The heart of a convex body

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THE HEART OF A CONVEX BODY

LORENZO BRASCO AND ROLANDO MAGNANINI

ABSTRACT. We investigate some basic properties of the *heart* $\heartsuit(\mathcal{K})$ of a convex set $\mathcal{K}$. It is a subset of $\mathcal{K}$, whose definition is based on mirror reflections of euclidean space, and is a non-local object. The main motivation of our interest for $\heartsuit(\mathcal{K})$ is that this gives an estimate of the location of the hot spot in a convex heat conductor with boundary temperature grounded at zero. Here, we investigate on the relation between $\heartsuit(\mathcal{K})$ and the mirror symmetries of $\mathcal{K}$; we show that $\heartsuit(\mathcal{K})$ contains many (geometrically and phisically) relevant points of $\mathcal{K}$; we prove a simple geometrical lower estimate for the diameter of $\heartsuit(\mathcal{K})$; we also prove an upper estimate for the area of $\heartsuit(\mathcal{K})$, when $\mathcal{K}$ is a triangle.

1. INTRODUCTION

Let $\mathcal{K}$ be a convex body in the euclidean space $\mathbb{R}^N$, that is $\mathcal{K}$ is a compact convex set with non-empty interior. In [1] we defined the *heart* $\heartsuit(\mathcal{K})$ of $\mathcal{K}$ as follows. Fix a unit vector $\omega \in \mathbb{S}^{N-1}$ and a real number λ; for each point $x \in \mathbb{R}^N$, let $T_{\lambda,\omega}(x)$ denote the reflection of x in the hyperplane $\pi_{\lambda,\omega}$ of equation $\langle x, \omega\rangle = \lambda$ (here, $\langle x, \omega\rangle$ denotes the usual scalar product of vectors in $\mathbb{R}^N$); then set

$$\mathcal{K}_{\lambda,\omega} = \{x \in \mathcal{K} : \langle x, \omega\rangle \geq \lambda\}$$

(see Figure 1). The heart of $\mathcal{K}$ is thus defined as

$$\heartsuit(\mathcal{K}) = \bigcap_{\omega\in\mathbb{S}^{N-1}} \{\mathcal{K}_{-\lambda,-\omega} : T_{\lambda,\omega}(\mathcal{K}_{\lambda,\omega}) \subset \mathcal{K}\}.$$

Our interest in $\heartsuit(\mathcal{K})$ was motivated in [1] in connection to the problem of locating the (unique) point of maximal temperature — the *hot spot* — in a convex heat conductor with boundary temperature grounded at zero. There, by means of A.D. Aleksandrov's reflection principle, we showed that $\heartsuit(\mathcal{K})$ must contain the hot spot at each time and must also contain the maximum point of the first Dirichlet eigenfunction of the Laplacian, which is known to control the asymptotic behaviour of temperature for large times. By the same arguments, we showed in [1] that $\heartsuit(\mathcal{K})$ must also contain the maximum point of positive solutions of nonlinear equations belonging to a quite large class. By these observations, the set $\heartsuit(\mathcal{K})$ can be viewed as a geometrical means to estimate the positions of these important points.

Another interesting feature of $\heartsuit(\mathcal{K})$ is the non-local nature of its definition. We hope that the study of $\heartsuit(\mathcal{K})$ can help, in a relatively simple setting, to develop techniques that may be useful in the study of other objects and properties of non-local nature, which have lately raised interest in the study of partial differential equations.

A further reason of interest is that the shape of $\heartsuit(\mathcal{K})$ seems to be related to the mirror symmetry of $\mathcal{K}$. By means of a numerical algorithm, developed in [1], that (approximately) constructs $\heartsuit(\mathcal{K})$

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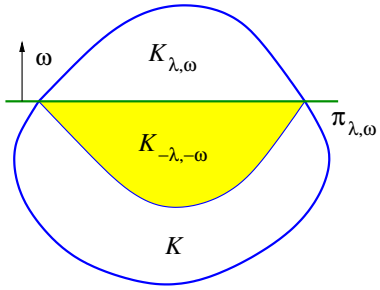

FIGURE 1. The sets $\mathcal{K}_{\lambda,\omega}$ and $\mathcal{K}_{-\lambda,-\omega}$.

for any given convex polyhedron $\mathcal{K}$, one can observe that relationship — and other features of $\heartsuit(\mathcal{K})$ — and raise some questions.

(i) We know that, if $\mathcal{K}$ has a hyperplane of symmetry, then $\heartsuit(\mathcal{K})$ is contained in that hyperplane; is the converse true?
(ii) How small $\heartsuit(\mathcal{K})$ can be? Can we estimate from below the ratio between the diameters of $\heartsuit(\mathcal{K})$ and $\mathcal{K}$?
(iii) How big $\heartsuit(\mathcal{K})$ can be? Can we estimate from above the ratio between the volumes of $\heartsuit(\mathcal{K})$ and $\mathcal{K}$?

The purpose of this note is to collect results in that direction.

In Section 2, we give a positive answer to question (i) (see Theorem 2.4).

In Section 3, we start by showing that many relevant points related to a convex set lie in its heart. For instance, we shall prove that, besides the center of mass $M_{\mathcal{K}}$ of $\mathcal{K}$ (as seen in [1, Proposition 4.1]), $\heartsuit(\mathcal{K})$ must also contain the center $C_{\mathcal{K}}$ of the smallest ball containing $\mathcal{K}$ — the so-called *circumcenter* —and the center of mass of the set of all the *incenters* of $\mathcal{K}$,

$$\mathcal{M}(\mathcal{K}) = \{x \in \mathcal{K} \,:\, \mathrm{dist}(x, \partial\mathcal{K}) = r_{\mathcal{K}}\}\,;$$

here $r_{\mathcal{K}}$ is the *inradius* of $\mathcal{K}$, i.e. the radius of the largest balls contained in $\mathcal{K}$. This information gives a simple estimate from below of the diameter of $\heartsuit(\mathcal{K})$, thus partially answering to question (ii) (see Theorem 3.5).

By further exploring in this direction, we prove that $\heartsuit(\mathcal{K})$ must also contain the points minimizing each p-moment of the set $\mathcal{K}$ (see Subsection 3.2) and other more general moments associated to $\heartsuit(\mathcal{K})$. As a consequence of this general result, we relate $\heartsuit(\mathcal{K})$ to a problem in spectral optimization considered in [5] and show that $\heartsuit(\mathcal{K})$ must contain the center of a ball realizing the so-called *Fraenkel asymmetry* (see [4] and Section 3 for a definition).

Finally, in Section 4, we begin an analysis of problem (iii). Therein, we discuss the shape optimization problem (4.1) and prove that an optimal shape does not exist in the subclass of triangles. In fact, in Theorem 4.2 we show that

$$|\heartsuit(\mathcal{K})| < \frac{3}{8}\,|\mathcal{K}| \quad \text{for every triangle } \mathcal{K};$$

the constant $3/8$ is not attained but is only approached by choosing a sequence of obtuse triangles.

2. Dimension of the heart and symmetries

Some of the results in Subsection 2.1 where proved in [1] but, for the reader's convenience, we reproduce them here.

2.1. **Properties of the maximal folding function.** The *maximal folding function* $\mathcal{R}_{\mathcal{K}} : \mathbb{S}^{N-1} \to \mathbb{R}$ of a convex body $\mathcal{K} \subset \mathbb{R}^N$ was defined in [1] by

$$\mathcal{R}_{\mathcal{K}}(\omega) = \min\{\lambda \in \mathbb{R} \ : \ T_{\lambda,\omega}(\mathcal{K}_{\lambda,\omega}) \subseteq \mathcal{K}\}, \quad \omega \in \mathbb{S}^{N-1}.$$

The heart of $\mathcal{K}$ can be defined in terms of $\mathcal{R}_{\mathcal{K}}$:

$$\heartsuit(\mathcal{K}) = \left\{x \in \mathcal{K} \ : \ \langle x, \omega\rangle \leq \mathcal{R}_{\mathcal{K}}(\omega), \text{ for every } \omega \in \mathbb{S}^{N-1}\right\}. \tag{2.1}$$

It is important to remark that we have

$$\max_{x \in \heartsuit(\mathcal{K})} \langle x, \omega\rangle \leq \mathcal{R}_{\mathcal{K}}(\omega), \qquad \omega \in \mathbb{S}^{N-1}$$

and in general the two terms do not coincide: in other words, $\mathcal{R}_{\mathcal{K}}$ does not coincide with the support function of $\heartsuit(\mathcal{K})$ (see [1, Example 4.8]).

Lemma 2.1. *The maximal folding function $\mathcal{R}_{\mathcal{K}} : \mathbb{S}^{N-1} \to \mathbb{R}$ is lower semicontinuous. In particular, $\mathcal{R}_{\mathcal{K}}$ attains its minimum on $\mathbb{S}^{N-1}$.*

Proof. Fix $\lambda \in \mathbb{R}$ and let $\omega_0 \in \{\omega \in \mathbb{S}^{N-1} : \mathcal{R}_{\mathcal{K}}(\omega) > \lambda\}$. Then

$$|T_{\lambda,\omega_0}(\mathcal{K}_{\lambda,\omega_0}) \cap (\mathbb{R}^N \setminus \mathcal{K})| > 0.$$

The continuity of the function $\omega \mapsto |T_{\lambda,\omega}(\mathcal{K}_{\lambda,\omega}) \cap (\mathbb{R}^N \setminus \mathcal{K})|$ implies that, for every ω in some neighborhood of ω_0, we have that $\mathcal{R}_{\mathcal{K}}(\omega) > \lambda$. □

Remark 2.2. In general $\mathcal{R}_{\mathcal{K}}$ is not continuous on $\mathbb{S}^{N-1}$: it sufficient to take a rectangle $\mathcal{K} = [-a, a] \times [-b, b]$ and observe that we have $\mathcal{R}_{\mathcal{K}}(1, 0) = 0$, since $\mathcal{K}$ is symmetric with respect to the y-axis but, for a sufficiently small ϑ, $\mathcal{R}_{\mathcal{K}}(\cos\vartheta, \sin\vartheta) = a\,\cos\vartheta - b\,\sin\vartheta$, so that

$$\lim_{\vartheta \to 0} \mathcal{R}_{\mathcal{K}}(\vartheta) = a > \mathcal{R}_{\mathcal{K}}(1, 0).$$

Observe that here the lack of continuity of the maximal folding function is not due to the lack of smoothness of the boundary of $\mathcal{K}$, but rather to the presence of non-strictly convex subsets of $\partial\mathcal{K}$.

Proposition 2.3. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body and define its center of mass by*

$$M_{\mathcal{K}} = \frac{1}{|\mathcal{K}|} \int_{\mathcal{K}} y\, dy.$$

Then we have that

$$\mathcal{R}_{\mathcal{K}}(\omega) \geq \langle M_{\mathcal{K}}, \omega\rangle, \quad \text{for every } \omega \in \mathbb{S}^{N-1}, \tag{2.2}$$

and the equality sign can hold for some $\omega \in \mathbb{S}^{N-1}$ if and only if $\mathcal{K}$ is $\omega-$symmetric, i.e. if $T_{\lambda,\omega}(\mathcal{K}) = \mathcal{K}$ for some $\lambda \in \mathbb{R}$. In particular, $M_{\mathcal{K}} \in \heartsuit(\mathcal{K})$.

Proof. Set $\lambda = \mathcal{R}_{\mathcal{K}}(\omega)$ and consider the set $\Omega = \mathcal{K}_{\lambda,\omega} \cup T_{\lambda,\omega}(\mathcal{K}_{\lambda,\omega})$; by the definition of center of mass and since Ω is symmetric with respect to the hyperplane $\pi_{\lambda,\omega}$, we easily get that

$$|\mathcal{K}|\,[\mathcal{R}_{\mathcal{K}}(\omega) - \langle M_{\mathcal{K}}, \omega\rangle] = \int_{\mathcal{K}} \left[\mathcal{R}_{\mathcal{K}}(\omega) - \langle y, \omega\rangle\right] dy = \int_{\mathcal{K}\setminus\Omega} \left[\mathcal{R}_{\mathcal{K}}(\omega) - \langle y, \omega\rangle\right] dy.$$

Observe that the last integral contains a positive quantity to be integrated on the region $\mathcal{K} \setminus \Omega$: this already shows (2.2). Moreover, the same identity implies:

$$\mathcal{R}_{\mathcal{K}}(\omega) - \langle M_{\mathcal{K}}, \omega\rangle = 0 \quad \Longleftrightarrow \quad |\mathcal{K} \setminus \Omega| = 0,$$

and the latter condition is equivalent to say that $\mathcal{K}$ is $\omega-$symmetric. Finally, by combining (2.2) with the definition (2.1) of $\heartsuit(\mathcal{K})$, it easily follows $M_{\mathcal{K}} \in \heartsuit(\mathcal{K})$. □

2.2. **On the mirror symmetries of $\mathcal{K}$.** A first application of Proposition 2.3 concerns the relation betwwen $\heartsuit(\mathcal{K})$ and the mirror symmetries of $\mathcal{K}$.

Theorem 2.4. *Let $\mathcal{K}\subset\mathbb{R}^N$ be a convex body.*

(i) If $\mathcal{K}$ there exist k $(1\le k\le N)$ independent directions $\omega_1,\dots,\omega_k\in\mathbb{S}^{N-1}$ such that $\mathcal{K}$ is ω_j-symmetric for $j=1,\dots,k$, then $\mathcal{R}_{\mathcal{K}}(\omega_j)=\langle M_{\mathcal{K}},\omega_j\rangle$ for $j=1,\dots,k$ and

$$\heartsuit(\mathcal{K})\subseteq\bigcap_{j=1}^{k}\pi_{\mathcal{R}(\omega_j),\omega_j}.$$

In particular, the co-dimension of $\heartsuit(\mathcal{K})$ is at least k.

(ii) If $\heartsuit(\mathcal{K})$ has dimension k $(1\le k\le N-1)$, then there exists at least a direction $\theta\in\mathbb{S}^{N-1}$ such that $\mathcal{R}_{\mathcal{K}}(\theta)=\langle M_{\mathcal{K}},\theta\rangle$, and $\mathcal{K}$ is θ-symmetric.

Proof. (i) The assertion is a straightforward consequence of Proposition 2.3.

(ii) By Lemma 2.1, the function $\omega\mapsto\mathcal{R}_{\mathcal{K}}(\omega)-\langle M_{\mathcal{K}},\omega\rangle$ attains its minimum for some $\theta\in\mathbb{S}^{N-1}$. Set $r=\mathcal{R}_{\mathcal{K}}(\theta)-\langle M_{\mathcal{K}},\theta\rangle$ and suppose that $r>0$.

Then, for every $x\in B(M_{\mathcal{K}},r)$, we have that

$$\langle x,\omega\rangle=\langle M_{\mathcal{K}},\omega\rangle+\langle x-M_{\mathcal{K}},\omega\rangle<\langle M_{\mathcal{K}},\omega\rangle+r\le\mathcal{R}_{\mathcal{K}}(\omega)-r+r=\mathcal{R}_{\mathcal{K}}(\omega),$$

for every $\omega\in\mathbb{S}^{N-1}$, and hence $x\in\heartsuit(\mathcal{K})$ by (2.1). Thus, $B(x,r)\subset\heartsuit(\mathcal{K})$ — a contradiction to the fact that $1\le k\le N-1$. Hence, $\mathcal{R}_{\mathcal{K}}(\theta)=\langle M_{\mathcal{K}},\theta\rangle$ and $\mathcal{K}$ is θ-symmetric by Proposition 2.3. □

Remark 2.5. It is clear that the dimension of the heart only gives information on the *minimal* number of symmetries of a convex body: the example of a ball is quite explicative.

We were not able to prove the following result:

> *if $\heartsuit(\mathcal{K})$ has co-dimension m $(1\le m\le N)$, then there exist at least m independent directions $\theta_1,\dots,\theta_N\in\mathbb{S}^{N-1}$ such that $\mathcal{R}_{\mathcal{K}}(\theta_j)=\langle M_{\mathcal{K}},\theta_j\rangle$, $j=1,\dots,m$, and $\mathcal{K}$ is θ_j-symmetric for $j=1,\dots,m$.*

We leave it as a conjecture.

3. Relevant points contained in the heart

In this section, we will show that many relevant points of a convex set are contained in its heart (e.g. the incenter and the circumcenter, besides the center mass). This fact will give us a means to estimate from below the diameter of the heart.

3.1. **An estimate of the heart's diameter.**

Proposition 3.1. *Let $\mathcal{K}\subset\mathbb{R}^N$ be a convex body. Then its circumcenter $C_{\mathcal{K}}$ belongs to $\heartsuit(\mathcal{K})$.*

Proof. Suppose that $C_{\mathcal{K}}\notin\heartsuit(\mathcal{K})$ and let $q\in\heartsuit(\mathcal{K})$ be its (unique) projection on the set $\heartsuit(\mathcal{K})$; then, define:

$$H^+=\{x\in\mathbb{R}^N:\langle x-q,C_{\mathcal{K}}-q\rangle>0\}\qquad\text{and}\qquad\Pi=\partial H^+.$$

Since $C_{\mathcal{K}}\in\mathcal{K}$, then $H^+\cap\mathcal{K}$ is non-empty and its reflection in Π is contained in $\mathcal{K}$, by the definition of $\heartsuit(\mathcal{K})$, and hence is contained in $B_R(C_{\mathcal{K}})$. Thus, $\mathcal{K}$ must be contained in the set $B_R(C_{\mathcal{K}})\cap B_R(2q-C_{\mathcal{K}})$.

This is a contradiction, since the smallest ball containig $\mathcal{K}$ would have a radius that is at most $\sqrt{R^2-|C_{\mathcal{K}}-q|^2}<R$. □

We now consider the *incenters* of $\mathcal{K}$: these are the centers of the balls of largest radius $r_{\mathcal{K}}$ inscribed in $\mathcal{K}$. Needless to say, a convex body may have many incenters. We start with the simpler case of a convex body with a unique incenter.

Proposition 3.2. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body; if its incenter $I_{\mathcal{K}}$ is unique, then $I_{\mathcal{K}} \in \heartsuit(\mathcal{K})$. In particular, $I_{\mathcal{K}} \in \heartsuit(\mathcal{K})$ if $\mathcal{K}$ is strictly convex.*

Proof. Consider the unique maximal ball $B(I_{\mathcal{K}}, r_{\mathcal{K}})$ inscribed in $\mathcal{K}$ and suppose that $I_{\mathcal{K}} \notin \heartsuit(\mathcal{K})$; this implies that there exists $\omega \in \mathbb{S}^{N-1}$ such that

$$\mathcal{R}_{\mathcal{K}}(\omega) - \langle I_{\mathcal{K}}, \omega \rangle < 0.$$

Set $\lambda = \mathcal{R}_{\mathcal{K}}(\omega)$ and define $I'_{\mathcal{K}} = T_{\lambda,\omega}(I_{\mathcal{K}})$; then $I'_{\mathcal{K}} \neq I_{\mathcal{K}}$ and $\langle I'_{\mathcal{K}}, \omega \rangle < \langle I_{\mathcal{K}}, \omega \rangle$. Now, the half-ball $B^+ = \{x \in B(I_{\mathcal{K}}, r_{\mathcal{K}}) : \langle x, \omega \rangle \geq \lambda)\}$ and its reflection $T_{\lambda,\omega}(B^+)$ in the hyperplane $\pi_{\lambda,\omega}$ are contained in $\mathcal{K}$, since B^+ is contained in the maximal cap $\mathcal{K}_{\lambda,\omega}$.

This fact implies in particular that the reflection of the whole ball B is contained in $\mathcal{K}$: but the latter is still a maximal ball of radius $r_{\mathcal{K}}$, with center $I'_{\mathcal{K}}$ different from $I_{\mathcal{K}}$. This is a contradiction, since it violates the assumed uniqueness of the incenter. □

To treat the general case, we need the following simple result.

Lemma 3.3. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body and let us set*

$$\mathcal{M}(\mathcal{K}) = \{x \in \mathcal{K} : \operatorname{dist}(x, \partial\mathcal{K}) = r_{\mathcal{K}}\}.$$

Then $\mathcal{M}(\mathcal{K})$ is a closed convex set with $|\mathcal{M}(\mathcal{K})| = 0$; in particular, the dimension of $\mathcal{M}(\mathcal{K})$ is at most $N - 1$.

Proof. The quasi-convexity[1] of $\operatorname{dist}(x, \partial\mathcal{K})$ (being $\mathcal{K}$ convex) immediately implies that $\mathcal{M}(\mathcal{K})$ is convex. For the reader's convenience, here we give a proof anyway. Let us take $x, z \in \mathcal{M}(\mathcal{K})$ two distinct points, then by definition of inradius we have

$$B(x, r_{\mathcal{K}}) \cup B(z, r_{\mathcal{K}}) \subset \mathcal{K}.$$

Since $\mathcal{K}$ is convex, it must contain the convex hull of $B(x, r_{\mathcal{K}}) \cup B(z, r_{\mathcal{K}})$, as well: hence, for every $t \in [0, 1]$ we have

$$B((1-t)\,x + t\,z, r_{\mathcal{K}}) \subset \mathcal{K},$$

that is $(1-t)\,x + t\,w \in \mathcal{M}(\mathcal{K})$, which proves the convexity of $\mathcal{M}(\mathcal{K})$.

Now, suppose that $|\mathcal{M}(\mathcal{K})| > 0$; since $\mathcal{M}(\mathcal{K})$ is convex, it must contain a ball $B(x, \varrho)$. The balls of radius $r_{\mathcal{K}}$ having centers on $\partial B(x, \varrho/2)$ are all contained in $\mathcal{K}$, so that their whole union is contained in $\mathcal{K}$ as well. Observing that this union is given by $B(x, \varrho/2 + r_{\mathcal{K}})$, we obtain the desired contradiction, since we violated the maximality of $r_{\mathcal{K}}$. □

Proposition 3.4. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body and let us suppose that $\mathcal{M}(\mathcal{K})$ has dimension $k \in \{1, \dots, N-1\}$.*

Then the center of mass of $\mathcal{M}(\mathcal{K})$, defined by

$$I_{\mathcal{M}} = \frac{\int_{\mathcal{M}(\mathcal{K})} y \, d\mathcal{H}^k(y)}{\mathcal{H}^k(\mathcal{M}(\mathcal{K}))},$$

belongs to $\heartsuit(\mathcal{K})$. Here, $\mathcal{H}^k$ denotes the standard k-dimensional Hausdorff measure.

[1] This means that the superlevel sets of the function are convex.

Proof. The proof is based on the observation that

$$\mathcal{R}_{\mathcal{M}(\mathcal{K})}(\omega) \le \mathcal{R}_{\mathcal{K}}(\omega), \qquad \omega \in \mathbb{S}^{N-1}, \tag{3.1}$$

where $\mathcal{R}_{\mathcal{M}(\mathcal{K})}$ is the maximal folding function of $\mathcal{M}(\mathcal{K})$, thought as a subset of $\mathbb{R}^N$. Assuming (3.1) to be true, we can use the definition of heart and Proposition 2.3 to obtain that the center of mass $I_{\mathcal{M}}$ belongs to $\heartsuit(\mathcal{M}(\mathcal{K}))$ and hence to $\heartsuit(\mathcal{K})$, which would conclude the proof.

Now, suppose by contradiction that there is an $\omega \in \mathbb{S}^{N-1}$ such that $\mathcal{R}_{\mathcal{M}(\mathcal{K})}(\omega) > \mathcal{R}_{\mathcal{K}}(\omega)$, and set $\lambda = \mathcal{R}_{\mathcal{K}}(\omega)$, as usual. Then, there exists $x \in \mathcal{M}(\mathcal{K})$ with $\langle x, \omega\rangle \ge \lambda$ such that its reflection $x^\lambda = T_{\lambda,\omega}(x)$ in the hyperplane $\pi_{\lambda,\omega}$ falls outside $\mathcal{M}(\mathcal{K})$: this would imply in particular that $B(x^\lambda, r_{\mathcal{K}}) \not\subset \mathcal{K}$. Observe that by definition of $\mathcal{M}(\mathcal{K})$, the ball $B(x, r_{\mathcal{K}})$ lies inside $\mathcal{K}$, so that the cap $B(x, r_{\mathcal{K}}) \cap \{\langle y, \omega\rangle \ge \lambda\}$ is reflected in $\mathcal{K}$; thus, as before, we obtain that the union of this cap and its reflection is contained in $\mathcal{K}$. This shows that the ball $B(x^\lambda, r_{\mathcal{K}})$ is contained in $\mathcal{K}$, thus giving a contradiction. □

The result here below follows at once from Propositions 2.3, 3.1 and 3.4.

Theorem 3.5. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body, then*

$$\operatorname{diam}[\heartsuit(\mathcal{K})] \ge \max(|M_{\mathcal{K}} - C_{\mathcal{K}}|, |C_{\mathcal{K}} - I_{\mathcal{M}}|, |I_{\mathcal{M}} - M_{\mathcal{K}}|).$$

Remark 3.6. Notice that, when $\heartsuit(\mathcal{K})$ degenerates to a single point, then clearly

$$\heartsuit(\mathcal{K}) = \{M_{\mathcal{K}}\} = \{I_{\mathcal{K}}\} = \{C_{\mathcal{K}}\}.$$

Needless to say, it may happen that the three points $M_{\mathcal{K}}, I_{\mathcal{K}}$ and $C_{\mathcal{K}}$ coincide, but $\heartsuit(\mathcal{K})$ is not a point. For example, take an ellipse parametrized in polar coordinates as

$$E = \left\{(\varrho, \vartheta) \,:\, 0 \le \varrho \le \sqrt{a^2\cos^2\vartheta + b^2\,\sin^2\vartheta},\ \vartheta \in [-\pi, \pi]\right\}$$

and a π-periodic, smooth non-negative function η on $[-\pi, \pi]$, having its support in two small neighborhoods of $-3\pi/4$ and $\pi/4$. Then, if ε is sufficiently small, the deformed set

$$E_\varepsilon = \left\{(\varrho, \vartheta) \,:\, 0 \le \varrho \le \sqrt{a^2\cos^2\vartheta + b^2\,\sin^2\vartheta} - \varepsilon\,\eta(\vartheta),\ \vartheta \in [-\pi, \pi]\right\}$$

is still convex and centrally symmetric. Moreover, it is easy to convince oneself that $\{M_{E_\varepsilon}\} = \{I_{E_\varepsilon}\} = \{C_{E_\varepsilon}\} = \{(0,0)\}$, whereas, by Theorem 2.4, $\heartsuit(E_\varepsilon)$ is not a point, since E_ε has no mirror symmetries.

3.2. **The p-moments of $\mathcal{K}$ and more.** We recall that the point $M_{\mathcal{K}}$ can also be characterized as the unique point in $\mathcal{K}$ which minimizes the function

$$x \mapsto \int_{\mathcal{K}} |x - y|^2\, dy, \qquad x \in \mathcal{K},$$

that can be viewed as the moment of inertia (or 2-moment) of $\mathcal{K}$ about the point x. In this subsection, we will extend the results of Subsection 3.1 to more general moments of $\mathcal{K}$, that include as special cases the p-moments $\int_{\mathcal{K}} |x - y|^p\, dy$.

We first establish a preliminary lemma.

Lemma 3.7. *Let $\varphi, \psi : [0, \infty) \to \mathbb{R}$ be, respectively, an increasing and a decreasing function and suppose that ψ is also integrable in $[0, \infty)$. Define the two functions*

$$F(t) = \int_a^b \varphi(|s - t|)\, ds, \quad t \in [a, b],$$

and

$$G(t) = \int_{-\infty}^{a} \psi(t-s)\,ds + \int_{b}^{\infty} \psi(s-t)\,ds, \quad t \in [a,b],$$

where a and b are two numbers with $0 \le a \le b$.

Then, both F and G are convex and attain their minimum at the midpoint $(a+b)/2$ of $[a,b]$. If φ is strictly increasing (resp. ψ is strictly decreasing), then F (resp. G) is strictly convex and the minimum point is unique.

Proof. (i) Let us introduce a primitive of φ, i.e.

$$\Phi(t) = \int_0^t \varphi(s)\,ds;$$

observe that this is a convex function, since its derivative is increasing. Now, we can write

$$F(t) = \int_a^t \varphi(t-s)\,ds + \int_t^b \varphi(s-t)\,ds = \Phi(t-a) + \Phi(b-t),$$

and notice that both functions on the right-hand side are convex, thus F is convex as well. It is not difficult to see that the graph of F is symmetric with respect to $(a+b)/2$, indeed for every $t \in [a,b]$ we have that

$$F(a+b-t) = \Phi(a+b-t-a) + \Phi(b-a-b+t) = \Phi(b-t) + \Phi(t-a) = F(t).$$

This shows that F attains its minimum at the midpoint. Clearly, if φ is strictly increasing, then F is strictly convex and the minimum point is unique.

(ii) It is enough to rewrite the function G as follows

$$G(t) = \int_{\mathbb{R}} \psi(|t-s|)\,ds - \int_a^b \psi(|t-s|)\,ds, \qquad t \in [a,b];$$

then we notice that the first term is a constant, while the second one gives a convex function, thanks to the first part of this lemma, since $-\psi$ is increasing. □

Theorem 3.8. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body and let $\varphi : [0,\infty) \to \mathbb{R}$ be an increasing function. The function defined by*

$$\mu_\varphi(x) = \int_{\mathcal{K}} \varphi(|x-y|)\,dy, \quad x \in \mathcal{K},$$

is convex in $\mathcal{K}$. Moreover, if we set

$$\mathbf{m}(\mu_\varphi) = \{x \in \mathcal{K} \,:\, \mu_\varphi(x) = \min \mu_\varphi\},$$

then

$$\mathbf{m}(\mu_\varphi) \cap \heartsuit(\mathcal{K}) \neq \varnothing. \tag{3.2}$$

Proof. We shall refer to $\mu_\varphi(x)$ as the *φ-moment* of $\mathcal{K}$ about the point x.

We first prove the convexity of μ_φ on $\mathcal{K}$. It is clearly enough to prove that for every fixed direction, μ_φ is convex along each chord of $\mathcal{K}$ having such a direction. By a rotation of coordinates, we can always suppose that our fixed direction is that of the x_1-axis. Then, we observe that we can parametrize $\mathcal{K}$ as

$$\mathcal{K} = \left\{(y_1, y') \,:\, y' \in \mathcal{S},\, a(y') \le y_1 \le b(y')\right\},$$

for some convex subset $\mathcal{S}$ of $\mathbb{R}^{N-1}$ and functions $a, b : \mathcal{S} \to \mathbb{R}$. In this way, we fix $x' \in \mathcal{S}$ and, by Fubini's theorem, for $a(x') \le x_1 \le b(x')$ we compute:

$$\mu_\varphi(x) = \mu_\varphi(x_1, x') = \int_{\mathcal{S}} \left\{ \int_{a(y')}^{b(y')} \varphi\left(\sqrt{|x'-y'|^2 + |x_1-y_1|^2}\right) \, dy_1 \right\} dy'.$$

Now, notice that, for each fixed x', y', the integral in the braces is a convex function in x_1 for $x_1 \in [a(x'), b(x')]$; in fact, we can apply Lemma 3.7 with the function φ replaced by the function $t \mapsto \varphi\left(\sqrt{|x'-y'|^2 + t^2}\right)$, that is still increasing. Thus, we conclude that, for fixed x', $\mu_\varphi(x_1, x')$ is a convex function of x_1, being an integral of convex functions.

The existence of a minimum point follows from the continuity of μ_φ and the compactness of $\mathcal{K}$: as a consequence of the convexity of μ_φ, we get that the closed set $\mathbf{m}(\mu_\varphi)$ is convex. In order to show that $\mathbf{m}(\mu_\varphi) \cap \heartsuit(\mathcal{K}) \neq \varnothing$, we will prove that

$$\mathcal{R}_{\mathbf{m}(\mu_\varphi)}(\omega) \le \mathcal{R}_{\mathcal{K}}(\omega), \qquad \text{for every } \omega \in \mathbb{S}^{N-1}. \tag{3.3}$$

Indeed this will show that $\heartsuit(\mathbf{m}(\mu_\varphi)) \subset \heartsuit(\mathcal{K})$ and thus $\mathbf{m}(\mu_\varphi) \cap \heartsuit(\mathcal{K}) \neq \varnothing$.

Let us suppose that (3.3) is not true: then, there exist $\omega \in \mathbb{S}^{N-1}$ and $X_\varphi \in \mathbf{m}(\mu_\varphi)$ such that

$$\mathcal{R}_{\mathcal{K}}(\omega) < \mathcal{R}_{\mathbf{m}(\mu_\varphi)}(\omega) \le \langle X_\varphi, \omega \rangle$$

We set $\lambda = \mathcal{R}_{\mathcal{K}}(\omega)$, $\lambda' = \langle X_\varphi, \omega \rangle$, so that $\lambda < \lambda'$, and define the new point

$$Y_\varphi = T_{\lambda,\omega}(X_\varphi) = X_\varphi - 2\,(\lambda' - \lambda)\,\omega.$$

We notice that

$$Y_\varphi \in \mathcal{K} \setminus \mathbf{m}(\mu_\varphi), \tag{3.4}$$

by construction and by the definition of $\heartsuit(\mathbf{m}(\mu_\varphi))$.

Now, we write the function μ_φ as

$$\mu_\varphi(x) = \int_\Omega \varphi(|x-y|)\,dy + \int_{\mathcal{K}\setminus\Omega} \varphi(|x-y|)\,dy,$$

where Ω is the $\omega-$symmetric set $\mathcal{K}_{\lambda,\omega} \cup T_{\lambda,\omega}(\mathcal{K}_{\lambda,\omega})$ contained in $\mathcal{K}$. Thanks to the symmetry of Ω, we can easily infer that

$$\int_\Omega \varphi(|X_\varphi - y|)\,dy = \int_\Omega \varphi(|Y_\varphi - y|)\,dy. \tag{3.5}$$

On the other hand, if $y \in \mathcal{K} \setminus \Omega$, then $\langle y, \omega \rangle \le \lambda$, and hence

$$\begin{aligned} |\langle Y_\varphi, \omega \rangle - \langle y, \omega \rangle| &= |\langle X_\varphi, \omega \rangle - 2(\lambda' - \lambda) - \langle y, \omega \rangle| \\ &\le |\langle X_\varphi, \omega \rangle - \langle y, \omega \rangle|. \end{aligned}$$

Since we also have $\langle Y_\varphi, \xi \rangle = \langle X_\varphi, \xi \rangle$ for every direction ξ orthogonal to ω, we get

$$|Y_\varphi - y| \le |X_\varphi - y|, \qquad \text{for every } y \in \mathcal{K} \setminus \Omega,$$

and, by the monotonicity of φ, we infer:

$$\int_{\mathcal{K}\setminus\Omega} \varphi(|Y_\varphi - y|)\,dy \le \int_{\mathcal{K}\setminus\Omega} \varphi(|X_\varphi - y|)\,dy,$$

(observe that it may happen $|\mathcal{K} \setminus \Omega| = 0$). By adding this inequality to (3.5), we obtain that

$$\mu_\varphi(Y_\varphi) = \int_{\mathcal{K}} \varphi(|Y_\varphi - y|)\, dy \leq \int_{\mathcal{K}} \varphi(|X_\varphi - y|)\, dy = \mu_\varphi(X_\varphi),$$

which implies that Y_φ is still a minimum point for μ_φ.

Thus, $Y_\varphi \in \mathbf{m}(\mu_\varphi)$: this contradicts (3.4), and hence (3.3) holds true. □

In the case of a strictly increasing function φ, the conclusions of Theorem 3.8 can be strengthened.

Corollary 3.9. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body and $\varphi : [0, \infty) \to \mathbb{R}$ be a strictly increasing function. Then the φ-moment μ_φ is strictly convex on $\mathcal{K}$ and its unique minimum point belongs to $\heartsuit(\mathcal{K})$.*

The analogous of Theorem 3.5 is readily proved.

Theorem 3.10. *Let $\mathcal{K}$ be a convex body. Then the convex hull of the set*

$$\bigcup \{\mathbf{m}(\mu_\varphi) \cap \heartsuit(\mathcal{K}) : \varphi \text{ is increasing on } [0, \infty)\}$$

is contained in $\heartsuit(\mathcal{K})$.

Remark 3.11. By similar arguments, we can prove that, if ψ is decreasing and such that the function

$$\nu_\psi(x) = \int_{\mathbb{R}^N \setminus \mathcal{K}} \psi(|x - y|)\, dy,$$

is finite for every $x \in \mathcal{K}$, then ν_ψ is convex and $\mathbf{m}(\nu_\psi) \cap \heartsuit(\mathcal{K}) \neq \varnothing$. Here, as well, if ψ is strictly decreasing, then ν_ψ is strictly convex and its unique minimum point belongs to $\heartsuit(\mathcal{K})$.

Particularly interesting are the cases in which $\varphi(t) = t^p$ with $p > 0$ and $\psi(t) = t^{-p}$ with $p > N$.

Corollary 3.12. *Consider the functions*

$$\mu_p(x) = \int_{\mathcal{K}} |x - y|^p\, dy, \quad x \in \mathcal{K},$$

for $p > 0$ or

$$\nu_p(x) = \int_{\mathbb{R}^N \setminus \mathcal{K}} |x - y|^{-p}\, dy, \quad x \in \mathcal{K},$$

for $p > N$.

Then, their unique minimum points in $\mathcal{K}$ belong to $\heartsuit(\mathcal{K})$.

Remark 3.13. Propositions 2.3, 3.1 and 3.2 can be re-proved by means of Corollary 3.12 by choosing $p = 2$ or, respectively, by taking limits as $p \to \infty$.

Notice, in fact, that

$$\lim_{p \to \infty} \mu_p(x)^{1/p} = \max_{y \in \mathcal{K}} |x - y|$$

and

$$\lim_{p \to \infty} \nu_p(x)^{-1/p} = \min_{y \in \mathcal{K}} |x - y|,$$

and hence the *circumradius* $\rho_{\mathcal{K}}$ and inradius $r_{\mathcal{K}}$ are readily obtained as

$$\rho_{\mathcal{K}} = \min_{x \in \mathcal{K}} \max_{y \in \partial\mathcal{K}} |x - y| = \lim_{p \to +\infty} \min_{x \in \mathcal{K}} \mu_p(x)^{1/p} \quad \text{and} \quad r_{\mathcal{K}} = \max_{x \in \mathcal{K}} \min_{y \in \partial\mathcal{K}} |x - y| = \lim_{p \to +\infty} \max_{x \in \mathcal{K}} \nu_p(x)^{-1/p}.$$

These observations quite straightforwardly imply that $C_{\mathcal{K}}$ and $I_{\mathcal{K}}$ belong to $\heartsuit(\mathcal{K})$.

A final remark concerns the case $p = 0$. It is well-known that

$$\lim_{p\to 0^+} \left(\frac{\mu_p(x)}{|\mathcal{K}|}\right)^{1/p} = \exp\left\{\int_{\mathcal{K}} \log|x-y|\,\frac{dy}{|\mathcal{K}|}\right\} = \exp\{\mu_{\log}(x)/|\mathcal{K}|\},$$

that can be interpreted as the *geometric mean* of the function $y \mapsto |x-y|$ on $\mathcal{K}$; needless to say, the set of its minimum points intersects $\heartsuit(\mathcal{K})$.

3.3. **On Fraenkel's asymmetry.**

Lemma 3.14. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body. For $r > 0$, define the function*

$$\gamma(x) = |\mathcal{K} \cap B(x,r)|, \qquad x \in \mathbb{R}^N.$$

Then γ is log-concave and, if $\mathbf{M}(\gamma) = \{x \in \mathbb{R}^N, :\ \gamma(x) = \max\gamma\}$, then

$$\mathbf{M}(\gamma) \cap \heartsuit(\mathcal{K}) \neq \varnothing. \tag{3.6}$$

Proof. The log-concavity of G is a consequence of Prékopa–Leindler's inequality, that we recall here for the reader's convenience: let $0 < t < 1$ and let f, g and h be nonnegative integrable functions on $\mathbb{R}^N$ satisfying

$$h((1-t)\,x + t\,y) \geq f(x)^{1-t}\,g(y)^t, \qquad \text{for every } x, y \in \mathbb{R}^N; \tag{3.7}$$

then

$$\int_{\mathbb{R}^N} h(x)\,dx \geq \left(\int_{\mathbb{R}^N} f(x)\,dx\right)^{1-t} \left(\int_{\mathbb{R}^N} g(x)\,dx\right)^t. \tag{3.8}$$

(For a proof and a discussion on the links between (3.8) and the Brunn-Minkowski inequality, the reader is referred to [3].)

Indeed, we pick two points $z, w \in \mathbb{R}^N$ and a number $t \in (0,1)$, and apply Prékopa–Leindler's inequality to the triple of functions

$$f = 1_{\mathcal{K}\cap B(z,r)}, \quad g = 1_{\mathcal{K}\cap B(w,r)}, \quad h = 1_{\mathcal{K}\cap B((1-t)\,z+t\,w,r)};$$

then, (3.7) is readily satisfied. Thus, (3.8) easily implies

$$\begin{aligned}\gamma((1-t)\,z + t\,w) =& |\mathcal{K} \cap B((1-t)\,z + t\,w, r)| \geq \\ & |\mathcal{K} \cap B(z,r)|^{1-t}\,|\mathcal{K} \cap B(w,r)|^t = \gamma(z)^{1-t}\,\gamma(w)^t,\end{aligned}$$

and, by taking the logarithm on both sides, we get the desired convexity. A straightforward consequence is that the set $\mathbf{M}(\gamma)$ is convex.

Once again, the validity of (3.6) will be a consequence of the inequality

$$\mathcal{R}_{\mathbf{M}(\gamma)}(\omega) \leq \mathcal{R}_{\mathcal{K}}(\omega), \quad \text{for every } \omega \in \mathbb{S}^{N-1}. \tag{3.9}$$

By contradiction: let us suppose that there exist $\omega \in \mathbb{S}^{N-1}$ and $x \in \mathbf{M}(\gamma)$ such that

$$\mathcal{R}_{\mathcal{K}}(\omega) < \mathcal{R}_{\mathbf{M}(\gamma)}(\omega) \leq \langle x, \omega\rangle.$$

In particular, this implies that the point $x^\lambda = T_{\lambda,\omega}(x)$, with $\lambda = \mathcal{R}_{\mathcal{K}}(\omega)$, does not belong to $\mathbf{M}(\gamma)$ – i.e. the reflection of x with respect to the hyperplane $\pi_{\lambda,\omega}$ falls outside $\mathbf{M}(\gamma)$.

We set for brevity $B = B(x,r)$ and $B^\lambda = B(x^\lambda, r)$, and we again consider the $\omega-$symmetric set $\Omega = \mathcal{K}_{\lambda,\omega} \cup T_{\lambda,\omega}(\mathcal{K}_{\lambda,\omega}) \subseteq \mathcal{K}$. Then, observe that

$$\begin{aligned}B \cap (\mathcal{K} \setminus \Omega) &= \{x \in B :\ \langle x, \omega\rangle < \lambda\} \cap (\mathcal{K} \setminus \Omega) \\ &\subseteq (B \cap B^\lambda) \cap (\mathcal{K} \setminus \Omega) \subseteq B^\lambda \cap (\mathcal{K} \setminus \Omega),\end{aligned}$$

which implies that $|B \cap (\mathcal{K} \setminus \Omega)| \leq |B^\lambda \cap (\mathcal{K} \setminus \Omega)|$. Also, notice that since Ω is symmetric in the hyperplane $\pi_{\lambda,\omega}$ and $B^\lambda = T_{\lambda,\omega}(B)$, we have that $|B^\lambda \cap \Omega| = |B \cap \Omega|$.

By using these informations and the maximality of x, we can infer that

$$\begin{aligned}\gamma(x^\lambda) = |\mathcal{K} \cap B^\lambda| &= |\Omega \cap B^\lambda| + |(\mathcal{K} \setminus \Omega) \cap B^\lambda| \\ &\geq |\Omega \cap B| + |(\mathcal{K} \setminus \Omega) \cap B| = |\Omega \cap B| = \gamma(x),\end{aligned}$$

that is x^λ is also a maximum point, i.e. $x^\lambda \in \mathbf{M}(\gamma)$ — a contradiction. □

As a consequence of this lemma, we obtain a result concerning the so-called *Fraenkel asymmetry* of $\mathcal{K}$:

$$\mathcal{A}(\mathcal{K}) = \min_{x \in \mathbb{R}^N} \frac{|\mathcal{K} \Delta B(x, r^*_{\mathcal{K}})|}{|\mathcal{K}|},$$

where Δ denotes the symmetric difference of the two sets and the radius $r^*_{\mathcal{K}}$ is determined by $|B(x, r^*_{\mathcal{K}})| = |\mathcal{K}|$. This is a measure of how a set is far from being spherically symmetric and was introduced in [4]; we refer the reader to [2] for a good account on $\mathcal{A}(\mathcal{K})$.

Corollary 3.15. *Let $\mathcal{K} \subset \mathbb{R}^N$ be a convex body. Then $\mathcal{A}(\mathcal{K})$ is attained for at least one ball centered at a point belonging to $\heartsuit(\mathcal{K})$.*

Proof. It is sufficient to observe that $|\mathcal{K} \Delta B(x, r^*_{\mathcal{K}})| = 2(|\mathcal{K}| - |\mathcal{K} \cap B(x, r^*_{\mathcal{K}})|)$, since $|B(x, r^*_{\mathcal{K}})| = |\mathcal{K}|$, and hence

$$\frac{|\mathcal{K} \Delta B(x, r^*_{\mathcal{K}})|}{|\mathcal{K}|} = 2\left(1 - \frac{\gamma(x)}{|\mathcal{K}|}\right).$$

Thus, $\mathcal{A}(\mathcal{K})$ is attained by points that maximize γ; hence, Lemma 3.14 provides the desired conclusion. □

Remark 3.16. Observe in particular that if $\mathcal{K}$ has N hyperplanes of symmetry, then an optimal ball can be placed at their intersection. However, in general, even under this stronger assumption, such optimal ball is not unique. For example, take the rectangle $Q_\varepsilon = [-\pi/4\varepsilon, \pi/4\varepsilon] \times [-\varepsilon, \varepsilon]$ with $0 < \varepsilon < \pi/4$; any unit ball centered at a point in the segment $(-\pi/4\varepsilon + 1, \pi/4\varepsilon - 1) \times \{0\}$ realizes the Fraenkel asymmetry $\mathcal{A}(Q_\varepsilon)$. Thus, in general *it is not true that all optimal balls are centered in the heart.*

Remark 3.17. The following problem in spectral optimization was considered in [5]: given a (convex) set $\mathcal{K} \subset \mathbb{R}^N$ and a radius $0 < r < r_{\mathcal{K}}$, find the ball $B(x_0, r) \subset \mathcal{K}$ which maximizes the quantity

$$\lambda_1(\mathcal{K} \setminus B(x, r))$$

as a function of x: here, $\lambda_1(\Omega)$ stands for the first Dirichlet-Laplacian eigenvalue of a set Ω. By considerations similar to the ones used in this section and remarks contained in [5, Theorem 2.1], it can be proved that $x_0 \in \heartsuit(\mathcal{K})$.

4. Estimating the volume of the heart

In this section, we begin an analysis of the following problem in shape optimization:

$$\text{maximize the ratio } \frac{|\heartsuit(\mathcal{K})|}{|\mathcal{K}|} \text{ among all convex bodies } \mathcal{K} \subset \mathbb{R}^N; \tag{4.1}$$

solving (4.1) would give an answer to question (iii) in the introduction. Since this ratio is scaling invariant, (4.1) is equivalent to the following problem:

$$\text{maximize the ratio } \frac{|\heartsuit(\mathcal{K})|}{|\mathcal{K}|} \text{ among all convex bodies } \mathcal{K}\subset[0,1]^N; \tag{4.2}$$

here, $[0,1]^N$ is the unit cube in $\mathbb{R}^N$.

We notice that the class of the competing sets in problem (4.2) is relatively compact in the topology induced by the *Hausdorff distance* (see [6, Chapter 2]) — the most natural topology when one deals with the constraint of convexity. This fact implies, in particular, that any maximizing sequence $\{\mathcal{K}_n\}_{n\in\mathbb{N}}\subset[0,1]^N$ of convex bodies converges – up to a subsequence – to a compact convex set $\mathcal{K}\subset[0,1]^N$.

However, there are two main obstructions to the existence of a maximizing set for (4.2): (a) in general, the limit set $\mathcal{K}$ may not be a convex body, i.e. $\mathcal{K}$ could have empty interior; in other words, maximizing sequences could "collapse" to a lower dimensional object; (b) it is not clear whether the shape functional $\mathcal{K}\mapsto|\heartsuit(\mathcal{K}))|$ is upper semicontinuous or not in the aforementioned topology.

The next example assures that the foreseen semicontinuity property *fails to be true* in general.

Example 4.1. Let $Q=[-2,2]\times[-1,1]$ and take the points

$$p^1_\varepsilon=(1,1+\varepsilon)\qquad\text{and}\qquad p^2_\varepsilon=(-2-\varepsilon,1/2),$$

and define Q_ε as the convex hull of $Q\cup\{p^1_\varepsilon,\,p^2_\varepsilon\}$. As ε vanishes, $\heartsuit(Q_\varepsilon)$ shrinks to the quadrangle having vertices

$$(0,0)\qquad(1,0)\qquad(1/2,1/2)\qquad\text{and}\qquad(0,1/2),$$

while clearly $\heartsuit(Q)=\{0\}$: indeed, observe that due to the presence of the new corners p^1_ε and p^2_ε, it is no more possible to use $\{x=0\}$ and $\{y=0\}$ as maximal axis of reflection in the directions $\mathbf{e}_1=(1,0)$ and $\mathbf{e}_2=(0,1)$. In particular, we get that

$$0=|\heartsuit(Q)|<\lim_{\varepsilon\to0^+}|\heartsuit(Q_\varepsilon)|.$$

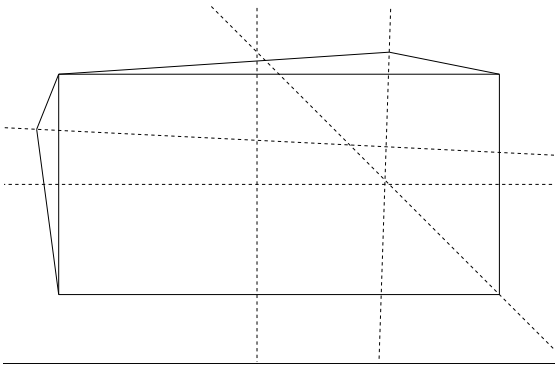

FIGURE 2. The heart of Q_ε.

These considerations show that the existence of a solution of (4.1) is not a trivial issue. Indeed, we are able to show that *an optimal shape does not exist* in the class of triangles. This is the content of the main result of this section.

Theorem 4.2. *It holds that*

$$\sup\left\{\frac{|\heartsuit(\mathcal{K})|}{|\mathcal{K}|} : \mathcal{K} \text{ is a triangle}\right\} = \frac{3}{8},$$

and the supremum is attained by a sequence of obtuse triangles.

The proof of Theorem 4.2 is based on the following lemma, in which we exactly determine $\heartsuit(\mathcal{K})$, when $\mathcal{K}$ is a triangle.

Lemma 4.3. *Let $\mathcal{K}$ be a triangle. Then the following assertions hold:*

(i) if $\mathcal{K}$ is acute, $\heartsuit(\mathcal{K})$ is contained in the triangle formed by the segments joining the midpoints of the sides of $\mathcal{K}$; also, $\heartsuit(\mathcal{K})$ equals the quadrangle $\mathcal{Q}$ formed by the bisectors of the smallest and largest angles and the axes of the shortest and longest sides of $\mathcal{K}$;

(ii) if $\mathcal{K}$ is obtuse, $\heartsuit(\mathcal{K})$ is contained in the parallelogram whose vertices are the midpoints mentioned in (i) and the vertex of the smallest angle in $\mathcal{K}$; also, $\heartsuit(\mathcal{K})$ equals the polygon $\mathcal{P}$ formed by the largest side of $\mathcal{K}$ and the bisectors and axes mentioned in (i); $\mathcal{P}$ may be either a pentagon or a quadrangle.

Proof. Observe that bisectors of angles and axes of sides are admissible axes of reflection. If $\mathcal{K}$ is acute, $C_{\mathcal{K}}$ and $I_{\mathcal{K}}$ fall in its interior and are the intersection of the axes and bisectors, respectively. If $\mathcal{K}$ is obtuse, $I_{\mathcal{K}}$ still falls in the interior of $\mathcal{K}$, while $C_{\mathcal{K}}$ is the midpoint of the largest side of $\mathcal{K}$ and is no longer the intersection of the axes. These remarks imply that $\heartsuit(\mathcal{K}) \subseteq \mathcal{Q}$ in case (i) and $\heartsuit(\mathcal{K}) \subseteq \mathcal{P}$ in case (ii); also $C_{\mathcal{K}}, I_{\mathcal{K}} \in \mathcal{Q} \cap \heartsuit(\mathcal{K})$ and $C_{\mathcal{K}}, I_{\mathcal{K}} \in \mathcal{P} \cap \heartsuit(\mathcal{K})$.

The segments specified in (i) are also admissible axes of reflection if $\mathcal{K}$ is acute; thus, the inclusion in the triangle mentioned in (i) easily follows. If $\mathcal{K}$ is obtuse, only the segment joining the midpoints of the smallest and intermediate side is an axis of reflection. However, we can still claim that $\heartsuit(\mathcal{K})$ is contained in the parallelogram mentioned in (ii), since $C_{\mathcal{K}}$ is now the midpoint of the largest side from which one of the axes is issued: thus, $\heartsuit(\mathcal{K})$ must stay below that axis.

Now, if $\heartsuit(\mathcal{K})$ were smaller than $\mathcal{Q}$ (or $\mathcal{P}$), then there would be an axis of reflection that cuts off one of the vertices of $\mathcal{Q}$ (or $\mathcal{P}$) different from $C_{\mathcal{K}}$ and $I_{\mathcal{K}}$ (that always belong to $\heartsuit(\mathcal{K})$). In any case, such an axis would violate the maximality that axes of sides and bisectors of angles enjoy with respect of reflections. □

We are now ready to prove Theorem 4.2.

Proof. First of all, thanks to the inclusion mentioned in Lemma 4.3, we get $|\heartsuit(\mathcal{K})| < 1/4\,|\mathcal{K}|$ when $\mathcal{K}$ is acute. Thus, we can restrict ourselves to the case of $\mathcal{K}$ obtuse.

Here, we refer to Figure 4. We observe that, by what we proved in Lemma 4.3, $\heartsuit(\mathcal{K})$ is always contained in the quadrangle $DEFG$ (when the angle in B is much larger than $\pi/2$, $DEFG$ and $\heartsuit(\mathcal{K})$ coincide), which is contained in the trapezoid $DELH$. Thus, $|\heartsuit(\mathcal{K})| \leq |DELH|$; hence it is enough to prove that, if the angle in B increases, $|DELH|$ increases and both ratios $|\heartsuit(\mathcal{K})|/|\mathcal{K}|$ and $|DELH|/|\mathcal{K}|$ tend to $3/8$.

We proceed to compute $|\heartsuit(\mathcal{K})|$, when the angle in B is large. We fix a base and a height of $\mathcal{K}$: as a base we choose the smallest side and we suppose it has length b; h will denote the length of its corresponding height. In this way, $|\mathcal{K}| = b\,h/2$.

In Figure 4, the lines through the points B and G, and C and L bisect the angles in B and C, respectively. The line through D and E is the only axis that contributes to form $\heartsuit(\mathcal{K})$, that equals the quadrangle $DEFG$; thus, $\heartsuit(\mathcal{K})$ is obtained as

$$\heartsuit(\mathcal{K}) = T_1 \setminus (T_2 \cup T_3),$$

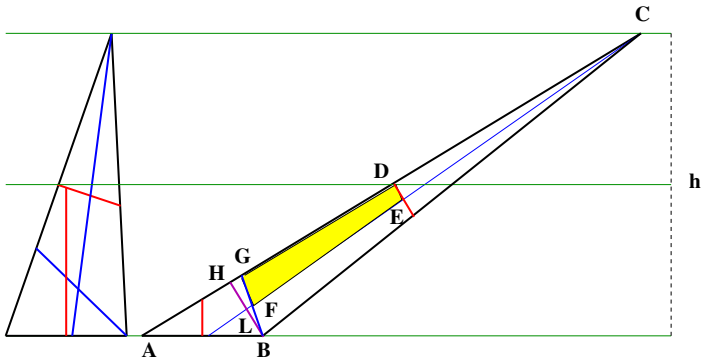

FIGURE 3. The heart of a triangle.

where the T_i's are triangles:

$$T_1 = CBG,\ T_2 = CBF,\ T_3 = CED.$$

We place the origin of cartesian axes in A and set $B = (b, 0)$; we also set $C = (t, h)$. Finally, we denote by α, β and γ the respective measures of the angles in A, B and C.

Trigonometric formulas imply that

$$\begin{aligned} |T_1| &= \frac{1}{2}\,[h^2 + (t-b)^2]\,\frac{\sin(\beta/2)\sin(\gamma)}{\sin(\beta/2+\gamma)}, \\ |T_2| &= \frac{1}{2}\,[h^2 + (t-b)^2]\,\frac{\sin(\beta/2)\sin(\gamma/2)}{\sin(\beta/2+\gamma/2)}, \\ |T_3| &= \frac{1}{8}\,[h^2 + t^2]\,\tan(\gamma/2), \end{aligned} \tag{4.3}$$

where the angles β and γ are related by the *theorem of sines:*

$$\frac{h}{\sqrt{h^2+t^2}\,\sqrt{h^2+(t-b)^2}} = \frac{\sin(\beta)}{\sqrt{h^2+t^2}} = \frac{\sin(\gamma)}{b}.$$

The area of $DELH$ is readily computed as

$$|DELH| = \frac{1}{2}\,\frac{b^2\,h^2}{(h^2+t^2)\,\tan(\gamma/2)} - \frac{1}{8}\,(h^2+t^2)\,\tan(\gamma/2). \tag{4.4}$$

Now, observe that this quantity increases with t, since it is the composition of two decreasing functions: $s \mapsto b^2\,h^2/(2s) - s/8$ and $t \mapsto (h^2+t^2)\,\tan(\gamma/2)$.

As $t \to \infty$, $|\mathcal{K}|$ does not change, the angle γ vanishes and the angle β tends to π; moreover, we have that

$$t\,\sin(\beta) \to h, \quad t^2\,\sin(\gamma) \to bh \quad \text{as} \quad t \to \infty.$$

Formulas (4.3) then yield:

$$|T_1| \to \frac{1}{2}\,bh = |\mathcal{K}|, \quad |T_2| \to \frac{1}{4}\,bh = \frac{1}{2}\,|\mathcal{K}|, \quad |T_3| \to \frac{1}{16}\,bh = \frac{1}{8}\,|\mathcal{K}|.$$

Thus, since $|\heartsuit(\mathcal{K})| = |T_1| - |T_2| - |T_3|$, we have that $|\heartsuit(\mathcal{K})| \to \frac{3}{8}\,|\mathcal{K}|$; by (4.4), $|DELH| \to \frac{3}{8}\,|\mathcal{K}|$ as well.

The proof is complete. □

Remark 4.4. Thus, Theorem 4.2 sheds some light on problem (4.1). In fact, observe that the maximizing sequence, once properly re-scaled, gives a maximizing sequence for the equivalent problem (4.2), that precisely collapses to a one-dimensional object.

Numerical evidence based on the algorithm developed in [1] suggests that, the more $\mathcal{K}$ is *round,* the more $\heartsuit(\mathcal{K})$ is small compared to $\mathcal{K}$. We conjecture that

$$\sup\left\{\frac{|\heartsuit(\mathcal{K})|}{|\mathcal{K}|} : \mathcal{K}\subset\mathbb{R}^2 \text{ is a convex body}\right\} = \frac{3}{8},$$

and the supremum is realized by a sequence of obtuse triangles.

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