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# Gaussian multiplicative chaos and KPZ duality

Julien Barral <sup>\*</sup>, Xiong Jin<sup>†</sup>, Rémi Rhodes <sup>‡</sup>, Vincent Vargas <sup>§¶</sup>

March 7, 2012

## Abstract

This paper is concerned with the KPZ formula. On the first hand, we give a simplified (in comparison with the existing literature) proof of the classical KPZ formula. On the other hand, we construct purely atomic random measures corresponding to values of the parameter  $\gamma^2$  beyond the transition phase (i.e.  $\gamma^2 > 2d$ ). We prove the dual KPZ formula for these measures and check the duality relation. In particular, this framework allows to construct singular Liouville measures and to understand the duality relation in Liouville quantum gravity.

## 1. Introduction

Log-normal multiplicative martingales were introduced by Mandelbrot [26] in order to build random measures describing energy dissipation and contribute explaining intermittency effects in Kolmogorov's theory of fully developed turbulence (see [7, 34, 36, 8, 16] and references therein). However, his model was difficult to define mathematically and this is why he proposed in [27] the simpler model of random multiplicative cascades whose detailed study started with Kahane's and Peyrière's notes [17, 29], gathered in their joint paper [19].

From that moment on, multiplicative cascades have been widely used as reference models in many applications. However, they possess many drawbacks related to their discrete scale invariance, mainly they involve a particular scale ratio and they do not possess stationary fluctuations (this comes from the fact that they are constructed

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<sup>\*</sup>Université Paris 13, Institut Galilée, LAGA, UMR CNRS 7539, 99 rue Jean-Baptiste Clément 93430 Villetaneuse, France.

<sup>†</sup>University of St Andrews, Mathematics Institute, North Haugh, St Andrews, KY16 9SS, Scotland.

<sup>‡</sup>Université Paris-Dauphine, Ceremade, UMR 7534, Place du marechal de Lattre de Tassigny, 75775 Paris Cedex 16, France.

<sup>§</sup>Université Paris-Dauphine, Ceremade, UMR 7534, Place du marechal de Lattre de Tassigny, 75775 Paris Cedex 16, France.

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on a  $p$ -adic tree structure). In the eighties, Kahane [18] came back to Mandelbrot's initial model and developed a continuous parameter theory of suitable stationary multifractal random measures, called Gaussian multiplicative chaos. His efforts were followed by several authors [3, 34, 2, 30, 32, 15, 1, 33] coming up with various generalizations at different scales. This family of random fields has found many applications in various fields of science like mathematical finance, turbulence, etc... Recently, the authors in [11] have drawn attention on the fact that 2d-Gaussian multiplicative chaos should be considered as a natural model for Liouville Quantum Gravity (see [23, 9, 11] among many others). In this context, the KPZ formula has been proved rigorously [5, 11, 31] below the transition phase arising at  $\gamma^2 = 4$ , where the constant  $\gamma$  is related to the central charge  $c \leq 1$  of the underlying conformal field theory by the relation (see [23])

$$\gamma = \frac{\sqrt{25 - c} - \sqrt{1 - c}}{\sqrt{6}}.$$

However the issue of mathematically constructing singular Liouville measures beyond the transition phase (i.e. for  $\gamma^2 > 4$ ) and proving the KPZ duality has never been solved mathematically (see [20, 21, 22] for an account of physical motivations).

Let us draw up the framework a bit more precisely. Fix a simply connected domain  $D \subset \mathbb{C}$ . For  $\gamma^2 < 4$ , the Liouville measure can formally be written as

$$M_\gamma(A) = \int_A e^{\gamma X_x - \frac{\gamma^2}{2} \mathbb{E}[X_x^2]} dx \quad (1)$$

where  $X$  is the Gaussian Free Field (GFF) over the domain  $D$ . For a given compact set  $K \subset D$ , it has been proved that the Hausdorff dimension of  $K$  computed with the Euclidian metric, call it  $\dim_{Leb}(K)$ , is related to the Hausdorff dimension of  $K$  computed with the measure  $M_\gamma$ , call it  $\dim_\gamma(K)$ . The connection is the so-called KPZ formula

$$\dim_{Leb}(K) = \left(1 + \frac{\gamma^2}{4}\right) \dim_\gamma(K) - \frac{\gamma^2}{4} \dim_\gamma(K)^2.$$

Based on the physics literature, the purpose of this paper is to propose a construction in the spirit of (1) of (purely atomic) random measures  $M_{\bar{\gamma}}$ , for parameter values  $\bar{\gamma}^2 > 4$  (i.e. beyond the transition phase), that satisfy the KPZ relation

$$\dim_{Leb}(K) = \left(1 + \frac{\bar{\gamma}^2}{4}\right) \dim_{\bar{\gamma}}(K) - \frac{\bar{\gamma}^2}{4} \dim_{\bar{\gamma}}(K)^2.$$

Then, by considering the dual value  $\gamma = \frac{4}{\bar{\gamma}}$  of the parameter  $\bar{\gamma}$ , we want to establish the duality relation

$$\dim_{\bar{\gamma}}(K) = \frac{\gamma^2}{4} \dim_\gamma(K).$$

We point out that physicists can recover the (more classical) relation between the scaling exponents by setting  $\Delta_\gamma = 1 - \dim_\gamma(K)$  and  $\Delta_{\bar{\gamma}} = 1 - \dim_{\bar{\gamma}}(K)$ .

Our construction for dual measures is roughly the following. Consider a couple of exponents  $(\gamma, \bar{\gamma})$  such that  $\gamma^2 < 4$  and  $\gamma\bar{\gamma} = 4$ . We introduce an independently scattered random measure  $n_\alpha$  characterized by its Laplace transform ( $|A|$  stands for the Lebesgue measure of  $A$ )

$$\forall A \subset \mathbb{R}^2 \text{ Borelian, } \mathbb{E}[e^{-un_\alpha(A)}] = e^{-u^\alpha|A|}$$

where  $\alpha = \frac{\gamma^2}{4}$ . The considered dual measure is then formally defined by (see below for a rigorous construction)

$$M_{\bar{\gamma}}(A) = \int_A e^{\bar{\gamma}X_x - 2\mathbb{E}[X_x^2]} n_\alpha(dx). \quad (2)$$

We point out that the above expression is not a Gaussian multiplicative chaos in the usual sense. Indeed the lognormal weight is not normalized to have expectation 1. Actually, the expectation explodes giving rise to a strong competition between the atoms produced by the random measure  $n_\alpha$  and the ability of the lognormal weight to kill these atoms. Since the big atoms produced by the measure  $n_\alpha$  are not numerous enough, the fact that the measure  $M_{\bar{\gamma}}$  is not trivial is only due to the production of small atoms. Notice that the production of atoms is directly connected to the parameter  $\gamma$  (and therefore to the central charge) by the relation  $\alpha = \frac{\gamma^2}{4} = \frac{4}{\bar{\gamma}^2}$ . We illustrate these remarks in Section 5.

We prove that the measure  $M_{\bar{\gamma}}$  can be obtained as an almost sure limit of suitably regularized versions, giving sense to a new and exciting theory of (non-standard) multiplicative chaos with respect to atomic measures. Beyond the applications in Liouville Quantum Gravity, we have the feeling that this approach offers new perspectives in the theory of Gaussian multiplicative chaos that we develop in Section 6.

Let us finally mention that our measures  $(M_\gamma, M_{\bar{\gamma}})$  are approximately  $\star$ -scale invariant random measures in the sense of [1]. As a consequence, they satisfy the scaling heuristics developed in [12] to quantify the measure of a Euclidean ball of size  $\epsilon$  (see in particular the section *Liouville quantum duality*). In fact, such heuristics amount to considering  $\star$ -scale invariant random measures (see [1, 33]). We conjecture that the  $\star$ -scale invariance property characterizes the measures that one can consider in this context (work in progress; see [13] for a precise statement and a rigorous proof in the case of random multiplicative cascades).

In the present paper, we will tackle the above problem in great generality: we will not restrict ourselves to the 2-dimensional case and we will not consider the only GFF but more generally log-correlated Gaussian distributions.

## 2. Background

In this section, we will briefly explain Kahane's theory of multiplicative chaos in  $\mathbb{R}^d$ . In fact, Kahane's theory is valid in any open domain  $D \subset \mathbb{R}^d$  with no substantial

change. At the end of the section, we will also roughly recall the connection with measures formally given by the exponential of the GFF.

## 2.1 Sigma positive kernels

We consider a covariance kernel  $K$  of  $\sigma$ -positive type ([18]), namely that  $K$  can be rewritten as a sum

$$\forall x, y \in \mathbb{R}^d, \quad K(x, y) = \sum_{n \geq 1} q_n(x, y) \quad (3)$$

where  $(q_n)_n$  is a sequence of continuous positive kernels of positive type. We further assume that

$$\forall x \in \mathbb{R}^d, \quad K(x, y) = \ln_+ \frac{T}{|x - y|} + g(x, y) \quad (4)$$

where  $g$  is a bounded continuous function over  $\mathbb{R}^d \times \mathbb{R}^d$  (and  $\ln_+(x) = \max(0, \ln(x))$ ). We can consider a sequence of independent centered Gaussian processes  $(Y^n)_{n \geq 1}$  where, for each  $n \geq 1$ ,  $(Y_x^n)_{x \in \mathbb{R}^d}$  is a centered continuous Gaussian field with covariance function given by

$$\forall x, y \in \mathbb{R}^d, \quad \text{Cov}(Y_x^n, Y_y^n) = q_n(x, y).$$

Finally, for  $n \geq 1$ , we define:

$$X_x^n = \sum_{p=1}^n Y_x^p.$$

It is a centered continuous Gaussian process with covariance function:

$$\forall x, y \in \mathbb{R}^d, \quad k_n(x, y) \stackrel{\text{def}}{=} \text{Cov}(X_x^n, X_y^n) = \sum_{k=1}^n q_k(x, y). \quad (5)$$

The reader may find several important examples of sigma-positive kernels in Appendix A.

## 2.2 Gaussian multiplicative chaos

For each  $n \geq 1$ , we can define a Radon measure  $M_n$  on the Borelian subsets of  $\mathbb{R}^d$  by

$$M_n(A) = \int_A e^{\gamma X_x^n - \frac{\gamma^2}{2} \mathbb{E}[(X_x^n)^2]} dx. \quad (6)$$

For each Borelian set  $A$ , the sequence  $(M_n(A))_n$  is a positive martingale. Thus it converges almost surely towards a random variable denoted by  $M(A)$ . One can deduce that the sequence of measures  $(M_n)_n$  weakly converges towards a random Radon measure  $M$ , commonly denoted by

$$M(A) = \int_A e^{\gamma X_x - \frac{\gamma^2}{2} \mathbb{E}[X_x^2]} dx \quad (7)$$

and called Gaussian multiplicative chaos associated to the kernel  $\gamma^2 K$ . Roughly speaking, (7) can be understood as a measure admitting as density the exponential of a Gaussian distribution  $X$  with covariance kernel  $\gamma^2 K$ . Of course, this is purely formal because the exponential of a random distribution cannot be directly defined. Kahane proved that the martingale  $(M_n(A))_n$ , for some Borelian set  $A$  with non-null finite Lebesgue measure, is uniformly integrable if and only if  $\gamma^2 < 2d$ . This condition is necessary and sufficient in order for the limiting measure  $M$  to be non identically null. Furthermore, he proved that the law of the limiting measure  $M$  does not depend on the decomposition (3) of  $K$  into a sum of positive continuous kernels. For kernels  $K$  that cannot be written as a sum of nonnegative terms as (3), we refer to the extended Gaussian multiplicative theory developed in [30].

### 2.3 Application to the construction of Liouville measures

Formally, the GFF (or Euclidian bosonic massless free field) in a bounded domain  $D \subset \mathbb{R}^2$  is a "Gaussian Field"  $X$  with covariance given by:

$$\mathbb{E}[X_x X_y] = G(x, y),$$

where  $G$  is the Green function of  $D$  with zero boundary condition (see for instance [35] or chapter 2.4 in [24] for the definition and main properties). Let  $B$  be a Brownian motion starting from  $x \in D$  under the measure  $P^x$  and consider the stopping time  $T_D = \inf\{t \geq 0, B_t \notin D\}$ . If we denote  $p_D(t, x, y) = P^x(B_t \in dy, T_D > t)$ , we have:

$$G(x, y) = \pi \int_0^\infty p_D(t, x, y) dt.$$

Note that, for each  $t > 0$ ,  $p_D(t, x, y)$  is a continuous positive and positive definite kernel on  $D$ . Therefore, following Kahane's theory, we can define the Gaussian multiplicative chaos  $M$  associated to the kernel  $\gamma^2 G$ . Since the Green function takes on the form (4), this measure is not trivial provided that  $\gamma^2 < 4$ . We point out that the authors in [11] have suggested a slightly different construction of the Liouville measure. Based on the uniqueness criterion in [30], it can be proved that their construction has the same law as that originally proposed by Kahane.

## 3. Multiplicative chaos and atomic chaos

We stick to the notations of the previous section. We nevertheless assume that the considered Gaussian fields are stationary. Though it may appear as a restriction, the proofs in the general case work exactly the same. Actually, being stationary or not is just hidden in the "small noise  $g$ " appearing in (4).

So, we consider the Gaussian multiplicative chaos  $M$  understood as the limit (in the sense previously described) as  $n \rightarrow \infty$  of the following sequence of random measures

$$M_n(dx) = e^{\gamma X_x^n - \frac{\gamma^2}{2} \mathbb{E}[(X_x^n)^2]} dx. \tag{8}$$

$M$  is a non trivial random measure for  $\gamma^2 < 2d$  with no atoms. Its power-law spectrum  $\xi$ , defined through the relation

$$\mathbb{E}[M(B(0, \lambda))^q] \simeq C_q \lambda^{\xi(q)}, \quad \lambda \rightarrow 0$$

for all  $q \geq 0$  such that the expectation makes sense (i.e. for  $0 \leq q < \frac{2d}{\gamma^2}$ , see [18]), is given by

$$\xi(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2.$$

Now we introduce what we call **atomic Gaussian multiplicative chaos**:

**Theorem 1.** *For  $\alpha \in ]0, 1[$ , we consider a Poisson random measure  $N_\alpha$  distributed on  $\mathbb{R}^d \times \mathbb{R}_+^*$  with intensity  $dx \frac{dz}{z^{1+\alpha}}$  and independent of the sequence  $(Y_x^n)_{x \in \mathbb{R}^d}$ . We introduce the random measure*

$$n_\alpha(dx) = \int_0^{+\infty} z N_\alpha(dx, dz).$$

Then we define the sequence of random measures

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \overline{M}_n(A) = \int_A e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} n_\alpha(dx). \quad (9)$$

For each bounded Borelian set  $A$ , the sequence  $(\overline{M}_n(A))_n$  converges in probability towards a non trivial random variable. Therefore, for each subsequence, we can extract a (deterministic) subsequence such that, almost surely, the sequence of random measures  $(\overline{M}_n(dx))_n$  weakly converges towards a random Radon measure  $\overline{M}$ , the law of which is characterized by the following relation:

$$\mathbb{E}[e^{-u_1 \overline{M}(A_1) + \dots - u_n \overline{M}(A_p)}] = \mathbb{E}[e^{-\frac{\Gamma(1-\alpha)}{\alpha} (u_1^\alpha M(A_1) + \dots + u_n^\alpha M(A_n))}] \quad (10)$$

valid for all  $u_1, \dots, u_p \in \mathbb{R}_+$  and all disjoint Borelian subsets  $A_1, \dots, A_p \subset \mathbb{R}^d$ . In particular, the limiting measure  $\overline{M}$  is non trivial for  $\gamma^2 < 2d$  and all  $\alpha \in ]0, 1[$ .

The above theorem justifies to write formally the law of  $\overline{M}$  as (with  $\overline{\gamma} = \frac{\gamma}{\alpha}$ )

$$\overline{M}(\cdot) = \int e^{\overline{\gamma} X_x - \frac{\alpha \overline{\gamma}^2}{2} \mathbb{E}[X_x^2]} n_\alpha(dx) \quad (11)$$

where  $X$  is a stationary Gaussian distribution with covariance kernel  $K$ . This expression also justifies the fact that the measure  $\overline{M}$  can be seen as a non standard Gaussian multiplicative chaos since the weight has not expectation 1. Furthermore, it can be defined for values of  $\overline{\gamma}^2$  beyond the critical value  $\overline{\gamma}^2 = 2d$ . Notice that the renormalization (i.e.  $\frac{\alpha \overline{\gamma}^2}{2} \mathbb{E}[X_x^2]$ ) differs from the standard Gaussian multiplicative chaos.

**Theorem 2.** For  $\gamma^2 < 2d$  and  $\alpha \in ]0, 1[$ , the law of the random measure  $\overline{M}$  does not depend on the decomposition of  $K$  into a sum of positive continuous kernels of positive type. Furthermore,  $\overline{M}$  is almost surely a purely atomic measure.

There is another way of seeing the law of the measure  $\overline{M}$ . We introduce a positive Radon random measure  $N_M$  distributed on  $\mathbb{R}^d \times \mathbb{R}_+^*$ , whose law conditionally to  $M$  is that of a Poisson random measure with intensity

$$\frac{M(dx) dz}{z^{1+\alpha}}.$$

Then we introduce the family of purely atomic positive random measures

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \overline{M}(A) = \int_A \int_{\mathbb{R}_+} z N_M(dx, dz). \quad (12)$$

Then the law of the random measure  $\overline{M}$  is the same as that of Theorem 1.

### 3.1 Power-law spectrum and moments of the atomic chaos

In this subsection, we assume that  $\gamma^2 < 2d$ . Let us define

$$\forall q \in \mathbb{R}, \quad \bar{\xi}(q) = \left(\frac{d}{\alpha} + \frac{\gamma^2}{2\alpha}\right)q - \frac{\gamma^2}{2\alpha^2}q^2.$$

We will show below that this function coincides with the power law spectrum of the measure  $\overline{M}$ . In particular, we see that  $\bar{\xi}(q) = \xi(\frac{q}{\alpha})$ .

Now we precise the existence of moments for the measure  $\overline{M}$ :

**Proposition 3.** For all Borelian set  $A$  with finite (not null) Lebesgue measure, the random variable  $\overline{M}(A)$  possesses a moment of order  $\beta \geq 0$  if and only if  $\beta < \alpha$ .

Furthermore, we can make explicit the connection between the moments of  $M$  and  $\overline{M}$ : for all  $0 \leq \beta < \alpha$ ,

$$\mathbb{E}[(\overline{M}(A))^\beta] = \frac{\Gamma(1 - \beta/\alpha)\Gamma(1 - \alpha)^{\beta/\alpha}}{\Gamma(1 - \beta)\alpha^{\beta/\alpha}} \mathbb{E}[(M(A))^{\frac{\beta}{\alpha}}] \quad (13)$$

**Theorem 4. (Perfect scaling).** If the kernel  $K$  is given by

$$K(x) = \ln_+ \frac{T}{|x|} + g(x)$$

where  $g$  is a continuous bounded function that is constant in a neighborhood of 0 then, for some  $R > 0$ :

$$\forall 0 < \lambda < 1, \quad (\overline{M}(\lambda A))_{A \subset B(0,R)} \stackrel{\text{law}}{=} \lambda^{d/\alpha} e^{\frac{\Omega_\lambda}{\alpha}} (\overline{M}(A))_{A \subset B(0,R)} \quad (14)$$



where  $\Omega_\lambda$  is a Gaussian random variable independent of the measure  $(\overline{M}(A))_{A \subset B(0,R)}$  the law of which is characterized by:

$$\mathbb{E}[e^{q\Omega_\lambda}] = \lambda^{\frac{\gamma^2}{2}q - \frac{\gamma^2}{2}q^2}.$$

In particular, for all  $0 \leq q < \alpha$ :

$$\mathbb{E}[\overline{M}(B(0, \lambda R))^q] = \lambda^{\bar{\xi}(q)} \mathbb{E}[\overline{M}(B(0, R))^q].$$

**Corollary 5.** Assume that the kernel  $K$  takes on the form (4). Then, for all  $0 \leq q < \alpha$ :

$$\mathbb{E}[\overline{M}(B(0, \lambda R))^q] \simeq C_{q,R} \lambda^{\bar{\xi}(q)}$$

as  $\lambda \rightarrow 0$  for some positive constant  $C_{q,R}$  only depending on  $q, R$ .

## 4. KPZ formula and duality

In this section, we adjust the parameters to stick to Liouville quantum gravity issues. We consider  $\gamma^2 < 2d$  and define  $\bar{\gamma}$  by the relation

$$\gamma \bar{\gamma} = 2d. \tag{15}$$

This gives  $\bar{\gamma} = \frac{\gamma}{\alpha}$  with

$$\alpha = \frac{\gamma^2}{2d} \in ]0, 1[. \tag{16}$$

With this value of  $(\bar{\gamma}, \alpha)$ , the power law spectrum of the corresponding random measure  $\overline{M}$

$$\begin{aligned} \forall A \in \mathcal{B}(\mathbb{R}^d), \quad \overline{M}(A) &= \int_A e^{\frac{\gamma}{\alpha} X_x - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x)^2]} n_\alpha(dx) \\ &= \int_A e^{\bar{\gamma} X_x - d \mathbb{E}[(X_x)^2]} n_\alpha(dx). \end{aligned} \tag{17}$$

can be rewritten as:

$$\bar{\xi}(q) = \left(d + \frac{\bar{\gamma}^2}{2}\right)q - \frac{\bar{\gamma}^2}{2}q^2. \tag{18}$$

Given  $\gamma^2 < 2d$ , we stress that this value of  $\alpha$  is the only possible value of  $0 < \alpha < 1$  ensuring the (statistically) volume preserving condition  $\bar{\xi}(1) = d$ .

The KPZ formula is a relation between the Hausdorff dimensions of a given set  $A$  as measured by the Lebesgue measure,  $M$  or  $\overline{M}$ . So we first recall how to define these dimensions. Given a Radon measure  $\mu$  on  $\mathbb{R}^d$  and  $s \in [0, 1]$ , we define

$$H_\mu^{s,\delta}(A) = \inf \left\{ \sum_k \mu(B_k)^s \right\}$$

where the infimum runs over all the covering  $(B_k)_k$  of  $A$  with open Euclidean balls centered at  $A$  with radius  $r_k \leq \delta$ . Clearly, the mapping  $\delta > 0 \mapsto H_\mu^{s,\delta}(A)$  is decreasing. Hence we can define the  $s$ -dimensional  $\mu$ -Hausdorff outer measure:

$$H_\mu^s(A) = \lim_{\delta \rightarrow 0} H_\mu^{s,\delta}(A).$$

The limit exists but may be infinite.  $H_\mu^s$  is a metric outer measure on  $\mathbb{R}^d$  (see [14] for the definitions). We point out that the fact that  $\mu$  possesses atoms or not does not give rise to any additional difficulty. Thus  $H_\mu^s$  is a measure on the  $\sigma$ -field of  $H_\mu^s$ -measurable sets, which contains all the Borelian sets.

The  $\mu$ -Hausdorff dimension of the set  $A$  is then defined as the value

$$\dim_\mu(A) = \inf\{s \geq 0; H_\mu^s(A) = 0\}. \quad (19)$$

Notice that  $\dim_\mu(A) \in [0, 1]$ . However, it is not clear, in great generality, that we have the classical property:

$$\dim_\mu(A) = \sup\{s \geq 0; H_\mu^s(A) = +\infty\}. \quad (20)$$

This is due to the possible presence of atoms for the measure  $\mu$ . However we claim

**Proposition 6.** *If we take  $\mu = \text{Leb}$  then (20) holds. If we take  $\mu = M$  then, almost surely, (20) holds for every bounded Borelian set. If we take  $\mu = \overline{M}$  and  $A$  a compact set with null Lebesgue measure then (20) holds almost surely.*

This proposition allows to characterize the Hausdorff dimension as the critical value at which the mapping  $s \mapsto H_\mu^s(A)$  jumps from  $+\infty$  to 0.

In what follows, given a compact set  $K$  of  $\mathbb{R}^d$  with null Lebesgue measure, we define its Hausdorff dimensions  $\dim_{\text{Leb}}(K)$ ,  $\dim_M(K)$ ,  $\dim_{\overline{M}}(K)$  computed as indicated above with  $\mu$  respectively equal to the Lebesgue measure,  $M$  and  $\overline{M}$ .

**Theorem 7. KPZ duality.** *Let  $K$  be a compact set of  $\mathbb{R}^d$  with null Lebesgue measure. Almost surely, we have the relations*

$$\dim_{\text{Leb}}(K) = \frac{\xi(\dim_M(K))}{d} \quad \dim_{\text{Leb}}(K) = \frac{\bar{\xi}(\dim_{\overline{M}}(K))}{d}$$

where  $\xi(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2$  and  $\bar{\xi}(q) = (d + \frac{\bar{\gamma}^2}{2})q - \frac{\bar{\gamma}^2}{2}q^2$ . In particular, we have the duality relation between the scaling exponents

$$\dim_{\overline{M}}(K) = \frac{\gamma^2}{2d} \dim_M(K). \quad (21)$$

**Remark 8.** Note that, in the classical physics literature (in particular  $d = 2$ ), it is more usual to focus on the scaling exponents

$$\Delta_\gamma = 1 - \dim_M(K), \quad \Delta_{\bar{\gamma}} = 1 - \dim_{\bar{M}}(K), \quad x = 1 - \dim_{Leb}(K),$$

instead of  $\dim_M(K)$ ,  $\dim_{\bar{M}}(K)$ ,  $\dim_{Leb}(K)$ . Then the KPZ relations read

$$x = \frac{\gamma^2}{4} \Delta_\gamma^2 + (1 - \frac{\gamma^2}{4}) \Delta_\gamma \quad \text{and} \quad x = \frac{\bar{\gamma}^2}{4} \Delta_{\bar{\gamma}}^2 + (1 - \frac{\bar{\gamma}^2}{4}) \Delta_{\bar{\gamma}}.$$

The duality relation then becomes

$$\Delta_{\bar{\gamma}} - 1 = \frac{\gamma^2}{2d} (\Delta_\gamma - 1) = \frac{2d}{\bar{\gamma}^2} (\Delta_\gamma - 1).$$

**Remark 9.** If one looks for random measures satisfying the duality relation (21), it is plain to deduce that such a relation implies that the power law spectrum is necessarily given by (18). Such a power law spectrum indicates that the searched random measures cannot be defined by (7) in the sense that the integrating measure ( $dx$  in (7)) cannot be the Lebesgue measure. Indeed, otherwise Kahane's theory ensures that such measure is identically null. So one has to look for other integrating measures in (7) than the Lebesgue measure. By noticing that  $\bar{\xi}(q) = \xi(\frac{q}{\alpha})$ , one can intuitively recover our construction, namely that the searched measures should be Gaussian multiplicative chaos integrated against independently scattered  $\alpha$ -stable random measures, as stated in Theorem 1.

## 5. Simulations

In this section, we present a few simulations to understand more intuitively the structure of the dual chaos as introduced in section 4, i.e.  $\gamma^2 < 2d$ ,  $\gamma\bar{\gamma} = 4$  and  $\alpha = \frac{\gamma^2}{4}$ .

For each figure 1, 2 and 3 we plot on the left hand side the "density" of the usual chaos. The two other figures (middle and right) are concerned with the corresponding dual measures. In the middle, we plot the position and weights of the atoms of the dual measure. Notice that there are only a very small quantity of atoms with a very big weight. The other atoms have much smaller weights. To have a better picture of the values of these weights, we plot on the right-hand side the same picture with a logarithmic ordinate scale.

Notice that the chaos tends to kill the atoms. The bigger  $\gamma$  is, the stronger the chaos is. We observe that for small values of  $\gamma$  (i.e. large values of  $\bar{\gamma}$ ) the corresponding value of  $\alpha$  is small. Hence the Poisson random measure produces small jumps with a weak intensity and this is sufficient to compensate for the killing of the chaos. When  $\gamma$  becomes larger (close to 2), the chaos is much more powerful to kill the atoms. So the Poisson random measure needs to produce much more atoms ( $\alpha$  is close to 1) to survive to the killing of the chaos (see Figure 3).

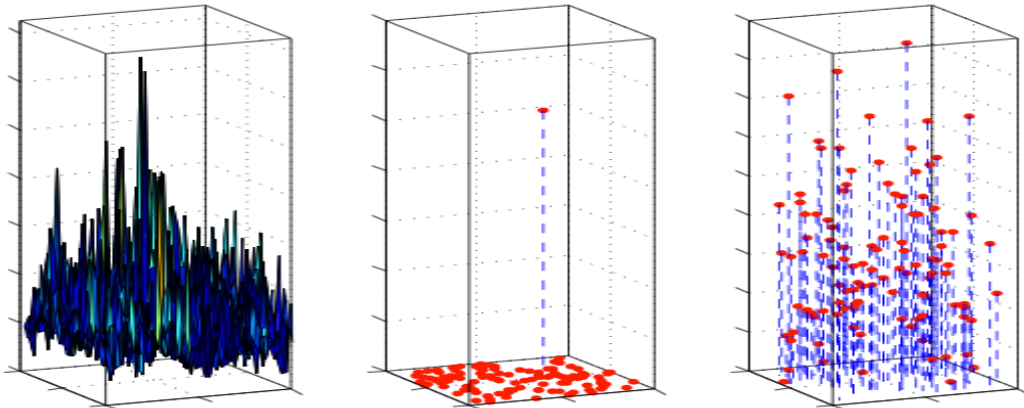


Figure 1: Chaos and dual chaos for the value  $\gamma^2 = 0,25$  (and then  $\alpha = 0.051$ )

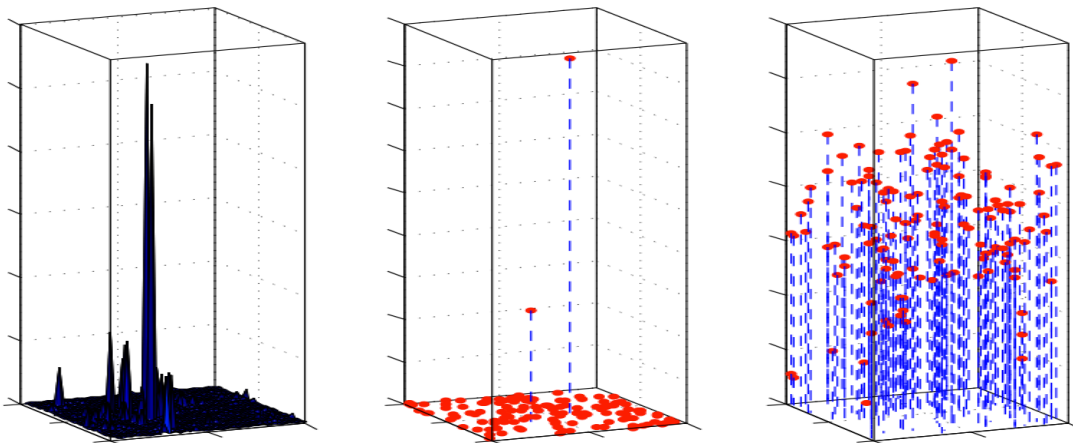


Figure 2: Chaos and dual chaos for the value  $\gamma^2 = 1$  (and then  $\alpha = 0.25$ )

## 6. Perspectives

Here we develop a few comments and open problems related to this work.

### 6.1 Dual chaos and possible renormalisations of degenerate Gaussian multiplicative chaos

We continue to assume to be under exact scale invariance. For  $\theta \geq 0$  consider the associated sequence of measures

$$M_{\theta,n}(A) = \int_A e^{\theta X_x^n - \frac{\theta^2}{2} \mathbb{E}((X_x^n)^2)} dx, \quad n \geq 1, \quad A \in B(\mathbb{R}^d).$$

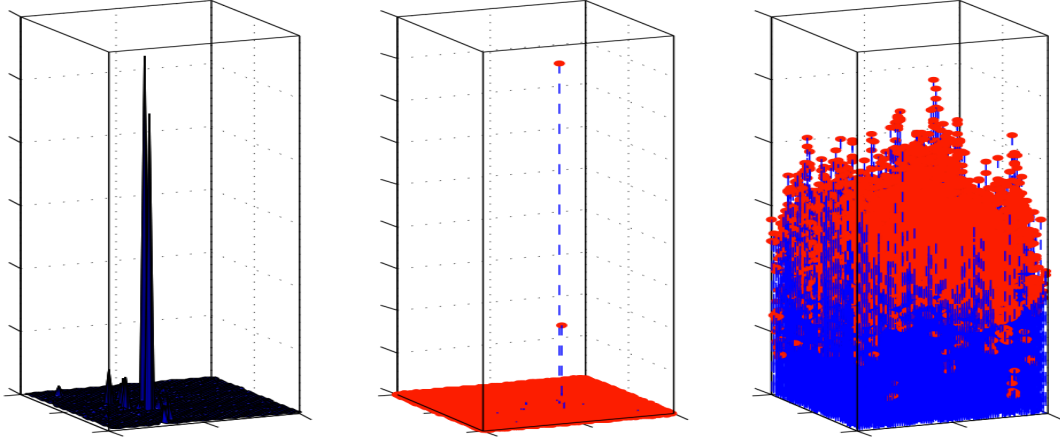


Figure 3: Chaos and dual chaos for the value  $\gamma^2 = 2,56$  (and then  $\alpha = 0.64$ )

Also define

$$\xi_\theta(q) = (d + \frac{\theta^2}{2})q - \frac{\theta^2}{2}q^2, \quad q \geq 0.$$

Recall that  $(M_{\theta,n})_{n \geq 1}$  converges almost surely in the weak-star topology to a Radon measure  $M_\theta$ , which is almost surely positive or null according to whether  $\theta^2 < 2d$  or  $\theta^2 \geq 2d$ .

By analogy with the study of Mandelbrot cascades and the fixed points of the associated smoothing transformation [13, 25], we may conjecture that when  $\theta^2 = 2d$ , the signed measures  $-\frac{d}{d\theta}M_{\theta,n}|_{\theta=\sqrt{2d}}$  weakly converge to a non-degenerate positive measure  $\widetilde{M}_{\sqrt{2d}}$ .

If  $\theta^2 > 2d$ , we have  $\xi'_\theta(1) < 0$  so that there exists a unique  $\alpha \in (0, 1)$  and a unique  $\tilde{\alpha} \in (\alpha, 1)$  such that

$$\xi_\theta(\alpha) = d \quad \text{and} \quad \xi_\theta^*(\xi'_\theta(\tilde{\alpha})) = -d,$$

where  $\xi_\theta^*(s) = \inf_{q \geq 0} sq - \xi_\theta(q)$ . Indeed, the concavity of  $\xi_\theta$  and the fact that  $\xi_\theta(0) = 0 < \xi_\theta(1) = d$  and  $\xi'_\theta(1) < 0$  yields the existence and uniqueness of  $\alpha$ , at which we necessarily have  $\xi'_\theta(\alpha) > \xi_\theta(\alpha)/\alpha = d/\alpha$ . Then, we have  $\xi_\theta^*(\xi'_\theta(\alpha)) = \xi'_\theta(\alpha)\alpha - d > 0$ . Since  $\xi_\theta^*(\xi'_\theta(1)) = \xi'_\theta(1) - d < -d$  and  $\xi_\theta^*$  is concave, we get the existence and uniqueness of  $\tilde{\alpha}$ .

Calculations show that  $\alpha = \frac{2d}{\theta^2}$  and  $\tilde{\alpha} = \sqrt{\alpha}$ . Consequently, if we set  $\bar{\gamma} = \theta$  and  $\gamma = \alpha\bar{\gamma}$ , we see that  $\gamma\bar{\gamma} = 2d$  and  $\alpha$  is exactly the exponent used in the previous sections to establish the duality formula starting from the measure  $M_\gamma$ . Moreover, continuing the analogy with Mandelbrot cascades, the dual chaos  $\overline{M}_\gamma$  is the expected non trivial solution, in “replacement” of  $M_{\bar{\gamma}}$  which vanishes, of the equation

$$\forall 0 < \lambda < 1, \quad (\overline{M}_\gamma(\lambda A))_{A \subset B(0,R)} \stackrel{law}{=} e^{\Omega'_\lambda}(\overline{M}(A))_{A \subset B(0,R)} \quad (22)$$

where  $\Omega'_\lambda$  is a Gaussian random variable independent of the measure  $(\overline{M}_\gamma(A))_{A \subset B(0,R)}$  the law of which is characterized by:

$$\mathbb{E}[e^{q\Omega_\lambda}] = \lambda^{\xi_{\overline{\gamma}}(q)}$$

(for Mandelbrot cascades, in dimension 1, such measures have been identified as stable Lévy subordinators in Mandelbrot time in [4]).

Thus, from the exact scale invariance point of view, the dual chaos of  $M_\gamma$  provides a first way to renormalize  $M_{\overline{\gamma}}$ , by giving a non trivial solution to the functional equation that would satisfy  $M_{\overline{\gamma}}$  if it was not degenerate.

Another way to build a non degenerate object from  $M_{\theta,n}$  when  $\theta^2 > 2d$  is to consider the sequence of normalized measures  $M_{\theta,n}/\|M_{\theta,n}\|$  (the equilibrium Gibbs measures considered for instance in [6]). Examining the behavior of  $\|M_{\theta,n}\|$  shows that it approximately goes to 0 like  $\exp(-n\xi'_\theta(\tilde{\alpha}))$  when  $\mathbb{E}[(X_0^n)^2] = n$ . Then, a tempting conjecture is that  $M_{\theta,n}/\|M_{\theta,n}\|$  converges weakly, in law, to  $\widetilde{M}_{\sqrt{2d}}^{(\tilde{\alpha})}/\|\widetilde{M}_{\sqrt{2d}}^{(\tilde{\alpha})}\|$ , where  $\widetilde{M}_{\sqrt{2d}}^{(\tilde{\alpha})}$  is defined in the same way as  $\overline{M}$  was in (12): fix a random measure  $N_{\widetilde{M}_{\sqrt{2d}}}$  distributed on  $\mathbb{R}^d \times \mathbb{R}_+^*$ , and whose law conditionnally on  $\widetilde{M}$  is that of a Poisson random measure with intensity  $\frac{\widetilde{M}(dx) dz}{z^{1+\tilde{\alpha}}}$ . Then,

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \widetilde{M}_{\sqrt{2d}}^{(\tilde{\alpha})}(A) = \int_A \int_{\mathbb{R}_+} z N_{\widetilde{M}_{\sqrt{2d}}}(dx, dz).$$

Notice, however, that this second family of random measures cannot satisfy the duality relation since their power law spectrum, say  $\xi_{\tilde{\alpha}}$ , satisfies  $\xi_{\tilde{\alpha}}(1) < d$ .

**Remark 10. To continue the interpretation of duality as renormalization:**

Consider an exact scale invariant log-infinitely divisible random measure (see [2, 32]) for which we have enough exponential moments to discuss. We have a parameterized family of multiplicative chaos  $Q_{\theta,\epsilon}(r) = \exp(\theta\Lambda(C_r(t)))/\mathbb{E}(\exp(\theta\Lambda(C_r(t)))$  and the associated measures  $M_{\theta,\epsilon}$ .

Let  $\psi(q) = \lim_{r \rightarrow 0} \log(\mathbb{E}(Q_{1,r}^q(x)))/\log(r)$ . Now we have ,

$$\xi_\theta(q) = (d + \psi(\theta))q - \psi(\theta q)$$

and  $\xi'_\theta(1) = d + \psi(\theta) - \theta\psi'(\theta)$ . There is at most one positive (and at most one negative) solution  $\theta_0$  to the equation  $\xi'_\theta(1) = 0$ , and if  $\theta_0 > 0$  exists, then for  $\theta > 0$  we have  $\xi'_\theta(1) > 0$  iff  $\theta < \theta_0$ .

If  $\overline{\gamma} = \theta > \theta_0$ , we can have exactly the same discussion as in the Gaussian case, by considering the unique root  $\alpha$  of  $\xi_\theta(\alpha) = 0$  and defining the dual chaos associated with  $\gamma = \alpha\theta$ . The dual KPZ relation is then naturally expressed via  $\dim_{\overline{M}_\gamma}(K) = \alpha \dim_{M_\gamma}(K)$ , of which the Gaussian case is a special case.

## 6.2 Singularity spectrum

It would be interesting to compute the free energy of the measure  $\overline{M}$ , namely proving that the following limit is not trivial:

$$\lim_{n \rightarrow \infty} 2^{n(\overline{\xi}(q)-d)} \sum_{I \in C_n} \overline{M}(I)^q \quad (q \in \mathbb{R}),$$

where  $C_n$  stands for the set of all dyadic cubes (included in the unit cube) with side length  $2^{-n}$ . This thermodynamic point of view is closely related to the calculation of the  $L^q$ -spectrum of the measure  $\overline{M}$ , defined as

$$q \in \mathbb{R} \mapsto \tau_{\overline{M}}(q) = \liminf_{r \rightarrow 0^+} \frac{\log \sup \left\{ \sum_i \overline{M}(B(x_i, r))^q \right\}}{\log(r)},$$

where the supremum is taken over all the centered packing of  $[0, 1]^d$  by closed balls of radius  $r$ . By analogy with the study achieved in [4], we conjecture that on the one hand,

$$\tau_{\overline{M}}(q) = \begin{cases} \overline{\xi}'(q_-)q & \text{if } q \leq q_-, \\ \overline{\xi}(q) - d & \text{if } q_- \leq q \leq \alpha, \\ 0 & \text{if } q \geq \alpha, \end{cases}$$

where  $q_-$  is the unique negative solution of  $\overline{\xi}^*(\overline{\xi}'(q)) = -d$ , and on the other hand that the multifractal formalism holds for  $\overline{M}$ : defining

$$E_\delta = \left\{ x \in [0, 1]^d; \liminf_{r \rightarrow 0^+} \frac{\ln \overline{M}(B(x, r))}{\ln(r)} = \delta \right\} \quad (\delta \geq 0),$$

with probability 1, the singularity spectrum of  $\overline{M}$ , i.e. the mapping  $\delta \geq 0 \mapsto \dim E_\delta$ , is given by  $\delta \geq 0 \mapsto \tau_{\overline{M}}^*(\delta) = \inf\{\delta q - \tau_{\overline{M}}(q) : q \in \mathbb{R}\}$ , a negative dimension meaning that  $E_\delta = \emptyset$ .

## A. Examples of sigma-positive kernels

In this section, we detail a few examples of sigma-positive kernels, apart from the Green function already explained in subsection 2.3. More precisely, we give two different classes of sigma-positive kernels, which yield two different notions of stochastic scale invariance for the associated Gaussian multiplicative chaos.

### A.1 Exact stochastic scale invariance

In this section, we describe how to construct kernels yielding the exact scale invariance relations of Theorem 4. This is useful in computations and it is possible to deduce all the other situations from this one.

We define on  $\mathbb{R}_+$  the measure  $\nu_T(dt) = \mathbf{1}_{[0,T]}(t) \frac{dt}{t^2} + \frac{1}{T} \delta_T(dt)$  where  $\delta_x$  denotes the Dirac mass at  $x$ . For  $\mu > 0$ , it is straightforward to check that

$$\forall x \in \mathbb{R}^d, \quad \ln_+ \frac{T}{|x|} = \frac{1}{\mu} \int_0^{+\infty} (t - |x|^\mu)_+ \nu_{T^\mu}(dt). \quad (23)$$

-In dimension  $d = 1$ , it is straightforward to check that the function  $x \mapsto (t - |x|)_+$  is of positive type. So, the kernel  $K(x) = \gamma^2 \ln_+ \frac{T}{|x|}$  is of sigma positive type. The kernels  $k_n$  can be easily computed:

$$k_n(x) = \begin{cases} 0 & \text{if } |x| > T, \\ \gamma^2 \ln_+ \frac{T}{|x|} & \text{if } \frac{T}{n} \leq |x| \leq T, \\ \gamma^2 \ln n + (1 - \frac{n|x|}{T}) & \text{if } 0 \leq |x| \leq \frac{T}{n}. \end{cases}$$

-In dimension  $d = 2$ , Pasenchenko [28] proved that the function  $(1 - |x|^{1/2})_+$  is positive definite in dimension 2. Choosing  $\mu = 2$  in (23), we can thus write

$$\forall x \in \mathbb{R}^2, \quad \gamma^2 \ln_+ \frac{T}{|x|} = \sum_{n \geq 1} q_n(x),$$

where  $q_n$  is the continuous positive and positive definite kernel

$$\forall x \in \mathbb{R}^2, \quad q_n(x) = 2\gamma^2 \int_{\frac{T^{1/2}}{n^{1/2}}}^{\frac{T^{1/2}}{(n-1)^{1/2}}} (t - |x|^\mu)_+ \nu_{T^{1/2}}(dt).$$

A simple computation shows that

$$k_n(x) = \begin{cases} 0 & \text{if } |x| > T, \\ \gamma^2 \ln_+ \frac{T}{|x|} & \text{if } \frac{T}{n} \leq |x| \leq T, \\ \gamma^2 \ln n + 2(1 - \sqrt{\frac{n|x|}{T}}) & \text{if } 0 \leq |x| \leq \frac{T}{n}. \end{cases}$$

-In dimension  $d \geq 3$ , it is proved in [32] that there exists a continuous bounded function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ , constant in a neighborhood of 0 such that

$$K(x) = \gamma^2 \ln_+ \frac{T}{|x|} + g(x) \quad (24)$$

is of sigma positive type.

## A.2 $\star$ -scale invariance

A simple way of constructing sigma positive kernels is given by

$$\forall x \in \mathbb{R}^d, \quad K(x) = \int_1^\infty \frac{k(xu)}{u} du, \quad (25)$$



where  $k$  is a continuous positive kernel of positive type. Such kernel is of sigma positive type since the decomposition can be realized by

$$q_n(x) = \int_{2^n}^{2^{n+1}} \frac{k(xu)}{u} du.$$

Furthermore,  $K$  takes on the form (4) with  $\gamma^2 = k(0)$ . Such kernels are related to the notion of  $\star$ -scale invariance (see [1, 33]).

## B. Proofs of Section 3

### Preliminary computations

We will use the following relation valid for any  $0 < \beta < 1$  and  $x \geq 0$ :

$$x^\beta = \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1 - e^{-xz}) \frac{dz}{z^{1+\beta}}. \quad (26)$$

Thus we have for all  $u \geq 0$ :

$$\mathbb{E}[e^{-u\overline{M}(A)}] = \mathbb{E}\left[e^{\int_A \int_{\mathbb{R}_+} (e^{-zu}-1) \frac{1}{z^{1+\alpha}} dz M(dx)}\right] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} u^\alpha M(A)}\right]. \quad (27)$$

Similarly, we have

$$\mathbb{E}[e^{-u_1\overline{M}(A_1)+\dots-u_n\overline{M}(A_p)}] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} (u_1^\alpha M(A_1)+\dots+u_n^\alpha M(A_n))}\right] \quad (28)$$

valid for all  $u_1, \dots, u_p \in \mathbb{R}_+$  and all disjoint Borelian subsets  $A_1, \dots, A_p \subset \mathbb{R}^d$ .

Then we have for  $0 < \beta < \alpha$ :

$$\begin{aligned} \mathbb{E}[(\overline{M}(A))^\beta] &= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1 - \mathbb{E}[e^{-w\overline{M}(A)}]) \frac{dw}{w^{1+\beta}} \\ &= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1 - \mathbb{E}[e^{-\frac{\Gamma(1-\alpha)}{\alpha} w^\alpha M(A)}]) \frac{dw}{w^{1+\beta}}. \end{aligned}$$

We make the change of variables  $y = w^\alpha$  to get:

$$\begin{aligned} \mathbb{E}[(\overline{M}(A))^\beta] &= \frac{\beta}{\alpha\Gamma(1-\beta)} \int_0^\infty (1 - \mathbb{E}[e^{-\frac{\Gamma(1-\alpha)}{\alpha} y M(A)}]) \frac{dy}{y^{1+\frac{\beta}{\alpha}}} \\ &= \frac{\Gamma(1-\beta/\alpha)\Gamma(1-\alpha)^{\beta/\alpha}}{\Gamma(1-\beta)\alpha^{\beta/\alpha}} \mathbb{E}[(M(A))^{\frac{\beta}{\alpha}}]. \end{aligned} \quad (29)$$

### Proof of Theorem 2.

We first stress that  $N$  can be constructed as the limit in law of a sequence  $(N_n)_n$  of Poisson random measures distributed on  $\mathbb{R}^d \times \mathbb{R}_+^*$  with intensity  $M_n(dx) \otimes \frac{dz}{z^{1+\alpha}}$ . From (28), we deduce that the law of  $\overline{M}$  is characterized by that of  $M$ , which does not depend on the chosen decomposition (see [18]). Furthermore, since  $N$  is a Poisson random measure conditionally to  $M$ , it is clear that it is almost surely purely atomic.  $\square$

## Proofs of Theorem 1

We have for all  $u \geq 0$ :

$$\mathbb{E}[e^{-u\overline{M}_n(A)}] = \mathbb{E}\left[e^{\int_A \int_{\mathbb{R}_+} (e^{-zu}-1) \frac{1}{z^{1+\alpha}} dz M_n(dx)}\right] = \mathbb{E}[e^{-\frac{\Gamma(1-\alpha)}{\alpha} u^\alpha M_n(A)}]. \quad (30)$$

Similarly, we have

$$\mathbb{E}[e^{-u_1\overline{M}_n(A_1)+\dots-u_p\overline{M}_n(A_p)}] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} (u_1^\alpha M_n(A_1)+\dots+u_p^\alpha M_n(A_p))}\right] \quad (31)$$

valid for all  $u_1, \dots, u_p \in \mathbb{R}_+$  and all disjoint Borelian subsets  $A_1, \dots, A_p \subset \mathbb{R}^d$ . Since  $(M_n)_n$  almost surely weakly converges towards  $M$ , it is obvious to check the convergence in law as well as to characterize the law of the limiting measure.

Now we tackle the convergence in probability. We will prove that we can extract from each subsequence a subsequence converging in probability. So we consider a subsequence  $(\phi(n))_n$ . Since  $\text{Var}(X_0^{\phi(n)}) = \sum_{k=1}^{\phi(n)} \text{Var}(Y_0^k) \rightarrow \infty$  as  $n \rightarrow \infty$ , we can find a subsequence  $(\psi(n))_n$  such that

$$\sum_{k=\psi(n-1)+1}^{\psi(n)} \text{Var}(Y_0^k) \geq \rho \ln n \quad (32)$$

where  $\rho > \frac{8}{\gamma^2}$ . So, it remains to prove that the sequence  $(\overline{M}_{\psi(n)}(A))_n$  almost surely converges for every bounded Borelian subset  $A$  of  $\mathbb{R}^d$ .

We can rearrange the  $(Y^n)_n$  and set

$$Y'^n = \sum_{k=\psi(n-1)+1}^{\psi(n)} Y^k, \quad X'^n = \sum_{k=1}^n Y'^k. \quad (33)$$

For the sake of clarity, we will omit the superscript  $'$  from the notations. So we will just assume below that the processes  $(X^n)_n, (Y^n)_n$  satisfy the usual properties of Section 2 together with the constraint (32).

We fix  $c > 0$ . We denote by  $\mathcal{F}_n$  ( $n \geq 0$ ) the sigma algebra generated by the random measure  $n_\alpha$  and the random processes  $(X^p)_p \leq n$ . We set

$$\begin{aligned} \overline{M}_n^c(A) &= \int_A \int_0^{ce^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]}} e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} z n_\alpha(dx, dz) \\ &= \int_A \int_0^{+\infty} \mathbf{1}_{\{0 \leq z \leq ce^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]}\}} e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} z n_\alpha(dx, dz) \end{aligned} \quad (34)$$

We have

$$\begin{aligned}
& \mathbb{E} \left[ (\overline{M}_{n+1}^c(A))^\alpha | \mathcal{F}_n \right] \\
&= \mathbb{E} \left[ \left( \int_A \int_0^{ce^{\frac{\gamma}{\alpha} X_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^{n+1])^2]}} e^{\frac{\gamma}{\alpha} X_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^{n+1])^2]} z n_\alpha(dx, dz) \right)^\alpha | \mathcal{F}_n \right] \\
&\geq \mathbb{E} \left[ \left( \int_A \int_0^{ce^{\frac{\gamma}{\alpha} X_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^{n+1])^2]}} \mathbf{1}_{\left\{ \frac{\gamma}{\alpha} Y_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(Y_x^{n+1])^2]} \geq 0 \right\}} e^{\frac{\gamma}{\alpha} X_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^{n+1])^2]} z n_\alpha(dx, dz) \right)^\alpha | \mathcal{F}_n \right] \\
&\geq \mathbb{E} \left[ \left( \int_A \int_0^{ce^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]}} \mathbf{1}_{\left\{ \frac{\gamma}{\alpha} Y_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(Y_x^{n+1])^2]} \geq 0 \right\}} e^{\frac{\gamma}{\alpha} X_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^{n+1])^2]} z n_\alpha(dx, dz) \right)^\alpha | \mathcal{F}_n \right],
\end{aligned}$$

using the fact that  $\frac{\gamma}{\alpha} X_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^{n+1])^2] = \frac{\gamma}{\alpha} Y_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(Y_x^{n+1])^2] + \frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]$ . Since the mapping  $x \in \mathbb{R}_+ \mapsto x^\alpha$  is concave, we apply Jensen's inequality with respect the restriction to  $A$  of the measure  $\overline{M}_n^c / \overline{M}_n^c(A)$ , conditionally on  $\overline{M}_n^c(A) \neq 0$ , and get:

$$\begin{aligned}
& \mathbb{E} \left[ (\overline{M}_{n+1}^c(A))^\alpha | \mathcal{F}_n \right] \\
&\geq \int_A \int_0^{ce^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]}} \mathbb{E} \left[ e^{\gamma Y_x^{n+1} - \frac{\gamma^2}{2} \mathbb{E}[(Y_x^{n+1])^2]} \mathbf{1}_{\left\{ \frac{\gamma}{\alpha} Y_x^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(Y_x^{n+1])^2]} \geq 0 \right\}} \right] \times \dots \\
&\quad \dots \times e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} z n_\alpha(dx, dz) \mathbf{1}_{\{\overline{M}_n^c(A) \neq 0\}} (\overline{M}_n^c(A))^{\alpha-1} \\
&= \mathbf{1}_{\{\overline{M}_n^c(A) \neq 0\}} (\overline{M}_n^c(A))^\alpha \mathbb{E} \left[ e^{\gamma Y_0^{n+1} - \frac{\gamma^2}{2} \mathbb{E}[(Y_0^{n+1])^2]} \mathbf{1}_{\left\{ \frac{\gamma}{\alpha} Y_0^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(Y_0^{n+1])^2]} \geq 0 \right\}} \right] \\
&= (\overline{M}_n^c(A))^\alpha \mathbb{E} \left[ e^{\gamma Y_0^{n+1} - \frac{\gamma^2}{2} \mathbb{E}[(Y_0^{n+1])^2]} \mathbf{1}_{\left\{ \frac{\gamma}{\alpha} Y_0^{n+1} - \frac{\gamma^2}{2\alpha} \mathbb{E}[(Y_0^{n+1])^2]} \geq 0 \right\}} \right].
\end{aligned}$$

By using a Girsanov transform we deduce

$$\begin{aligned}
\mathbb{E} \left[ (\overline{M}_{n+1}^c(A))^\alpha | \mathcal{F}_n \right] &\geq (\overline{M}_n^c(A))^\alpha \mathbb{P} \left( \gamma Y_0^{n+1} + \frac{\gamma^2}{2} \mathbb{E}[(Y_0^{n+1])^2] \geq 0 \right) \\
&= (\overline{M}_n^c(A))^\alpha \left( 1 - \mathbb{P}(\mathcal{N}(0, 1) \geq \frac{\gamma}{2} \text{Var}(Y_0^{n+1})^{\frac{1}{2}}) \right).
\end{aligned}$$

Let us set  $b_n = \mathbb{P}(\mathcal{N}(0, 1) \geq \frac{\gamma}{2} \text{Var}(Y_0^{n+1})^{1/2})$  in such a way that

$$\mathbb{E} \left[ (\overline{M}_{n+1}^c(A))^\alpha | \mathcal{F}_n \right] \geq (\overline{M}_n^c(A))^\alpha (1 - b_n).$$

We further stress that (32) implies that the series  $\sum_n b_n$  is absolutely convergent. Let us define

$$c_n = \prod_{k=1}^n \frac{1}{1 - b_k}$$

and

$$A_n = c_n (\overline{M}_n^c(A))^\alpha.$$

The sequence  $(A_n)_n$  is a positive sub-martingale. Let us prove that it is bounded in  $L^1$ . We have

$$\begin{aligned} \mathbb{E}[A_n] &= c_n \mathbb{E} \left[ (\overline{M}_n^c(A))^\alpha \right] \\ &= \frac{\alpha c_n}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{1 - \mathbb{E}[e^{-u\overline{M}_n^c(A)}]}{u^{1+\alpha}} du \\ &= \frac{\alpha c_n}{\Gamma(1-\alpha)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{\int_A \int_0^c \frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} \left( e^{-u z e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} - 1 \right) \frac{1}{z^{1+\alpha}} dz dx} \right] \right) \frac{1}{u^{1+\alpha}} du \\ &= \frac{\alpha c_n}{\Gamma(1-\alpha)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{\int_A \int_0^c (e^{-uy} - 1) \frac{1}{y^{1+\alpha}} dy e^{\gamma X_x^n - \frac{\gamma^2}{2} \mathbb{E}[(X_x^n)^2]} dx} \right] \right) \frac{1}{u^{1+\alpha}} du \\ &= \frac{\alpha c_n}{\Gamma(1-\alpha)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{-M_n(A)L_c(u)} \right] \right) \frac{1}{u^{1+\alpha}} du, \end{aligned}$$

where  $L_c(u) = \int_0^c (1 - e^{-uy}) \frac{1}{y^{1+\alpha}} dy$ . Notice that  $L_c(u) \geq 0$  for each  $u \geq 0$ . Since the mapping  $x \mapsto -e^{-xL_c(u)}$  is concave, we can use Jensen's inequality to get

$$\begin{aligned} \mathbb{E}[A_n] &\leq \frac{\alpha c_n}{\Gamma(1-\alpha)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{-|A|L_c(u)} \right] \right) \frac{1}{u^{1+\alpha}} du \\ &= c_n \mathbb{E} \left[ \left( \int_A \int_0^c z n_\alpha(dx, dz) \right)^\alpha \right]. \end{aligned}$$

The expectation in the above right-hand side is finite. Furthermore, the convergence of the series  $\sum_n b_n$  implies the convergence of the sequence  $(c_n)_n$  towards  $\prod_{k=1}^{+\infty} \frac{1}{1-b_k} \in ]0, +\infty[$ . Therefore, the sub-martingale  $(A_n)_n$  almost surely converges. So does  $(\overline{M}_n^c(A))_n$ . Let us denote by  $\overline{M}^c(A)$  its limit. Obviously, the mapping  $c \mapsto \overline{M}^c(A)$  is increasing and thus converges as  $c$  goes to  $\infty$ . Let us denote by  $\overline{M}^\infty(A)$  the limit.

Now we prove that the sequence  $(\overline{M}_n(A))_n$  almost surely converges towards  $\overline{M}^\infty(A)$ . We have for  $\delta > 0$ :

$$\begin{aligned} &\mathbb{P}(|\overline{M}_n(A) - \overline{M}^\infty(A)| > \delta) \\ &\leq \mathbb{P}(|\overline{M}_n(A) - \overline{M}_n^c(A)| > \frac{\delta}{3}) + \mathbb{P}(|\overline{M}_n^c(A) - \overline{M}^c(A)| > \frac{\delta}{3}) + \mathbb{P}(|\overline{M}^c(A) - \overline{M}^\infty(A)| > \frac{\delta}{3}) \\ &\leq \frac{3^\beta}{\delta^\beta} \mathbb{E}[|\overline{M}_n(A) - \overline{M}_n^c(A)|^\beta] + \mathbb{P}(|\overline{M}_n^c(A) - \overline{M}^c(A)| > \frac{\delta}{3}) + \mathbb{P}(|\overline{M}^c(A) - \overline{M}^\infty(A)| > \frac{\delta}{3}) \end{aligned} \tag{35}$$

where  $\beta < \alpha$ . We evaluate the first quantity. We have:

$$\begin{aligned}
& \mathbb{E}[|\overline{M}_n(A) - \overline{M}_n^c(A)|^\beta] \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^{+\infty} \frac{1 - \mathbb{E}[e^{-u(\overline{M}_n(A) - \overline{M}_n^c(A))}]}{u^{1+\beta}} du \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{\int_A \int_c^{\frac{\gamma}{c} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} \left( e^{-uz e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} - 1 \right) \frac{1}{z^{1+\alpha}} dz dx} \right] \right) \frac{1}{u^{1+\beta}} du \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{\int_A \int_c^{\frac{\gamma}{c} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} \left( e^{-uy} - 1 \right) \frac{1}{y^{1+\alpha}} dy e^{\gamma X_x^n - \frac{\gamma^2}{2} \mathbb{E}[(X_x^n)^2]} dx} \right] \right) \frac{1}{u^{1+\beta}} du \\
&= \frac{\beta}{\Gamma(1-\beta)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{-M_n(A) U_c(u)} \right] \right) \frac{1}{u^{1+\beta}} du
\end{aligned}$$

where  $U_c(u) = \int_c^{+\infty} (1 - e^{-uy}) \frac{1}{y^{1+\alpha}} dy$ . Notice that  $U_c(u) \geq 0$  for each  $u \geq 0$ . From Jensen's inequality again, we have

$$\begin{aligned}
\mathbb{E}[|\overline{M}_n(A) - \overline{M}_n^c(A)|^\beta] &\leq \frac{\beta}{\Gamma(1-\beta)} \int_0^{+\infty} \left( 1 - \mathbb{E} \left[ e^{-|A| U_c(u)} \right] \right) \frac{1}{u^{1+\beta}} du \\
&= \mathbb{E} \left[ \left( \int_A \int_c^{\frac{\gamma}{c} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} z n_\alpha(dx, dz) \right)^\beta \right]. \tag{36}
\end{aligned}$$

This latter quantity converges to 0 as  $c$  goes to  $\infty$  (uniformly with respect to  $n$ ).

Now we come back to (35) to complete the proof. We can fix  $c > 0$  so as to make the first and third quantity as small as we please. Indeed, concerning the first quantity, it results from the bound just above (36) and concerning the third quantity, it results from the almost sure convergence of  $\overline{M}_n^c(A)$  towards  $\overline{M}^\infty(A)$ . For such a  $c$ , we can find  $N$  such that the second quantity is also as small as we please for  $n \geq N$ .

□

**Remark 11.** *The reader may find the above proof more tricky than expected. Actually, the truncation suggested in (34) is not the more natural way that we may think of to tackle the problem. The first idea that we may come up with is rather to define*

$$\overline{M}_n^c(A) = \int_A \int_0^c e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} z n_\alpha(dx, dz).$$

*It is straightforward to check that  $(\overline{M}_n^c(A)^\alpha)_n$  is a submartingale. But it is not bounded in  $L^1$ . Indeed, if it was, its limit would be  $\overline{M}(A)^\alpha$  regardless of the value of  $c$  because the chaos "kills" the big jumps, i.e.  $\int_A \int_c^\infty e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} z n_\alpha(dx, dz) \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Thus,  $\overline{M}(A)$  would admit a moment of order  $\alpha$ , which is impossible (see below).*

### Proofs of Proposition 3

For  $\beta < \alpha$ , we can use relation (29) to show the existence of the moments and the dual relation (13). If  $\overline{M}$  possesses a moment of order  $\alpha$  then the left-hand side of equation (13) must converge as  $\beta \rightarrow \alpha$ . But it is equal to the right-hand side, which diverges because of the term  $\Gamma(1 - \beta/\alpha)$  and the fact that the measure  $M$  possesses a non trivial moment of order 1.

### Proof of Theorem 4

First we stress that it has already been proved that the chaos measure  $M$ , associated to the given kernel  $K$ , satisfies the scale invariance relation (see [32]) for some  $R > 0$ :

$$\forall 0 < \lambda < 1, \quad (M(\lambda A))_{A \subset B(0,R)} \stackrel{law}{=} \lambda^d e^{\Omega_\lambda} (M(A))_{A \subset B(0,R)}$$

where  $\Omega_\lambda$  is a Gaussian random variable independent of the measure  $(\overline{M}(A))_{A \subset B(0,R)}$  the law of which is characterized by:

$$\mathbb{E}[e^{q\Omega_\lambda}] = \lambda^{\frac{\gamma^2}{2}q - \frac{\gamma^2}{2}q^2}.$$

The results then easily follows from the relation

$$\mathbb{E}[e^{iu_1 \overline{M}(A_1) + \dots + iu_n \overline{M}(A_n)}] = \mathbb{E}[e^{-u_1^\alpha M(A_1) + \dots - u_n^\alpha M(A_n)}]$$

valid for all  $u_1, \dots, u_n \in \mathbb{R}$  and all disjoint Borelian subsets  $A_1, \dots, A_n \subset \mathbb{R}^d$ .  $\square$

### Proof of Corollary 5

Let us write the kernel  $K$  as

$$K(x) = K_p(x) + h(x)$$

where  $K_p$  is the "perfect kernel" given by (24) and  $g$  is some continuous bounded function over  $\mathbb{R}^d$ . Even if it means adding to  $K$  a constant, we may assume that  $h(0) = 0$  and, without loss of generality, we assume  $R = 1$ . For  $t > 0$ , we define

$$G_t = \sup_{|x| \leq t} |h(x)|.$$

Let us also consider the measures  $M^p, \overline{M}^p$  associated to the perfect kernel  $K_p$ . Let us denote by  $B_\lambda$  the ball centered at 0 with radius  $\lambda$ . From Kahane's concentration inequalities [18], we have for all  $q \leq 1$ :

$$\mathbb{E}[(M(B_\lambda))^q] \geq \mathbb{E}[(M^p(B_\lambda) e^{\gamma \sqrt{G_\lambda} Z - \frac{\gamma^2}{2} G_\lambda})^q]$$

where  $Z$  is a standard Gaussian random variable independent of  $M^p$ . Hence, by using Theorem 4, we have:

$$\begin{aligned}\mathbb{E}[(M(B_\lambda))^q] &\geq \mathbb{E}[(M^p(B_\lambda))^q] \mathbb{E}[(e^{\gamma\sqrt{G_\lambda}Z - \frac{\gamma^2}{2}G_\lambda})^q] \\ &= \lambda^{\xi(q)} \mathbb{E}[M^p(B_1)^q] e^{q^2 \frac{\gamma^2}{2} G_\lambda - q \frac{\gamma^2}{2} G_\lambda}.\end{aligned}$$

With the same argument we prove

$$e^{q^2 \frac{\gamma^2}{2} G_\lambda - q \frac{\gamma^2}{2} G_\lambda} \mathbb{E}[(M(B_\lambda))^q] \leq \lambda^{\xi(q)} \mathbb{E}[M^p(B_1)^q].$$

Because  $G_\lambda \rightarrow 0$  as  $\lambda \rightarrow 0$ , the result follows from relation (13).  $\square$

## C. Proofs of Section 4.

### C.1 Proof of Proposition 6

We assume that  $A$  is bounded, say included in the ball  $B(0, 1)$ . We have for  $s < t$ :

$$H_\mu^{t,\delta}(A) \leq H_\mu^{s,\delta}(A) \sup_{\substack{B \text{ ball centered in } A, \\ B \subset B(0,1), \text{diam}(B) \leq \delta}} \mu(B)^{t-s}.$$

Obviously, it suffices to prove that the quantity  $\sup_{\substack{B \text{ ball centered in } A, \\ B \subset B(0,1), \text{diam}(B) \leq \delta}} \mu(B)$  converges to 0 as  $\delta \rightarrow 0$ . It is clear if  $\mu$  is the Lebesgue measure. If  $\mu = M$ , this results from the fact that  $M$  does not possess any atom (see Lemma 12 below). It remains to investigate the situation when  $\mu = \overline{M}$ . Let  $A$  be a compact subset included in the ball  $B(0, 1)$  with null Lebesgue measure. For  $0 < \beta < \alpha$ , we have  $\mathbb{E}[\overline{M}(A)^\beta] = c_{\alpha,\beta} \mathbb{E}[M(A)^\frac{\beta}{\alpha}] = 0$  since  $M(A) = 0$  almost surely. Therefore, almost surely, the measure  $\overline{M}$  does not possess any atom on the set  $A$ . Now we prove that, almost surely,

$$\sup_{\substack{B \text{ ball centered in } A, \\ B \subset B(0,1), \text{diam}(B) \leq \delta}} \overline{M}(B) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We argue by contradiction. Assume that this quantity does not converge towards 0. We can find  $\epsilon > 0$ , a sequence  $(x_n)_n$  of points in  $A$  and a sequence  $(r_n)_n$  of positive radius such that  $\overline{M}(B(x_n, r_n)) \geq \epsilon$ . Even if it means extracting a subsequence, we may assume that the sequence  $(x_n)_n$  converges towards  $x \in A$ . We deduce  $\overline{M}(\{x\}) \geq \epsilon$ . This means that  $\overline{M}$  possesses an atom on  $A$ . Contradiction.  $\square$

**Lemma 12.** *Almost surely, the measure  $M$  does not possess any atom.*

*Proof.* By stationarity, it is enough to prove that, almost surely, the measure  $M$  does not possess any atom on the cube  $[0, 1]^d$ . For  $n \in \mathbb{N}^*$  and  $k_1, \dots, k_d \in \{1, \dots, n\}$ ,

let us denote by  $I_{k_1, \dots, k_d}^n$  the cube  $\prod_{i=1}^d [\frac{k_i-1}{n}, \frac{k_i}{n}]$ . From [10, Corollary 9.3 VI], it is enough to check that for each  $\eta > 0$ :

$$\sum_{k_1, \dots, k_d=1}^n \mathbb{P}\left(M(I_{k_1, \dots, k_d}^n) > \eta\right) = n^d \mathbb{P}\left(M(I_{0, \dots, 0}^n) > \eta\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This is a direct consequence of the Markov inequality

$$n^d \mathbb{P}\left(M(I_{0, \dots, 0}^n) > \eta\right) \leq \frac{n^d}{\eta^q} \mathbb{E}[M(I_{0, \dots, 0}^n)^q]$$

and the relation, for  $1 < q < \frac{2d}{\gamma^2}$  (see the proof of corollary 5),

$$\mathbb{E}[M(I_{0, \dots, 0}^n)^q] \leq C n^{-\xi(q)}.$$

Indeed, for  $1 < q < \frac{2d}{\gamma^2}$ , we have  $\xi(q) > d$ . □

## C.2 Proof of the usual KPZ formula

The usual KPZ relation has already been proved in [11, 31]. For the sake of clarity and completeness, we sketch here a simple proof in the Gaussian case. We further have the feeling that this short proof is worth being written as it helps to understand the KPZ formula in an easy way. It relies on the intensive use of the scaling properties of the Gaussian multiplicative chaos as well as the use of the Girsanov transform, which much simplifies the computations in comparison with [11, 31]. For the sake of simplicity of notations, we make the proof in dimension  $d = 1$  but the proof in higher dimensions can be identically reproduced word for word. We also assume that  $M$  is the perfect measure, namely the measure with associated kernel given by  $\gamma^2 \ln_+ \frac{T}{|x|}$ . Actually, it can easily be proved with the Kahane convexity inequalities (see [18] or [30, cor. 6.2]) that this is not a restriction. We also mention that  $M$  can be constructed as the limit

$$M(dx) = \lim_{l \rightarrow 0} M_l(dx) \stackrel{def}{=} e^{\gamma X_x^l - \frac{\gamma^2}{2} \mathbb{E}[(X_x^l)^2]} dx$$

where  $X_l$  is a stationary Gaussian process with covariance kernel given by:

$$k_l(x) = \begin{cases} 0 & \text{if } |x| > T, \\ \ln_+ \frac{T}{|x|} & \text{if } lT \leq |x| \leq T, \\ \ln \frac{1}{l} + (1 - \frac{|x|}{lT}) & \text{if } 0 \leq |x| \leq lT. \end{cases}$$

Such a family of kernels possesses useful scaling properties, namely that for  $|x| \leq T$  and  $0 < \lambda < 1$ ,  $k_{\lambda l}(\lambda x) = k_l(x) + \ln \frac{1}{\lambda}$ . In particular, we have the following scaling relation for all  $0 < l < 1$  and all  $0 < \lambda < 1$ :

$$\left( (X_{\lambda x}^{\lambda l})_{x \in B(0, T)}, (M_{\lambda l}(\lambda A))_{A \subset B(0, T)} \right) \stackrel{law}{=} \left( (X_x^l + \Omega_\lambda)_{x \in B(0, T)}, (\lambda e^{\gamma \Omega_\lambda - \frac{\gamma^2}{2} \ln \frac{1}{\lambda}} M_l(A))_{A \subset B(0, T)} \right). \quad (37)$$



where  $\Omega_\lambda$  is a centered Gaussian random variable with variance  $\ln \frac{1}{\lambda}$  and independent of the couple  $((X_x^l)_{x \in B(0,T)}, (M_l(A))_{A \subset B(0,T)})$ . We will use the above relation throughout the proof.

Now we begin with the proof. Without loss of generality we assume that  $T = 1$ . Let  $K$  be a compact subset of  $\mathbb{R}$ , included in  $[0, 1]$ , with Hausdorff dimension  $\dim_{Leb}(K)$ . Let  $q \in [0, 1]$  be such that  $\xi(q) > \dim_{Leb}(K)$ . For  $\epsilon > 0$ , there is a covering of  $K$  by a countable family of balls  $(B(x_n, r_n))_n$  such that

$$\sum_n r_n^{\xi(q)} < \epsilon.$$

By using in turn the stationarity and the power law spectrum of the measure, we have

$$\begin{aligned} \mathbb{E} \left[ \sum_n M(B(x_n, r_n))^q \right] &= \sum_n \mathbb{E} \left[ M(B(0, r_n))^q \right] \\ &\leq C_q \sum_n r_n^{\xi(q)} \\ &\leq C_q \epsilon, \end{aligned}$$

we deduce by the Markov inequality

$$\mathbb{P} \left( \sum_n M(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon} \right) \geq 1 - \sqrt{\epsilon}.$$

Thus, with probability  $1 - \sqrt{\epsilon}$ , there is a covering of balls of  $K$  such that  $\sum_n M(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon}$ . So  $q \geq \dim_M(K)$  almost surely.

Conversely, consider  $q \geq 0$  such that  $\xi(q) < \dim_{Leb}(K)$ . By the Frostman Lemma, there is a probability measure  $\gamma$  supported by  $K$  such that

$$\int_{[0,1]^2} \frac{1}{|x-y|^{\xi(q)}} \gamma(dx) \gamma(dy) < +\infty.$$

Let us define the random measure  $\tilde{\gamma}$  as the almost sure limit of the following family of positive random measures:

$$\tilde{\gamma}(dx) = \lim_{l \rightarrow 0} e^{q\gamma X_x^l - \frac{q^2 \gamma^2}{2} \mathbb{E}[(X_x^l)^2]} \gamma(dx). \quad (38)$$

For  $\gamma^2 < 2$ , the limit is non trivial because  $q^2 \gamma^2 / 2 < \xi(q)$  and supported by  $K$  (see [18]). From the Frostman lemma again, we just have to prove that the quantity

$$\int_{[0,1]^2} \frac{1}{M([x,y])^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy)$$

is finite almost surely. It suffices to prove that the above quantity has a finite expectation. Moreover, by using the Fatou lemma and the stationarity of the measure  $M$ , we have

$$\begin{aligned} \mathbb{E} \left[ \int_{[0,1]^2} \frac{1}{M([x,y])^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] &\leq \liminf_l \int_{[0,1]^2} \mathbb{E} \left[ \frac{e^{q\gamma X_x^l + q\gamma X_y^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([x,y])^q} \right] \gamma(dx) \gamma(dy) \\ &= \liminf_l 2 \int_{y \geq x} \mathbb{E} \left[ \frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy). \end{aligned}$$

We decompose the last integral into two terms:

$$\begin{aligned} &\int_{y \geq x} \mathbb{E} \left[ \frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{0 \leq y-x \leq l} \mathbb{E} \left[ \frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy) \\ &\quad + \int_{y-x \geq l} \mathbb{E} \left[ \frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy) \\ &\stackrel{def}{=} A_l^1 + A_l^2. \end{aligned}$$

For each of the above terms, we will use an appropriate scaling relation.

By using (37), we deduce

$$\begin{aligned} A_l^2 &= \int_{y-x \geq l} \mathbb{E} \left[ \frac{e^{2q\gamma \Omega_{y-x} - q^2 \gamma^2 \ln \frac{1}{y-x} e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}}}{(y-x)^q e^{q\Omega_{y-x} - q \frac{\gamma^2}{2} \ln \frac{1}{y-x}} M_{\frac{l}{y-x}}([0,1])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{y-x \geq l} \mathbb{E} \left[ \frac{e^{q\gamma \Omega_{y-x} - (q^2 \gamma^2 - q \frac{\gamma^2}{2}) \ln \frac{1}{y-x}}}{(y-x)^q} \right] \mathbb{E} \left[ \frac{e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}}{M_{\frac{l}{y-x}}([0,1])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{y-x \geq l} \frac{1}{(y-x)^{\xi(q)}} \mathbb{E} \left[ \frac{e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}}{M_{\frac{l}{y-x}}([0,1])^q} \right] \gamma(dx) \gamma(dy) \end{aligned}$$

By using a Girsanov transform, we have

$$\begin{aligned} \mathbb{E} \left[ \frac{e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}}{M_{\frac{l}{y-x}}([0,1])^q} \right] &= \mathbb{E} \left[ \frac{e^{q^2 \gamma^2 k \frac{l}{y-x} (1)}}{\left( \int_0^1 e^{\gamma X_r^{\frac{l}{y-x}} - \frac{\gamma^2}{2} \mathbb{E}[(X_x^{\frac{l}{y-x})^2]} + q\gamma^2 k \frac{l}{y-x} (1-r) + q\gamma^2 k \frac{l}{y-x} (r)} dr \right)^q} \right] \\ &\leq C \mathbb{E} \left[ \frac{1}{\left( \int_0^1 e^{\gamma X_r^{\frac{l}{y-x}} - \frac{\gamma^2}{2} \mathbb{E}[(X_x^{\frac{l}{y-x})^2]} } dr \right)^q} \right] \end{aligned}$$

for some positive constant  $C$ . Notice that we have just used the fact that  $k_{\frac{l}{y-x}}(1) = 0$  and that  $k_{\frac{l}{y-x}}$  is positive. It is a standard fact that the measure  $M_{\frac{l}{y-x}}$  possesses moments of negative order so that we have proved

$$\lim_l A_l^2 \leq C \int_{[0,1]^2} \frac{1}{|y-x|^{\xi(q)}} \gamma(dx) \gamma(dy) < +\infty.$$

To treat the term  $A_l^1$ , we use quite a similar argument excepted that we use the scaling relation on  $l$  instead of  $y-x$ , and the Girsanov transform again:

$$\begin{aligned} A_l^2 &= \int_{0 \leq y-x \leq l} \mathbb{E} \left[ \frac{e^{2q\gamma\Omega_l - q^2\gamma^2 \ln \frac{l}{l}} e^{q\gamma X_0^1 + q\gamma X_{\frac{y-x}{l}}^1 - q^2\gamma^2 \mathbb{E}[(X_x^1)^2]}}{l^q e^{q\Omega_l - q\frac{\gamma^2}{2} \ln \frac{l}{l}} M_1([0, \frac{y-x}{l}])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{0 \leq y-x \leq l} \mathbb{E} \left[ \frac{e^{q\gamma\Omega_l - (q^2\gamma^2 - q\frac{\gamma^2}{2}) \ln \frac{l}{l}}}{l^q} \right] \mathbb{E} \left[ \frac{e^{q\gamma X_{\frac{y-x}{l}}^1 + q\gamma X_0^1 - q^2\gamma^2 \mathbb{E}[(X_x^1)^2]}}{M_1([0, \frac{y-x}{l}])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{0 \leq y-x \leq l} \frac{1}{l^{\xi(q)}} \mathbb{E} \left[ \frac{e^{q^2\gamma^2 k_1(\frac{y-x}{l})}}{\left( \int_0^{\frac{y-x}{l}} e^{\gamma X_r^1 - \frac{\gamma^2}{2} \mathbb{E}[(X_r^1)^2] + q\gamma^2 k_1(\frac{y-x}{l} - r) + q\gamma^2 k_1(r)} dr \right)^q} \right] \gamma(dx) \gamma(dy). \end{aligned}$$

By using the fact that  $k_1$  is positive and bounded by 1, we have (for some positive constant  $C$  independent of  $l$ )

$$A_l^2 \leq C \int_{0 \leq y-x \leq l} \frac{1}{l^{\xi(q)}} \mathbb{E} \left[ \frac{1}{\left( \int_0^{\frac{y-x}{l}} e^{\gamma X_r^1 - \frac{\gamma^2}{2} \mathbb{E}[(X_r^1)^2]} dr \right)^q} \right] \gamma(dx) \gamma(dy).$$

Since  $\mathbb{E}[X_r^1 X_0^1] \leq \mathbb{E}[(X_0^1)^2]$ , we can use Kahane's convexity inequalities to the convex mapping  $x \mapsto \frac{1}{x^q}$ . We deduce (for some positive constant  $C'$ )

$$\begin{aligned} A_l^2 &\leq C \int_{0 \leq y-x \leq l} \frac{1}{l^{\xi(q)}} \mathbb{E} \left[ \frac{1}{\left( \int_0^{\frac{y-x}{l}} e^{\gamma X_0^1 - \frac{\gamma^2}{2} \mathbb{E}[(X_0^1)^2]} dr \right)^q} \right] \gamma(dx) \gamma(dy) \\ &\leq C' \int_{0 \leq y-x \leq l} \frac{l^q}{l^{\xi(q)} (y-x)^q} \gamma(dx) \gamma(dy) \\ &\leq C' \int_{0 \leq y-x \leq l} \frac{1}{(y-x)^{\xi(q)}} \gamma(dx) \gamma(dy). \end{aligned}$$

Hence

$$\lim_l A_l^1 \leq C' \int_{B(0,T)^2} \frac{1}{|y-x|^{\xi(q)}} \gamma(dx) \gamma(dy) < +\infty.$$

The KPZ formula is proved (by using scaling relations only).  $\square$

### C.3 Proof of the dual KPZ formula

This time, we do not restrict to the dimension 1. Let  $K$  be a compact subset of  $\mathbb{R}^d$ , included in the ball  $B(0, 1)$  with Hausdorff dimension  $0 \leq \dim_{Leb}(K) < 1$ . Let  $\delta_0$  be the unique solution in  $[0, \alpha[$  such that  $\frac{\bar{\xi}(\delta_0)}{d} = \dim_{Leb}(K)$ . We want to prove  $\delta_0 = \dim_{\bar{M}}(K)$ .

Let  $0 \leq q < \alpha$  be such that  $\frac{\bar{\xi}(q)}{d} > \dim_{Leb}(K)$ . For  $\epsilon > 0$ , there is a covering of  $K$  by a countable family of balls  $(B(x_n, r_n))_n$  such that

$$\sum_n r_n^{\bar{\xi}(q)} < \epsilon.$$

Since we have (see Theorem 4)

$$\begin{aligned} \mathbb{E} \left[ \sum_n \bar{M}(B(x_n, r_n))^q \right] &= \sum_n \mathbb{E} \left[ \bar{M}(B(0, r_n))^q \right] \\ &\leq C_q \sum_n r_n^{\bar{\xi}(q)} \\ &\leq C_q \epsilon, \end{aligned}$$

we deduce by the Markov inequality

$$\mathbb{P} \left( \sum_n \bar{M}(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon} \right) \geq 1 - \sqrt{\epsilon}.$$

Thus, with probability  $1 - \sqrt{\epsilon}$ , there is a covering of balls of  $K$  such that  $\sum_n \bar{M}(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon}$ . So  $q \geq \dim_{\bar{M}}(K)$  almost surely.

Conversely, consider  $p \in [0, \alpha[$  such that  $\frac{\bar{\xi}(p)}{d} < \dim_{Leb}(K)$ . Since  $\bar{\xi}(p) = \xi(\frac{p}{\alpha})$ , we can set  $q = \frac{p}{\alpha} \in [0, 1[$  and we have  $\frac{\xi(q)}{d} < \dim_{Leb}(K)$ . As we proved above, we can consider the measure  $\tilde{\gamma}$  introduced in (38). It is almost surely supported by  $K$  and non trivial. Furthermore, it satisfies

$$\mathbb{E} \left[ \int_{B(0, T)^2} \frac{1}{\bar{M}(B(x, |y-x|))^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] < +\infty.$$

Let us prove that

$$\mathbb{E} \left[ \int_{B(0, T)^2} \frac{1}{\bar{M}(B(x, |y-x|))^p} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] < +\infty. \quad (39)$$

By using the relation for  $p, x > 0$

$$\Gamma(p) = x^p \int_0^{+\infty} u^{p-1} e^{-ux} du,$$

we deduce:

$$\begin{aligned}
& \mathbb{E} \left[ \int_{B(0,T)^2} \frac{1}{\overline{M}(B(x, |y-x|))^p} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] \\
&= \frac{1}{\Gamma(p)} \mathbb{E} \left[ \int_0^{+\infty} u^{p-1} \int_{B(0,T)^2} e^{-u\overline{M}(B(x, |y-x|))} \tilde{\gamma}(dx) \tilde{\gamma}(dy) du \right] \\
&= \frac{1}{\Gamma(p)} \mathbb{E} \left[ \int_0^{+\infty} u^{p-1} \int_{B(0,T)^2} \mathbb{E} \left[ e^{-u\overline{M}(B(x, |y-x|))} | Y^n, n \geq 1 \right] \tilde{\gamma}(dx) \tilde{\gamma}(dy) du \right] \\
&= \frac{1}{\Gamma(p)} \mathbb{E} \left[ \int_0^{+\infty} u^{p-1} \int_{B(0,T)^2} e^{-u^\alpha M(B(x, |y-x|))} \tilde{\gamma}(dx) \tilde{\gamma}(dy) du \right]
\end{aligned}$$

Now we make the change of variables  $y = u^\alpha M(B(x, |y-x|))$  to obtain:

$$\begin{aligned}
& \mathbb{E} \left[ \int_{B(0,T)^2} \frac{1}{\overline{M}(B(x, |y-x|))^p} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] \\
&= \frac{1}{\alpha \Gamma(p)} \mathbb{E} \left[ \int_{B(0,T)^2} \frac{1}{M(B(x, |y-x|))^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] \int_0^{+\infty} y^{\frac{p}{\alpha}-1} e^{-y} dy \\
&= \frac{\Gamma(\frac{p}{\alpha} + 1)}{\Gamma(p + 1)} \mathbb{E} \left[ \int_{B(0,T)^2} \frac{1}{M(B(x, |y-x|))^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right].
\end{aligned}$$

Hence, the above quantity is finite and (39) is proved. As usually, we conclude by using the Frostman lemma that  $p < \dim_{\overline{M}}(K)$ . The dual KPZ formula is proved. Notice that we have also proved We point out that this duality relation can be directly recovered from the relation  $\dim_{\overline{M}}(K) = \alpha \dim_M(K)$ , which is nothing but the duality relation.  $\square$

**Remark 13.** Notice that the above proof is much simplified by the fact our construction of  $M, \overline{M}$  allows to construct these measures on the same probability space. Therefore the proof of the dual KPZ formula boils down to a conditioning argument and a small computation.

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