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Gaussian multiplicative chaos and KPZ duality

Julien Barral ^{*}, Xiong Jin[†], Rémi Rhodes [‡], Vincent Vargas ^{§¶}

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Abstract

This paper is concerned with the KPZ formula. On the first hand, we give a simplified (in comparison with the existing literature) proof of the classical KPZ formula. On the other hand, we construct purely atomic random measures corresponding to values of the parameter γ^2 beyond the transition phase (i.e. $\gamma^2 > 2d$). We prove the dual KPZ formula for these measures and check the duality relation. In particular, this framework allows to construct singular Liouville measures and to understand the duality relation in Liouville quantum gravity.

1. Introduction

Log-normal multiplicative martingales were introduced by Mandelbrot [23] in order to build random measures describing energy dissipation and contribute explaining intermittency effects in Kolmogorov's theory of fully developed turbulence (see [5, 31, 33, 6, 14] and references therein). However, his model was difficult to define mathematically and this is why he proposed in [24] the simpler model of random multiplicative cascades whose detailed study started with Kahane's and Peyrière's notes [15, 26], gathered in their joint paper [17].

From that moment on, multiplicative cascades have been widely used as reference models in many applications. However, they possess many drawbacks related to their discrete scale invariance, mainly they involve a particular scale ratio and they do not possess stationary fluctuations (this comes from the fact that they are

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constructed on a p -adic tree structure). In the eighties, Kahane [16] came back to Mandelbrot's initial model and developed a continuous parameter theory of suitable stationary multifractal random measures, called Gaussian multiplicative chaos. His efforts were followed by several authors [3, 31, 2, 27, 29, 13, 1, 30] coming up with various generalizations at different scales. This family of random fields has found many applications in various fields of science like mathematical finance, turbulence, etc... Recently, the authors in [9] have drawn attention on the fact that Gaussian multiplicative chaos should be considered as a natural model for Liouville Quantum Gravity (see [21, 7, 9] among many others). In this context, the KPZ formula has been proved rigorously [4, 9, 28] below the transition phase arising at $\gamma^2 = 4$, where the constant γ is related to the central charge $c \leq 1$ of the underlying conformal field theory by the relation (see [21])

$$\gamma = \frac{\sqrt{25 - c} - \sqrt{1 - c}}{\sqrt{6}}.$$

However the issue of mathematically constructing singular Liouville measures beyond the transition phase (i.e. for $\gamma^2 > 4$) and proving the KPZ duality has never been solved mathematically (see [18, 19, 20] for an account of physical motivations).

Let us draw up the framework a bit more precisely. Fix a simply connected domain $D \subset \mathbb{C}$. For $\gamma^2 < 4$, the Liouville measure can formally be written as

$$M_\gamma(A) = \int_A e^{\gamma X_x - \frac{\gamma^2}{2} \mathbb{E}[X_x^2]} dx \quad (1)$$

where X is the Gaussian Free Field (GFF) over the domain D . For a given compact set $K \subset D$, it has been proved that the Hausdorff dimension of K computed with the Euclidian metric, call it $\dim_{Leb}(K)$, is related to the Hausdorff dimension of K computed with the measure M_γ , call it $\dim_\gamma(K)$. The connection is the so-called KPZ formula

$$\dim_{Leb}(K) = \left(1 + \frac{\gamma^2}{4}\right) \dim_\gamma(K) - \frac{\gamma^2}{4} \dim_\gamma(K)^2.$$

Based on the physics literature, the purpose of this paper is to propose a construction in the spirit of (1) of (purely atomic) random measures $M_{\bar{\gamma}}$, for some parameter values $\bar{\gamma} > 4$ (i.e. beyond the transition phase), that satisfies the KPZ relation

$$\dim_{Leb}(K) = \left(1 + \frac{\bar{\gamma}^2}{4}\right) \dim_{\bar{\gamma}}(K) - \frac{\bar{\gamma}^2}{4} \dim_{\bar{\gamma}}(K)^2.$$

Then, by considering the dual value $\gamma = \frac{\bar{\gamma}}{4}$ of the parameter $\bar{\gamma}$, we want to establish the duality relation

$$\dim_{\bar{\gamma}}(K) = \frac{\gamma^2}{4} \dim_\gamma(K).$$

We point out that physicists can recover the (more classical) relation between the scaling exponents by setting $\Delta_\gamma = 1 - \dim_\gamma(K)$ and $\Delta_{\bar{\gamma}} = 1 - \dim_{\bar{\gamma}}(K)$.

Our construction for dual measures is roughly the following. Consider a couple of exponents $(\gamma, \bar{\gamma})$ such that $\gamma^2 < 4$ and $\gamma\bar{\gamma} = 4$. We introduce an independently scattered random measure n_α characterized by its Laplace transform ($|A|$ stands for the Lebesgue measure of A)

$$\forall A \subset \mathbb{R}^2 \text{ Borelian, } \mathbb{E}[e^{-un_\alpha(A)}] = e^{-u^\alpha|A|}$$

where $\alpha = \frac{\gamma^2}{4}$. The considered dual measure is then formally defined by (see below for a rigorous construction)

$$M_{\bar{\gamma}}(A) = \int_A e^{\bar{\gamma}X_x - 2\mathbb{E}[X_x^2]} n_\alpha(dx). \quad (2)$$

Let us mention that our measures $(M_\gamma, M_{\bar{\gamma}})$ are approximately \star -scale invariant random measures in the sense of [1]. As a consequence, they satisfy the scaling heuristics developed in [10] to quantify the measure of a Euclidean ball of size ϵ (see in particular the section *Liouville quantum duality*). In fact, such heuristics amount to considering \star -scale invariant random measures (see [1, 30]). We conjecture that the \star -scale invariance property characterizes the measures that one can consider in this context (work in progress; see [11] for a precise statement and a rigorous proof in the case of random multiplicative cascades).

In the present paper, we will tackle the above problem in great generality: we will not restrict ourselves to the 2-dimensional case and we will not consider the only GFF but more generally log-correlated Gaussian distributions.

2. Background

In this section, we will briefly explain Kahane's theory of multiplicative chaos in \mathbb{R}^d . In fact, Kahane's theory is valid in any open domain $D \subset \mathbb{R}^d$ with no substantial change. At the end of the section, we will also roughly recall the connection with measures formally given by the exponential of the GFF.

2.1 Sigma positive kernels

We consider a covariance kernel K of σ -positive type ([16]), namely that K can be rewritten as a sum

$$\forall x, y \in \mathbb{R}^d, \quad K(x, y) = \sum_{n \geq 1} q_n(x, y) \quad (3)$$

where $(q_n)_n$ is a sequence of continuous positive kernels of positive type. We further assume that

$$\forall x \in \mathbb{R}^d, \quad K(x, y) = \ln_+ \frac{T}{|x - y|} + g(x, y) \quad (4)$$

where g is a bounded continuous function over $\mathbb{R}^d \times \mathbb{R}^d$ (and $\ln_+(x) = \max(0, \ln(x))$). We can consider a sequence of independent centered Gaussian processes $(Y^n)_{n \geq 1}$

where, for each $n \geq 1$, $(Y_x^n)_{x \in \mathbb{R}^d}$ is a centered continuous Gaussian field with covariance function given by

$$\forall x, y \in \mathbb{R}^d, \quad \text{Cov}(Y_x^n, Y_y^n) = q_n(x, y).$$

Finally, for $n \geq 1$, we define:

$$X_x^n = \sum_{p=1}^n Y_x^p.$$

It is a centered continuous Gaussian process with covariance function:

$$\forall x, y \in \mathbb{R}^d, \quad k_n(x, y) \stackrel{\text{def}}{=} \text{Cov}(X_x^n, X_y^n) = \sum_{k=1}^n q_k(x, y). \quad (5)$$

The reader may find several important examples of sigma-positive kernels in Appendix A.

2.2 Gaussian multiplicative chaos

For each $n \geq 1$, we can define a Radon measure M_n on the Borelian subsets of \mathbb{R}^d by

$$M_n(A) = \int_A e^{\gamma X_x^n - \frac{\gamma^2}{2} \mathbb{E}[(X_x^n)^2]} dx.$$

For each Borelian set A , the sequence $(M_n(A))_n$ is a positive martingale. Thus it converges almost surely towards a random variable denoted by $M(A)$. One can deduce that the sequence of measures $(M_n)_n$ weakly converges towards a random Radon measure M , commonly denoted by

$$M(A) = \int_A e^{\gamma X_x - \frac{\gamma^2}{2} \mathbb{E}[X_x^2]} dx \quad (6)$$

and called Gaussian multiplicative chaos associated to the kernel $\gamma^2 K$. Roughly speaking, (6) can be understood as a measure admitting as density the exponential of a Gaussian distribution X with covariance kernel $\gamma^2 K$. Of course, this is purely formal because the exponential of a random distribution cannot be directly defined. Kahane proved that the martingale $(M_n(A))_n$, for some Borelian set A with non-null finite Lebesgue measure, is uniformly integrable if and only if $\gamma^2 < 2d$. This condition is necessary and sufficient in order for the limiting measure M to be non identically null. Furthermore, he proved that the law of the limiting measure M does not depend on the decomposition (3) of K into a sum of positive continuous kernels. For kernels K that cannot be written as a sum of nonnegative terms as (3), we refer to the extended Gaussian multiplicative theory developed in [27].

2.3 Application to the construction of Liouville measure

Formally, the GFF (or Euclidian bosonic massless free field) in a bounded domain $D \subset \mathbb{R}^2$ is a "Gaussian Field" X with covariance given by:

$$\mathbb{E}[X_x X_y] = G(x, y),$$

where G is the Green function of D with zero boundary condition (see for instance [32] or chapter 2.4 in [22] for the definition and main properties). Let B be a Brownian motion starting from $x \in D$ under the measure P^x and consider the stopping time $T_D = \inf\{t \geq 0, B_t \notin D\}$. If we denote $p_D(t, x, y) = P^x(B_t \in dy, T_D > t)$, we have:

$$G(x, y) = \pi \int_0^\infty p_D(t, x, y) dt.$$

Note that, for each $t > 0$, $p_D(t, x, y)$ is a continuous positive and positive definite kernel on D . Therefore, following Kahane's theory, we can define the Gaussian multiplicative chaos M associated to the kernel $\gamma^2 G$. Since the Green function takes on the form (4), this measure is not trivial provided that $\gamma^2 < 4$. We point out that the authors in [9] have suggested a slightly different construction of the Liouville measure. Based on the uniqueness criterion in [27], it can be proved that their construction has the same law as that originally proposed by Kahane.

3. Multiplicative chaos and dual chaos

We stick to the notations of the previous section. We nevertheless assume that the considered Gaussian fields are stationary. Though it may appear as a restriction, the proofs in the general case work exactly the same. Actually, being stationary or not is just hidden in the "small noise g " appearing in (4).

So, we consider the Gaussian multiplicative chaos M understood as the limit (in the sense previously described) as $n \rightarrow \infty$ of the following sequence of random measures

$$M_n(dx) = e^{\gamma X_x^n - \frac{\gamma^2}{2} \mathbb{E}[(X_x^n)^2]} dx \quad (7)$$

M is a non trivial random measure for $\gamma^2 < 2d$ with no atoms. Its power-law spectrum ξ , defined through the relation

$$\mathbb{E}[M(B(0, \lambda))^q] \simeq C_q \lambda^{\xi(q)}, \quad \lambda \rightarrow 0$$

for all $q \geq 0$ such that the expectation makes sense (i.e. for $0 \leq q < \frac{2d}{\gamma^2}$, see [16]), is given by

$$\xi(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2.$$

We introduce a positive Radon random measure N_M distributed on $\mathbb{R}^d \times \mathbb{R}_+^*$, whose law conditionally to M is that of a Poisson random measure with intensity

$$\frac{M(dx) dz}{z^{1+\alpha}}$$

for some $0 < \alpha < 1$, which is chosen to be $\alpha = \frac{\gamma^2}{2d}$. Then we introduce the family of purely atomic positive random measures

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \overline{M}(A) = \int_A \int_{\mathbb{R}_+} z N_M(dx, dz). \quad (8)$$

Theorem 1. *For $\gamma < 2d$, the law of the random measure \overline{M} does not depend on the decomposition of K into a sum of positive continuous kernels of positive type. Furthermore, \overline{M} is almost surely a purely atomic measure.*

The law of \overline{M} is characterized by the following relation

$$\mathbb{E}[e^{-u_1 \overline{M}(A_1) + \dots - u_n \overline{M}(A_p)}] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} (u_1^\alpha M(A_1) + \dots + u_n^\alpha M(A_n))}\right] \quad (9)$$

valid for all $u_1, \dots, u_p \in \mathbb{R}_+$ and all disjoint Borelian subsets $A_1, \dots, A_p \subset \mathbb{R}^d$.

There is another way of seeing the law of the measure \overline{M} , which highlights the presentation that we made in introduction. We define $\overline{\gamma}$ by the relation

$$\gamma \overline{\gamma} = 2d. \quad (10)$$

We have:

Theorem 2. *Consider a Poisson random measure N_α distributed on $\mathbb{R}^d \times \mathbb{R}_+^*$ with intensity $dx \frac{dz}{z^{1+\alpha}}$ and independent of the sequence $(Y_x^n)_{x \in \mathbb{R}^d}$. Set*

$$n_\alpha(dx) = \int \int_0^{+\infty} z N_\alpha(dx, dz).$$

Define the sequence of random measures

$$\begin{aligned} \forall A \in \mathcal{B}(\mathbb{R}^d), \quad \overline{M}_n(A) &= \int_A e^{\frac{\gamma}{\alpha} X_x^n - \frac{\gamma^2}{2\alpha} \mathbb{E}[(X_x^n)^2]} n_\alpha(dx) \\ &= \int_A e^{\overline{\gamma} X_x^n - d \mathbb{E}[(X_x^n)^2]} n_\alpha(dx). \end{aligned} \quad (11)$$

The sequence of random measures $(\overline{M}_n(dx))_n$ weakly converges in law towards a random measure \overline{M} , the law of which is given by (8).

The above theorem justifies to write formally the law of \overline{M} as

$$\overline{M}(dx) = \int e^{\overline{\gamma} X_x - d \mathbb{E}[X_x^2]} n_\alpha(dx) \quad (12)$$

where X is a stationary Gaussian distribution with covariance kernel K . This expression also justifies the fact that the measure \overline{M} can be seen as a Gaussian multiplicative chaos. Furthermore, it is defined for values of $\overline{\gamma}^2$ beyond the critical value $\overline{\gamma}^2 = 2d$. Notice that the renormalization (i.e. $d \mathbb{E}[X_x^2]$) differs from the sub-critical Gaussian multiplicative chaos (i.e. $\gamma^2 < 2d$).

3.1 Power-law spectrum and moments of the dual chaos

Let us define

$$\forall q \in \mathbb{R}, \quad \bar{\xi}(q) = \left(\frac{d}{\alpha} + \frac{\gamma^2}{2\alpha}\right)q - \frac{\gamma^2}{2\alpha^2}q^2.$$

We will show below that this function coincides with the power law spectrum of the measure \bar{M} . The choice of α can be understood now since it is the only possible value of $0 < \alpha < 1$ ensuring the renormalization condition $\xi(1) = d$. The function $\bar{\xi}$ can also be rewritten as:

$$\bar{\xi}(q) = \left(d + \frac{\bar{\gamma}^2}{2}\right)q - \frac{\bar{\gamma}^2}{2}q^2. \quad (13)$$

In particular, we see that $\bar{\xi}(q) = \xi(\frac{q}{\alpha})$.

Now we precise the existence of moments for the measure \bar{M} :

Proposition 3. *For all Borelian set A with finite (not null) Lebesgue measure, the random variable $\bar{M}(A)$ possesses a moment of order $\beta \geq 0$ if and only if $\beta < \alpha$.*

Furthermore, we can make explicit the connection between the moments of M and \bar{M} : for all $0 \leq \beta < \alpha$,

$$\mathbb{E}[(\bar{M}(A))^\beta] = \frac{\Gamma(1 - \beta/\alpha)\Gamma(1 - \alpha)^{\beta/\alpha}}{\Gamma(1 - \beta)\alpha^{\beta/\alpha}} \mathbb{E}[(M(A))^{\frac{\beta}{\alpha}}] \quad (14)$$

Theorem 4. (Perfect scaling). *If the kernel K is given by*

$$K(x) = \ln_+ \frac{T}{|x|} + g(x)$$

where g is a continuous bounded function that is constant in a neighborhood of 0 then, for some $R > 0$:

$$\forall 0 < \lambda < 1, \quad (\bar{M}(\lambda A))_{A \subset B(0,R)} \stackrel{\text{law}}{=} \lambda^{d/\alpha} e^{\frac{\Omega_\lambda}{\alpha}} (\bar{M}(A))_{A \subset B(0,R)} \quad (15)$$

where Ω_λ is a Gaussian random variable independent of the measure $(\bar{M}(A))_{A \subset B(0,R)}$ the law of which is characterized by:

$$\mathbb{E}[e^{q\Omega_\lambda}] = \lambda^{\frac{\gamma^2}{2}q - \frac{\gamma^2}{2}q^2}.$$

In particular, for all $0 \leq q < \alpha$:

$$\mathbb{E}[\bar{M}(B(0, \lambda R))^q] = \lambda^{\bar{\xi}(q)} \mathbb{E}[\bar{M}(B(0, R))^q].$$

Corollary 5. *Assume that the kernel K takes on the form (4) and that $\bar{\gamma} > 2d$. Then, for all $0 \leq q < \alpha$:*

$$\mathbb{E}[\bar{M}(B(0, \lambda R))^q] \simeq C_{q,R} \lambda^{\bar{\xi}(q)}$$

as $\lambda \rightarrow 0$ for some positive constant $C_{q,R}$ only depending on q, R .

4. KPZ formula and duality

The KPZ formula is a relation between the Hausdorff dimensions of a given set A as measured by the Lebesgue measure, M or \overline{M} . So we first recall how to define these dimensions. Given a Radon measure μ on \mathbb{R}^d and $s \in [0, 1]$, we define

$$H_\mu^{s,\delta}(A) = \inf \left\{ \sum_k \mu(B_k)^s \right\}$$

where the infimum runs over all the covering $(B_k)_k$ of A with open Euclidean balls centered at A with radius $r_k \leq \delta$. Clearly, the mapping $\delta > 0 \mapsto H_\mu^{s,\delta}(A)$ is decreasing. Hence we can define the s -dimensional μ -Hausdorff outer measure:

$$H_\mu^s(A) = \lim_{\delta \rightarrow 0} H_\mu^{s,\delta}(A).$$

The limit exists but may be infinite. H_μ^s is a metric outer measure on \mathbb{R}^d (see [12] for the definitions). We point out that the fact that μ possesses atoms or not does not give rise to any additional difficulty. Thus H_μ^s is a measure on the σ -field of H_μ^s -measurable sets, which contains all the Borelian sets.

The μ -Hausdorff dimension of the set A is then defined as the value

$$\dim_\mu(A) = \inf \{s \geq 0; H_\mu^s(A) = 0\}. \quad (16)$$

Notice that $\dim_\mu(A) \in [0, 1]$. However, it is not clear, in great generality, that we have the classical property:

$$\dim_\mu(A) = \sup \{s \geq 0; H_\mu^s(A) = +\infty\}. \quad (17)$$

This is due to the possible presence of atoms for the measure μ . However we claim

Proposition 6. *If we take $\mu = \text{Leb}$ then (17) holds. If we take $\mu = M$ then, almost surely, (17) holds for every bounded Borelian set. If we take $\mu = \overline{M}$ and A a compact set with null Lebesgue measure then (17) holds almost surely.*

This proposition allows to characterize the Hausdorff dimension as the critical value at which the mapping $s \mapsto H_\mu^s(A)$ jumps from $+\infty$ to 0.

In what follows, given a compact set K of \mathbb{R}^d with null Lebesgue measure, we define its Hausdorff dimensions $\dim_{\text{Leb}}(K)$, $\dim_M(K)$, $\dim_{\overline{M}}(K)$ computed as indicated above with μ respectively equal to the Lebesgue measure, M and \overline{M} .

Theorem 7. KPZ duality. *Let K be a compact set of \mathbb{R}^d with null Lebesgue measure. Almost surely, we have the relations*

$$\dim_{\text{Leb}}(K) = \frac{\xi(\dim_M(K))}{d} \quad \dim_{\text{Leb}}(K) = \frac{\overline{\xi}(\dim_{\overline{M}}(K))}{d}$$

where $\xi(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2$ and $\overline{\xi}(q) = (d + \frac{\overline{\gamma}^2}{2})q - \frac{\overline{\gamma}^2}{2}q^2$. In particular, we have the duality relation between the scaling exponents

$$\dim_{\overline{M}}(K) = \frac{\gamma^2}{2d} \dim_M(K). \quad (18)$$

Remark 8. Note that, in the classical physics literature (in particular $d = 2$), it is more usual to focus on the scaling exponents

$$\Delta_\gamma = 1 - \dim_M(K), \quad \Delta_{\bar{\gamma}} = 1 - \dim_{\bar{M}}(K), \quad x = 1 - \dim_{Leb}(K),$$

instead of $\dim_M(K)$, $\dim_{\bar{M}}(K)$, $\dim_{Leb}(K)$. Then the KPZ relations read

$$x = \frac{\gamma^2}{4}\Delta_\gamma^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta_\gamma \quad \text{and} \quad x = \frac{\bar{\gamma}^2}{4}\Delta_{\bar{\gamma}}^2 + \left(1 - \frac{\bar{\gamma}^2}{4}\right)\Delta_{\bar{\gamma}}.$$

The duality relation then becomes

$$\Delta_{\bar{\gamma}} - 1 = \frac{\gamma^2}{2d}(\Delta_\gamma - 1) = \frac{2d}{\bar{\gamma}^2}(\Delta_\gamma - 1).$$

Remark 9. If one looks for random measures satisfying the duality relation (18), it is plain to deduce that such a relation implies that the power law spectrum is necessarily given by (13). Such a power law spectrum indicates that the searched random measures cannot be defined by (7) in the sense that the integrating measure (dx in (7)) cannot be the Lebesgue measure. Indeed, otherwise Kahane's theory ensures that such measure is identically null. So one has to look for other integrating measures in (7) than the Lebesgue measure. By noticing that $\bar{\xi}(q) = \xi(\frac{q}{\alpha})$, one can intuitively recover our construction, namely that the searched measures should be Gaussian multiplicative chaos integrated against independently scattered α -stable random measures, as stated in Theorem 2.

A. Examples of sigma-positive kernels

In this section, we detail a few examples of sigma-positive kernels, apart from the Green function already explained in subsection 2.3. More precisely, we give two different classes of sigma-positive kernels, which yield two different notions of stochastic scale invariance for the associated Gaussian multiplicative chaos.

A.1 Exact stochastic scale invariance

In this section, we describe how to construct kernels yielding the exact scale invariance relations of Theorem 4. This is useful in computations and it is possible to deduce all the other situations from this one.

We define on \mathbb{R}_+ the measure $\nu_T(dt) = \mathbf{1}_{[0,T]}(t)\frac{dt}{t^2} + \frac{1}{T}\delta_T(dt)$ where δ_x denotes the Dirac mass at x . For $\mu > 0$, it is straightforward to check that

$$\forall x \in \mathbb{R}^d, \quad \ln_+ \frac{T}{|x|} = \frac{1}{\mu} \int_0^{+\infty} (t - |x|^\mu)_+ \nu_{T^\mu}(dt). \quad (19)$$

-In dimension $d = 1$, it is straightforward to check that the function $x \mapsto (t - |x|)_+$ is of positive type. So, the kernel $K(x) = \gamma^2 \ln_+ \frac{T}{|x|}$ is of sigma positive type. The kernels k_n can be easily computed:

$$k_n(x) = \begin{cases} 0 & \text{if } |x| > T, \\ \gamma^2 \ln_+ \frac{T}{|x|} & \text{if } \frac{T}{n} \leq |x| \leq T, \\ \gamma^2 \ln n + \left(1 - \frac{n|x|}{T}\right) & \text{if } 0 \leq |x| \leq \frac{T}{n}. \end{cases}$$

-In dimension $d = 2$, Pasenchenko [25] proved that the function $(1 - |x|^{1/2})_+$ is positive definite in dimension 2. Choosing $\mu = 2$ in (19), we can thus write

$$\forall x \in \mathbb{R}^2, \quad \gamma^2 \ln_+ \frac{T}{|x|} = \sum_{n \geq 1} q_n(x),$$

where q_n is the continuous positive and positive definite kernel

$$\forall x \in \mathbb{R}^2, \quad q_n(x) = 2\gamma^2 \int_{\frac{T^{1/2}}{n^{1/2}}}^{\frac{T^{1/2}}{(n-1)^{1/2}}} (t - |x|^\mu)_+ \nu_{T^{1/2}}(dt).$$

A simple computation shows that

$$k_n(x) = \begin{cases} 0 & \text{if } |x| > T, \\ \gamma^2 \ln_+ \frac{T}{|x|} & \text{if } \frac{T}{n} \leq |x| \leq T, \\ \gamma^2 \ln n + 2\left(1 - \sqrt{\frac{n|x|}{T}}\right) & \text{if } 0 \leq |x| \leq \frac{T}{n}. \end{cases}$$

-In dimension $d \geq 3$, it is proved in [29] that there exists a continuous bounded function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, constant in a neighborhood of 0 such that

$$K(x) = \gamma^2 \ln_+ \frac{T}{|x|} + g(x) \tag{20}$$

is of sigma positive type.

A.2 \star -scale invariance

A simple way of constructing sigma positive kernels is given by

$$\forall x \in \mathbb{R}^d, \quad K(x) = \int_1^\infty \frac{k(xu)}{u} du, \tag{21}$$

where k is a continuous positive kernel of positive type. Such kernel is of sigma positive type since the decomposition can be realized by

$$q_n(x) = \int_{2^n}^{2^{n+1}} \frac{k(xu)}{u} du.$$

Furthermore, K takes on the form (4) with $\gamma^2 = k(0)$. Such kernels are related to the notion of \star -scale invariance (see [1, 30]).

B. Proofs of Section 3

Preliminary computations

We will use the following relation valid for any $0 < \beta < 1$ and $x \geq 0$:

$$x^\beta = \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1 - e^{-xz}) \frac{dz}{z^{1+\beta}}. \quad (22)$$

Thus we have for all $u \geq 0$:

$$\mathbb{E}[e^{-u\overline{M}(A)}] = \mathbb{E}\left[e^{\int_A \int_{\mathbb{R}_+} (e^{-zu}-1) \frac{1}{z^{1+\alpha}} dz M(dx)}\right] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} u^\alpha M(A)}\right]. \quad (23)$$

Similarly, we have

$$\mathbb{E}[e^{-u_1\overline{M}(A_1)+\dots-u_n\overline{M}(A_p)}] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} (u_1^\alpha M(A_1)+\dots+u_n^\alpha M(A_n))}\right] \quad (24)$$

valid for all $u_1, \dots, u_p \in \mathbb{R}_+$ and all disjoint Borelian subsets $A_1, \dots, A_p \subset \mathbb{R}^d$.

Then we have for $0 < \beta < \alpha$:

$$\begin{aligned} \mathbb{E}[(\overline{M}(A))^\beta] &= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1 - \mathbb{E}[e^{-w\overline{M}(A)}]) \frac{dw}{w^{1+\beta}} \\ &= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1 - \mathbb{E}[e^{-\frac{\Gamma(1-\alpha)}{\alpha} w^\alpha M(A)}]) \frac{dw}{w^{1+\beta}}. \end{aligned}$$

We make the change of variables $y = w^\alpha$ to get:

$$\begin{aligned} \mathbb{E}[(\overline{M}(A))^\beta] &= \frac{\beta}{\alpha\Gamma(1-\beta)} \int_0^\infty (1 - \mathbb{E}[e^{-\frac{\Gamma(1-\alpha)}{\alpha} y M(A)}]) \frac{dy}{y^{1+\frac{\beta}{\alpha}}} \\ &= \frac{\Gamma(1-\beta/\alpha)\Gamma(1-\alpha)^{\beta/\alpha}}{\Gamma(1-\beta)\alpha^{\beta/\alpha}} \mathbb{E}[(M(A))^{\frac{\beta}{\alpha}}]. \end{aligned} \quad (25)$$

Proof of Theorem 1.

We first stress that N can be constructed as the limit in law of a sequence $(N_n)_n$ of Poisson random measures distributed on $\mathbb{R}^d \times \mathbb{R}_+^*$ with intensity $M_n(dx) \otimes \frac{dz}{z^{1+\alpha}}$. From (24), we deduce that the law of \overline{M} is characterized by that of M , which does not depend on the chosen decomposition (see [16]). Furthermore, since N is a Poisson random measure conditionally to M , it is clear that it is almost surely purely atomic. \square

Proofs of Theorem 2

We have for all $u \geq 0$:

$$\mathbb{E}[e^{-u\overline{M}_n(A)}] = \mathbb{E}\left[e^{\int_A \int_{\mathbb{R}_+} (e^{-zu}-1) \frac{1}{z^{1+\alpha}} dz M_n(dx)}\right] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} u^\alpha M_n(A)}\right]. \quad (26)$$

Similarly, we have

$$\mathbb{E}[e^{-u_1 \overline{M}_n(A_1) + \dots - u_n \overline{M}_n(A_p)}] = \mathbb{E}\left[e^{-\frac{\Gamma(1-\alpha)}{\alpha} (u_1^\alpha M_n(A_1) + \dots + u_n^\alpha M_n(A_p))}\right] \quad (27)$$

valid for all $u_1, \dots, u_p \in \mathbb{R}_+$ and all disjoint Borelian subsets $A_1, \dots, A_p \subset \mathbb{R}^d$. Since $(M_n)_n$ almost surely weakly converges towards M , it is obvious to check the convergence in law as well as to characterize the law of the limiting measure.

Proofs of Proposition 3

For $\beta < \alpha$, we can use relation (25) to show the existence of the moments and the dual relation (14). If \overline{M} possesses a moment of order α then the left-hand side of equation (14) must converge as $\beta \rightarrow \alpha$. But it is equal to the right-hand side, which diverges because of the term $\Gamma(1 - \beta/\alpha)$ and the fact that the measure M possesses a non trivial moment of order 1.

Proof of Theorem 4

First we stress that it has already been proved that the chaos measure M , associated to the given kernel K , satisfies the scale invariance relation (see [29]) for some $R > 0$:

$$\forall 0 < \lambda < 1, \quad (M(\lambda A))_{A \subset B(0,R)} \stackrel{law}{=} \lambda^d e^{\Omega_\lambda} (M(A))_{A \subset B(0,R)}$$

where Ω_λ is a Gaussian random variable independent of the measure $(\overline{M}(A))_{A \subset B(0,R)}$ the law of which is characterized by:

$$\mathbb{E}[e^{q\Omega_\lambda}] = \lambda^{\frac{\gamma^2}{2}q - \frac{\gamma^2}{2}q^2}.$$

The results then easily follows from the relation

$$\mathbb{E}[e^{iu_1 \overline{M}(A_1) + \dots + iu_n \overline{M}(A_n)}] = \mathbb{E}[e^{-u_1^\alpha M(A_1) + \dots - u_n^\alpha M(A_n)}]$$

valid for all $u_1, \dots, u_n \in \mathbb{R}$ and all disjoint Borelian subsets $A_1, \dots, A_n \subset \mathbb{R}^d$. \square

Proof of Corollary 5

Let us write the kernel K as

$$K(x) = K_p(x) + h(x)$$

where K_p is the "perfect kernel given by (20) and g is some continuous bounded function over \mathbb{R}^d . Even if it means adding to K a constant, we may assume that $h(0) = 0$ and, without loss of generality, we assume $R = 1$. For $t > 0$, we define

$$G_t = \sup_{|x| \leq t} |h(x)|.$$

Let us also consider the measures M^p, \overline{M}^p associated to the perfect kernel K_p . Let us denote by B_λ the ball centered at 0 with radius λ . From Kahane's concentration inequalities, we have for all $q \leq 1$:

$$\mathbb{E}[(M(B_\lambda))^q] \geq \mathbb{E}[(M^p(B_\lambda)e^{\gamma\sqrt{G_\lambda}Z - \frac{\gamma^2}{2}G_\lambda})^q]$$

where Z is a standard Gaussian random variable independent of M^p . Hence, by using Theorem 4, we have:

$$\begin{aligned} \mathbb{E}[(M(B_\lambda))^q] &\geq \mathbb{E}[(M^p(B_\lambda))^q] \mathbb{E}[(e^{\gamma\sqrt{G_\lambda}Z - \frac{\gamma^2}{2}G_\lambda})^q] \\ &= \lambda^{\xi(q)} \mathbb{E}[\overline{M}^p(B_1)^q] e^{q^2 \frac{\gamma^2}{2\alpha^2} G_\lambda - q \frac{\gamma^2}{2\alpha^2} G_\lambda}. \end{aligned}$$

With the same argument we prove

$$e^{q^2 \frac{\gamma^2}{2} G_\lambda - q \frac{\gamma^2}{2} G_\lambda} \mathbb{E}[(M(B_\lambda))^q] \leq \lambda^{\xi(q)} \mathbb{E}[M^p(B_1)^q].$$

Because $G_\lambda \rightarrow 0$ as $\lambda \rightarrow 0$, the result follows from relation (14). \square

C. Proofs of Section 4.

C.1 Proof of Proposition 6

We assume that A is bounded, say included in the ball $B(0, 1)$. We have for $s < t$:

$$H_\mu^{t, \delta}(A) \leq H_\mu^{s, \delta}(A) \sup_{\substack{B \text{ ball centered in } A, \\ B \subset B(0, 1), \text{diam}(B) \leq \delta}} \mu(B)^{t-s}.$$

Obviously, it suffices to prove that the quantity $\sup_{\substack{B \text{ ball centered in } A, \\ B \subset B(0, 1), \text{diam}(B) \leq \delta}} \mu(B)$ converges to 0 as $\delta \rightarrow 0$. It is clear if μ is the Lebesgue measure. If $\mu = M$, this results from the fact that M does not possess any atom (see Lemma 10 below). It remains to investigate the situation when $\mu = \overline{M}$. Let A be a compact subset included in the ball $B(0, 1)$ with null Lebesgue measure. For $0 < \beta < \alpha$, we have $\mathbb{E}[\overline{M}(A)^\beta] = c_{\alpha, \beta} \mathbb{E}[M(A)^{\frac{\beta}{\alpha}}] = 0$ since $M(A) = 0$ almost surely. Therefore, almost surely, the measure \overline{M} does not possess any atom on the set A . Now we prove that, almost surely,

$$\sup_{\substack{B \text{ ball centered in } A, \\ B \subset B(0, 1), \text{diam}(B) \leq \delta}} \overline{M}(B) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We argue by contradiction. Assume that this quantity does not converge towards 0. We can find $\epsilon > 0$, a sequence $(x_n)_n$ of points in A and a sequence $(r_n)_n$ of positive radius such that $\overline{M}(B(x_n, r_n)) \geq \epsilon$. Even if it means extracting a subsequence, we may assume that the sequence $(x_n)_n$ converges towards $x \in A$. We deduce $\overline{M}(\{x\}) \geq \epsilon$. This means that \overline{M} possesses an atom on A . Contradiction. \square

Lemma 10. *Almost surely, the measure M does not possess any atom.*

Proof. By stationarity, it is enough to prove that, almost surely, the measure M does not possess any atom on the cube $[0, 1]^d$. For $n \in \mathbb{N}^*$ and $k_1, \dots, k_d \in \{1, \dots, n\}$, let us denote by I_{k_1, \dots, k_d}^n the cube $\prod_{i=1}^d [\frac{k_i-1}{n}, \frac{k_i}{n}]$. From [8, Corollary 9.3 VI], it is enough to check that for each $\eta > 0$:

$$\sum_{k_1, \dots, k_d=1}^n \mathbb{P}\left(M(I_{k_1, \dots, k_d}^n) > \eta\right) = n^d \mathbb{P}\left(M(I_{0, \dots, 0}^n) > \eta\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This is a direct consequence of the Markov inequality

$$n^d \mathbb{P}\left(M(I_{0, \dots, 0}^n) > \eta\right) \leq \frac{n^d}{\eta^q} \mathbb{E}[M(I_{0, \dots, 0}^n)^q]$$

and the relation, for $1 < q < \frac{2d}{\gamma^2}$ (see the proof of corollary 5),

$$\mathbb{E}[M(I_{0, \dots, 0}^n)^q] \leq C n^{-\xi(q)}.$$

Indeed, for $1 < q < \frac{2d}{\gamma^2}$, we have $\xi(q) > d$. □

C.2 Proof of the usual KPZ formula

We first prove the usual KPZ relation though it has already been proved in [9, 28]. For the sake of clarity and completeness, we sketch here a simple proof in the Gaussian case. It relies on the intensive use of the scaling properties of the Gaussian multiplicative chaos as well as the use of the Girsanov transform, which much simplifies the computations in comparison with [9, 28]. For the sake of simplicity of notations, we make the proof in dimension $d = 1$ but the proof in higher dimension can be identically reproduced word for word. We also assume that M is the perfect measure, namely the measure with associated kernel given by $\gamma^2 \ln_+ \frac{T}{|x|}$. Actually, it can easily be proved with the Kahane concentration inequalities (see [16] or [27, cor. 6.2]) that this is not a restriction. We also mention that M can be constructed as the limit

$$M(dx) = \lim_{l \rightarrow 0} M_l(dx) \stackrel{def}{=} e^{\gamma X_x^l - \frac{1}{2} \mathbb{E}[(X_x^l)^2]} dx$$

where X_l is a stationary Gaussian process with covariance kernel given by:

$$k_l(x) = \begin{cases} 0 & \text{if } |x| > T, \\ \ln_+ \frac{T}{|x|} & \text{if } lT \leq |x| \leq T, \\ \ln \frac{1}{l} + \left(1 - \frac{|x|}{lT}\right) & \text{if } 0 \leq |x| \leq lT. \end{cases}$$

Such a family of kernels possesses useful scaling properties, namely that for $|x| \leq T$ and $0 < \lambda < 1$, $k_{\lambda l}(\lambda x) = k_l(x) + \ln \frac{1}{\lambda}$. In particular, we have the following scaling relation for all $0 < l < 1$ and all $0 < \lambda < 1$:

$$\left((X_{\lambda x}^{\lambda l})_{x \in B(0, T)}, (M_{\lambda l}(\lambda A))_{A \subset B(0, T)} \right) \stackrel{law}{=} \left((X_x^l + \Omega_\lambda)_{x \in B(0, T)}, (\lambda e^{\gamma \Omega_\lambda - \frac{\gamma^2}{2} \ln \frac{1}{\lambda}} M_l(A))_{A \subset B(0, T)} \right). \quad (28)$$

where Ω_λ is a centered Gaussian random variable with variance $\ln \frac{1}{\lambda}$ and independent of the couple $((X_x^l)_{x \in B(0,T)}, (M_l(A))_{A \subset B(0,T)})$. We will use the above relation throughout the proof.

Now we begin with the proof. Without loss of generality we assume that $T = 1$. Let K be a compact subset of \mathbb{R} , included in the ball $B(0, 1)$, with Hausdorff dimension $\dim_{Leb}(K)$. Let $1 > q > 0$ be such that $\xi(q) > \dim_{Leb}(K)$. For $\epsilon > 0$, there is a covering of K by a countable family of balls $(B(x_n, r_n))_n$ such that

$$\sum_n r_n^{\xi(q)} < \epsilon.$$

Since we have (we use the power law spectrum and the stationarity of the measure)

$$\begin{aligned} \mathbb{E} \left[\sum_n M(B(x_n, r_n))^q \right] &= \sum_n \mathbb{E} \left[M(B(0, r_n))^q \right] \\ &\leq C_q \sum_n r_n^{\xi(q)} \\ &\leq C_q \epsilon, \end{aligned}$$

we deduce by the Markov inequality

$$\mathbb{P} \left(\sum_n M(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon} \right) \geq 1 - \sqrt{\epsilon}.$$

Thus, with probability $1 - \sqrt{\epsilon}$, there is a covering of balls of K such that $\sum_n M(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon}$. So $q \geq \dim_M(K)$ almost surely.

Conversely, consider $q \geq 0$ such that $\xi(q) < \dim_{Leb}(K)$. By the Frostman Lemma, there is a probability measure γ supported by K such that

$$\int_{B(0,T)^2} \frac{1}{|x-y|^{\xi(q)}} \gamma(dx) \gamma(dy) < +\infty.$$

Let us define the random measure $\tilde{\gamma}$ as the almost sure limit of the following family of positive random measures:

$$\tilde{\gamma}(dx) = \lim_{l \rightarrow 0} e^{q\gamma X_x^l - \frac{q^2 \gamma^2}{2} \mathbb{E}[(X_x^l)^2]} \gamma(dx). \quad (29)$$

The limit is non trivial because $q^2 \gamma^2 / 2 < \xi(q)$ for $\gamma^2 < 2$ (see [16]) and supported by K . From the Frostman lemma again, we just have to prove that the quantity

$$\int_{B(0,T)^2} \frac{1}{M([x,y])^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy)$$

is finite almost surely. It suffices to prove that the above quantity has a finite expectation. Moreover, by using the Fatou lemma and the stationarity of the measure

M , we have

$$\begin{aligned} \mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{M([x,y])^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] &\leq \liminf_l \int_{B(0,T)^2} \mathbb{E} \left[\frac{e^{q\gamma X_x^l + q\gamma X_y^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([x,y])^q} \right] \gamma(dx) \gamma(dy) \\ &= \liminf_l 2 \int_{y \geq x} \mathbb{E} \left[\frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy). \end{aligned}$$

We decompose the last integral into two terms:

$$\begin{aligned} &\int_{y \geq x} \mathbb{E} \left[\frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{0 \leq y-x \leq l} \mathbb{E} \left[\frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy) + \int_{y-x \geq l} \mathbb{E} \left[\frac{e^{q\gamma X_0^l + q\gamma X_{y-x}^l - q^2 \gamma^2 \mathbb{E}[(X_x^l)^2]}}{M_l([0,y-x])^q} \right] \gamma(dx) \gamma(dy) \\ &\stackrel{def}{=} A_l^1 + A_l^2. \end{aligned}$$

For each of the above terms, we will use an appropriate scaling relation.

By using (28), we deduce

$$\begin{aligned} A_l^2 &= \int_{y-x \geq l} \mathbb{E} \left[\frac{e^{2q\gamma \Omega_{y-x} - q^2 \gamma^2 \ln \frac{1}{y-x} e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}}{(y-x)^q e^{q\Omega_{y-x} - q \frac{\gamma^2}{2} \ln \frac{1}{y-x}} M_{\frac{l}{y-x}}([0,1])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{y-x \geq l} \mathbb{E} \left[\frac{e^{q\gamma \Omega_{y-x} - (q^2 \gamma^2 - q \frac{\gamma^2}{2}) \ln \frac{1}{y-x}}}{(y-x)^q} \right] \mathbb{E} \left[\frac{e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}{M_{\frac{l}{y-x}}([0,1])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{y-x \geq l} \frac{1}{(y-x)^{\xi(q)}} \mathbb{E} \left[\frac{e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}{M_{\frac{l}{y-x}}([0,1])^q} \right] \gamma(dx) \gamma(dy) \end{aligned}$$

By using a Girsanov transform, we have

$$\begin{aligned} \mathbb{E} \left[\frac{e^{q\gamma X_0^{\frac{l}{y-x}} + q\gamma X_1^{\frac{l}{y-x}} - q^2 \gamma^2 \mathbb{E}[(X_x^{\frac{l}{y-x})^2]}}{M_{\frac{l}{y-x}}([0,1])^q} \right] &= \mathbb{E} \left[\frac{e^{\frac{k}{y-x}(1)}}{\left(\int_0^1 e^{\gamma X_r^{\frac{l}{y-x}} - \frac{\gamma^2}{2} \mathbb{E}[(X_x^{\frac{l}{y-x})^2]} + q\gamma^2 k \frac{l}{y-x} (1-r) + q\gamma^2 k \frac{l}{y-x} (r)} dr \right)^q} \right] \\ &\leq C \mathbb{E} \left[\frac{1}{\left(\int_{1/4}^{3/4} e^{\gamma X_r^{\frac{l}{y-x}} - \frac{\gamma^2}{2} \mathbb{E}[(X_x^{\frac{l}{y-x})^2]} dr \right)^q} \right] \\ &= C \mathbb{E} \left[\frac{1}{\left(M_{\frac{l}{y-x}}([1/4, 3/4]) \right)^q} \right] \end{aligned}$$

for some positive constant C . Notice that we have just used the fact that $k \frac{l}{y-x}(1) = 0$ and that $k \frac{l}{y-x}$ is bounded (independently of $\frac{l}{y-x}$) over the complement of any non-empty ball centered at 0. It is a standard fact that the measure M possesses moments

of negative order so that we have proved

$$\lim_l A_l^2 \leq C \int_{B(0,T)^2} \frac{1}{|y-x|^{\xi(q)}} \gamma(dx) \gamma(dy) < +\infty.$$

To treat the term A_l^1 , we use quite a similar argument excepted that we use the scaling relation on l instead of $y-x$, and the Girsanov transform again:

$$\begin{aligned} A_l^2 &= \int_{0 \leq y-x \leq l} \mathbb{E} \left[\frac{e^{2q\gamma\Omega_l - q^2\gamma^2 \ln \frac{1}{l}} e^{q\gamma X_0^1 + q\gamma X_{\frac{y-x}{l}}^1 - q^2\gamma^2 \mathbb{E}[(X_x^1)^2]}}{l^q e^{q\Omega_l - q\frac{\gamma^2}{2} \ln \frac{1}{l}} M_1([0, \frac{y-x}{l}])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{0 \leq y-x \leq l} \mathbb{E} \left[\frac{e^{q\gamma\Omega_l - (q^2\gamma^2 - q\frac{\gamma^2}{2}) \ln \frac{1}{l}}}{l^q} \right] \mathbb{E} \left[\frac{e^{q\gamma X_{\frac{y-x}{l}}^1 + q\gamma X_0^1 - q^2\gamma^2 \mathbb{E}[(X_x^1)^2]}}{M_1([0, \frac{y-x}{l}])^q} \right] \gamma(dx) \gamma(dy) \\ &= \int_{0 \leq y-x \leq l} \frac{1}{l^{\xi(q)}} \mathbb{E} \left[\frac{e^{q^2\gamma^2 k_1(\frac{y-x}{l})}}{\left(\int_0^{\frac{y-x}{l}} e^{q\gamma X_r^1 - \frac{q^2\gamma^2}{2} \mathbb{E}[(X_r^1)^2] + q\gamma^2 k_1(\frac{y-x}{l} - r) + q^2\gamma^2 k_1(r)} dr \right)^q} \right] \gamma(dx) \gamma(dy). \end{aligned}$$

By using the fact that k_1 is positive bounded by 1 we have

$$A_l^2 \leq \int_{0 \leq y-x \leq l} \frac{1}{l^{\xi(q)}} \mathbb{E} \left[\frac{1}{\left(\int_0^{\frac{y-x}{l}} e^{q\gamma X_r^1 - \frac{q^2\gamma^2}{2} \mathbb{E}[(X_r^1)^2]} dr \right)^q} \right] \gamma(dx) \gamma(dy).$$

By using Kahane's concentration inequalities to the convex mapping $x \mapsto \frac{1}{x^q}$, we deduce (for some positive constant C)

$$\begin{aligned} A_l^2 &\leq C \int_{0 \leq y-x \leq l} \frac{1}{l^{\xi(q)}} \mathbb{E} \left[\frac{1}{\left(\int_0^{\frac{y-x}{l}} dr \right)^q} \right] \gamma(dx) \gamma(dy) \\ &\leq C \int_{0 \leq y-x \leq l} \frac{l^q}{l^{\xi(q)} (y-x)^q} \gamma(dx) \gamma(dy) \\ &\leq C \int_{0 \leq y-x \leq l} \frac{1}{(y-x)^{\xi(q)}} \gamma(dx) \gamma(dy). \end{aligned}$$

Hence

$$\lim_l A_l^1 \leq C \int_{B(0,T)^2} \frac{1}{|y-x|^{\xi(q)}} \gamma(dx) \gamma(dy) < +\infty.$$

The KPZ formula is proved. \square

C.3 Proof of the dual KPZ formula

This time, we do not restrict to the dimension 1. Let K be a compact subset of \mathbb{R}^d , included in the ball $B(0,1)$ with Hausdorff dimension $0 \leq \dim_{Leb}(K) < 1$. Let δ_0 be the unique solution in $[0, \alpha[$ such that $\frac{\xi(\delta_0)}{d} = \dim_{Leb}(K)$. We want to prove $\delta_0 = \dim_{\overline{M}}(K)$.

Let $0 \leq q < \alpha$ be such that $\frac{\bar{\xi}(q)}{d} > \dim_{Leb}(K)$. For $\epsilon > 0$, there is a covering of K by a countable family of balls $(B(x_n, r_n))_n$ such that

$$\sum_n r_n^{\bar{\xi}(q)} < \epsilon.$$

Since we have (see Theorem 4)

$$\begin{aligned} \mathbb{E} \left[\sum_n \overline{M}(B(x_n, r_n))^q \right] &= \sum_n \mathbb{E} \left[\overline{M}(B(0, r_n))^q \right] \\ &\leq C_q \sum_n r_n^{\bar{\xi}(q)} \\ &\leq C_q \epsilon, \end{aligned}$$

we deduce by the Markov inequality

$$\mathbb{P} \left(\sum_n \overline{M}(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon} \right) \geq 1 - \sqrt{\epsilon}.$$

Thus, with probability $1 - \sqrt{\epsilon}$, there is a covering of balls of K such that $\sum_n \overline{M}(B(x_n, r_n))^q \leq C_q \sqrt{\epsilon}$. So $q \geq \dim_{\overline{M}}(K)$ almost surely.

Conversely, consider $p \in [0, \alpha[$ such that $\frac{\bar{\xi}(p)}{d} < \dim_{Leb}(K)$. Since $\bar{\xi}(p) = \xi(\frac{p}{\alpha})$, we can set $q = \frac{p}{\alpha} \in [0, 1[$ and we have $\frac{\xi(q)}{d} < \dim_{Leb}(K)$. As we proved above, we can consider the measure $\tilde{\gamma}$ introduced in (29). It is almost surely supported by K and non trivial. Furthermore, it satisfies

$$\mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{\overline{M}(B(x, |y-x|))^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] < +\infty.$$

Let us prove that

$$\mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{\overline{M}(B(x, |y-x|))^p} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] < +\infty. \quad (30)$$

By using the relation for $p, x > 0$

$$\Gamma(p) = x^p \int_0^{+\infty} u^{p-1} e^{-ux} du,$$

we deduce:

$$\begin{aligned} &\mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{\overline{M}(B(x, |y-x|))^p} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] \\ &= \frac{1}{\Gamma(p)} \mathbb{E} \left[\int_0^{+\infty} u^{p-1} \int_{B(0,T)^2} e^{-u \overline{M}(B(x, |y-x|))} \tilde{\gamma}(dx) \tilde{\gamma}(dy) du \right] \\ &= \frac{1}{\Gamma(p)} \mathbb{E} \left[\int_0^{+\infty} u^{p-1} \int_{B(0,T)^2} \mathbb{E} \left[e^{-u \overline{M}(B(x, |y-x|))} | Y_n, n \geq 1 \right] \tilde{\gamma}(dx) \tilde{\gamma}(dy) du \right] \\ &= \frac{1}{\Gamma(p)} \mathbb{E} \left[\int_0^{+\infty} u^{p-1} \int_{B(0,T)^2} e^{-u^\alpha \overline{M}(B(x, |y-x|))} \tilde{\gamma}(dx) \tilde{\gamma}(dy) du \right] \end{aligned}$$

Now we make the change of variable $y = u^\alpha M(B(x, |y - x|))$ to obtain:

$$\begin{aligned} & \mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{\overline{M}(B(x, |y - x|))^p} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] \\ &= \frac{1}{\alpha \Gamma(p)} \mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{M(B(x, |y - x|))^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right] \int_0^{+\infty} y^{\frac{p}{\alpha} - 1} e^{-y} dy \\ &= \frac{\Gamma(\frac{p}{\alpha} + 1)}{\Gamma(p + 1)} \mathbb{E} \left[\int_{B(0,T)^2} \frac{1}{M(B(x, |y - x|))^q} \tilde{\gamma}(dx) \tilde{\gamma}(dy) \right]. \end{aligned}$$

Hence, the above quantity is finite and (30) is proved. As usually, we conclude by using the Frostman lemma that $p < \dim_{\overline{M}}(K)$. The dual KPZ formula is proved. \square

C.4 Proof of the duality relation

By inverting the KPZ relations, we have

$$\dim_M(K) = \frac{1 + \frac{\gamma^2}{2d} - \sqrt{(1 + \frac{\gamma^2}{2d})^2 - 4\frac{\gamma^2}{2d} \dim_{Leb}(K)}}{\frac{\gamma^2}{d}}$$

and

$$\begin{aligned} \dim_{\overline{M}}(K) &= \frac{1 + \frac{\overline{\gamma}^2}{2d} - \sqrt{(1 + \frac{\overline{\gamma}^2}{2d})^2 - 4\frac{\overline{\gamma}^2}{2d} \dim_{Leb}(K)}}{2\frac{\overline{\gamma}^2}{2d}} \\ &= \frac{1}{2} \left(1 + \frac{2d}{\overline{\gamma}^2} - \sqrt{(1 + \frac{2d}{\overline{\gamma}^2})^2 - 4\frac{2d}{\overline{\gamma}^2} \dim_{Leb}(K)} \right) \end{aligned}$$

By using the relation $\frac{2d}{\overline{\gamma}^2} = \frac{\gamma^2}{2d}$, we deduce:

$$\dim_{\overline{M}}(K) = \frac{\gamma^2}{2d} \dim_M(K).$$

We point out that this duality relation can be directly recovered from the relation $\dim_{\overline{M}}(K) = \alpha \dim_M(K)$ we just proved above. \square

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