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$\mathbf{W}^{1,q}$ stability of the Fortin operator for the MAC scheme

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Abstract. We prove in this paper the continuity of the natural projection operator from $W_0^{1,q}(\Omega)^d$ to the MAC discrete space of piecewise constant functions over the dual cells, endowed with the finite volume $W^{1,q}$-discrete norm. Since this projection operator is also a Fortin operator (that is an operator which "preserves" the divergence), this result may be applied to control the pressure in mixed problems where the test function for the velocity must be more regular than usual (*i.e.* more regular than $H_0^1(\Omega)^d$).

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1. Introduction

To illustrate the motivation of the work presented in this paper, let us consider the following model problem:

$$-\operatorname{div}\big(\mu \boldsymbol{\nabla} \boldsymbol{u}\big) + \boldsymbol{\nabla} p = \boldsymbol{f} \text{ in } \Omega, \quad \boldsymbol{u} = 0 \text{ on } \partial\Omega, \tag{1a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1b}$$

where Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$, and μ is a scalar function defined on Ω, non-necessarily bounded from above but bounded by below by a positive real number, let us say:

$$\mu \in \mathrm{L}^q(\Omega) \text{ with some } q > 2, \quad \mu(\boldsymbol{x}) \geq \mu_0 > 0 \text{ for a.e. } \boldsymbol{x} \in \Omega.$$

Problems of this type (of course, switching from Stokes to Navier-Stokes equations) arise in the modelling of turbulent flows, when using a so-called Reynolds-Averaged (RANS) model, as the wellknown $k-\epsilon$ model. In this context, μ is the turbulent viscosity, derived from other unknowns of the problem (precisely, k and ϵ), and μ_0 is the laminar (or molecular) viscosity, *i.e.* the intrinsic viscosity of the considered fluid.

Let us suppose that (1) is solved with a numerical scheme which may be set under a variational form, *i.e.* searching for $(\boldsymbol{u},p) \in \boldsymbol{V} \times Q$ such that:

$$(\mu \boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\nabla} \boldsymbol{v}) - (p, \mathrm{div}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2a}$$

$$(\mathrm{div}\boldsymbol{u}, \lambda) = 0, \quad \forall \lambda \in Q. \tag{2b}$$

In these relations, the differential operators $\boldsymbol{\nabla}$ and div, the inner product $(\cdot,\cdot)$ and the discrete spaces $\boldsymbol{V}$ and Q need to be defined; let us postpone this for a while, and make some (formal) steps toward a stability analysis, as if we were working at the continuous level. Taking $\boldsymbol{v}=\boldsymbol{u}$ in (2a) yields:

$$\mu_0\ (\boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\nabla} \boldsymbol{u}) \le (\boldsymbol{f}, \boldsymbol{u}), \tag{3}$$

which yields a control of $\boldsymbol{u}$ in $\mathrm{H}^1(\Omega)^d$. To control the pressure, we now use the following "L^r *inf-sup* inequality":

$$\exists\ C(\Omega) > 0 \text{ s.t.} \qquad \sup_{\boldsymbol{v} \in \mathrm{W}^{1,r'}(\Omega)^d} \frac{(p, \mathrm{div}\boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathrm{W}^{1,r'}(\Omega)^d}} \ge C(\Omega)\, \|p\|_{\mathrm{L}^r(\Omega)} \tag{4}$$

where $r \in (1,\infty)$ and $1/r + 1/r' = 1$. Thus, from (2a), we get:

$$\|p\|_{\mathrm{L}^r(\Omega)} \le \frac{1}{C(\Omega)} \sup_{\boldsymbol{v} \in \mathrm{W}^{1,r'}(\Omega)^d} \frac{1}{\|\boldsymbol{v}\|_{\mathrm{W}^{1,r'}(\Omega)^d}} \Big[(\mu \boldsymbol{\nabla} \boldsymbol{u}, \boldsymbol{\nabla} \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}) \Big] \tag{5}$$

and the right-hand side of this equation is bounded as soon as $1/r' + 1/q + 1/2 \le 1$, since μ is bounded in $\mathrm{L}^q(\Omega)$ and $\boldsymbol{u}$ in $\mathrm{H}^1(\Omega)^d$.

Let us now assume that (1) is discretized with the classical Marker And Cell (or MAC) scheme [2], and try to make the preceding computation rigorous in the discrete setting. Discrete counterparts of the operators $\boldsymbol{\nabla}$ and div, let us say $\boldsymbol{\nabla}_{\mathcal{T}}$ and $\mathrm{div}_{\mathcal{T}}$, may be defined, the scheme may be written under variational form, and the discrete counterpart of $\boldsymbol{v} \mapsto \|\boldsymbol{v}\|_{\mathcal{T}} = (\boldsymbol{\nabla}_{\mathcal{T}} \boldsymbol{v}, \boldsymbol{\nabla}_{\mathcal{T}} \boldsymbol{v})^{1/2}$ defines a norm over the velocity discretization space $\boldsymbol{V}$, which controls the $\mathrm{L}^2(\Omega)^d$ norm by a Poincaré estimate [1]: it is thus possible to reproduce (3) at the discrete level, *i.e.* to control $\boldsymbol{u}$ in the $\|\cdot\|_{\mathcal{T}}$ norm. In addition, we are also able to define a discrete $\mathrm{W}^{1,r}(\Omega)^d$ norm for $r \geq 1$, denoted by $\|\cdot\|_{1,r,\mathcal{T}}$, consistent with the discrete H^1 norm (*i.e.* with $\|\boldsymbol{v}\|_{1,2,\mathcal{T}} = (\boldsymbol{\nabla}_{\mathcal{T}}\boldsymbol{v}, \boldsymbol{\nabla}_{\mathcal{T}}\boldsymbol{v})^{1/2}$), satisfying $\|\boldsymbol{v}\|_{1,2,\mathcal{T}} \leq (d\,|\Omega|)^{1/2-1/r'}\,\|\boldsymbol{v}\|_{1,r',\mathcal{T}}$, and to prove:

$$(\mu \boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{v}) \leq \|\mu\|_{\mathrm{L}^q(\Omega)}\ \|\boldsymbol{u}\|_{1,2,\mathcal{T}}\ \|\boldsymbol{v}\|_{1,r',\mathcal{T}}$$

for any $r' \geq 2$ such that $1/r' + 1/q + 1/2 \leq 1$ and any $\boldsymbol{u}$ and $\boldsymbol{v} \in \boldsymbol{V}$.

The last point to conclude to the stability of the scheme is to prove a discrete analogue of (4). To this purpose, a possible strategy is to build a projection operator $\boldsymbol{\Pi}$ which is continuous from $\mathrm{W}^{1,r}(\Omega)^d$ to the discrete space endowed with the discrete $\mathrm{W}^{1,r}(\Omega)^d$ norm, and preserves the divergence with respect to the discrete pressures, that is:

$$(i)\qquad \|\boldsymbol{\Pi}(\boldsymbol{u})\|_{1,r,\mathcal{T}} \leq C\,\|\boldsymbol{u}\|_{\mathrm{W}^{1,r}(\Omega)^d}, \quad \forall \boldsymbol{u} \in \mathrm{W}^{1,r}(\Omega)^d,$$

$$(ii)\qquad \int_\Omega \mathrm{div}_{\mathcal{T}}(\boldsymbol{\Pi}(\boldsymbol{u}))\ q\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}(\boldsymbol{u})\ q\,\mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{u} \in \mathrm{W}^{1,r}(\Omega)^d,\ \forall q \in Q.$$

where C may depend on r, Ω and, possibly, on the regularity of the mesh but not on the mesh step. Such a projection is often referred to as a Fortin operator; building such an operator for the MAC scheme is the issue addressed in this paper. The obtained stability result provides a control of the pressure in $\mathrm{L}^r(\Omega)$, $r < 2$, from the discrete $\mathrm{W}^{-1,r}$ norm of its gradient; besides the model problem detailed here, it may be applied to tackle various situations, as the compressible Navier-Stokes equations.

2. Discrete spaces and norms

We suppose that Ω is an open bounded domain of $\mathbb{R}^d$ with $d = 2$ or 3, adapted to the MAC discretization, that is Ω is any finite union of rectangles, if $d = 2$, or rectangular parallelipeds if $d = 3$. For simplicity (but the generalization is easy to understand even if it is cumbersome to precisely state), we take in this short note $d = 2$ and $\Omega = (0,1)^d$.

Let $N, M \in \mathbb{N}$, $N, M \geq 2$ and let $\{h^x_i,\ i = 1,\ldots,N\}$ and $\{h^y_j,\ j = 1,\ldots,M\}$ be two families of positive numbers such that $\sum_{i=1}^N h^x_i =$

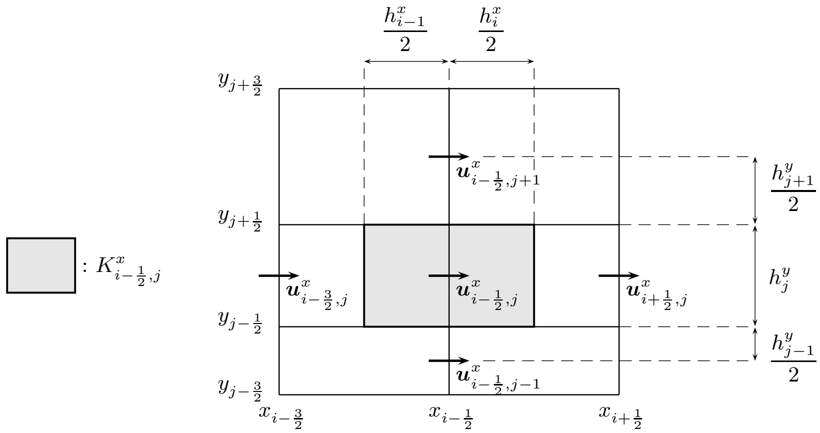

Fig. 1. Mesh and unknowns.

$\sum_{j=1}^M h^y_j = 1$. Let $(x_{i-\frac{1}{2}})_{1\leq i\leq N+1}$ and $(y_{j-\frac{1}{2}})_{1\leq j\leq M+1}$ be the families of real numbers defined as follows:

$$x_{\frac{1}{2}} = 0, \quad x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = h^x_i \quad (\text{so that } x_{N+\frac{1}{2}} = 1),$$
$$y_{\frac{1}{2}} = 0, \quad y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} = h^y_j \quad (\text{so that } y_{M+\frac{1}{2}} = 1).$$

Let $\boldsymbol{u}$ and p be a discrete velocity and pressure field respectively. In the MAC scheme, the degrees of freedom for the first component of the velocity are associated to the vertical edges of the mesh, and read:

$$\{\boldsymbol{u}^x_{i+\frac{1}{2},j},\ 0 \leq i \leq N,\ 1 \leq j \leq M\}.$$

Similarly, the degrees of freedom for the second component of the velocity read:

$$\{\boldsymbol{u}^y_{i,j+\frac{1}{2}},\ 1 \leq i \leq N,\ 0 \leq j \leq M\}.$$

The pressure is piecewise constant over each

$$K_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}),\ 1 \leq i \leq N,\ 1 \leq j \leq M$$

and so is the discrete divergence, which, on each $K_{i,j}$, is defined by:

$$(\mathrm{div}_{\mathcal{T}}\boldsymbol{u})_{i,j} = \frac{1}{h^x_i\, h^y_j}\ (h^y_j\, \boldsymbol{u}^x_{i+\frac{1}{2},j} + h^x_i\, \boldsymbol{u}^y_{i,j+\frac{1}{2}} - h^y_j\, \boldsymbol{u}^x_{i-\frac{1}{2},j} - h^x_i\, \boldsymbol{u}^y_{i,j-\frac{1}{2}}).$$

The projector onto the discrete space associated to the first component of the velocity is defined by:

$$\forall u \in \mathrm{W}^{1,p}_0(\Omega), \quad u_{i+\frac{1}{2},j} = \frac{1}{h^y_j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x_{i+\frac{1}{2}}, s)\,\mathrm{d}s,$$

i.e. is obtained by just taking for each degree of freedom the average of the function over the associated edge of the mesh. It is an easy task to check that, with the same definition for the second component of the velocity, the integral of the divergence of a vector-valued function $\boldsymbol{u} \in \mathrm{W}_0^{1,p}(\Omega)^d$ over each element $K_{i,j}$ is equal to the expression of the discrete divergence of its projection $\boldsymbol{\Pi}(\boldsymbol{u})$ onto the same element $K_{i,j}$, and thus, for any discrete (thus constant over each element $K_{i,j}$) pressure p:

$$\int_\Omega p \ \mathrm{div}\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_\Omega p \ \mathrm{div}_{\mathcal{T}}(\boldsymbol{\Pi}(\boldsymbol{u})) \, \mathrm{d}\boldsymbol{x},$$

that is that we have indeed built a Fortin operator.

In order to simplify the expression of the discrete norm now introduced, we also set:

$$\begin{aligned}
&u_{i+\frac{1}{2},0} = u_{i+\frac{1}{2},M+1} = 0, && \text{for } 0 \le i \le N,\\
&h^x_{i+\frac{1}{2}} = \frac{1}{2}\,(h^x_i + h^x_{i+1}), && \text{for } 1 \le i \le N-1,\\
&h^y_{j+\frac{1}{2}} = \frac{1}{2}\,(h^y_j + h^y_{j+1}) && \text{for } 0 \le i \le M,
\end{aligned}$$

setting for this definition $h^y_0 = h^y_{M+1} = 0$. Then, the norm in the discrete space associated to the first component of the velocity is, as usual in the Finite Volume setting:

$$\begin{aligned}
\|\Pi(u)\|^q_{1,q} \ &= \sum_{i=1}^{N}\sum_{j=1}^{M} \Big|\frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{h^x_i}\Big|^q \ h^x_i h^y_j \\
&\qquad + \sum_{i=2}^{N}\sum_{j=0}^{M} \Big|\frac{u_{i-\frac{1}{2},j+1} - u_{i-\frac{1}{2},j}}{h^y_{j+\frac{1}{2}}}\Big|^q \ h^y_{j+\frac{1}{2}} h^x_{i-\frac{1}{2}}.
\end{aligned}$$

We are now going to prove that the discrete norm of the projection $\Pi(u)$ is controlled by the $\mathrm{W}^{1,q}$ norm of u, with a proportionality coefficient independent on the mesh step, but depending on a positive parameter characterizing the regularity of the mesh, which we denote by ζ and which satisfies:

$$\zeta \le \frac{h^x_i}{h^y_j} \le \frac{1}{\zeta} \quad \text{ for } 0 \le i \le N, \ 1 \le j \le M.$$

Of course, the proof may easily be transposed to the second component of the velocity.

3. The stability result

Theorem 1. *Let $q \in [1,+\infty)$ and $u \in \mathrm{W}_0^{1,q}(\Omega)$. Then:*

$$\|\Pi(u)\|_{1,q} \le C_2 \, \|u\|_{W_0^{1,q}(\Omega)} \,, \tag{6}$$

where C_2 depends only on ζ, q and Ω.

Proof. We will only prove Inequality (6) for $u \in C_c^\infty(\Omega)$ (then, by density, Inequality (6) is also true for $u \in W_0^{1,q}(\Omega)$). We recall that $\|\Pi(u)\|_{1,q}^q = A + B$ with:

$$A = \sum_{i=1}^{N} \sum_{j=1}^{M} \Big| \frac{u_{i+\frac12,j} - u_{i-\frac12,j}}{h_i^x} \Big|^q \; h_i^x h_j^y,$$
$$B = \sum_{i=2}^{N} \sum_{j=0}^{M} \Big| \frac{u_{i-\frac12,j+1} - u_{i-\frac12,j}}{h_{j+\frac12}^y} \Big|^q \; h_{j+\frac12}^y h_{i-\frac12}^x.$$

The sums A and B corresponds to the discrete derivative of $\Pi(u)$ in the x-direction and y-direction respectively. The proof is divided in 3 steps: we successively obtain a bound for A, then for B, and conclude.

Step 1 – Estimate for A.

Let $i \in \{1,\dots,N\}$, $j \in \{1,\dots,M\}$ and $y \in (y_{i-\frac12}, y_{i+\frac12})$. One has:

$$u(x_{i+\frac12}, y) - u(x_{i-\frac12}, y) = \int_{x_{i-\frac12}}^{x_{i+\frac12}} \frac{\partial u}{\partial x}(x,y) \, \mathrm{d}\boldsymbol{x}.$$

Integrating with respect to $y \in (y_{i-\frac12}, y_{i+\frac12})$ leads to:

$$h_j^y (u_{i+\frac12,j} - u_{i-\frac12,j}) = \int_{K_{i,j}} \frac{\partial u}{\partial x}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

Then, with Hölder Inequality,

$$\begin{aligned}(h_j^y)^q \; |u_{i+\frac12,j} - u_{i-\frac12,j}|^q &\le |K_{i,j}|^{q-1} \int_{K_{i,j}} \Big| \frac{\partial u}{\partial x}(\boldsymbol{x}) \Big|^q \; \mathrm{d}\boldsymbol{x} \\ &= (h_i^x \, h_j^y)^{q-1} \int_{K_{i,j}} \Big| \frac{\partial u}{\partial x}(\boldsymbol{x}) \Big|^q \; \mathrm{d}\boldsymbol{x},\end{aligned}$$

which gives

$$\Big| \frac{u_{i+\frac12,j} - u_{i-\frac12,j}}{h_i^x} \Big|^q \; h_i^x \, h_j^y \le \int_{K_{i,j}} \Big| \frac{\partial u}{\partial x}(\boldsymbol{x}) \Big|^q \; \mathrm{d}\boldsymbol{x}.$$

Summing for $i \in \{1,\dots,N\}$ and $j \in \{1,\dots,M\}$ leads to

$$A \le \int_\Omega \left|\frac{\partial u}{\partial x}(\boldsymbol{x})\right|^q \,\mathrm{d}\boldsymbol{x} \le \int_\Omega |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x}.$$

Step 2 – Estimate for B.

Let $i \in \{1,\dots,N\}$ and $j \in \{1,\dots,M\}$. In Step 1, we compared $u_{i-\frac12,j}$ with $u_{i+\frac12,j}$. In this step, we will first compare $u_{i-\frac12,j}$ with $u_{i,j-\frac12}$ and $u_{i,j+\frac12}$ which we define as the mean values of u on the sets $\{(x,y_{j-\frac12}),\ x \in [x_{i-\frac12},x_{i+\frac12}]\}$ and $\{(x,y_{j+\frac12}),\ x \in [x_{i-\frac12},x_{i+\frac12}]\}$. This comparison is given by Lemma 1 which reads:

$$|u_{i-\frac12,j} - u_{i,j-\frac12}|^q \le 2^{1-q/2}\ (h_i^x)^{q-2}\frac{1}{\zeta^{q+1}} \int_{T^-_{i,j}} |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x},$$

and

$$|u_{i-\frac12,j} - u_{i,j+\frac12}|^q \le 2^{1-q/2}\ (h_i^x)^{q-2}\frac{1}{\zeta^{q+1}} \int_{T^+_{i,j}} |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x},$$

where $T^\pm_{i,j}$ are the triangles the vertices of which are the points $(x_{i-\frac12},y_{j-\frac12})$, $(x_{i-\frac12},y_{j+\frac12})$ and $(x_{i+\frac12},y_{j\pm\frac12})$. We now turn to the terms appearing in B. For $i \in \{2,\dots,N\}$ and $j \in \{0,\dots,M\}$, one has:

$$|u_{i-\frac12,j+1} - u_{i-\frac12,j}|^q \le 2^q\,|u_{i-\frac12,j+1} - u_{i,j+\frac12}|^q + 2^q\,|u_{i,j+\frac12} - u_{i-\frac12,j}|^q.$$

Thus:

$$|u_{i-\frac12,j+1} - u_{i-\frac12,j}|^q \le 2^{1+q/2}\ (h_i^x)^{q-2}\frac{1}{\zeta^{q+1}} \Big(\int_{T^-_{i,j+1}} |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x} + \int_{T^+_{i,j}} |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x}\Big),$$

setting, for the limit cases of j, $T^+_{i,0} = T^-_{i,M+1} = \emptyset$. Summing, we obtain:

$$B = \sum_{i=2}^N \sum_{j=0}^M \left|\frac{u_{i-\frac12,j+1} - u_{i-\frac12,j}}{h^y_{j+\frac12}}\right|^q\ h^y_{j+\frac12} h^x_{i-\frac12} \le C_1 \int_\Omega |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x},$$

with $C_1 = \dfrac{2^{2+q/2}}{\zeta^{2(q+1)}}$.

Step 3 – Conclusion. Summing the estimates on A and B obtained in Step 1 and 2, we have:

$$\|\Pi(u)\|_{1,q}^q = A + B \leq (1 + C_1)\, \|u\|_{W_0^{1,q}(\Omega)}^q ,$$

i.e. Inequality (6) with $C_2 = (1 + C_1)^{1/q}$, which depends only on ζ, q and Ω (which does not appear explicitely here because we have performed the computations for the specific domain $\Omega = (0,1)^2$).

Lemma 1 (Comparison of mean values). *Let $h^x > 0$, $h^y > 0$ and T be the triangle whose vertices are $(0,0)$, $(h^x, 0)$ and $(0, h^y)$. Let $q \in [1, +\infty)$ and $u \in W^{1,q}(T)$. Let u_ℓ be the mean value of u on the segment $\{(0, t\, h^y),\ t \in [0,1]\}$ and u_b be the mean value of u on the segment $\{(t\, h^x, 0),\ t \in [0,1]\}$. Then:*

$$|u_\ell - u_b|^q \leq \frac{1}{2^{q-1}}\ \frac{\big((h^x)^2 + (h^y)^2\big)^{q/2}}{h^x\, h^y} \int_T |\boldsymbol{\nabla} u|^q \,\mathrm{d}\boldsymbol{x}. \qquad (7)$$

Proof. As usual we prove Inequality (7) for $u \in C^\infty(\mathbb{R}^2)$ and we conclude by density. Let $u \in C^\infty(\mathbb{R}^2)$, and $t \in [0,1]$. For $s \in [0,1]$ we set $\varphi(s) = u\big((1-s)\, t\, h^x, s\, t\, h^y\big)$ so that:

$$\begin{aligned} u(0, t\, h^y) - u(t\, h^x, 0) = \varphi(1) - \varphi(0) &= \int_0^1 \varphi'(s) \,\mathrm{d}s \\ &= \int_0^1 \boldsymbol{\nabla} u\big((1-s)\, t\, h^x, s\, t\, h^y\big) \cdot t \begin{bmatrix} -h^x \\ h^y \end{bmatrix} \mathrm{d}s. \end{aligned}$$

We now integrate this equality for $t \in [0,1]$. It leads to:

$$u_\ell - u_b = \int_0^1 \int_0^1 \boldsymbol{\nabla} u\big((1-s)\, t\, h^x, s\, t\, h^y\big) \cdot t \begin{bmatrix} -h^x \\ h^y \end{bmatrix} \mathrm{d}s\, \mathrm{d}t.$$

In the integral in the right hand side, we now perform a change of variable, setting $\boldsymbol{z} = (z_1, z_2)^t$, $z_1 = (1-s)\, t\, h^x$, $z_2 = s\, t h^y$. The Jacobian of this change of variable is $1/(z_1 h^y + z_2 h^x)$ and one has $t h^x = (z_1 h^y + z_2 h^x)/h^y$ and $t h^y = (z_1 h^y + z_2 h^x)/h^x)$. Then, we obtain:

$$\begin{aligned} |u_\ell - u_b| &= \Big| \int_T \boldsymbol{\nabla} u(\boldsymbol{z}) \cdot \begin{bmatrix} -1/h^y \\ 1/h^x \end{bmatrix} \mathrm{d}\boldsymbol{z} \Big| \\ &\leq \frac{\big[(h^x)^2 + (h^y)^2\big]^{1/2}}{h^x\, h^y} \int_T |\boldsymbol{\nabla} u(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z}. \end{aligned}$$

It remains to use the Hölder inequality and the fact that $|T| = h^x h^y/2$ to conclude:

$$|u_\ell - u_b|^q \leq \frac{\big[(h^x)^2 + (h^y)^2\big]^{q/2}}{\big[h^x\, h^y\big]^q}\ \Big[\frac{h^x h^y}{2}\Big]^{q-1} \int_T |\boldsymbol{\nabla} u(\boldsymbol{z})|^q \,\mathrm{d}\boldsymbol{z}.$$

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