  implies

$$\|u(\sigma)\|_{\mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})} \leq C. \quad (1.2)$$

In the present paper we are concerned with the inverse problem consisting in the determination of the time dependent coefficient  $\sigma(t)$  from Neumann boundary data  $\partial_\nu u(\sigma)$  on  $\Sigma$ , where  $\partial_\nu$  is the derivative in the direction of the unit outward normal vector to  $\Gamma$ .

We prove the following theorem, where  $B(M)$  is the ball of  $C^1[0, T]$  centered at 0 and with radius  $M > 0$ .

**Theorem 1.** *Assume that (H1) and (H2) are fulfilled. For  $i = 1, 2$ , let  $\sigma_i \in B(M)$  and  $u_i = u(\sigma_i)$ . Then there exists a constant  $C > 0$ , depending only on data, such that*

$$\|\sigma_2 - \sigma_1\|_{L^\infty(0, T)} \leq C \|\partial_t \partial_\nu u_2 - \partial_t \partial_\nu u_1\|_{L^\infty(\Sigma)}. \quad (1.3)$$

Following [COY], it is quite natural to extend Theorem 1 when the linear parabolic equation is changed to a semi-linear parabolic equation. To this end, introduce the following semi-linear initial-boundary value problem :

$$\begin{cases} \partial_t u - \Delta_x u = F(x, t, \sigma(t), u(x, t)), & (x, t) \in Q, \\ u(x, 0) = h(x), & x \in \Omega, \\ u(x, t) = g(x, t), & (x, t) \in \Sigma \end{cases} \quad (1.4)$$

and consider the following assumptions

(H3)  $h \in C^{2, \alpha}(\overline{\Omega})$ ,  $g \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Sigma})$ , for some  $0 < \alpha < 1$ , and satisfy the compatibility condition

$$\partial_t g(x, 0) - \Delta_x h(x) = F(0, x, \sigma(0), h(x)), \quad x \in \Gamma.$$

(H4)  $F \in C^1(\overline{\Omega}_x \times \mathbb{R}_t \times \mathbb{R}_\sigma \times \mathbb{R}_u)$  is such that  $\partial_u F$  and  $\partial_\sigma F$  are  $C^1$ ,  $F$  and  $\partial_\sigma F$  are  $C^2$  with respect to  $x$  and  $u$ .

(H5) There exist  $M > 0$  and  $x_0 \in \Gamma$  such that

$$\inf_{t \in [0, T], \sigma \in [-M, M]} |\partial_\sigma F(x_0, t, \sigma, g(x_0, t))| > 0.$$

(H6) There exist two non negative constants  $c$  and  $d$  such that

$$uF(x, t, \sigma(t), u) \leq cu^2 + d, \quad t \in [0, T], \quad x \in \overline{\Omega}, \quad u \in \mathbb{R}.$$

Under the above mentioned conditions, for any  $\sigma \in C^1[0, T]$ , the initial-boundary value problem (1.4) admits a unique solution  $u = u(\sigma) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q})$  (see Theorem 6.1 in [LSU]) and, given  $M > 0$ , there exists a constant  $C > 0$  depending only on data (that is  $\Omega$ ,  $T$ ,  $F$ ,  $g$  and  $h$ ) such that  $\|\sigma\|_{W^{1, \infty}(0, T)} \leq M$  implies

$$\|u(\sigma)\|_{C^{2+\alpha, 1+\alpha/2}(\overline{Q})} \leq C. \quad (1.5)$$

We have the following extension of Theorem 1.

**Theorem 2.** *Assume that (H3), (H4), (H5) and (H6) are fulfilled. For  $i = 1, 2$ , let  $\sigma_i \in B(M)$  and  $u_i = u(\sigma_i)$ . Then there exists a constant  $C > 0$ , depending only on data, such that*

$$\|\sigma_2 - \sigma_1\|_{L^\infty(0, T)} \leq C \|\partial_t \partial_\nu u_2 - \partial_t \partial_\nu u_1\|_{L^\infty(\Sigma)}. \quad (1.6)$$

**Remark 1.** *Let us observe that we can generalize the results in Theorems 1 and 2 as follows:*

i) In (1.1), we can replace  $\sigma(t)f(x)$  by  $\sum_{k=1}^p \sigma_k(t)f_k(x)$ , where  $f_k$ ,  $1 \leq k \leq p$ , are known. Assume that (H1) is satisfied, with  $f = f_k$  for each  $k$ , where the compatibility condition is changed to

$$\partial_t g(x, 0) - \Delta_x h(x) + \sum_{k=1}^p \sigma_k(0)f_k(x)h(x) = 0, \quad x \in \Gamma.$$

Therefore, to each  $(\sigma_1, \dots, \sigma_p) \in C[0, T]^p$  corresponds a unique solution  $u = u(\sigma_1, \dots, \sigma_p) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q})$  and  $\max\{\|\sigma_k\|_{W^{1, \infty}(0, T)}; 1 \leq k \leq p\} \leq M$  implies

$$\|u(\sigma_1, \dots, \sigma_p)\|_{C^{2+\alpha, 1+\alpha/2}(\overline{Q})} \leq C,$$

for some positive constant  $C$  depending only on data.

Following the proof of Theorem 1, we prove that, under the following conditions : there exists  $x_1, \dots, x_p \in \Gamma$  such that the matrix  $M(t) = (f_k(x_l)g(x_l, t))$  is invertible for any  $t \in [0, T]$ ,

$$\max_{1 \leq k \leq p} \|\sigma_k^1 - \sigma_k^2\|_{L^\infty(0, T)} \leq C \|\partial_t \partial_\nu u_2 - \partial_t \partial_\nu u_1\|_{L^\infty(\Sigma)},$$

if  $\sigma_k^j \in B(M)$ ,  $1 \leq k \leq p$  and  $j = 1, 2$ . Here  $C$  is a constant that can depend only on data and  $u_j = u(\sigma_1^j, \dots, \sigma_p^j)$ ,  $j = 1, 2$ .

ii) We can replace the semi-linear parabolic equation in (1.4) by a semi-linear integro-differential equation. In other words,  $F$  can be changed to

$$F_1(x, t, \sigma(t), u(x, t)) + \int_0^t F_2(x, s, \sigma(t-s), u(x, s)) ds.$$

Under appropriate assumptions on  $F_1$  and  $F_2$ , one can establish that Theorem 2 is still valid in the present case.

ii) Both in (1.1) and (1.4), the Laplace operator can be replaced by a second order elliptic operator in divergence form :

$$E = \nabla \cdot A(x) \nabla + B(x) \cdot \nabla,$$

where  $A(x) = (a_{ij}(x))$  is a symmetric matrix with coefficients in  $C^{1+\alpha}(\overline{\Omega})$ ,  $B(x) = (b_i(x))$  is a vector with components in  $C^\alpha(\overline{\Omega})$  and the following ellipticity condition holds

$$A(x)\xi \cdot \xi \geq \lambda |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega}.$$

Actually, the normal derivative associated to  $E$  is the boundary operator  $\partial_{\nu_E} = \nu(x) \cdot A(x) \nabla$ .

To our knowledge, there are only few results concerning the determination of a time-dependent coefficient in an initial-boundary value problem for a parabolic equation from a single measurement. The determination of a source term of the form  $f(t)\chi_D(x)$ , where  $\chi_D$  the characteristic function of the known subdomain  $D$ , was considered by J. R. Canon and S. P. Esteva. They established in [CE86-1] a logarithmic stability estimate in 1D case in a half line when the overdetermined data is the trace at the end point. A similar inverse problem in 3D case was studied by these authors in [CE86-2], where they obtained a Lipschitz stability estimate in weighted spaces of continuous functions. The case of a non local measurement was considered by J. R. Canon and Y. Lin in [CL88] and [CL90], where they proved existence and uniqueness for both quasilinear and semi-linear parabolic equations. The determination of a time dependent coefficient in an abstract integrodifferential equation was studied by the first author in [Ch91-1]. He proved existence, uniqueness and Lipschitz stability estimate, extending earlier results by [Ch91-2], [LS87], [LS88], [PO85-1] and [PO85-2]. In [CY06], the first author and M. Yamamoto obtained a stability result, in a restricted class, for the inverse problem of determining a source term  $f(x, t)$ , appearing in a Dirichlet initial-boundary value problem for the heat equation, from Neumann boundary data. In a recent work, the first author and M. Yamamoto [CY11] considered the inverse problem of finding a control parameter  $p(t)$  that reach a desired temperature  $h(t)$  along a curve  $\gamma(t)$  for a parabolic semi-linear equation with homogeneous Neumann boundary data and they established existence, uniqueness as well as Lipschitz stability. Using geometric optic solutions, the first author [Ch09] proved uniqueness as well as stability for the inverse problem of determining a general time dependent coefficient of order zero for parabolic equations from Dirichlet to Neumann map. In [E07] and [E08], G. Eskin considered the same inverse problem for hyperbolic and the Schrödinger equations with time-dependent electric and magnetic potential and he established uniqueness by gauge invariance. Recently, R. Salazar [Sa] extended the result of [E07] and obtained a stability result for compactly supported coefficients.

We would like to mention that the determination of space dependent coefficient  $f(x)$ , in the source term  $\sigma(t)f(x)$ , from Neumann boundary data was already considered by the first author and M. Yamamoto [CY06]. But, it seems that our paper is the first work where one treats the determination of a time dependent coefficient, appearing in a parabolic initial-boundary value problem, from Neumann boundary data.

This paper is organized as follows. In section 2 we come back to the construction of the Neumann fundamental solution by [It] and establish time-differentiability of some potential-type functions, necessary for proving Theorems 1 and 2. Section 3 is devoted to the proof of Theorems 1 and 2.

## 2. TIME-DIFFERENTIABILITY OF POTENTIAL-TYPE FUNCTIONS

In this section, we establish time-differentiability of some potential-type functions, needed in the proof of our stability estimates. In our analysis we follow the construction of the fundamental solution by S. Itô [It].

First of all, we recall the definition of fundamental solution associate to the heat equation plus a time-dependent coefficient of order zero, in the case of Neumann boundary condition. Consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta_x u + q(x, t)u, & (x, t) \in \Omega \times (s, t_0), \\ \lim_{t \rightarrow s} u(x, t) = u_0(x), & x \in \Omega, \\ \partial_\nu u(x, t) = 0, & (x, t) \in \Gamma \times (s, t_0). \end{cases} \quad (2.1)$$

Here  $s_0 < t_0$  are fixed,  $s \in (s_0, t_0)$ ,  $u_0$  and  $q(x, t)$  are continuous respectively in  $\overline{\Omega}$  and in  $\overline{\Omega} \times [s, t_0]$ . Let  $U(x, t; y, s)$  be a continuous function in the domain  $s_0 < s < t < t_0$ ,  $x \in \overline{\Omega}$ ,  $y \in \overline{\Omega}$ . We recall that  $U$  is the fundamental solution of (2.1) if for any  $u_0 \in \mathcal{C}(\overline{\Omega})$ ,

$$u(x, t) = \int_{\Omega} U(x, t; y, s) u_0(y) dy$$

is the solution of (2.1). We refer to [It] for the existence and uniqueness of this fundamental solution.

We start with time-differentiability of volume potential-type functions<sup>1</sup>.

**Lemma 1.** *Fix  $s \in (s_0, t_0)$ . Let  $f \in \mathcal{C}(\overline{\Omega} \times [s, t_0])$  be  $\mathcal{C}^2$  with respect to  $x$ ,  $q \in \mathcal{C}^1(\overline{\Omega} \times [s, t_0])$  and define, for  $(x, t) \in \overline{\Omega} \times (s, t_0)$ ,*

$$f^1(x, t; \tau) = \int_{\Omega} U(x, t; y, \tau) f(y, \tau) dy, \quad t > \tau > s.$$

*Then,  $f^1$  admits a derivative with respect to  $t$  and*

$$\begin{aligned} \frac{\partial f^1}{\partial t}(x, t; \tau) &= \int_{\Omega} U(x, t; y, \tau) (\Delta_y + q(x, \tau)) f(y, \tau) dy \\ &+ \int_{\tau}^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau') \partial_t q(z, \tau') U(z, \tau'; y, \tau) f(y, \tau) dz dy d\tau'. \end{aligned} \quad (2.2)$$

*Moreover,  $F$  given by*

$$F(x, t) = \int_s^t f^1(x, t; \tau) d\tau, \quad (x, t) \in \Gamma \times (s_0, t_0),$$

*possesses a derivative with respect to  $t$ ,*

$$\frac{\partial F}{\partial t}(x, t) = f(x, t) + \int_s^t \frac{\partial f^1}{\partial t}(x, t; \tau) d\tau \quad (2.3)$$

*and*

$$\left| \int_s^t \frac{\partial f^1}{\partial t}(x, t; \tau) d\tau \right| \leq C \int_s^t \|f(\cdot, \tau)\|_{C_x^2(\overline{\Omega})} d\tau. \quad (2.4)$$

---

<sup>1</sup>Recall that if  $\varphi = \varphi(x, t)$  is a continuous function then the corresponding volume potential is given by

$$\psi(x, t) = \int_s^t \int_{\Omega} U(x, t; y, \tau) \varphi(y, \tau) dy d\tau.$$

*Proof.* We have only to prove (2.2) and (2.4), because (2.3) follows immediately from (2.2).

Let then  $u_0 \in \mathcal{C}^2(\overline{\Omega})$  and consider the function

$$u(x, t) = \int_{\Omega} U(x, t; y, s) u_0(y) dy, \quad x \in \overline{\Omega}, \quad s < t < t_0.$$

We show that  $u$  admits a derivative with respect to  $t$  and

$$\begin{aligned} \partial_t u(x, t) &= \partial_t \left( \int_{\Omega} U(x, t; y, s) u_0(y) dy \right) \\ &= \int_{\Omega} U(x, t; y, s) (\Delta_y + q(x, s)) u_0(y) dy \\ &\quad - \int_s^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau) q_t(z, \tau) U(z, \tau; y, s) u_0(y) dz dy d\tau. \end{aligned} \quad (2.5)$$

We need to consider first the case  $u_0 = w_0 \in \mathcal{C}^\infty(\overline{\Omega})$ . Set

$$w(x, t) = \int_{\Omega} U(x, t; z, s) w_0(y) dy, \quad x \in \overline{\Omega}, \quad s < t < t_0.$$

Clearly,  $w(x, t)$  is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t w - \Delta_x w - q(x, t)w = 0, & (x, t) \in \Omega \times (s, t_0), \\ \lim_{t \rightarrow s} w(x, t) = w_0(x), & x \in \Omega, \\ \partial_\nu w(x, t) = 0, & (x, t) \in \Gamma \times (s, t_0) \end{cases}$$

and  $w_1 = \partial_t w$  satisfies

$$\begin{cases} \partial_t w_1 - \Delta_x w_1 - q(x, t)w_1 = -\partial_t q w, & (x, t) \in \Omega \times (s, t_0), \\ \lim_{t \rightarrow s} w_1(x, t) = (\Delta_x + q(x, s))w_0(x), & x \in \Omega, \\ \partial_\nu w_1(x, t) = 0, & (x, t) \in \Gamma \times (s, t_0). \end{cases}$$

Therefore, (2.5), with  $w$  in place of  $u$ , is a consequence of Theorem 9.1 of [It].

Next, let  $(w_0^n)_n$  be a sequence in  $\mathcal{C}^\infty(\overline{\Omega})$  converging to  $u_0$  in  $\mathcal{C}^2(\overline{\Omega})$  and  $v(x, t)$  given by

$$\begin{aligned} v(x, t) &= \int_{\Omega} U(x, t; y, s) (\Delta_x + q(x, s)) u_0(y) dy \\ &\quad - \int_s^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau) \partial_t q(z, \tau) U(z, \tau; y, s) u_0(y) dz dy d\tau. \end{aligned}$$

Consider  $(w_n)_n$ , the sequence of functions, defined by

$$w_n(x, t) = \int_{\Omega} U(x, t; z, s) w_0^n(y) dy.$$

We proved that, for any  $n \in \mathbb{N}$ ,

$$\begin{aligned} \partial_t w_n(x, t) &= \int_{\Omega} U(x, t; y, s) (\Delta_y + q(x, s)) w_0^n(y) dy \\ &\quad - \int_s^t \int_{\Omega} \int_{\Omega} U(x, t; z, \tau) \partial_t q(z, \tau) U(z, \tau; y, s) w_0^n(y) dz dy d\tau. \end{aligned} \quad (2.6)$$

From the proof of Theorem 7.1 of [It],

$$\int_{\Omega} |U(x, t; y, s)| dy \leq C e^{C(t-s)}, \quad (x, t) \in \overline{\Omega} \times (s, t_0). \quad (2.7)$$

Therefore, we can pass to the limit, as  $n$  goes to infinity, in (2.6). We deduce that  $\partial_t w_n$  converges to  $v$  in  $\mathcal{C}(\overline{\Omega} \times [s, t_0])$ . But,  $w_n$  converges to  $u$  in  $\mathcal{C}(\overline{\Omega} \times [s, t_0])$ . Hence  $u$  admits a derivative with respect to  $t$  and

$\partial_t u = v$ . That is we proved (2.5) and consequently (2.2) holds true. Finally, we note that (2.4) is deduced easily from (2.7).  $\square$

Next, we consider time-differentiability a single layer potential-type function<sup>2</sup>.

**Lemma 2.** *Fix  $s \in (s_0, t_0)$ . Let  $f \in \mathcal{C}(\Gamma \times [s, t_0])$  be  $\mathcal{C}^1$  with respect to  $t \in [s, t_0]$  with  $f(x, s) = 0$ . Define, for  $(x, t) \in \Gamma \times (s, t_0)$ ,*

$$f^1(x, t; \tau) = \int_{\Gamma} U(x, t; y, \tau) f(y, \tau) d\sigma(y), \quad t > \tau > s.$$

Then

$$F(x, t) = \int_s^t f^1(x, t; \tau) d\tau$$

is well defined, admits a derivative with respect to  $t$  and we have

$$\left\| \frac{\partial F}{\partial t} \right\|_{L^\infty(\Gamma \times (s, t_0))} \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (s, t_0))}. \quad (2.8)$$

Contrary to Lemma 1, for Lemma 2 we cannot use directly the general properties of the fundamental solutions developed in [It]. We need to come back to the construction of the fundamental solution of (2.1) introduced by [It]. First, consider the heat equation  $\partial_t u = \Delta_x u$  in the half space  $\Omega_1 = \{x = (x_1, \dots, x_n); x_1 > 0\}$  in  $\mathbb{R}^n$  with the boundary condition  $\partial_{x_1} u = 0$  on  $\Gamma_1 = \{x = (0, x_2, \dots, x_n); (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}$ . For any  $y = (y_1, y_2, \dots, y_n)$ , we define  $\bar{y}$  by  $\bar{y} = (-y_1, y_2, \dots, y_n)$ . Let

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{t}}$$

denotes the Gaussian kernel and set

$$G_1(x, t; y) = G(x - y, t) + G(x - \bar{y}, t).$$

Then, the fundamental solution  $U_0(x, t; y, s)$  of

$$\begin{cases} \partial_t u = \Delta_x u, & (x, t) \in \Omega_1 \times (s, t_0), \\ \lim_{t \rightarrow s} u(x, t) = u_0(x), & x \in \Omega_1, \\ \partial_\nu u(x, t) = 0, & (x, t) \in \Gamma_1 \times (s, t_0) \end{cases} \quad (2.9)$$

is given by

$$U_0(x, t; y, s) = G_1(x, t - s; y).$$

In order to construct the fundamental solution in the case of an arbitrary domain  $\Omega$ , Itô introduced the following local coordinate system around each point  $z \in \Gamma$ .

**Lemma 3.** (Lemma 6.1 and its corollary, Chapter 6 of [It]) *For every point  $z \in \Gamma$ , there exist a coordinate neighborhood  $W_z$  of  $z$  and a coordinate system  $(x_1^*, \dots, x_n^*)$  satisfying the following conditions:*

1) *the coordinate transformation between the coordinate system  $(x_1^*, \dots, x_n^*)$  and the original coordinate system in  $W_z$  is of class  $C^2$  and the partial derivatives of the second order of the transformation functions are Hölder continuous ;*

2)  *$\Gamma \cap W_z$  is represented by the equation  $x_1^* = 0$  and  $\Omega \cap W_z$  is represented by  $x_1^* > 0$  ;*

3) *let  $\mathcal{L}$  be the diffeomorphism from  $W_z$  to  $\mathcal{L}(W_z)$  defined by*

$$\mathcal{L} : W_z \rightarrow \mathcal{L}(W_z)$$

$$x \mapsto (x_1^*(x), \dots, x_n^*(x)).$$

<sup>2</sup>The single-layer potential corresponding to a continuous function  $\varphi = \varphi(x, t)$  is given by

$$\psi(x, t) = \int_s^t \int_{\Gamma} U(x, t; y, \tau) \varphi(y, \tau) d\sigma(y) d\tau.$$

Then, for any  $u \in C^1(\overline{\Omega})$  we have

$$\partial_\nu u(\xi) = -\partial_{x_1}(u \circ \mathcal{L}^{-1})(x), \quad \xi \in \Gamma \cap W_z \quad \text{and} \quad x = \mathcal{L}(\xi).$$

From now on, for any  $z \in \Gamma$ , we view coordinate system  $(x_1^*, \dots, x_n^*)$  as a rectangular coordinate system. Moreover, using the local coordinate system of Lemma 3, for any  $y = (y_1, y_2, \dots, y_n) \in \mathcal{L}(W_z)$ , we define  $\overline{y} = (-y_1, y_2, \dots, y_n)$  and, without loss of generality, we assume that, for any  $y \in \mathcal{L}(W_z)$ , we have  $\overline{y} \in \mathcal{L}(W_z)$ . For any interior point  $z$  of  $\Omega$ , we fix an arbitrary local coordinate system and a coordinate neighborhood  $W_z$  contained in  $\Omega$ . For any  $z \in \overline{\Omega}$  and  $\delta > 0$ , we set  $W(z, \delta) = \{x : |x - z|^2 < \delta\}$  and  $\delta_z > 0$  such that, for any  $z \in \overline{\Omega}$  we have  $\overline{W(z, \delta_z)} \subset W_z$ .

Recall the following partition of unity lemma.

**Lemma 4.** (Lemma 7.1, Chapter 7 of [It]) *There exist a finite subset  $\{z_1, \dots, z_m\}$  of  $\overline{\Omega}$  and a finite sequence of functions  $\{\omega_1, \dots, \omega_m\}$  with the following properties:*

- 1) *supp  $\omega_l \subset W(z_l, \delta_{z_l})$ ,  $l = 1, \dots, m$ , and each  $\omega_l$  is of class  $\mathcal{C}^3$  with respect to the local coordinates in  $W_{z_l}$ ;*
- 2)  *$\{\omega_l(x)^2; l = 1, \dots, m\}$  forms a partition of unity in  $\overline{\Omega}$ ;*
- 3)  *$\partial_\nu \omega_l(\xi) = 0$ ,  $l = 1, \dots, m$ ,  $\xi \in \Gamma$ .*

Let  $\{z_1, \dots, z_m\}$  be the finite subset of  $\overline{\Omega}$ , introduced in the previous lemma. For any  $k \in \{1, \dots, m\}$ , let  $\mathcal{L}_k$  denotes the diffeomorphism from  $W_{z_k}$  to  $\mathcal{L}_k(W_{z_k})$  defined by

$$\begin{aligned} \mathcal{L}_k : W_{z_k} &\rightarrow \mathcal{L}_k(W_{z_k}) \\ x &\mapsto (x_1^*(x), \dots, x_n^*(x)), \end{aligned}$$

where  $(x_1^*, \dots, x_n^*)$  is the local coordinate system of Lemma 3 defined in  $W_{z_k}$ . For any  $k \in \{1, \dots, m\}$ , the differential operator

$$\partial_t - \Delta_x - q(x, t)$$

becomes, in terms of local coordinate system  $x^* = (x_1^*, \dots, x_n^*)$ ,

$$\mathcal{L}_{t,x^*}^k = \partial_t - \frac{1}{\sqrt{a_k(x^*)}} \sum_{i,j=1}^n \partial_{x_i^*} \left( \sqrt{a_k(x^*)} a_k^{ij}(x^*) \partial_{x_j^*} \right) - q_k(x^*, t)$$

in  $\mathcal{L}_k(W_{z_k}) \times (s_0, t_0)$ . Here  $q_k(x^*, t)$  is Hölder continuous on  $\mathcal{L}_k(W_{z_k}) \times (s_0, t_0)$  and  $(a^{ij}(x^*))$  is the contravariant tensor of degree 2 defined by

$$(a_k^{ij}(x^*)) = (J_{\mathcal{L}_k}(\mathcal{L}_k^{-1}(x^*)))^T (J_{\mathcal{L}_k}(\mathcal{L}_k^{-1}(x^*))),$$

with

$$J_{\mathcal{L}_k}(x) = \left( \frac{\partial x_j^*(x)}{\partial x_i} \right).$$

According to the construction of [It] given in Chapter 6 (see pages 42 to 45 of [It]),  $(a_k^{ij}(x^*))$  is of class  $\mathcal{C}^2$  in  $\mathcal{L}_k(\overline{\Omega} \cap W_{z_k})$  and it is a positive definite symmetric matrix at every point  $x^* \in \mathcal{L}_k(W_{z_k})$ . We set  $(a_{ij}^k(x^*)) = (a_k^{ij}(x^*))^{-1}$  and  $a_k(x^*) = \det(a_{ij}^k(x^*))$ . Consider the volume element  $dx^* = \sqrt{a_k(x^*)} dx_1^* \dots dx_n^*$  on  $\mathcal{L}_k(W_{z_k})$  and  $dx' = \sqrt{a(0, x')} dx_2^* \dots dx_n^*$  on  $\mathcal{L}_k(W_{z_k} \cap \Gamma)$  with  $x' = (x_2^*, \dots, x_n^*)$ . Note that, by the construction of S. Itô [It] (see page 45), for any  $k = 1, \dots, m$ , we have

$$a_k^{ij}(\mathcal{L}_k(x)) = a_k^{ij}(\overline{\mathcal{L}_k(x)}), \quad x \in \overline{\Omega} \cap W_{z_k}, \quad \text{for } i = j = 1 \text{ or } i, j = 2, \dots, n, \quad (2.10)$$

$$a_k^{1j}(\mathcal{L}_k(x)) = a_k^{j1}(\mathcal{L}_k(x)) = -a_k^{1j}(\overline{\mathcal{L}_k(x)}), \quad x \in \overline{\Omega} \cap W_{z_k}, \quad \text{for } j = 1, \dots, n \quad (2.11)$$

and

$$a_k^{1j}(\mathcal{L}_k(\xi)) = a_k^{j1}(\mathcal{L}_k(\xi)) = \delta_{j1}, \quad \xi \in \Gamma \cap W_{z_k}, \quad j = 1, \dots, n, \quad (2.12)$$

where  $\delta_{j1}$  denotes the kronecker's symbol. For any  $k \in \{1, \dots, m\}$ , let  $G_k(x, t; y)$  be defined, in the region

$$D_k = \{(x, t, y); x, y \in \mathcal{L}_k(W_{z_k}), 0 < t < t_0 - s\},$$



by

$$G_k(x, t; y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\sum_{i,j=1}^n \frac{a_{ij}^k(y)(x_i - y_i)(x_j - y_j)}{4t}}.$$

Next, define  $H_{z_k}(x, t; y) = G_k(\mathcal{L}_k(x), t; \mathcal{L}_k(y))$ , for  $k \in \{1, \dots, m\}$  and  $z_k \in \Omega$ ;  $H_{z_k}(x, t; y) = G_k(\mathcal{L}_k(x), t; \mathcal{L}_k(y)) + G_k(\mathcal{L}_k(x), t; \overline{\mathcal{L}_k(y)})$ , for  $k \in \{1, \dots, m\}$ ,  $z_k \in \Gamma$ ,  $x \in W_{z_k}$  and  $y \in W_{z_k}$ ;  $H_{z_k}(x, t; y) = 0$  if  $x \notin W_{z_k}$  or  $y \notin W_{z_k}$ . Consider also  $H(x, t; y)$ , defined in the region

$$D = \{(x, t, y); x \in \overline{\Omega}, y \in \overline{\Omega}, 0 < t < t_0 - s\},$$

as follows

$$H(x, t; y) = \sum_{l=1}^m \omega_l(x) H_{z_l}(x, t; y) \omega_l(y).$$

As in Lemma 7.2 of [It], we define successively:

$$J_0(x, t; y, s) = (\partial_t - \Delta_x - q(x, t))(H(x, t - s; y)),$$

$$J_k(x, t; y, s) = \int_s^t \int_{\Omega} J_0(x, t; z, \tau) J_{k-1}(z, \tau; y, s) dz d\tau,$$

$$K(x, t; y, s) = \sum_{k=0}^{+\infty} J_k(x, t; y, s).$$

Then, following [It] (see page 53), the fundamental solution of (2.1) is given by

$$U(x, t; s, y) = H(x, t - s; y) + \int_s^t \int_{\Omega} H(x, t - \tau; z) K(z, \tau; y, s) dz d\tau. \quad (2.13)$$

We are now able to prove Lemma 2 with the help of representation (2.13), the properties of  $H(x, t; y)$  and  $K(x, t; y, s)$ .

*Proof of Lemma 2.* Without loss of generality, we assume that  $s = 0$ . Set

$$F_1(x, t) = \int_0^t \int_{\Gamma} H(x, t - s; y) f(y, s) d\sigma(y) ds,$$

$$F_2(x, t) = \int_0^t \int_{\Gamma} \int_s^t \int_{\Omega} H(x, t - \tau; z) K(z, \tau; y, s) f(y, s) dz d\tau d\sigma(y) ds.$$

According to representation (2.13), one needs to show that  $F_1$  and  $F_2$  admit a derivative with respect to  $t$  and

$$|\partial_t F_1(x, t)| + |\partial_t F_2(x, t)| \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))} \quad (2.14)$$

for  $(x, t) \in \Gamma \times (0, t_0)$ . We start by considering  $F_1$ . Applying a simple substitution, we obtain

$$F_1(x, t) = \int_0^t \int_{\Gamma} H(x, s; y) f(y, t - s) d\sigma(y) ds. \quad (2.15)$$

Next, for  $x \in \Gamma$ , there exist  $l_1, \dots, l_r \subset \{1, \dots, m\}$  such that  $x \in \text{supp } \omega_l$  for  $l \in \{l_1, \dots, l_r\}$  and  $x \notin \text{supp } \omega_l$  for  $l \notin \{l_1, \dots, l_r\}$ . Moreover, since  $x \in \Gamma$ , we have  $z_{l_1}, \dots, z_{l_r} \in \Gamma$ . Then, from the construction of  $H(x, t; y)$ , we obtain

$$\begin{aligned} \int_{\Gamma} H(x, s; y) d\sigma(y) &= \int_{\Gamma} \sum_{k=1}^r \omega_{l_k}(x) H_{z_{l_k}}(x, s; y) \omega_{l_k}(y) d\sigma(y) \\ &= 2 \sum_{k=1}^r \int_{\mathbb{R}^{n-1}} \chi_{l_k}(0, x') \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-\sum_{i,j=1}^n \frac{a_{ij}^{l_k}(0, y')(x'_i - y'_i)(x'_j - y'_j)}{4s}} \chi_{l_k}(0, y') \sqrt{a_{l_k}(0, y')} dy' \end{aligned}$$

with, for  $l \in \{1, \dots, m\}$ ,  $\chi_l \in C_0^3(\mathcal{L}_l(\text{supp } \omega_l))$  such that  $\chi_l(x) = \omega_l(\mathcal{L}_l^{-1}(x))$  and with  $(x'_1, \dots, x'_n) = (0, x')$ ,  $(y'_1, \dots, y'_n) = (0, y')$ . Using the substitution  $y' \rightarrow z' = \frac{x' - y'}{\sqrt{s}}$ , we derive

$$\begin{aligned} & \int_{\Gamma} H(x, s; y) d\sigma(y) \\ & \leq C \sum_{k=1}^r \int_{\mathbb{R}^{n-1}} \chi_{l_k}(0, x') \frac{1}{\sqrt{s}} e^{-\sum_{i,j=1}^n a_{ij}^{l_k}(0, x' - \sqrt{s}z') z'_i z'_j} \chi_{l_k}(0, x' - \sqrt{s}z') \sqrt{a_{l_k}(0, x' - \sqrt{s}z')} dz'. \end{aligned} \quad (2.16)$$

Therefore,

$$\int_{\Gamma} |H(x, s; y)| d\sigma(y) \leq \frac{C}{\sqrt{s}} \int_{\mathbb{R}^{n-1}} e^{-a_0|z'|^2} dz' \leq \frac{C}{\sqrt{s}},$$

where  $a_0 > 0$  is a constant. From this estimate, we deduce that

$$\int_{\Gamma} |H(x, s; y) f(y, t - s)| d\sigma(y) \leq C \frac{\|f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{s}}$$

and

$$\left| \partial_t \left( \int_{\Gamma} H(x, s; y) f(y, t - s) d\sigma(y) \right) \right| \leq C \frac{\|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{s}}.$$

Thus,  $F_1$  admits a derivative with respect to  $t$ ,

$$\partial_t F_1(x, t) = \int_{\Gamma} H(x, t; y) f(y, 0) d\sigma(y) + \int_0^t \int_{\Gamma} H(x, s; y) \partial_t f(y, t - s) d\sigma(y) ds$$

and, since  $f(y, 0) = 0$  for  $y \in \Gamma$ , we obtain

$$|\partial_t F_1(x, t)| \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}, \quad (x, t) \in \Gamma \times (0, t_0). \quad (2.17)$$

Let us now consider  $F_2$ . We want to show that  $\partial_t F_2$  exists and the following estimate holds:

$$|\partial_t F_2(x, t)| \leq C \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}, \quad (x, t) \in \Gamma \times (0, t_0). \quad (2.18)$$

For this purpose, using the local coordinate system, it suffices to prove

$$|\partial_t F_2(\mathcal{L}_l^{-1}(0, x'), t)| \leq C_l \|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}, \quad ((0, x'), t) \in \mathcal{L}_l(\Gamma \cap W_{z_l}) \times (0, t_0), \quad l \in \{1, \dots, m\}. \quad (2.19)$$

From now on we set  $x = \mathcal{L}_l^{-1}(0, x')$  with  $(0, x') \in \mathcal{L}_l(\Gamma \cap W_{z_l}) \subset \{0\} \times \mathbb{R}^{n-1}$  and we will show (2.19). First, note that

$$\begin{aligned} J_0(z, \tau; s, y) &= (\partial_\tau - \Delta_z - q(z, t)) H(z, \tau - s; y) \\ &= \sum_{l=1}^m \omega_l(\mathcal{L}_l^{-1}(z^*)) L_{\tau, z^*}^l H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; y) \omega_l(y) \\ &\quad + \sum_{l=1}^m [L_{\tau, z^*}^l, \omega_l(\mathcal{L}_l^{-1}(z^*))] H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; y) \omega_l(y). \end{aligned}$$

According to the results in Chapter 4 of [It] (pages 26 and 27), using the local coordinate system, we obtain

$$\begin{aligned} L_{\tau, z^*}^l H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; \mathcal{L}_l^{-1}(y^*)) &= \sum_{i,j=1}^n (a_i^{ij}(z^*) - a_i^{ij}(y^*)) \frac{\partial^2 H_{z_l}}{\partial z_i^* \partial z_j^*}(\mathcal{L}_l^{-1}(z^*), \tau - s; \mathcal{L}_l^{-1}(y^*)) \\ &\quad + [B_l(z^*, y^*, \partial_{z^*}) + q_l(z^*, t)] H_{z_l}(\mathcal{L}_l^{-1}(z^*), \tau - s; \mathcal{L}_l^{-1}(y^*)), \end{aligned}$$

where  $B_l(z^*, y^*, \partial_{z^*})$  is a differential operator of order  $\leq 1$  in  $z^*$  with continuous coefficients in  $z^*, y^* \in \mathcal{L}_l(\text{supp } \omega_l)$ . In view of the results in Chapter 4 of [It] (see pages 26 and 27), combining (2.10), (2.11), (2.12)

and (2.16), applying the substitution  $y'' = \frac{z' - y'}{\sqrt{\tau - s}}$ , with  $z^* = (z_1^*, z')$  and  $y^* = (0, y')$ , we obtain

$$\int_{\Gamma} J_0(\mathcal{L}_l^{-1}(z^*), \tau; y, s) d\sigma(y) = \sum_{j=0}^2 P_j \left( \frac{z_1^*}{\sqrt{\tau - s}} \right) e^{-\frac{(z_1^*)^2}{\tau - s}} \left[ \int_{\mathbb{R}^{n-1}} \frac{J_0^j(z^*, \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right],$$

for  $0 < s < \tau < t_0$  and  $z^* \in \mathcal{L}_l(\Omega \cap W_{z_k})$ , where, for  $j = 0, 1, 2$ ,  $P_j$  are polynomials and  $J_0^j$  are continuous functions,  $\mathcal{C}^1$  with respect to  $\tau$ ,  $s \in (0, t_0)$  and satisfy

$$\max_{\substack{i=0,1,2 \\ \alpha_1+\alpha_2 \leq 1}} \int_{\mathbb{R}^{n-1}} \left| \partial_{\tau}^{\alpha_1} \partial_s^{\alpha_2} J_0^j(z^*, \tau; y'', s; v_1) \right| dy'' \leq C_l, \quad 0 < s < \tau < t_0, \quad z_1^* > 0, \quad 0 < v_1 < t_0,$$

for some constant  $C_l > 0$ . We note that  $\partial_{v_1} J_0^j((z_1', z''), \tau; y'', s; v_1)$  is not necessarily bounded. Indeed, we show

$$\left| \partial_{v_1} J_0^j((z_1', z''), \tau; y'', s; v_1) \right| \leq \frac{C_l}{\sqrt{v_1}}, \quad 0 < v_1 < t_0, \quad j = 0, 1, 2.$$

This representation and the construction of  $K(z, \tau; y, s)$  in Chapter 5 of [It] (see pages 31 to 32 for the construction in  $\mathbb{R}^n$  and page 53 for the construction in a bounded domain) lead

$$\int_{\Gamma} K(\mathcal{L}_l^{-1}(z^*), \tau; y, s) d\sigma(y) = \sum_{j=0}^2 Q_j \left( \frac{z_1^*}{\sqrt{\tau - s}} \right) e^{-\frac{(z_1^*)^2}{\tau - s}} \left[ \int_{\mathbb{R}^{n-1}} \frac{K_j(z^*, \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right], \quad (2.20)$$

for  $0 < s < \tau < t_0$  and  $z^* \in \mathcal{L}_l(\Omega \cap W_{z_k})$ , where, for  $j = 0, 1, 2$ ,  $Q_j$  are polynomials and  $K_j$  are continuous functions,  $\mathcal{C}^1$  with respect to  $\tau$ ,  $s \in (0, t_0)$  and satisfy

$$\max_{\substack{i=0,1,2 \\ \alpha_1+\alpha_2 \leq 1}} \int_{\mathbb{R}^{n-1}} \left| \partial_{\tau}^{\alpha_1} \partial_s^{\alpha_2} K_j(z^*, \tau; y'', s; v_1) \right| dy'' \leq C_l, \quad 0 < s < \tau < t_0, \quad z_1^* > 0, \quad 0 < v_1 < t_0,$$

where  $C_l > 0$  is a constant. Furthermore, using representation (2.20), we have, for  $s < \tau < t < t_0$ ,

$$\begin{aligned} & \int_{\Omega} H(x, t - \tau; z) \int_{\Gamma} K(\tau, z; y, s) f(y, s) d\sigma(y) dz \\ &= \sum_{j=0}^2 \sum_{l=1}^m \int_{\mathbb{R}_+^n} \omega_l(x) H_{z_l}(x, t - \tau; \mathcal{L}_l^{-1}(z^*)) \chi_l(z^*) Q_j \left( \frac{z_1^*}{\sqrt{\tau - s}} \right) e^{-\frac{(z_1^*)^2}{\tau - s}} \left[ \int_{\mathbb{R}^{n-1}} \frac{K_j(z^*, \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right] dz^* \end{aligned}$$

with  $\mathbb{R}_+^n = \{(z_1^*, \dots, z_n^*) \in \mathbb{R}^n; z_1^* > 0\}$ . Then, applying the substitutions  $z'' = \frac{x' - z'}{\sqrt{t - \tau}}$  and  $z'_1 = \frac{z_1^*}{\sqrt{\tau - s}}$ , we deduce, in view of the form of the functions  $K_j$ , the following

$$\begin{aligned} & \int_{\Omega} H(x, t - \tau; z) \int_{\Gamma} K(z, \tau; y, s) d\sigma(y) dz \\ &= \sum_{j=0}^2 \int_{\mathbb{R}_+^n} \frac{H'_l(x', t - \tau; (z'_1, z''), \tau - s)}{\sqrt{t - \tau}} \left[ \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} dy'' \right] dz'' dz'_1, \quad s < \tau < t < t_0, \end{aligned} \quad (2.21)$$

for some continuous functions  $K'_0$ ,  $K'_1$  and  $H'_l$  such that  $K'_0$ ,  $K'_1$  are  $\mathcal{C}^1$ , with respect to  $s$  and  $\tau$ , and the following estimates hold:

$$\int_{\mathbb{R}_+^n} |H'_l(x', t - \tau; (z'_1, z''), \tau - s)| dz'' \leq C_l, \quad 0 < s < \tau < t < t_0, \quad (2.22)$$

$$\max_{\substack{j=0,1 \\ \alpha_1+\alpha_2 \leq 1}} \int_{\mathbb{R}^{n-1}} \left| \partial_{\tau}^{\alpha_1} \partial_s^{\alpha_2} K'_j((z'_1, z''), \tau; y'', s; v_1) \right| dy'' \leq C_l, \quad 0 < s < \tau < t_0, \quad 0 < v_1 < t_0, \quad (2.23)$$

for some constant  $C_l > 0$ . Repeating the arguments used for (2.21) and applying some results of page 31 of [It], we obtain, for  $0 < t < t_0$ ,

$$\begin{aligned} \int_0^t \int_s^t \int_{\Omega} |H(x, t - \tau; z)| \int_{\Gamma} |K(z, \tau; y, s)| d\sigma(y) dz d\tau ds &\leq C_l \int_0^t \int_s^t \left[ \sum_{j=0}^1 \frac{1}{\sqrt{t - \tau}} \cdot \frac{1}{(\tau - s)^{\frac{j}{2}}} \right] d\tau ds \\ &\leq C_l \sum_{j=0}^1 \int_0^t (t - s)^{1 - \frac{j}{2}} ds \leq C_l. \end{aligned} \quad (2.24)$$

This estimate and Fubini's theorem imply

$$F_2(x, t) = \int_0^t \int_s^t \int_{\Omega} H(x, t - \tau; z) \int_{\Gamma} K(z, \tau; y, s) f(y, s) d\sigma(y) dz d\tau ds.$$

Then, in view of representation (2.21), for all  $0 < t < t_0$ ,

$$\begin{aligned} F_2(x, t) &= \int_0^t \int_s^t \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', t - \tau; (z'_1, z''), \tau - s)}{\sqrt{t - \tau}} \\ &\quad \times \left[ \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), \tau; y'', s; \tau - s)}{(\tau - s)^{\frac{j}{2}}} f_1(x', s; y'', z'') dy'' \right] dz'' dz'_1 d\tau ds, \end{aligned}$$

where  $f_1(x', s; y'', z'') = f(\mathcal{L}_l^{-1}(0, x' - (\sqrt{t - s})z'' - (\sqrt{\tau - s})y''), s)$ . Making the substitution  $\tau' = t - \tau$ , we obtain

$$\begin{aligned} F_2(t, x) &= \int_0^t \int_0^{t-s} \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), t - s - \tau')}{\sqrt{\tau'}} \\ &\quad \times \left[ \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', s; t - s - \tau')}{(t - s - \tau')^{\frac{j}{2}}} f_1(x', s; y'', z'') dy'' \right] dz'' dz'_1 d\tau' ds. \end{aligned}$$

Then, the substitution  $s' = t - s$  yields

$$\begin{aligned} F_2(x, t) &= \int_0^t \int_0^{s'} \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \\ &\quad \times \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} f_1(x', t - s'; y'', z'') dy'' dz'_1 dz'' d\tau' ds'. \end{aligned}$$

But, for  $0 < \tau' < s' < t < t_0$ , estimates (2.22), (2.23) and  $f(y, 0) = 0$ ,  $y \in \Gamma$ , imply

$$\begin{aligned} \left| \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau', x'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} \right. \\ \left. \times f_1(x', t - s'; y'', z'') dy'' dz'_1 dz'' \right| \leq C_l \sum_{j=0}^1 \frac{\|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{\tau'}} \frac{1}{(s' - \tau')^{\frac{j}{2}}} \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} \left| \partial_t \left( \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} \right. \right. \\ \left. \left. \times f_1(x', t - s'; y'', z'') dy'' dz'_1 dz'' \right) \right| \leq C_l \sum_{j=0}^1 \frac{\|\partial_t f\|_{L^\infty(\Gamma \times (0, t_0))}}{\sqrt{\tau'}} \frac{1}{(s' - \tau')^{\frac{j}{2}}}. \end{aligned} \quad (2.26)$$

From estimates (2.25), (2.26) and  $f(y, 0) = 0$ ,  $y \in \Gamma$ , we conclude that  $F_2$  admits a derivative with respect to  $t$  and

$$\begin{aligned} \partial_t F_2(x, t) = & \int_0^t \int_0^{s'} \partial_t \left( \sum_{j=0}^1 \int_{\mathbb{R}_+^n} \frac{H'_l(x', \tau'; (z'_1, z''), s' - \tau')}{\sqrt{\tau'}} \int_{\mathbb{R}^{n-1}} \frac{K'_j((z'_1, z''), t - \tau'; y'', t - s'; s' - \tau')}{(s' - \tau')^{\frac{j}{2}}} \right. \\ & \left. \times f_1(x', t - s'; y''; z'') dy'' dz'_1 dz'' \right) d\tau' ds'. \end{aligned}$$

Moreover, (2.24) and (2.26) imply (2.19) and (2.18). Finally, we obtain (2.14) from (2.17) and (2.18). This completes the proof.  $\square$

### 3. PROOF OF THEOREMS 1 AND 2

*Proof of Theorem 1.* Let  $u = u_1 - u_2$  and  $\sigma = \sigma_2 - \sigma_1$ . Then  $u$  is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u + \sigma_2(t)f(x)u = \sigma(t)f(x)u_1(x, t), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & (x, t) \in \Sigma. \end{cases} \quad (3.1)$$

Let  $U(x, t; y, s)$  be the fundamental solution of (2.1) with  $q(x, t) = -\sigma_2(t)f(x)$ . Applying Theorem 9.1 of [It], we obtain

$$u(x, t) = \int_0^t \int_{\Omega} U(x, t; y, s) \sigma(s) f(y) u_1(y, s) dy ds + \int_0^t \int_{\Gamma} U(x, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds. \quad (3.2)$$

Now, since  $u(x, t) = 0$ ,  $(t, x) \in \Sigma$  and  $x \in \Gamma$ ,

$$\int_0^t \int_{\Omega} U(x, t; y, s) \sigma(s) f(y) u_1(y, s) dy ds = - \int_0^t \int_{\Gamma} U(x, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds. \quad (3.3)$$

In view of differentiability properties in Lemma 1 and 2, we can take the  $t$ -derivative of both sides of identity (3.2). We find

$$\begin{aligned} f(x)g(x, t)\sigma(t) = & - \int_0^t \partial_t \left( \int_{\Omega} U(x, t; y, s) \sigma(s) f(y) u_1(y, s) dy \right) ds \\ & - \partial_t \left( \int_0^t \int_{\Gamma} U(x, t; y, s) (\partial_{\nu} u_2(y, s) - \partial_{\nu} u_1(y, s)) d\sigma(y) ds \right) \end{aligned}$$

and, for  $x = x_0$ , condition (H2) implies

$$\begin{aligned} \sigma(t) = & h(t) \int_0^t \partial_t \left( \int_{\Omega} U(x_0, t; y, s) \sigma(s) f(y) u_1(y, s) dy \right) ds \\ & + h(t) \partial_t \left( \int_0^t \int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds \right), \end{aligned} \quad (3.4)$$

where  $h(t) = -1/(g(t, x_0)f(x_0))$ .

Since  $u(x, 0) = 0$ ,  $x \in \Omega$ , we have  $\partial_{\nu} u(x, 0) = 0$ ,  $x \in \Gamma$ . Thus, the estimates in Lemma 1 and 2 lead

$$\left| \int_0^t \partial_t \left( \int_{\Omega} U(x_0, t; y, s) \sigma(s) f(y) u_1(y, s) dy \right) ds \right| \leq C \int_0^t |\sigma(s)| ds, \quad (3.5)$$

$$\left| \partial_t \left( \int_0^t \int_{\Gamma} U(x, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) ds \right) \right| \leq C \|\partial_t \partial_{\nu} u\|_{L^{\infty}(\Sigma)}. \quad (3.6)$$

Therefore, representation (3.4) and estimates (1.5), (3.5), (3.6) imply

$$|\sigma(t)| \leq \int_0^t C |\sigma(s)| ds + C \|\partial_t \partial_\nu u\|_{L^\infty(\Sigma)}.$$

Here and henceforth,  $C > 0$  is a generic constant depending only on data. Hence, Gronwall's lemma yields

$$|\sigma(t)| \leq C \|\partial_t \partial_\nu u\|_{L^\infty(\Sigma)} e^{Ct} \leq C e^{CT} \|\partial_t \partial_\nu u\|_{L^\infty(\Sigma)}, \quad t \in (0, T).$$

Then (1.3) follows and the proof is complete.  $\square$

*Proof of Theorem 2.* Set  $u = u_1 - u_2 = u(\sigma_1) - u(\sigma_2)$ . Then, according to (H4) and (H6),  $u$  is the solution of the following initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta_x u - q(x, t)u = F(t, x, \sigma_1(t), u_2(x, t)) - F(t, x, \sigma_2(t), u_2(x, t)), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(t, x) = 0, & (t, x) \in \Sigma, \end{cases} \quad (3.7)$$

with

$$q(x, t) = \int_0^1 \partial_u F[t, x, \sigma_1(t), u_2(x, t) + \tau(u_1(x, t) - u_2(x, t))] d\tau. \quad (3.8)$$

Note that assumptions (H4) and (H6) imply that  $q \in \mathcal{C}^1(\overline{Q})$ .

On the other hand, in view of (H4),

$$F(t, x, \sigma_1(t), u_2(x, t)) - F(t, x, \sigma_2(t), u_2(x, t)) = (\sigma_1(t) - \sigma_2(t))G(x, t),$$

with

$$G(x, t) = \int_0^1 \partial_\sigma F(t, x, \sigma_2(t) + s(\sigma_1(t) - \sigma_2(t)), u_2(x, t)) ds.$$

Using this representation, we deduce that  $u$  is the solution of

$$\begin{cases} \partial_t u - \Delta_x u - q(x, t)u = (\sigma_1(t) - \sigma_2(t))G(x, t), & (x, t) \in Q, \\ u(x, 0) = 0, & x \in \Omega, \\ u(x, t) = 0, & (t, x) \in \Sigma. \end{cases} \quad (3.9)$$

Let us remark that (H4) and (H6) imply that  $G \in \mathcal{C}^{2,1}(\overline{Q})$ . Let  $U(t, x; s, y)$  be the fundamental solution of (2.1) with  $q(x, t)$  defined by (3.8). Then, according to Theorem 9.1 of [It], for  $\sigma(t) = \sigma_1(t) - \sigma_2(t)$ , we have the representation

$$u(x, t) = \int_0^t \int_\Omega U(x, t; y, s) \sigma(s) G(y, s) dy ds + \int_0^t \int_\Gamma U(x, t; y, s) \partial_\nu u(y, s) d\sigma(y) ds.$$

Since  $u(x, t) = 0$ ,  $(t, x) \in \Sigma$ , we obtain

$$\int_0^t \int_\Omega U(x_0, t; y, s) \sigma(s) G(y, s) dy ds = - \int_0^t \int_\Gamma U(x_0, t; y, s) \partial_\nu u(y, s) d\sigma(y) ds, \quad (3.10)$$

with  $x_0$  defined in assumption (H5). Combining Lemma 1 and Lemma 2 with some arguments used in the proof of Theorem 1, we prove that  $f_1$  and  $f_2$  defined respectively by

$$\begin{aligned} f_1(t) &= \int_0^t \int_\Omega U(x_0, t; y, s) \sigma(s) G(y, s) dy ds, \\ f_2(t) &= \int_0^t \int_\Gamma U(x_0, t; y, s) \partial_\nu u(y, s) d\sigma(y) ds, \end{aligned}$$

admit a derivative with respect to  $t$  and

$$f_1'(t) = \sigma(t)G(x_0, t) + \int_0^t \partial_t \left( \int_\Omega U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds,$$

$$\begin{aligned}
f_2'(t) &= \int_0^t \partial_t \left( \int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds, \\
\left| \int_0^t \partial_t \left( \int_{\Omega} U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds \right| &\leq C \int_0^t |\sigma(s)| \|G(\cdot, s)\|_{C_x^2(\overline{\Omega})} ds \leq C \int_0^t |\sigma(s)| ds
\end{aligned} \tag{3.11}$$

and

$$\left| \int_0^t \partial_t \left( \int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds \right| \leq C \|\partial_t \partial_{\nu} u\|_{L^{\infty}(\Sigma)}. \tag{3.12}$$

Here and in the sequel  $C > 0$  is a generic constant that can depend only on data.

Taking the  $t$ -derivative of both sides of identity (3.10), we obtain

$$\begin{aligned}
\sigma(t)G(x_0, t) &= - \int_0^t \partial_t \left( \int_{\Omega} U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds \\
&\quad - \int_0^t \partial_t \left( \int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds.
\end{aligned}$$

Let us observe that (H5) and  $\max(\|\sigma_1\|_{\infty}, \|\sigma_2\|_{\infty}) \leq M$  imply

$$\begin{aligned}
|G(x_0, t)| &= \int_0^1 |\partial_{\sigma} F(t, x_0, \sigma_2(t) + s(\sigma_1(t) - \sigma_2(t)), g(x_0, t))| ds \\
&\geq \inf_{t \in [0, T], \sigma \in [-M, M]} |\partial_{\sigma} F(t, x_0, \sigma, g(x_0, t))| > 0.
\end{aligned} \tag{3.13}$$

Then,

$$\begin{aligned}
\sigma(t) &= H(t) \int_0^t \partial_t \left( \int_{\Omega} U(x_0, t; y, s) \sigma(s) G(y, s) dy \right) ds \\
&\quad + H(t) \int_0^t \partial_t \left( \int_{\Gamma} U(x_0, t; y, s) \partial_{\nu} u(y, s) d\sigma(y) \right) ds,
\end{aligned}$$

where  $H(t) = -1/G(x_0, t)$ . Hence, (3.11), (3.12) and (3.13) imply

$$|\sigma(t)| \leq \int_0^t C |\sigma(s)| ds + C \|\partial_t \partial_{\nu} u\|_{L^{\infty}(\Sigma)}.$$

We complete the proof of Theorem 2 by applying Gronwall's lemma.  $\square$

## REFERENCES

- [CE86-1] J. R. CANNON AND S. P. ESTEVA, *An inverse problem for the heat equation*, Inverse Problems 2 (1986), 395-403.
- [CE86-2] J. R. CANNON AND S. P. ESTEVA, *A note on an inverse problem related to the 3-D heat equation*, Inverse problems (Oberwolfach, 1986), 133-137, Internat. Schriftenreihe Numer. Math. 77, Birkhäuser, Basel, 1986.
- [CL88] J. R. CANNON AND Y. LIN, *Determination of a parameter  $p(t)$  in some quasi-linear parabolic differential equations*, Inverse Problems 4 (1988), 35-45.
- [CL90] J. R. CANNON AND Y. LIN, *An Inverse Problem of Finding a Parameter in a Semi-linear Heat Equation*, J. Math. Anal. Appl. 145 (1990), 470-484.
- [Ch91-1] M. CHOULLI, *An abstract inverse problem*, J. Appl. Math. Stoc. Ana. 4 (2) (1991) 117-128.
- [Ch91-2] M. CHOULLI, *An abstract inverse problem and application*, J. Math. Anal. Appl. 160 (1) (1991), 190-202.
- [Ch09] M. CHOULLI, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Mathématiques et Applications, 65, Springer-Verlag, Berlin, 2009.
- [COY] M. CHOULLI, E. M. OUHABAZ AND M. YAMAMOTO, *Stable determination of a semilinear term in a parabolic equation*, Commun. Pure Appl. Anal. 5 (3) (2006), 447-462.
- [CY06] M. CHOULLI AND M. YAMAMOTO, *Some stability estimates in determining sources and coefficients*, J. Inv. Ill-Posed Problems 14 (4) (2006), 355-373.

- [CY11] M. CHOULLI AND M. YAMAMOTO, *Global existence and stability for an inverse coefficient problem for a semilinear parabolic equation*, Arch. Math (Basel), 97 (6) (2011), 587-597.
- [E07] G. ESKIN, *Inverse hyperbolic problems with time-dependent coefficients*, Commun. PDE, 32 (11) (2007), 1737-1758.
- [E08] G. ESKIN, *Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohm effect*, J. Math. Phys. 49 (2) (2008), 1-18.
- [It] S. ITÔ, *Diffusion equations*, Transaction of Mathematical Monographs 114, Providence, RI, 1991.
- [LSU] O. A. LADYZHENSKAJA, V. A. SOLONNIKOV AND N. N. URAL'TZEVA, *Linear and quasilinear equations of parabolic type*, Nauka, Moscow, 1967 in Russian ; English translation : American Math. Soc., Providence, RI, 1968.
- [LS88] A. LORENZI AND E. SINISTRARI, *An inverse problem in the theory of materials with memory*, J. Nonlinear Anal. TMA 12 (12) (1988), 1217-1333.
- [LS87] A. LORENZI AND E. SINISTRARI, *Stability results for a partial integrodifferential equation*, Proc. of the Meeting, Volterra Integrodifferential Equations in Banach Spaces; Trento, Pitman, London, (1987).
- [PO85-1] A. I. PRILEPKO AND D. G. ORLOVSKII, *Determination of evolution parameter of an equation, and inverse problems in mathematical physics I*, Translations from Diff. Uravn. 21 (1) (1985), 119-129.
- [PO85-2] A. I. PRILEPKO AND D. G. ORLOVSKII, *Determination of evolution parameter of an equation, and inverse problems in mathematical physics II*, Translations from Diff. Uranv. 21 (4) (1985), 694-701.
- [Sa] R. SALAZAR, *Determination of time-dependent coefficients for a hyperbolic inverse problem*, arXiv:1009.4003v1.

MOURAD CHOULLI, LMAM, UMR 7122, UNIVERSITÉ DE LORRAINE, ILE DU SAULCY, 57045 METZ CEDEX 1, FRANCE  
*E-mail address:* mourad.choulli@univ-lorraine.fr

YAVAR KIAN, UMR-7332, AIX MARSEILLE UNIVERSITÉ, CENTRE DE PHYSIQUE THÉORIQUE, CAMPUS DE LUMINY, CASE 907 13288 MARSEILLE CEDEX 9, FRANCE  
*E-mail address:* yavar.kian@univ-amu.fr