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# The Stretch Factor of $L_1$ - and $L_\infty$ -Delaunay Triangulations

Nicolas Bonichon  
LaBRI - INRIA Bordeaux Sud-Ouest  
bonichon@labri.fr

Cyril Gavoille\*  
LaBRI - University of Bordeaux  
gavoille@labri.fr

Nicolas Hanusse  
LaBRI - CNRS  
hanusse@labri.fr

Ljubomir Perković†  
DePaul University, Chicago  
lperkovic@cs.depaul.edu

## Abstract

In this paper we determine the stretch factor of the  $L_1$ -Delaunay and  $L_\infty$ -Delaunay triangulations, and we show that this stretch is  $\sqrt{4 + 2\sqrt{2}} \approx 2.61$ . Between any two points  $x, y$  of such triangulations, we construct a path whose length is no more than  $\sqrt{4 + 2\sqrt{2}}$  times the Euclidean distance between  $x$  and  $y$ , and this bound is best possible. This definitively improves the 25-year old bound of  $\sqrt{10}$  by Chew (SoCG '86).

To the best of our knowledge, this is the first time the stretch factor of the well-studied  $L_p$ -Delaunay triangulations, for any real  $p \geq 1$ , is determined exactly.

**Keywords:** Delaunay triangulations,  $L_1$ -metric,  $L_\infty$ -metric, stretch factor

## 1 Introduction

Given a set of points  $P$  on the plane, the Delaunay triangulation for  $P$  is a spanning subgraph of the Euclidean graph on  $P$  that is the dual of the Voronoï diagram of  $P$ . The Delaunay triangulation is a fundamental structure with many applications in computational geometry and other areas of Computer Science. In some applications (including on-line routing [BM04]), the triangulation is used as a spanner, defined as a spanning subgraph in which the distance between any pair of points is no more than a constant multiplicative ratio of the Euclidean distance between the points. The constant ratio is typically referred to as the stretch factor of the spanner. While Delaunay triangulations have been studied extensively, obtaining a tight bound on its stretch factor has been elusive even after decades of attempts.

In the mid-1980s, it was not known whether Delaunay triangulations were spanners at all. In order to gain an understanding of the spanning properties of Delaunay triangulations, Chew considered related, “easier” structures. In his seminal 1986 paper [Che86], he proved that an  $L_1$ -Delaunay triangulation — the dual of the Voronoï diagram of  $P$  based on the  $L_1$ -metric rather than the  $L_2$ -metric — has a stretch factor bounded by  $\sqrt{10}$ . Chew then continued on

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Paper	Graph	Stretch factor
[DFS87]	$L_2$ -Delaunay	$\pi(1 + \sqrt{5})/2 \approx 5.08$
[KG92]	$L_2$ -Delaunay	$4\pi/(3\sqrt{3}) \approx 2.41$
[Xia11]	$L_2$ -Delaunay	1.998
[Che89]	TD-Delaunay	<b>2</b>
[Che86]	$L_1, L_\infty$ -Delaunay	$\sqrt{10} \approx 3.16$
<b>[this paper]</b>	$L_1, L_\infty$ -Delaunay	$\sqrt{4 + 2\sqrt{2}} \approx \mathbf{2.61}$

Table 1: Key stretch factor upper bounds (optimal values are bold).

and showed that the a TD-Delaunay triangulation — the dual of a Voronoï diagram defined using a *Triangular Distance*, a distance function not based on a circle ( $L_2$ -metric) or a square ( $L_1$ -metric) but an equilateral triangle — has a stretch factor of 2 [Che89].

Finally, Dobkin et al. [DFS87] succeeded in showing that the (classical,  $L_2$ -metric) Delaunay triangulation of  $P$  is a spanner as well. The bound on the stretch factor they obtained was subsequently improved by Keil and Gutwin [KG92] as shown in Table 1. The bound by Keil and Gutwin stood unchallenged for many years until very recently when Xia improved the bound to below 2 [Xia11].

While progress has been made, none of the techniques developed so far lead to a tight bound on the spanning ratio. There has been some progress recently on the lower bound side. The trivial lower bound of  $\pi/2 \approx 1.5707$  has recently been improved to 1.5846 [BDL<sup>+</sup>11] and then to 1.5932 [XZ11].

While much effort has been made on understanding the stretch factor of Delaunay triangulations, little has been done on the  $L_p$ -Delaunay triangulations for  $p \neq 2$ . Lee and Wong [LW80] show that  $L_1, L_\infty$ -Delaunay triangulations have applications in scheduling problems for 2-dimensional storage, and how to construct, for all real  $p \geq 1$ , Voronoï diagrams in the  $L_p$ -metric in  $O(n \log n)$  time [Lee80]. Delaunay triangulations based on arbitrary convex distance functions have been studied in [BCCS08], which shows that such geometric graphs are indeed plane graphs and spanners whose stretch factor depends only on the shape of the convex body. However, due to the general approach, the bounds on the stretch factor remain loose. For instance the bounds they obtain for  $L_2$ -Delaunay triangulations are  $> 24$ .

The general picture is that, in spite of much effort, with the exception of the triangular distance the exact value of the stretch factor of Delaunay triangulations based on any convex function is unknown. In particular, the stretch factor of  $L_p$ -Delaunay triangulations is unknown for each  $p \geq 1$ .

**Our contributions.** We show that the exact stretch factor of  $L_1$ -Delaunay triangulations and  $L_\infty$ -Delaunay triangulations is  $\sqrt{4 + 2\sqrt{2}} \approx 2.61$ , ultimately improving the 3.16 bound of Chew [Che86].

Technically, we use rectangular coordinates to prove the upper bound. We show that the distance in the triangulation between any source-destination pair of points lying on the

border of a horizontal rectangle of length  $x$  and of depth  $y \leq x$  is no more than  $(1 + \sqrt{2})x + y$ . The stretch factor bound then simply follows. In our proof, we construct the route from the source to the destination by maintaining two possible short paths, until we reach some special point (called *inductive point*) where we can apply our main inductive hypothesis.

Despite the technical aspect of our contribution, we believe that our proof techniques may give insights into determining the stretch factor of other convex distance based Delaunay triangulations. For example, let  $P_k$  denote the convex distance function defined by a regular  $k$ -gon. We observe that the stretch factor of  $P_k$ -Delaunay triangulations is known for  $k = 3, 4$  since  $P_3$  is the triangular distance function of [Che89], and  $P_4$  is nothing else than the  $L_\infty$ -metric. Determining the stretch factor of  $P_k$ -Delaunay triangulations for larger  $k$  would undoubtedly be an important step towards understanding the stretch factor of classical Delaunay triangulations.

## 2 Preliminaries

Given a set  $P$  of points in the two-dimensional Euclidean space, the Euclidean graph  $\mathcal{E}$  is the complete weighted graph embedded in the plane whose nodes are identified with the points. We assume a Cartesian coordinate system is associated with the Euclidean space and thus every point can be specified with  $x$  and  $y$  coordinates. For every pair of nodes  $u$  and  $w$ , the edge  $(u, w)$  represents the segment  $[uw]$  and its weight is the edge length defined in Euclidean distance:  $d_2(u, w) = \sqrt{d_x(u, w)^2 + d_y(u, w)^2}$  where  $d_x(u, w)$  (resp.  $d_y(u, w)$ ) is the difference between the  $x$  (resp.  $y$ ) coordinates of  $u$  and  $w$ .

We say that a subgraph  $H$  of a graph  $G$  is a  $t$ -spanner of  $G$  if for any pair of vertices  $u, v$  of  $G$ , the distance between  $u$  and  $v$  in  $H$  is at most  $t$  times the distance between  $u$  and  $v$  in  $G$ ; the constant  $t$  is referred to as the *stretch factor* of  $H$  (with respect to  $G$ ).  $H$  is a  $t$ -spanner (or spanner for some  $t$  constant) if it is a  $t$ -spanner of  $\mathcal{E}$ .

In our paper, we deal with the construction of spanners based on Delaunay triangulations. As we saw in the introduction, the  $L_1$ -Delaunay triangulation is the dual of the Voronoi diagram based on the  $L_1$ -metric  $d_1(u, w) = d_x(u, w) + d_y(u, w)$ . A property of the  $L_1$ -Delaunay triangulations, actually shared by all  $L_p$ -Delaunay triangulations, is that all their triangles can be defined in terms of empty circumscribed convex bodies (squares for  $L_1$  or  $L_\infty$  and circles for  $L_2$ ). More precisely, let a *square* in the plane be a square whose sides are parallel to the  $x$  and  $y$  axis and let a *tipped square* be a square tipped at  $45^\circ$ . For every pair of points  $u, v \in P$ ,  $(u, v)$  is an edge in the  $L_1$ -Delaunay triangulation of  $P$  iff there is a tipped square that has  $u$  and  $v$  on its boundary and contains no point of  $P$  in its interior (cf. [Che89]).

If a *square* with sides parallel to the  $x$  and  $y$  axes, rather than a tipped square, is used in this definition then a different triangulation is defined; it corresponds to the dual of the Voronoi diagram based on the  $L_\infty$ -metric  $d_\infty(u, w) = \max\{d_x(u, w), d_y(u, w)\}$ . We refer to this triangulation as the  $L_\infty$ -Delaunay triangulation. This triangulation is nothing more than the  $L_1$ -Delaunay triangulation of the set of points  $P$  after rotating all the points by  $45^\circ$  around the origin. Therefore Chew's bound of  $\sqrt{10}$  on the stretch factor of the  $L_1$ -Delaunay triangulation ([Che86]) applies to  $L_\infty$ -Delaunay triangulations as well. In the remainder of this paper, we will be referring to  $L_\infty$ -Delaunay (rather than  $L_1$ ) triangulations because we will be (mostly) using the  $L_\infty$ -metric and squares, rather than tipped squares.

One issue with Delaunay triangulations is that there might not be a unique triangulation of a given set of points  $P$ . To insure uniqueness and keep our arguments simple, we make the usual assumption that the points in  $P$  are in *general position*, which for us means that no four points lie on the boundary of a square and no two points share the same abscissa or the same ordinate.

We end this section by giving a lower bound on the stretch factor of  $L_\infty$ -Delaunay triangulations.

**Proposition 1** *For every  $\varepsilon > 0$ , there exists a set of points  $P$  in the plane such that the  $L_\infty$ -Delaunay triangulation on  $P$  has stretch factor at least*

$$\sqrt{4 + 2\sqrt{2}} - \varepsilon .$$

This lower bound applies, of course, to  $L_1$ -Delaunay triangulations as well. The proof of this proposition relies on the example of Figure 1.

**Proof.** Given  $\delta > 0$ , we define the set of points  $P$  as follows. Let point  $a$  be the origin and let points  $b$ ,  $c_1$ , and  $c_2$  have coordinates  $(1, \sqrt{2} - 1)$ ,  $(\delta, \sqrt{2} - 2\delta)$ , and  $(1 - \delta, 1 - 2\delta)$ , respectively. Additional  $k = \frac{\sqrt{2}-2\delta}{\delta} - 1$  points are placed on line segment  $[ac_1]$  and another  $k$  on line segment  $[c_2b]$  in such a way that the difference in  $y$  coordinates between successive points on a segment is  $\delta$ , as shown in Figures 1. (W.l.o.g. assume that  $\frac{\sqrt{2}}{\delta}$  and thus  $k$  is an integer so that this can be done.) Let  $a = p_0, p_1, p_2, p_3, \dots, p_k, p_{k+1} = c_1$  be the labels, in order as they appear when moving from  $a$  to  $c_1$ , of the points on segment  $[ac_1]$  and let  $c_2 = q_0, q_1, q_2, q_3, \dots, q_{k+1} = b$  be the labels, in order as they appear when moving from  $c_2$  to  $b$ , of the points on segment  $[c_2b]$ , as illustrated in Figure 1.

Consider the square  $S_1$  of side length  $1 - \delta$  and having  $a$  and  $p_1$  on its west (left) and north sides, respectively (see Figure 1b)). Since  $d_\infty(a, c_2) = d_x(a, c_2) = 1 - \delta$  and  $d_\infty(p_1, c_2) = d_y(p_1, c_2) = 1 - \delta$ , point  $c_2$  is exactly the southeast vertex of square  $S_1$ . By symmetry, it

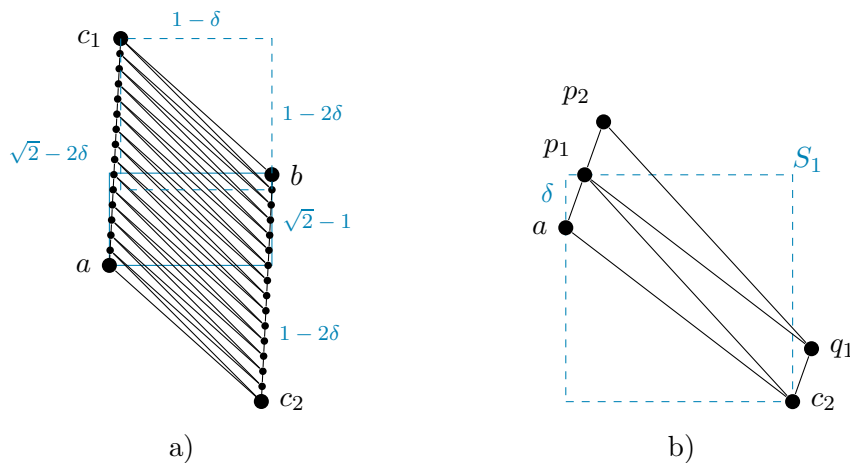


Figure 1: a) An  $L_\infty$ -Delaunay triangulation with stretch factor arbitrarily close to  $\sqrt{4 + 2\sqrt{2}}$ . b) A closer look at the first few faces of this triangulation.

follows that for every  $i = 0, 1, 2, \dots, k$ , if  $S_i$  is the square of side length  $1 - \delta$  with  $p_i$  and  $p_{i+1}$  on its west and north sides, then point  $q_i$  is exactly the southeast vertex of  $S_i$ . This means that all points  $q_j$  with  $j \neq i$  as well as all points  $p_j$  with  $j \neq i, i + 1$  must lie outside  $S_i$ . Therefore, for every  $i = 0, 1, 2, \dots, k$ , points  $p_i$ ,  $p_{i+1}$ , and  $q_i$  define a triangle in the  $L_\infty$ -Delaunay triangulation  $T$  on  $P$ . A similar argument shows that the path  $q_0, q_1, \dots, q_{k+1}$  is in triangulation  $T$  as well. The triangulation  $T$  is illustrated in Figure 1a).

Having defined the set of points  $P$  and described its  $L_\infty$ -Delaunay triangulation  $T$ , we now analyze the stretch factor of  $T$ . A shortest path from  $a$  to  $b$  in  $T$  is, for example,  $a, p_1, p_2, \dots, p_k, c_1, b$ . The length of this path is

$$\begin{aligned} d_2(a, c_1) + d_2(c_1, b) &= \sqrt{d_x(a, c_1)^2 + d_y(a, c_1)^2} + \sqrt{d_x(c_1, b)^2 + d_y(c_1, b)^2} \\ &= \sqrt{(\sqrt{2} - \delta)^2 + \delta^2} + \sqrt{(1 - \delta)^2 + (1 - 2\delta)^2} \end{aligned}$$

which tends to  $2\sqrt{2}$  as  $\delta$  tends to 0. The Euclidean distance between  $a$  and  $b$  is:

$$d_2(a, b) = \sqrt{1^2 + (\sqrt{2} - 1)^2} = \sqrt{4 - 2\sqrt{2}}.$$

Therefore, it is possible to construct a  $L_\infty$ -Delaunay triangulation whose stretch factor is arbitrarily close to:

$$\frac{2\sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} = \sqrt{4 + 2\sqrt{2}}.$$

□

### 3 Main result

In this section we obtain a tight upper bound on the stretch factor of an  $L_\infty$ -Delaunay triangulation. It follows from this key theorem:

**Theorem 2** *Let  $T$  be the  $L_\infty$ -Delaunay triangulation on a set of points  $P$  in the plane and let  $a$  and  $b$  be any two points of  $P$ . If  $x = d_\infty(a, b) = \max\{d_x(a, b), d_y(a, b)\}$  and  $y = \min\{d_x(a, b), d_y(a, b)\}$  then*

$$d_T(a, b) \leq (1 + \sqrt{2})x + y$$

where  $d_T(a, b)$  denotes the distance between  $a$  and  $b$  in triangulation  $T$ .

**Corollary 3** *The stretch factor of the  $L_1$ - and the  $L_\infty$ -Delaunay triangulation on a set of points  $P$  is at most*

$$\sqrt{4 + 2\sqrt{2}} \approx 2.6131259\dots$$

**Proof.** By Theorem 2, an upper-bound of the stretch factor of an  $L_\infty$ -Delaunay triangulation is the maximum of the function

$$\frac{(1 + \sqrt{2})x + y}{\sqrt{x^2 + y^2}}$$

over values  $x$  and  $y$  such that  $0 < y \leq x$ . The maximum is reached when  $x$  and  $y$  satisfy  $y/x = 1 + \sqrt{2}$ , and the maximum is equal to  $\sqrt{1 + (1 + \sqrt{2})^2} = \sqrt{4 + 2\sqrt{2}}$ . As  $L_1$ - and  $L_\infty$ -Delaunay triangulations have the same stretch factor, this result also holds for  $L_1$ -Delaunay triangulations.  $\square$

To prove Theorem 2, we will construct a bounded length path in  $T$  between two arbitrary points  $a$  and  $b$  of  $P$ . To simplify the notation and the discussion, we assume that point  $a$  has coordinates  $(0, 0)$  and point  $b$  has coordinates  $(x, y)$  with  $0 < y \leq x$ . The segment  $[ab]$  divides the Euclidean plane into two half-planes; a point in the same half-plane as point  $(0, 1)$  is said to be *above* segment  $[ab]$ , otherwise it is *below*. Let  $T_1, T_2, T_3, \dots, T_k$  be the sequence of triangles of triangulation  $T$  that line segment  $[ab]$  intersects when moving from  $a$  to  $b$ . Let  $h_1$  and  $l_1$  be the nodes of  $T_1$  other than  $a$ , with  $h_1$  lying above segment  $[ab]$  and  $l_1$  lying below. Every triangle  $T_i$ , for  $1 < i < k$ , intersects line segment  $[ab]$  twice; let  $h_i$  and  $l_i$  be the endpoints of the edge of  $T_i$  that intersects segment  $[ab]$  last, when moving on segment  $[ab]$  from  $a$  to  $b$ , with  $h_i$  being above and  $l_i$  being below segment  $[ab]$ . Note that either  $h_i = h_{i-1}$  and  $T_i = \triangle(h_i, l_i, l_{i-1})$  or  $l_i = l_{i-1}$  and  $T_i = \triangle(h_{i-1}, h_i, l_i)$ , for  $1 < i < k$ . We also set  $h_0 = l_0 = a$ ,  $h_k = b$ , and  $l_k = l_{k-1}$ . For  $1 \leq i \leq k$ , we define  $S_i$  to be the square whose sides pass through the three vertices of  $T_i$  (see Figure 2); since  $T$  is an  $L_\infty$ -Delaunay triangulation, the interior of  $S_i$  is devoid of points of  $P$ . We will refer to the sides of the square using the notation: N (north), E (east), S (south), and W (west). We will also use this notation to describe the position of an edge connecting two points lying on two sides a square: for example, a WN edge connects a point on the west and a point on the N side. We will say that an edge is *gentle* if the line segment corresponding to it in the graph embedding has a slope within  $[-1, 1]$ ; otherwise we will say that it is *steep*.

We will prove Theorem 2 by induction on the distance, using the  $L_\infty$ -metric, between  $a$  and  $b$ . Let  $R(a, b)$  be the rectangle with sides parallel to the  $x$  and  $y$  axes and with vertices at points  $a$  and  $b$ . If there is a point of  $P$  inside  $R(a, b)$ , we will easily apply induction. The case when  $R(a, b)$  does not contain points of  $P$  — and in particular the points  $h_i$  and  $l_i$  for  $0 < i < k$  — is more difficult and we need to develop tools to handle it. The following Lemma describes the structure of the triangles  $T_1, \dots, T_k$  when  $R(a, b)$  is empty. We need some additional terminology first though: we say that a point  $u$  is *above* (resp. *below*)  $R(a, b)$

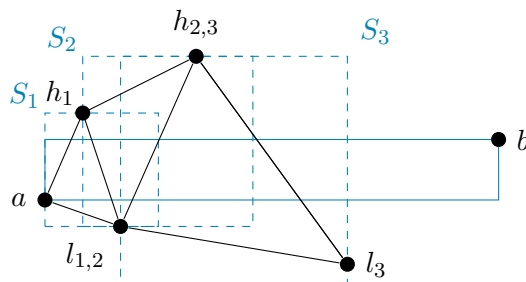


Figure 2: Triangles  $T_1$  (with points  $a, h_1, l_1$ ),  $T_2$  (with points  $h_1, h_2, l_2$ ), and  $T_3$  (with points  $l_2, h_3, l_3$ ) and associated squares  $S_1, S_2, S_3$ . When traveling from  $a$  to  $b$  along segment  $[a, b]$ , the edge that is hit when leaving  $T_i$  is  $(h_i, l_i)$ .

if  $0 < x_u < x$  and  $y_u > y$  (resp.  $y_u < 0$ ).

**Lemma 4** *If  $(a, b) \notin T$  and no point of  $P$  lies inside rectangle  $R(a, b)$ , then point  $a$  lies on the W side of square  $S_1$ , point  $b$  lies on the E side of square  $S_k$ , points  $h_1, \dots, h_k$  all lie above  $R(a, b)$ , and points  $l_1, \dots, l_k$  all lie below  $R(a, b)$ . Furthermore, for any  $i$  such that  $1 < i < k$ :*

- a) *Either  $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$ , points  $h_{i-1}$ ,  $h_i$ , and  $l_{i-1} = l_i$  lie on the sides of  $S_i$  in clockwise order, and  $(h_{i-1}, h_i)$  is a WN, WE, or NE edge in  $S_i$*
- b) *Or  $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$ , points  $h_{i-1} = h_i$ ,  $l_i$ , and  $l_{i-1}$  lie on the sides of  $S_i$  in clockwise order, and  $(l_{i-1}, l_i)$  is a WS, WE, or SE edge in  $S_i$ .*

These properties are illustrated in Figure 2.

**Proof.** Since points of  $P$  are in general position, points  $a$ ,  $h_1$ , and  $l_1$  must lie on 3 different sides of  $S_1$ . Because segment  $[ab]$  intersects the interior of  $S_1$  and since  $a$  is the origin and  $b$  is in cone 0 of  $a$ ,  $a$  can only lie on the W or S side of  $S_1$ . If  $a$  lies on the S side then  $l_1 \neq b$  would have to lie inside  $R(a, b)$ , which is a contradiction. Therefore  $a$  lies on the W side of  $S_1$  and, similarly,  $b$  lies on the E side of  $S_k$ .

Since points  $h_i$  ( $0 < i < k$ ) are above segment  $[ab]$  and points  $l_i$  ( $0 < i < k$ ) are below segment  $[ab]$ , and because all squares  $S_i$  ( $0 < i < k$ ) intersect  $[ab]$ , points  $h_1, \dots, h_k$  all lie above  $R(a, b)$ , and points  $l_1, \dots, l_k$  all lie below  $R(a, b)$ .

The three vertices of  $T_i$  can be either  $h_i = h_{i-1}$ ,  $l_{i-1}$ , and  $l_i$  or  $h_{i-1}$ ,  $h_i$ , and  $l_{i-1} = l_i$ . Because points of  $T$  are in general position, the three vertices of  $T_i$  must appear on three different sides of  $S_i$ . Finally, because  $h_{i-1}$  and  $h_i$  are above  $R(a, b)$ , they cannot lie on the S side of  $S_i$ , and because  $l_{i-1}$  and  $l_i$  are below  $R(a, b)$ , they cannot lie on the N side of  $S_i$ .

If  $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$ , points  $h_{i-1}$ ,  $h_i$ ,  $l_i$  must lie on the sides of  $S_i$  in clockwise order. The only placements of points  $h_{i-1}$  and  $h_i$  on the sides of  $S_i$  that satisfy all these constraints are as described in a). If  $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$ , points  $h_i$ ,  $l_i$ ,  $l_{i-1}$  must lie on the sides of  $S_i$  in clockwise order. Part b) lists the placements of points  $l_{i-1}$  and  $l_i$  that satisfy the constraints.  $\square$

In the following definition, we define the points on which induction can be applied in the proof of Theorem 2.

**Definition 5** *Let  $R(a, b)$  be empty. Square  $S_j$  is inductive if edge  $(l_j, h_j)$  is gentle. The point  $c = h_j$  or  $c = l_j$  with the larger abscissa is the inductive point of inductive square  $S_j$ .*

The following lemma will be the key ingredient of our inductive proof of Theorem 2.

**Lemma 6** *Assume that  $R(a, b)$  is empty. If no square  $S_1, \dots, S_k$  is inductive then*

$$d_T(a, b) \leq (1 + \sqrt{2})x + y .$$

*Otherwise let  $S_j$  be the first inductive square in the sequence  $S_1, S_2, \dots, S_k$ . If  $h_j$  is the inductive point of  $S_j$  then*

$$d_T(a, h_j) + (y_{h_j} - y) \leq (1 + \sqrt{2})x_{h_j} .$$



If  $l_j$  is the inductive point of  $S_j$  then

$$d_T(a, l_j) - y_{l_j} \leq (1 + \sqrt{2})x_{l_j} .$$

Given an inductive point  $c$ , we can use Lemma 6 to bound  $d_T(a, b)$  and then apply induction to bound  $d_T(b, c)$ , *but only if* the position of point  $c$  relative to the position of point  $b$  is *good*, i.e., if  $x - x_c \geq |y - y_c|$ . If that is not the case, we will use the following Lemma:

**Lemma 7** *Let  $R(a, b)$  be empty and let the coordinates of point  $c = h_i$  or  $c = l_i$  satisfy  $0 < x - x_c < |y - y_c|$ .*

a) *If  $c = h_i$ , and thus  $0 < x - x_{h_i} < y_{h_i} - y$ , then there exists  $j$ , with  $i < j \leq k$  such that all edges in path  $h_i, h_{i+1}, h_{i+2}, \dots, h_j$  are NE edges in their respective squares and  $x - x_{h_j} \geq y_{h_j} - y \geq 0$ .*

b) *If  $c = l_i$ , and thus  $0 < x - x_{l_i} < y - y_{l_i}$ , then there exists  $j$ , with  $i < j \leq k$  such that all edges in path  $l_i, l_{i+1}, l_{i+2}, \dots, l_j$  are SE edges and  $x - x_{l_j} \geq y - y_{l_j} \geq 0$ .*

**Proof.** We only prove the case  $c = h_j$  as the case  $c = l_i$  follows using a symmetric argument.

We construct the path  $h_i, h_{i+1}, h_{i+2}, \dots, h_j$  iteratively. If  $h_i = h_{i+1}$ , we just continue building the path from  $h_{i+1}$ . Otherwise,  $(h_i, h_{i+1})$  is an edge of  $T_{i+1}$  which, by Lemma 4, must be a WN, WE, or NE edge in square  $S_{i+1}$ . Since the S side of square  $S_{i+1}$  is below  $R(a, b)$  and because  $x - x_{h_i} < y_{h_i} - y$ , point  $h_i$  cannot be on the W side of  $S_{i+1}$  (otherwise  $b$  would be inside square  $S_{i+1}$ ). Thus  $(h_i, h_{i+1})$  is a NE edge. If  $x - x_{h_{i+1}} \geq y_{h_{i+1}} - y$  we stop, otherwise we continue the path construction from  $h_{i+1}$ .  $\square$

We can now prove the main theorem.

**Proof of Theorem 2.** The proof is by induction on the distance, using the  $L_\infty$ -metric, between points of  $P$  (since  $P$  is finite there is only a finite number of distances to consider).

Let  $a$  and  $b$  be the two points of  $P$  that are the closest points, using the  $L_\infty$ -metric. We assume w.l.o.g. that  $a$  has coordinates  $(0, 0)$  and  $b$  has coordinates  $(x, y)$  with  $0 < y \leq x$ . Since  $a$  and  $b$  are the closest points using the  $L_\infty$ -metric, the largest square having  $a$  as a southwest vertex and containing no points of  $P$  in its interior, which we call  $S_a$  must have  $b$  on its E side. Therefore  $(a, b)$  is an edge in  $T$  and  $d_T(a, b) = d_2(a, b) \leq x + y \leq (1 + \sqrt{2})x + y$ .

For the induction step, we again assume, w.l.o.g., that  $a$  has coordinates  $(0, 0)$  and  $b$  has coordinates  $(x, y)$  with  $0 < y \leq x$ .

**Case 1:  $R(a, b)$  is not empty.**

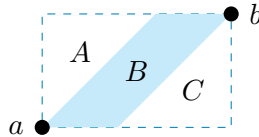


Figure 3: Partition of  $R(a, b)$  into three regions in Case 1 of the proof of Theorem 2.

We first consider the case when there is at least one point of  $P$  lying inside rectangle  $R(a, b)$ . If there is a point  $c$  inside  $R(a, b)$  such that  $y_c \leq x_c$  and  $y - y_c \leq x - x_c$  (i.e.,  $c$  lies in the region  $B$  shown in Figure 3 then we can apply induction to get  $d_T(a, c) \leq (1 + \sqrt{2})x_c + y_c$  and  $d_T(c, b) \leq (1 + \sqrt{2})(x - x_c) + y - y_c$  and use these to obtain the desired bound for  $d_T(a, b)$ .

We now assume that there is no point inside region  $B$ . If there is still a point in  $R(a, b)$  then there must be one that is on the border of  $S_a$ , the square we defined in the basis step, or  $S_b$ , defined as the largest square having  $b$  as a northeast vertex and containing no points of  $P$  in its interior. W.l.o.g., we assume the former and thus there is an edge  $(a, c) \in T$  such that either  $y_c > x_c$  (i.e.,  $c$  is inside region  $A$  shown in Figure 3 or  $y - y_c > x - x_c$  (i.e.,  $c$  is inside region  $C$ ). Either way,  $d_T(a, c) = d_2(a, c) \leq x_c + y_c$ . If  $c$  is in region  $A$ , since  $x - x_c \geq y - y_c$ , by induction we also have that  $d_T(c, b) \leq (1 + \sqrt{2})(x - x_c) + (y - y_c)$ . Then

$$\begin{aligned} d_T(a, b) &\leq d_T(a, c) + d_T(c, b) \\ &\leq x_c + y_c + (1 + \sqrt{2})(x - x_c) + (y - y_c) \leq (1 + \sqrt{2})x + y \end{aligned}$$

In the second case, since  $x - x_c < y - y_c$ , by induction we have that  $d_T(c, b) \leq (1 + \sqrt{2})(y - y_c) + (x - x_c)$ . Then, because  $y < x$ ,

$$\begin{aligned} d_T(a, b) &\leq d_T(a, c) + d_T(c, b) \\ &\leq x_c + y_c + (1 + \sqrt{2})(y - y_c) + (x - x_c) \leq (1 + \sqrt{2})x + y \end{aligned}$$

**Case 2: The interior of  $R(a, b)$  is empty.**

If no square  $S_1, S_2, \dots, S_k$  is inductive,  $d_T(a, b) \leq (1 + \sqrt{2})x + y$  by Lemma 6. Otherwise, let  $S_i$  be the first inductive square in the sequence and suppose that  $h_i$  is the inductive point of  $S_i$ . By Lemma 7, there is a  $j$ ,  $i \leq j \leq k$ , such that  $h_i, h_{i+1}, h_{i+2}, \dots, h_j$  is a path in  $T$  of length at most  $(x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j})$  and such that  $x - x_{h_j} \geq y_{h_j} - y \geq 0$ . Since  $h_j$  is closer to  $b$ , using the  $L_\infty$ -metric, than  $a$  is, we can apply induction to bound  $d_T(h_j, b)$ . Putting all this together with Lemma 6, we get:

$$\begin{aligned} d_T(a, b) &\leq d_T(a, h_i) + d_T(h_i, h_j) + d_T(h_j, b) \\ &\leq (1 + \sqrt{2})x_{h_i} - (y_{h_i} - y) + (x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j}) + (1 + \sqrt{2})(x - x_{h_j}) + (y_{h_j} - y) \\ &\leq (1 + \sqrt{2})x . \end{aligned}$$

If  $l_i$  is the inductive point of  $S_i$ , by Lemma 7 there is a  $j$ ,  $i \leq j \leq k$ , such that  $l_i, l_{i+1}, l_{i+2}, \dots, l_j$  is a path in  $T$  of length at most  $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$  and such that  $x - x_{h_j} \geq y - y_{h_j} \geq 0$ . Because the position of  $j$  with respect to  $b$  is good and since  $l_j$  is closer to  $b$ , using the  $L_\infty$ -metric, than  $a$  is, we can apply induction to bound  $d_T(l_j, b)$ . Putting all this together with Lemma 6, we get:

$$\begin{aligned} d_T(a, b) &\leq d_T(a, l_i) + d_T(l_i, l_j) + d_T(l_j, b) \\ &\leq (1 + \sqrt{2})x_{l_i} + y_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_j} - y_{l_i}) + (1 + \sqrt{2})(x - x_{l_j}) + (y - y_{l_j}) \\ &\leq (1 + \sqrt{2})x + y . \end{aligned}$$

□

What remains to be done is to prove Lemma 6. To do this, we need to develop some further terminology and tools. Let  $x_i$ , for  $1 \leq i \leq k$ , be the horizontal distance between point  $a$  and the E side of  $S_i$ , respectively. We also set  $x_0 = 0$ .

**Definition 8** A square  $S_i$  has potential if

$$d_T(a, h_i) + d_T(a, l_i) + d_{S_i}(h_i, l_i) \leq 4x_i$$

where  $d_{S_i}(h_i, l_i)$  is the Euclidean distance when moving from  $h_i$  to  $l_i$  along the sides of  $S_i$ , clockwise.

**Lemma 9** If  $R(a, b)$  is empty then  $S_1$  has potential. Furthermore, for any  $1 \leq i < k$ , if  $S_i$  has potential but is not inductive then  $S_{i+1}$  has potential.

**Proof.** If  $R(a, b)$  is empty then, by Lemma 4,  $a$  lies on the W side of  $S_1$  and  $x_1$  is the side length of square  $S_1$ . Also,  $h_1$  lies on the N or E side of  $S_1$ , and  $l_1$  lies on the S or E side of  $S_1$ . Then  $d_T(a, h_1) + d_T(a, l_1) + d_{S_1}(h_1, l_1)$  is bounded by the perimeter of square  $S_1$  which is  $4x_1$ .

Now assume that  $S_i$ , for  $1 \leq i < k$ , has potential but is not inductive. Squares  $S_i$  and  $S_{i+1}$  both contain points  $l_i$  and  $h_i$ . Because  $S_i$  is not inductive, edge  $(l_i, h_i)$  must be steep and thus  $d_x(l_i, h_i) < d_y(l_i, h_i)$ . To simplify the arguments, we assume that  $l_i$  is to the W of  $h_i$ , i.e.,  $x_{l_i} < x_{h_i}$ . The case  $x_{l_i} > x_{h_i}$  can be shown using equivalent arguments.

Since  $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$  or  $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$ , there has to be a side of  $S_i$  between the sides on which  $l_i$  and  $h_i$  lie, when moving clockwise from  $l_i$  to  $h_i$ . Using the constraints on the position of  $h_i$  and  $l_i$  within  $S_i$  from Lemma 4 and using assumptions that  $(l_i, h_i)$  is steep and that  $x_{l_i} < x_{h_i}$ , we deduce that  $l_i$  must be on the S side and  $h_i$  must be on the N or E side of  $S_i$ .

If  $h_i$  is on the N side of  $S_i$  then, because  $x_{l_i} < x_{h_i}$ ,  $h_i$  must also be on the N side of  $S_{i+1}$  and either  $l_i$  is on the S side of  $S_{i+1}$  and

$$d_{S_{i+1}}(h_i, l_i) - d_{S_i}(h_i, l_i) = 2(x_{i+1} - x_i) \quad (1)$$

as shown in Figure 4a) or  $l_i$  is on the W side of  $S_{i+1}$ , in which case

$$d_{S_{i+1}}(h_i, l_i) - d_{S_i}(h_i, l_i) \leq 4(x_{i+1} - x_i) \quad (2)$$

as shown in Figure 4b).

If  $h_i$  is on the E side of  $S_i$  then, because  $x_{i+1} > x_i$  (since  $(l_i, h_i)$  is steep),  $h_i$  must be on the N side of  $S_{i+1}$  and either  $l_i$  is on the S side of  $S_{i+1}$  and inequality (1) holds or  $l_i$  is on the W side of  $S_{i+1}$  and inequality (2) holds, as shown in Figure 4c).

Since  $S_i$  has potential, in all cases we obtain:

$$d_T(a, h_i) + d_T(a, l_i) + d_{S_{i+1}}(h_i, l_i) \leq 4x_{i+1} . \quad (3)$$

Assume  $T_{i+1} = \triangle(h_i, h_{i+1}, l_i = l_{i+1})$ ; in other words,  $(h_i, h_{i+1})$  is an edge of  $T$  with  $h_{i+1}$  lying somewhere on the boundary of  $S_{i+1}$  between  $h_i$  and  $l_i$ , when moving clockwise from  $h_i$  to  $l_i$ . By the triangular inequality,  $d_2(h_i, h_{i+1}) \leq d_{S_{i+1}}(h_i, h_{i+1})$  and we have that:

$$\begin{aligned} d_T(a, h_{i+1}) + d_T(a, l_{i+1}) + d_{S_{i+1}}(h_{i+1}, l_{i+1}) &\leq d_T(a, h_i) + d_T(a, l_i) + d_2(h_i, h_{i+1}) + d_{S_{i+1}}(h_{i+1}, l_i) \\ &\leq d_T(a, h_i) + d_T(a, l_i) + d_{S_{i+1}}(h_i, l_i) \\ &\leq 4x_{i+1} . \end{aligned}$$

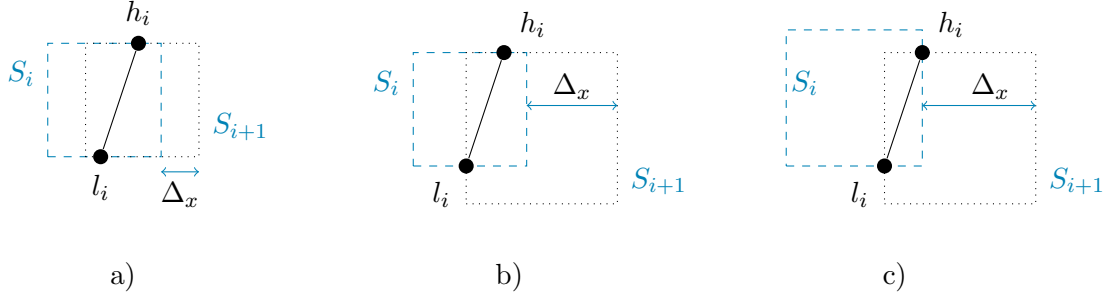


Figure 4: The first, second and fourth case in the proof of Lemma 9. In each case, the difference  $d_{S_{i+1}}(h_i, l_i) - d_{S_i}(h_i, l_i)$  is shown to be at most  $4\Delta_x$ , where  $\Delta_x = x_{i+1} - x_i$ .

Thus  $S_{i+1}$  has potential. The argument for the case when  $T_{i+1} = \triangle(h_i = h_{i+1}, l_i, l_{i+1})$  is symmetric.  $\square$

**Definition 10** A vertex  $c$  ( $h_i$  or  $l_i$ ) of  $T_i$  is promising in  $S_i$  if it lies on the  $E$  side of  $S_i$ .

**Lemma 11** If square  $S_i$  has potential and  $c = h_i$  or  $c = l_i$  is a promising point in  $S_i$  then

$$d_T(a, c) \leq 2x_c .$$

**Proof.** W.l.o.g., assume  $c = h_i$ . Since  $h_i$  is promising,  $x_c = x_{h_i} = x_i$ . Because  $S_i$  has potential, either  $d_T(a, h_i) \leq 2x_{h_i}$  or  $d_T(a, l_i) + d_{S_i}(l_i, h_i) \leq 2x_{h_i}$ . In the second case, we can use edge  $(l_i, h_i)$  and the triangular inequality to obtain  $d_T(a, h_i) \leq d_T(a, l_i) + |l_i h_i| \leq 2x_{h_i}$ .  $\square$

Here we define the maximal high and minimal low path.

**Definition 12**

- If  $h_j$  is promising in  $S_j$ , the maximal high path ending at  $h_j$  is simply  $h_j$ ; otherwise, it is the path  $h_i, h_{i+1}, \dots, h_j$  such that  $h_{i+1}, \dots, h_j$  are not promising and either  $i = 0$  or  $h_i$  is promising in  $S_i$ .
- If  $l_j$  is promising in  $S_j$ , the maximal low path ending at  $l_j$  is simply  $l_j$ ; otherwise, it is the path  $l_i, l_{i+1}, \dots, l_j$  such that  $l_{i+1}, \dots, l_j$  are not promising and either  $i = 0$  or  $l_i$  is promising in  $S_i$ .

Note that by Lemma 4, all edges on the path  $h_i, h_{i+1}, \dots, h_j$  are WN edges and thus the path length is bounded by  $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$ . Similarly, all edges in path  $l_i, l_{i+1}, \dots, l_j$  are WS edges and the length of the path is at most  $(x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$ .

We now have the tools to prove Lemma 6.

**Proof of Lemma 6.** If  $R(a, b)$  is empty then, by Lemma 4,  $b$  is promising. Thus, by Lemma 9 and Lemma 11, if no square  $S_1, \dots, S_k$  is inductive then  $d_T(a, b) \leq 2x < (1 + \sqrt{2})x + y$ .

Assume now that there is at least one inductive square in the sequence of squares  $S_1, \dots, S_k$ . Let  $S_j$  be the first inductive square and assume, for now, that  $h_j$  is the inductive point in  $S_j$ . By Lemma 9, every square  $S_i$ , for  $i < j$ , is a potential square.

Since  $(l_j, h_j)$  is gentle, it follows that  $d_2(l_j, h_j) \leq \sqrt{2}(x_{h_j} - x_{l_j})$ . Let  $l_i, l_{i+1}, \dots, l_{j-1} = l_j$  be the maximal low path ending at  $l_j$ . Note that  $d_T(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$ . Either  $l_i = l_0 = a$  or  $l_i$  is a promising point in potential square  $S_i$ ; either way, by Lemma 11, we have that  $d_T(a, l_i) \leq 2x_{l_i}$ . Putting all this together, we get

$$\begin{aligned} d_T(a, h_j) + (y_{h_j} - y) &\leq d_T(a, l_i) + d_T(l_i, l_j) + d_2(l_j, h_j) + y_{h_j} \\ &\leq 2x_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + \sqrt{2}(x_{h_j} - x_{l_j}) + y_{h_j} \\ &\leq \sqrt{2}x_{h_j} + x_{l_j} + y_{h_j} - y_{l_j} \\ &\leq (1 + \sqrt{2})x_{h_j} \end{aligned}$$

where the last inequality follows  $x_{l_j} + y_{h_j} - y_{l_j} \leq x_{h_j}$ , i.e., from the assumption that edge  $(l_j, h_j)$  is gentle.

If, instead,  $c = l_j$  is the inductive point in inductive square  $S_j$ , let  $h_i, h_{i+1}, \dots, h_{j-1} = h_j$  be the maximal high path ending at  $h_j$ . Then  $d_T(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$ . Just as in the first case, we have that  $d_T(a, h_i) \leq 2x_{h_i}$  and

$$\begin{aligned} d_T(a, l_j) - y_{l_j} &\leq d_T(a, h_i) + d_T(h_i, h_j) + d_2(h_j, l_j) - y_{l_j} \\ &\leq 2x_{h_i} + (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i}) + \sqrt{2}(x_{l_j} - x_{h_j}) - y_{l_j} \\ &\leq \sqrt{2}x_{l_j} + x_{h_j} + y_{h_j} - y_{l_j} \\ &\leq (1 + \sqrt{2})x_{l_j} . \end{aligned}$$

where the last inequality follows from  $x_{h_j} + y_{h_j} - y_{l_j} \leq x_{l_j}$ , i.e., from the assumption that  $(h_j, l_j)$  is gentle.  $\square$

## 4 Conclusion and perspectives

The  $L_1$ -Delaunay triangulation is the first type of Delaunay triangulation to be shown to be a spanner [Che86]. Progress on the spanning properties of the TD-Delaunay and the classical  $L_2$ -Delaunay triangulation soon followed. In this paper, we determine the precise stretch factor of an  $L_1$ - and  $L_\infty$ -Delaunay triangulation and close the problem for good. The techniques we develop in this paper have potential to be successfully applied to Delaunay triangulations defined by other regular polygons, and possibly even to the classical Delaunay triangulation.

From a routing perspective, it is of interest to construct routes in geometric graphs that can be determined *locally* from a neighbor's coordinates only [BCD09]. Unfortunately, the route that is implicitly constructed in our proof is built using non-local decisions. It would be interesting to know whether in the  $L_1$ -Delaunay triangulation a route with stretch  $\sqrt{4 + 2\sqrt{2}}$  can be constructed using a local routing algorithm. For TD-Delaunay triangulations, [BFvRV12] showed that there is no local routing algorithm that achieves a stretch that is less than  $5/\sqrt{3} \approx 2.88$ , whereas the stretch factor is actually 2. We leave open the questions regarding the gap between the stretch factor of  $L_1$ -Delaunay triangulations and the stretch that is possible using local routing.

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