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► **To cite this version:**

Nicolas Bonichon, Cyril Gavaille, Nicolas Hanusse, Ljubomir Perkovic. The Stretch Factor of  $L_1$ - and  $L_\infty$ -Delaunay Triangulations. 2012. hal-00673187v1

**HAL Id: hal-00673187**

**<https://hal.science/hal-00673187v1>**

Submitted on 23 Feb 2012 (v1), last revised 26 Feb 2012 (v2)

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# The Stretch Factor of $L_1$ - and $L_\infty$ -Delaunay Triangulations

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February 23, 2012

## Abstract

In this paper we determine the stretch factor of the  $L_1$ -Delaunay and  $L_\infty$ -Delaunay triangulations, and we show that this stretch is  $\sqrt{4 + 2\sqrt{2}} \approx 2.61$ . Between any two points  $x, y$  of such triangulations, we construct a path whose length is no more than  $\sqrt{4 + 2\sqrt{2}}$  times the Euclidean distance between  $x$  and  $y$ , and this bound is best possible. This definitively improves the 25-year old bound of  $\sqrt{10}$  by Chew (SoCG'86).

To the best of our knowledge, this is the first time the stretch factor of the well-studied  $L_p$ -Delaunay triangulations, for any real  $p \geq 1$ , is determined exactly.

Moreover, we show that the same short path exists in Yao $_4^\infty$  graphs, a subgraph of  $L_\infty$ -Delaunay triangulations, and therefore we determine the stretch factor for these graphs too.

**Keywords:** Delaunay triangulations,  $L_1$ -metric,  $L_\infty$ -metric, stretch factor, Yao graphs

## 1 Introduction

Given a set of points  $P$  on the plane, the Delaunay triangulation for  $P$  is a spanning subgraph of the Euclidean graph on  $P$  that is the dual of the Voronoi diagram of  $P$ . The Delaunay triangulation is a fundamental structure with many applications in computational geometry and other areas of Computer Science. In some applications (including on-line routing [BM04]), the triangulation is used as a spanner, defined as a spanning subgraph in which the distance between any pair of points is no more than a constant multiplicative ratio of the Euclidean distance between the points. The constant ratio is typically referred to as the stretch factor of the spanner. While Delaunay triangulations have been studied extensively, obtaining a tight bound on its stretch factor has been elusive even after decades of attempts.

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\*Member of the “Institut Universitaire de France”. Supported by the ANR-11-BS02-014 “DISPLEXITY” project and the équipe-projet INRIA “CEPAGE”.

†Supported by a Fulbright Aquitaine Regional grant and a DePaul University research grant.

Paper	Graph	Stretch factor
[DFS87]	$L_2$ -Delaunay	$\pi(1 + \sqrt{5})/2 \approx 5.08$
[KG92]	$L_2$ -Delaunay	$4\pi/(3\sqrt{3}) \approx 2.42$
[Xia11]	$L_2$ -Delaunay	1.998
[Che89]	TD-Delaunay	<b>2</b>
[Che86]	$L_1, L_\infty$ -Delaunay	$\sqrt{10} \approx 3.17$
[BDD <sup>+</sup> 10]	$Yao_4^\infty$	$8\sqrt{2} \approx 11.32$
<b>[this paper]</b>	$L_1, L_\infty$ -Delaunay and $Yao_4^\infty$	<b><math>\sqrt{4 + 2\sqrt{2}} \approx 2.61</math></b>

Table 1: Key stretch factor upper bounds (optimal values are bold).

In the mid-1980s, it was not known whether Delaunay triangulations were spanners at all. In order to gain an understanding of the spanning properties of Delaunay triangulations, Chew considered related, “easier” structures. In his seminal 1986 paper [Che86], he proved that an  $L_1$ -Delaunay triangulation — the dual of the Voronoï diagram of  $P$  based on the  $L_1$ -metric rather than the  $L_2$ -metric — has a stretch factor bounded by  $\sqrt{10}$ . Chew then continued on and showed that the a TD-Delaunay triangulation — the dual of a Voronoï diagram defined using a *Triangular Distance*, a distance function not based on a circle ( $L_2$ -metric) or a square ( $L_1$ -metric) but an equilateral triangle — has a stretch factor of 2 [Che89].

Finally, Dobkin et al. [DFS87] succeeded in showing that the (classical,  $L_2$ -metric) Delaunay triangulation of  $P$  is a spanner as well. The bound on the stretch factor they obtained was subsequently improved by Keil and Gutwin [KG92] as shown in Table 1. The bound by Keil and Gutwin stood unchallenged for many years until very recently when Xia improved the bound to below 2 [Xia11].

While progress has been made, none of the techniques developed so far lead to a tight bound on the spanning ratio. There has been some progress recently on the lower bound side. The trivial lower bound of  $\pi/2 \approx 1.5846$  has recently been improved to 1.5846 [BDL<sup>+</sup>11] and then to 1.5932 [XZ11].

While much effort has been made on understanding the stretch factor of Delaunay triangulations, little has been done on the  $L_p$ -Delaunay triangulations for  $p \neq 2$ . Lee and Wong [LW80] show that  $L_1, L_\infty$ -Delaunay triangulations have applications in scheduling problems for 2-dimensional storage, and how to construct, for all real  $p \geq 1$ , Voronoï diagrams in the  $L_p$ -metric in  $O(n \log n)$  time [Lee80]. Delaunay triangulations based on arbitrary convex distance functions have been studied in [BCCS08], which shows that such geometric graphs are indeed plane graphs and spanners whose stretch factor depends only on the shape of the convex body. However, due to the general approach, the bounds on the stretch factor remain loose. For instance the bounds they obtain for  $L_2$ -Delaunay triangulations are  $> 24$ .

The general picture is that, in spite of much effort, with the exception of the triangular distance the exact value of the stretch factor of Delaunay triangulations based on any convex function is unknown. In particular, the stretch factor of  $L_p$ -Delaunay triangulations is unknown for each  $p \geq 1$ .

**Our contributions.** We show that the exact stretch factor of  $L_1$ -Delaunay triangulations and  $L_\infty$ -Delaunay triangulations is  $\sqrt{4 + 2\sqrt{2}} \approx 2.61$ , ultimately improving the 3.17 bound of Chew [Che86].

Technically, we use rectangular coordinates to prove the upper bound. We show that the distance in the triangulation between any source-destination pair of points lying on the border of a horizontal rectangle of length  $x$  and of depth  $y \leq x$  is no more than  $(1 + \sqrt{2})x + y$ . The stretch factor bound then simply follows. In our proof, we construct the route from the source to the destination by maintaining two possible short paths, until we reach some special point (called inductive point) where we can apply our main inductive hypothesis.

As a by-product of our analysis, we show that all the edges we use for the final path exists also in the graph  $\text{Yao}_4^\infty$ , a subgraph of the  $L_\infty$ -Delaunay triangulation. The  $\text{Yao}_k^p$ -graph is locally constructed by connecting each point in  $P$  to its closest neighbor, according to the  $L_p$ -metric, in each  $2\pi/k$ -angle (see Definition 1 for  $k = 4$  and  $p = \infty$ ). As a consequence, we determine also the exact stretch factor of  $\text{Yao}_4^\infty$ , improving significantly the previous upper bound of 11.32. In addition, by integrating our bound into the analysis of [BDD<sup>+</sup>10], the upper bound on the stretch factor of  $\text{Yao}_4^2$ -graph is reduced from  $8\sqrt{2} \cdot (29 + 23\sqrt{2}) \approx 579$  to  $\sqrt{4 + 2\sqrt{2}} \cdot (29 + 23\sqrt{2}) \approx 160$ .

Despite the technical aspect of our contribution, we believe that our proof techniques may give insights into determining the stretch factor of other convex distance based Delaunay triangulations. For example, let  $P_k$  denote the convex distance function defined by a regular  $k$ -gon. We observe that the stretch factor of  $P_k$ -Delaunay triangulations is known for  $k = 3, 4$  since  $P_3$  is the triangular distance function of [Che89], and  $P_4$  is nothing else than the  $L_\infty$ -metric. Determining the stretch factor of  $P_k$ -Delaunay triangulations for larger  $k$  would undoubtedly be an important step towards understanding the stretch factor of classical Delaunay triangulations.

## 2 Preliminaries

Given a set  $P$  of points in the two-dimensional Euclidean space, the Euclidean graph  $\mathcal{E}$  is the complete weighted graph embedded in the plane whose nodes are identified with the points. We assume a Cartesian coordinate system is associated with the Euclidean space and thus every point can be specified with  $x$  and  $y$  coordinates. For every pair of nodes  $u$  and  $w$ , the edge  $(u, w)$  represents the segment  $[uw]$  and its weight is the edge length defined in Euclidean distance:  $d_2(u, w) = \sqrt{d_x(u, w)^2 + d_y(u, w)^2}$  where  $d_x(u, w)$  (resp.  $d_y(u, w)$ ) is the difference between the  $x$  (resp.  $y$ ) coordinates of  $u$  and  $w$ .

We say that a subgraph  $H$  of a graph  $G$  is a  $t$ -spanner of  $G$  if for any pair of vertices  $u, v$  of  $G$ , the distance between  $u$  and  $v$  in  $H$  is at most  $t$  times the distance between  $u$  and  $v$  in  $G$ ; the constant  $t$  is referred to as the *stretch factor* of  $H$  (with respect to  $G$ ).  $H$  is a  $t$ -spanner (or spanner for some  $t$  constant) if it is a  $t$ -spanner of  $\mathcal{E}$ .

In our paper, we deal with the construction of spanners based on Delaunay triangulations. As we saw in the introduction, the  $L_1$ -Delaunay triangulation is the dual of the Voronoi diagram based on the  $L_1$ -metric  $d_1(u, w) = d_x(u, w) + d_y(u, w)$ . A property of the  $L_1$ -Delaunay triangulations, actually shared by all  $L_p$ -Delaunay triangulations, is that all their triangles can be defined in terms of empty circumscribed convex bodies (squares for  $L_1$  or  $L_\infty$  and circles for  $L_2$ ). More precisely, let a *square* in the plane be a square whose sides

are parallel to the  $x$  and  $y$  axis and let a *tipped square* be a square tipped at  $45^\circ$ . For every pair of points  $u, v \in P$ ,  $(u, v)$  is an edge in the  $L_1$ -Delaunay triangulation of  $P$  iff there is a tipped square that has  $u$  and  $v$  on its boundary and contains no point of  $P$  in its interior (cf. [Che89]).

If a *square* with sides parallel to the  $x$  and  $y$  axes, rather than a tipped square, is used in this definition then a different triangulation is defined; it corresponds to the dual of the Voronoï diagram based on the  $L_\infty$ -metric  $d_\infty(u, w) = \max\{d_x(u, w), d_y(u, w)\}$ . We refer to this triangulation as the  $L_\infty$ -Delaunay triangulation. This triangulation is nothing more than the  $L_1$ -Delaunay triangulation of the set of points  $P$  after rotating all the points by  $45^\circ$  around the origin. Therefore Chew's bound of  $\sqrt{10}$  on the stretch factor of the  $L_1$ -Delaunay triangulation ([Che86]) applies to  $L_\infty$ -Delaunay triangulations as well. In the remainder of this paper, we will be referring to  $L_\infty$ -Delaunay (rather than  $L_1$ ) triangulations because we will be (mostly) using the  $L_\infty$ -metric and squares, rather than tipped squares.

One issue with Delaunay triangulations is that there might not be a unique triangulation of a given set of points  $P$ . To insure uniqueness and keep our arguments simple, we make the usual assumption that the points in  $P$  are in *general position*, which for us means that no four points lie on the boundary of a square and no two points share the same abscissa or the same ordinate.

In this paper we prove a tight bound on the stretch factor of the  $L_\infty$ -Delaunay triangulation. In fact the same bound also holds for a subgraph of the  $L_\infty$ -Delaunay triangulation that is a version of the Yao graph and described by Bose et al. [BDD<sup>+</sup>10]. In order to define this subgraph and its properties, we introduce some terminology. A *cone* is the region in the plane between two rays that emanate from the same point. With every point  $u$  of  $P$  we associate four disjoint  $90^\circ$  cones emanating from  $u$ : they are defined by the translation of the positive and negative  $x$ -axis and  $y$ -axis from the origin to point  $u$ . We label the cones 0, 1, 2, and 3, in counter-clockwise order and starting with the cone defined by the two positive axes. Given a cone  $i$ , the counter-clockwise next cone is cone  $i + 1$ , whereas the clockwise next cone is cone  $i - 1$ ; we assume a cyclic structure on the labels so that  $i + 1$  and  $i - 1$  are always defined. As the considered point sets are in general position, no point will ever lie on the boundary of another point's cone. For a point  $v$  and cone  $i$  of  $v$ , we denote by  $S_v^i(s)$  the  $s \times s$  square having  $v$  as a vertex and whose two sides match the boundary of cone  $i$  of  $v$ , and by  $S_v^i$  the square  $S_v^i(s)$  with the largest  $s$  that contains no points of  $P$  in its interior (see Figure 1.)

The following defines a version of the Yao subgraph of  $\mathcal{E}$  based on the  $L_\infty$ -metric proposed by [BDD<sup>+</sup>10].

**Definition 1** *Every node  $v$  of  $\mathcal{E}$  chooses in each one of its non-empty cones the shortest edge of  $\mathcal{E}$  out of  $v$  according to the  $L_\infty$ -metric. We name the resulting (undirected) graph  $\text{Yao}_4^\infty$  just as in [BDD<sup>+</sup>10]. (See Figure 1.)*

Using our notation,  $(v, w) \in \text{Yao}_4^\infty$  iff  $w$  lies on the boundary of  $S_v^i$  for some cone  $i$ . Because there is an empty square circumscribing the endpoints of every edge of  $\text{Yao}_4^\infty$ , it is a subgraph of the  $L_\infty$ -Delaunay triangulation  $T$ .

**Lemma 2** *If  $(u, v)$  is an edge in  $T$  but not in  $\text{Yao}_4^\infty$  and if  $S$  is any square circumscribing  $u$  and  $v$  whose interior is devoid of points of  $P$  then  $u$  and  $v$  must lie in the interior of*

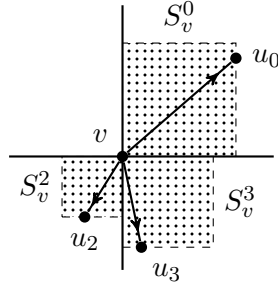


Figure 1: Definition of  $S_v^i$  and the construction of  $\text{Yao}_4^\infty$ .

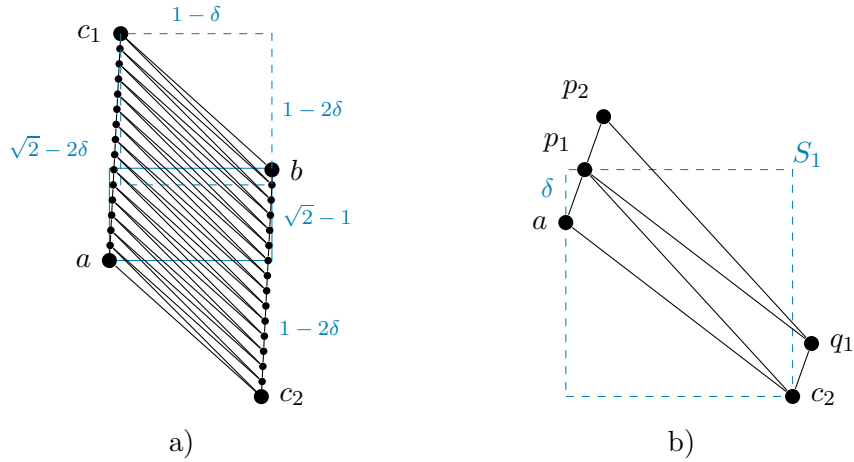


Figure 2: a) A  $L_\infty$ -Delaunay triangulation with stretch factor arbitrarily close to  $\sqrt{4 + 2\sqrt{2}}$ . b) A closer look at the first faces of this triangulation.

opposite sides of  $S$ . Furthermore, if  $T_1 = \triangle(u, v, w)$  is a triangle of  $T$  and  $S_1$  is the square circumscribing the vertices of  $T_1$ , then  $(u, w)$  and  $(v, w)$  are in  $\text{Yao}_4^\infty$  and connect adjacent sides of  $S_1$ .

**Proof.** W.l.o.g., we assume that  $v$  is in cone 0 of  $u$ . If  $u$  and  $v$  lie on adjacent sides of a square  $S$  devoid of points of  $P$  then either  $S_u^0$  is contained in  $S$  and has  $v$  on its boundary, or  $S_v^2$  is contained in  $S$  and has  $u$  on its boundary, contradicting the assumption that  $(u, v) \notin \text{Yao}_4^\infty$ . Therefore  $u$  and  $v$  must lie in the interior of opposite sides of the square  $S$ . This is particularly true for  $S_1$  and thus  $w$  must lie on a side of  $S_1$  that is adjacent to the sides of  $S_1$  containing  $u$  and  $v$ . By the above argument,  $(u, w)$  and  $(v, w)$  must be in  $\text{Yao}_4^\infty$ .  $\square$

We end this section by giving a lower bound on the stretch factor of  $L_\infty$ -Delaunay triangulation.

**Proposition 3** For every  $\epsilon > 0$ , there exists a set of points  $P$  in the plane such that the

$L_\infty$ -Delaunay triangulation on  $P$  has stretch factor at least

$$\sqrt{4 + 2\sqrt{2}} - \epsilon.$$

This result, of course, applies to  $L_1$ -Delaunay triangulations and Yao $_4^\infty$  graphs as well. The proof of this proposition relies on the example of Figure 2.

**Proof.** Given  $\delta > 0$ , we define the set of points  $P$  as follows. Let point  $a$  be the origin and let points  $b$ ,  $c_1$ , and  $c_2$  have coordinates  $(1, \sqrt{2} - 1)$ ,  $(\delta, \sqrt{2} - 2\delta)$ , and  $(1 - \delta, 1 - 2\delta)$ , respectively. Additional  $k = \frac{\sqrt{2}-2\delta}{\delta} - 1$  points are placed on line segment  $[ac_1]$  and another  $k$  on line segment  $[c_2b]$  in such a way that the difference in  $y$  coordinates between successive points on a segment is  $\delta$ , as shown in Figures 2 a) and b). (W.l.o.g. assume that  $\frac{\sqrt{2}}{\delta}$  is an integer so that this can be done.) Let  $a = p_0, p_1, p_2, p_3, \dots, p_k, p_{k+1} = c_1$  be the labels, in order as they appear when moving from  $a$  to  $c_1$ , of the points on segment  $[ac_1]$  and let  $c_2 = q_0, q_1, q_2, q_3, \dots, q_{k+1} = b$  be the labels, in order as they appear when moving from  $c_2$  to  $b$ , of the points on segment  $[c_2b]$ , as illustrated in Figure 2 b).

Consider the square  $S_1$  of side length  $1 - \delta$  and having  $a$  and  $p_1$  on its west (left) and north sides, respectively. Since  $d_\infty(a, c_2) = d_x(a, c_2) = 1 - \delta$  and  $d_\infty(p_1, c_2) = d_y(p_1, c_2) = 1 - \delta$ , point  $c_2$  is exactly the southeast vertex of square  $S_1$  (as shown in Figure 2 b)). By symmetry, it follows that for every  $i = 0, 1, 2, \dots, k$ , if  $S_i$  is the square of side length  $1 - \delta$  with  $p_i$  and  $p_{i+1}$  on its west and north sides, then point  $q_i$  is exactly the southeast vertex of  $S_i$ . This means that all points  $q_j$  with  $j \neq i$  as well as all points  $p_j$  with  $j \neq i, i + 1$  must lie outside  $S_i$ . Therefore, for every  $i = 0, 1, 2, \dots, k$ , points  $p_i$ ,  $p_{i+1}$ , and  $q_i$  define a triangle in the  $L_\infty$ -Delaunay triangulation  $T$  on  $P$ . A similar argument shows that the path  $q_0, q_1, \dots, q_{k+1}$  is in triangulation  $T$  as well. The triangulation  $T$  is illustrated in Figure 2 a).

Having defined the set of points  $P$  and described its  $L_\infty$ -Delaunay triangulation  $T$ , we now analyze the stretch factor of  $T$ . A shortest path from  $a$  to  $b$  in  $T$  is, for example,  $a, p_1, p_2, \dots, p_k, c_1, b$ . The length of this path is

$$\begin{aligned} d_2(a, c_1) + d_2(c_1, b) &= \sqrt{d_x(a, c_1)^2 + d_y(a, c_1)^2} + \sqrt{d_x(c_1, b)^2 + d_y(c_1, b)^2} \\ &= \sqrt{(\sqrt{2} - \delta)^2 + \delta^2} + \sqrt{(1 - \delta)^2 + (1 - 2\delta)^2} \end{aligned}$$

which tends to  $2\sqrt{2}$  as  $\delta$  tends to 0. The Euclidean distance between  $a$  and  $b$  is fixed as  $\delta$  goes to 0:

$$d_2(a, b) = \sqrt{1^2 + (\sqrt{2} - 1)^2} = \sqrt{4 - 2\sqrt{2}}.$$

Therefore, it is possible to construct a  $L_\infty$ -Delaunay triangulation whose stretch factor is arbitrarily close to:

$$\frac{2\sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} = \sqrt{4 + 2\sqrt{2}}.$$

□

### 3 Main result

In this section we obtain a tight upper bound on the stretch factor of the subgraph Yao $_4^\infty$  of an  $L_\infty$ -Delaunay triangulation. It follows from this key theorem:

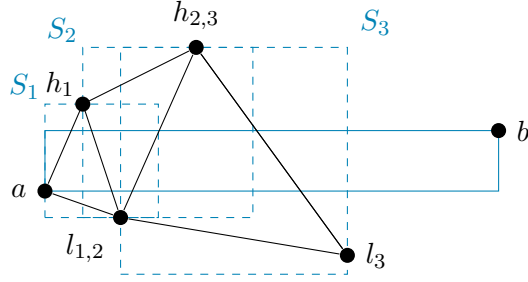


Figure 3: Triangles  $T_1$  (with points  $a, h_1, l_1$ ),  $T_2$  (with points  $h_1, h_2$ , and  $l_2$ ), and  $T_3$  (with points  $l_2, h_3$ , and  $l_3$ ) and associated squares  $S_1, S_2$ , and  $S_3$ . When traveling from  $a$  to  $b$  along segment  $[a, b]$ , the edge that is hit when leaving  $T_i$  is  $(h_i, l_i)$ .

**Theorem 4** *Let  $T$  be the  $L_\infty$ -Delaunay triangulation on a set of points  $P$  in the plane and let  $a$  and  $b$  be any two points of  $P$ . If  $x = d_\infty(a, b) = \max\{d_x(a, b), d_y(a, b)\}$  and  $y = \min\{d_x(a, b), d_y(a, b)\}$  then*

$$d_Y(a, b) \leq (1 + \sqrt{2})x + y,$$

where  $d_Y(a, b)$  denotes the distance between  $a$  and  $b$  in subgraph  $\text{Yao}_4^\infty$  of  $T$ .

**Corollary 5** *The stretch factor of the graph  $\text{Yao}_4^\infty$ , the  $L_1$ -, and the  $L_\infty$ -Delaunay triangulation on a set of points  $P$  is at most*

$$\sqrt{4 + 2\sqrt{2}} \approx 2.6131259 \dots$$

**Proof.** By Theorem 4, an upper-bound of the stretch factor of  $\text{Yao}_4^\infty$  is the maximum of the function

$$\frac{(1 + \sqrt{2})x + y}{\sqrt{x^2 + y^2}}$$

over values  $x$  and  $y$  such that  $x \leq y$ . This maximum is equal to  $\sqrt{4 + 2\sqrt{2}}$  and is reached when  $x$  and  $y$  satisfy  $\frac{y}{x} = \frac{1}{1 + \sqrt{2}}$ . Since  $\text{Yao}_4^\infty$  is a subgraph of  $L_\infty$ -Delaunay triangulation, the same result also holds for  $L_1$ - and  $L_\infty$ -Delaunay triangulations.  $\square$

To simplify the notation and the discussion, assume that point  $a$  has coordinates  $(0, 0)$  and point  $b$  has coordinates  $(x, y)$  with  $0 < y \leq x$ . The segment  $[ab]$  divides the Euclidean plane into two half-planes; a point in the same half-plane as point  $(0, 1)$  is said to be *above* segment  $[ab]$ , otherwise it is *below*.

Let  $T_1, T_2, T_3, \dots, T_k$  be the sequence of triangles of triangulation  $T$  that line segment  $[ab]$  intersects when moving from  $a$  to  $b$ . Let  $h_1$  and  $l_1$  be the nodes of  $T_1$  other than  $a$ , with  $h_1$  lying above segment  $[ab]$  and  $l_1$  lying below. Every triangle  $T_i$ , for  $1 < i < k$ , intersects line segment  $[ab]$  twice; let  $h_i$  and  $l_i$  be the endpoints of the edge of  $T_i$  that intersects segment  $[ab]$  last, when moving on segment  $[ab]$  from  $a$  to  $b$ , with  $h_i$  being above and  $l_i$  being below segment  $[ab]$ . Note that either  $h_i = h_{i-1}$  and  $T_i = \triangle(h_i, l_i, l_{i-1})$  or  $l_i = l_{i-1}$  and  $T_i = \triangle(h_{i-1}, h_i, l_i)$ , for  $1 < i < k$ . We also set  $h_0 = l_0 = a$  and  $h_k = l_k = b$ . For  $1 \leq i \leq k$ , we define  $S_i$  to be the square whose sides pass through the three vertices of  $T_i$  (see Figure 3);



since  $T$  is an  $L_\infty$ -Delaunay triangulation, the interior of  $S_i$  is devoid of points of  $P$ . We will refer to the sides of the square using the notation: N (north), E (east), S (south), and W (west). We will also use this notation to describe the position of an edge connecting two points lying on two sides a square: for example, a WN edge connects a point on the west and a point on the N side. We will say that a line segment is *gentle* if its slope is within  $[-1, 1]$ , otherwise we will say that it is *steep*.

We will prove Theorem 4 by induction on the distance, using the  $L_\infty$ -metric, between  $a$  and  $b$ , two points of  $P$ . Let  $R(a, b)$  be the rectangle with sides parallel to the  $x$  and  $y$  axes and with vertices at points  $a$  and  $b$ . If there is point of  $P$  inside  $R(a, b)$ , we will easily apply induction. The case when  $R(a, b)$  does not contain points of  $P$  — and in particular the points  $h_i$  and  $l_i$  for  $0 < i < k$  — is more difficult and we need to develop tools to handle it. The following Lemma describes the structure of the triangles  $T_1, \dots, T_k$  when  $R(a, b)$  is empty. We need some additional terminology first though: we say that a point  $u$  is *above* (resp. *below*)  $R(a, b)$  if  $0 < x_u < x$  and  $y_u > y$  (resp.  $y_u < y$ ).

**Lemma 6** *If  $(a, b) \notin T$  and no point of  $P$  lies inside rectangle  $R(a, b)$ , then point  $a$  lies on the W side of square  $S_1$ , point  $b$  lies on the E side of square  $S_k$ , points  $h_1, \dots, h_k$  all lie above  $R(a, b)$ , and points  $l_1, \dots, l_k$  all lie below  $R(a, b)$ . Furthermore, for any  $i$  such that  $1 < i < k$ :*

- a) *Either  $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$  and  $(h_{i-1}, h_i)$  is a WN, WE, or NE edge in  $S_i$*
- b) *Or  $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$  and  $(l_{i-1}, l_i)$  is a WS, WE, or SE edge in  $S_i$ .*

Figure 3 illustrates these properties.

**Proof.** Since points of  $P$  are in general position, points  $a$ ,  $h_1$ , and  $l_1$  must lie on 3 different sides of  $S_1$ . Because segment  $[ab]$  intersects the interior of  $S_1$  and since  $a$  is the origin and  $b$  is in cone 0 of  $a$ ,  $a$  can only lie on the W or S side of  $S_1$ . If  $a$  lies on the S side then  $l_1 \neq b$  would have to lie inside  $R(a, b)$ , which is a contradiction. Therefore  $a$  lies on the W side of  $S_1$  and, similarly,  $b$  lies on the E side of  $S_k$ .

Since points  $h_i$  ( $0 < i < k$ ) are above segment  $[ab]$  and points  $l_i$  ( $0 < i < k$ ) are below segment  $[ab]$ , and because all squares  $S_i$  ( $0 < i < k$ ) intersect  $[ab]$ , points  $h_1, \dots, h_k$  all lie above  $R(a, b)$ , and points  $l_1, \dots, l_k$  all lie below  $R(a, b)$ .

The three vertices of  $T_i$  can be either  $h_i = h_{i-1}$ ,  $l_{i-1}$ , and  $l_i$  or  $h_{i-1}$ ,  $h_i$ , and  $l_{i-1} = l_i$ . Because points of  $T$  are in general position, the three vertices of  $T_i$  must appear on three different sides of  $S_i$ . Finally, because  $h_{i-1}$  and  $h_i$  are above  $R(a, b)$ , they cannot lie on the S side of  $S_i$ , and because  $l_{i-1}$  and  $l_i$  are below  $R(a, b)$ , they cannot lie on the N side of  $S_i$ .

If  $T_i = \triangle(h_{i-1}, h_i, l_{i-1} = l_i)$ , points  $h_{i-1}$ ,  $h_i$ ,  $l_i$  must lie on the sides of  $S_i$  in clockwise order. The only placements of points  $h_{i-1}$  and  $h_i$  on the sides of  $S_i$  that satisfy all these constraints are as described in a). If  $T_i = \triangle(h_{i-1} = h_i, l_{i-1}, l_i)$ , points  $h_i$ ,  $l_i$ ,  $l_{i-1}$  must lie on the sides of  $S_i$  in clockwise order. Part b) lists the placements of points  $l_{i-1}$  and  $l_i$  that satisfy the constraints.  $\square$

In the following definition, we define the points on which induction can be applied in the proof of Theorem 4.

**Definition 7** *Let  $R(a, b)$  be empty and let  $c = h_j$  or  $c = l_j$  be a point lying on the E side of  $S_j$  such that line segment  $[l_j h_j]$  is gentle. Point  $c$  is inductive if either:*

- a)  $(l_j, h_j) \in \text{Yao}_4^\infty$  or
- b)  $(l_j, h_j) \notin \text{Yao}_4^\infty$ ,  $c = h_j$  and line segments  $[l_j h_{j-1}]$ ,  $[h_{j-1} h_j]$ ,  $[l_j l_{j+1}]$ , and  $[l_{j+1} h_j]$  are not steep, gentle, steep, and gentle, respectively, or
- c)  $(l_j, h_j) \notin \text{Yao}_4^\infty$ ,  $c = l_j$  and line segments  $[h_j l_{j-1}]$ ,  $[l_{j-1} l_j]$ ,  $[h_j h_{j+1}]$ , and  $[h_{j+1} l_j]$  are not steep, gentle, steep, and gentle, respectively.

The following lemma will be the key ingredient of our inductive proof of Theorem 4. We leave the proof of Lemma 8 until later.

**Lemma 8** *Assume that  $R(a, b)$  is empty. If no square  $S_1, \dots, S_k$  is inductive then*

$$d_Y(a, b) \leq (1 + \sqrt{2})x + y.$$

*Otherwise let  $S_j$  be the first inductive square in the sequence  $S_1, S_2, \dots, S_k$ . If  $h_j$  is the inductive point of  $S_j$  then*

$$d_Y(a, h_j) + (y_{h_j} - y) \leq (1 + \sqrt{2})x_{h_j}.$$

*If  $l_j$  is the inductive point of  $S_j$  then*

$$d_Y(a, l_j) - y_{l_j} \leq (1 + \sqrt{2})x_{l_j}.$$

If point  $c$  is inductive, we can apply induction to bound  $d_Y(b, c)$  and then use Lemma 8 to bound  $d_Y(a, b)$ , but only if the position of point  $c$  relative to the position of point  $b$  is *good*, i.e. if  $x - x_c > |y - y_c|$ . If that is not the case, we will use the following Lemma:

**Lemma 9** *Let  $R(a, b)$  be empty and let the coordinates of point  $c = h_i$  or  $c = l_i$  satisfy  $0 < x - x_c < |y - y_c|$ .*

- a) *If  $c = h_i$ , and thus  $0 < x - x_{h_i} < y_{h_i} - y$ , then there exists  $j$ , with  $i < j \leq k$  such that all edges in path  $h_i, h_{i+1}, h_{i+2}, \dots, h_j$  are NE edges in their respective squares and  $x - x_{h_j} > y_{h_j} - y > 0$ .*
- b) *If  $c = l_i$ , and thus  $0 < x - x_{l_i} < y - y_{l_i}$ , then there exists  $j$ , with  $i < j \leq k$  such that all edges in path  $l_i, l_{i+1}, l_{i+2}, \dots, l_j$  are SE edges and  $x - x_{l_j} > y - y_{l_j} > 0$ .*

Note that, since all edges in path  $h_i, h_{i+1}, \dots, h_j$  are NE edges in their respective squares, they must be in  $\text{Yao}_4^\infty$ . The same is true for edges in path  $l_i, l_{i+1}, \dots, l_j$ .

**Proof.** We only prove the case  $c = h_j$  as the case  $c = l_i$  follows using a symmetric argument.

We construct the path  $h_i, h_{i+1}, h_{i+2}, \dots, h_j$  iteratively. If  $h_i = h_{i+1}$ , we just continue building the path from  $h_{i+1}$ . Otherwise,  $(h_i, h_{i+1})$  is an edge of  $T_{i+1}$  which, by Lemma 6, must be a WN, WE, or NE edge in square  $S_{i+1}$ . Since the S side of square  $S_{i+1}$  is below  $R(a, b)$  and because  $x - x_{h_i} < y_{h_i} - y$ , point  $h_i$  cannot be on the W side of  $S_{i+1}$  (otherwise  $b$  would be inside square  $S_{i+1}$ ). Thus  $(h_i, h_{i+1})$  is a NE edge. If  $x - x_{h_{i+1}} > y_{h_{i+1}} - y$  we stop, otherwise we continue the path construction from  $h_{i+1}$ .  $\square$

We are now ready to prove Theorem 4.

**Proof of Theorem 4.** The proof is by induction on the distance, using the  $L_\infty$ -metric, between points of  $P$  (since  $P$  is finite there is only a finite number of distances to consider).

Let  $a$  and  $b$  be the two points of  $P$  that are the closest points, using the  $L_\infty$ -metric. We assume w.l.o.g. that  $a$  has coordinates  $(0,0)$  and  $b$  has coordinates  $(x,y)$  with  $0 < y \leq x$ . Since  $a$  and  $b$  are the closest points using the  $L_\infty$ -metric, square  $S_a^0$  (i.e., the largest square having  $a$  as a southwest vertex and containing no points of  $P$  in its interior) must have  $b$  on its E side. Therefore  $(a,b)$  is an edge in  $\text{Yao}_4^\infty$  and  $d_Y(a,b) = d_2(a,b) \leq x+y \leq (1+\sqrt{2})x+y$ .

For the induction step, we again assume, w.l.o.g., that  $a$  has coordinates  $(0,0)$  and  $b$  has coordinates  $(x,y)$  with  $0 < y \leq x$ .

**Case 1:  $R(a,b)$  is not empty.**

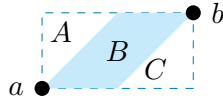


Figure 4: Partition of  $R(a,b)$  into three regions in Case 1 of the proof of Theorem 4.

We first consider the case when there is at least one point of  $P$  lying inside rectangle  $R(a,b)$ . If there is a point  $c$  inside  $R(a,b)$  such that  $y_c \leq x_c$  and  $y - y_c \leq x - x_c$  (i.e.,  $c$  lies in the region  $B$  shown in Figure 4) then we can apply induction to get  $d_Y(a,c) \leq (1+\sqrt{2})x_c + y_c$  and  $d_Y(c,b) \leq (1+\sqrt{2})(x - x_c) + y - y_c$  and use these to obtain the desired bound for  $d_Y(a,b)$ .

We now assume that there is no point inside region  $B$ . If there is still a point in  $R(a,b)$  then there must be one that is on the border of  $S_a^0$  or  $S_b^2$ . W.l.o.g., we assume the former and thus there is an edge  $(a,c) \in \text{Yao}_4^\infty$  such that either  $y_c > x_c$  (i.e.,  $c$  is inside region  $A$  shown in Figure 4) or  $y - y_c > x - x_c$  (i.e.,  $c$  is inside region  $C$ ). Either way,  $d_Y(a,c) = d_2(a,c) \leq x_c + y_c$ . If  $c$  is in region  $A$ , since  $x - x_c \geq y - y_c$ , by induction we also have that  $d_Y(c,b) \leq (1+\sqrt{2})(x - x_c) + (y - y_c)$ . Then

$$\begin{aligned} d_Y(a,b) &\leq d_Y(a,c) + d_Y(c,b) \\ &\leq x_c + y_c + (1+\sqrt{2})(x - x_c) + (y - y_c) \leq (1+\sqrt{2})x + y \end{aligned}$$

In the second case, since  $x - x_c < y - y_c$ , by induction we have that  $d_Y(c,b) \leq (1+\sqrt{2})(y - y_c) + (x - x_c)$ . Then, because  $y < x$ ,

$$\begin{aligned} d_Y(a,b) &\leq d_Y(a,c) + d_Y(c,b) \\ &\leq x_c + y_c + (1+\sqrt{2})(y - y_c) + (x - x_c) \leq (1+\sqrt{2})x + y \end{aligned}$$

**Case 2:  $R(a,b)$  is empty.** If no square  $S_1, S_2, \dots, S_k$  is inductive,  $d_Y(a,b) \leq (1+\sqrt{2})x + y$  by Lemma 8. Otherwise, let  $S_i$  be the first inductive square in the sequence.

Suppose that  $h_i$  is the inductive point of  $S_i$ . By Lemma 9, there is a  $j$ ,  $i \leq j \leq k$ , such that  $h_i, h_{i+1}, h_{i+2}, \dots, h_j$  is a path in  $\text{Yao}_4^\infty$  of length at most  $(x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j})$  and such that  $x - x_{h_j} > y_{h_j} - y > 0$ . Since  $h_j$  is closer to  $b$ , using the  $L_\infty$ -metric, than  $a$  is, we can apply induction to bound  $d_Y(h_j, b)$ . Putting all this together with Lemma 8, we get:

$$\begin{aligned} d_Y(a,b) &\leq d_Y(a, h_i) + d_Y(h_i, h_j) + d_Y(h_j, b) \\ &\leq (1+\sqrt{2})x_{h_i} - (y_{h_i} - y) + (x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j}) + (1+\sqrt{2})(x - x_{h_j}) + (y_{h_j} - y) \\ &\leq (1+\sqrt{2})x \end{aligned}$$

If  $l_i$  is the inductive point of  $S_i$ , by Lemma 9 there is a  $j$ ,  $i \leq j \leq k$ , such that  $l_i, l_{i+1}, l_{i+2}, \dots, l_j$  is a path in  $\text{Yao}_4^\infty$  of length at most  $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$  and such that  $x - x_{h_j} > y - y_{h_j} > 0$ . Because the position of  $j$  with respect to  $b$  is good and since  $l_j$  is closer to  $b$ , using the  $L_\infty$ -metric, than  $a$  is, we can apply induction to bound  $d_Y(l_j, b)$ . Putting all this together with Lemma 8, we get:

$$\begin{aligned} d_Y(a, b) &\leq d_Y(a, l_i) + d_Y(l_i, l_j) + d_Y(l_j, b) \\ &\leq (1 + \sqrt{2})x_{l_i} + y_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_j} - y_{l_i}) + (1 + \sqrt{2})(x - x_{l_j}) + (y - y_{l_j}) \\ &\leq (1 + \sqrt{2})x + y. \end{aligned}$$

□

What remains to be done is to prove Lemma 8. To do this, we need to develop some further terminology and a technical Lemma.

**Definition 10** A vertex  $c$  ( $h_i$  or  $l_i$ ) of  $T_i$  is promising in  $S_i$  if it lies on the  $E$  side of  $S_i$ . A square  $S_i$  is promising if  $h_i$  or  $l_i$  is promising in  $S_i$ . The sequence  $h_i, h_{i+1}, \dots, h_j$  (resp.,  $l_i, l_{i+1}, \dots, l_j$ ) is a maximal high (resp., low) sequence ending at  $h_j$  (resp.,  $l_j$ ) if  $l_{i+1}, \dots, l_j$  are not promising and either  $i = 0$  or  $h_i$  (resp.,  $l_i$ ) is promising in  $S_i$ .

Note that by Lemma 6, all edges on the path defined by a maximal high sequence  $h_i, h_{i+1}, \dots, h_j$  are WN edges and thus the path is in  $\text{Yao}_4^\infty$  and its length is bounded by  $(x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$ . Similarly, all edges in a maximal low sequence  $l_i, l_{i+1}, \dots, l_j$  are WS edges, and thus in  $\text{Yao}_4^\infty$ , and the length of the path is at most  $(x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$ .

**Lemma 11** Let  $S_i$  be a promising but not inductive square and let  $c$  ( $h_i$  or  $l_i$ ) be its promising point. If all previous squares are not inductive then  $d_Y(a, c) \leq (1 + \sqrt{2})x_c + |y_c|$ .

Note that this Lemma implies that if no square  $S_1, \dots, S_k$  is inductive, and since  $b$  is promising in  $S_k$  (by Lemma 6),  $d_Y(a, b) \leq (1 + \sqrt{2})x + y$ .

**Proof.** We prove Lemma 11 by induction on the number of promising squares before  $S_i$ . We thus start by assuming that  $S_i$  is the first promising square and that it is not inductive. We also assume that  $h_i$  is the promising point in  $S_i$  and thus  $T_i = \Delta(h_{i-1}, h_i, l_{i-1} = l_i)$ . The case when  $l_i$  is promising can be shown using symmetric arguments.

We consider 3 cases in the base step of the induction. In each case we construct two paths (specific to each case) in  $\text{Yao}_4^\infty$  from  $a$  to  $h_i$ . The first path starts with the maximal high sequence from  $a$  to  $h_{i-1}$  and the second starts with the maximal low sequence from  $a$  to  $l_i$ . To conclude, all we need to do is show that the sum of the lengths of these two paths is at most  $2(1 + \sqrt{2})x_{h_i} + 2y_{h_i}$ .

Before we start, we show how we can reduce the computation of the sum to a simpler geometric problem. Using the bounds on the lengths of a maximal high and low sequence starting at  $a$ , we have that  $d_Y(a, h_{i-1}) \leq y_{h_{i-1}} + x_{h_{i-1}}$  and  $d_Y(a, l_i) \leq -y_{l_i} + x_{l_i}$ . If we define point  $a'$  to be the abscissa of the W side of  $S_i$ , we can split the sum of  $d_Y(a, h_{i-1}) + d_Y(h_{i-1}, h_i)$  and  $d_Y(a, l_i) + d_Y(l_i, h_i)$  into two parts:  $2x_{a'}$  and

$$y_{h_{i-1}} + (x_{h_{i-1}} - x_{a'}) + -y_{l_i} + x_{l_i} - x_{a'} + d_Y(h_{i-1}, h_i) + d_Y(l_i, h_i)$$

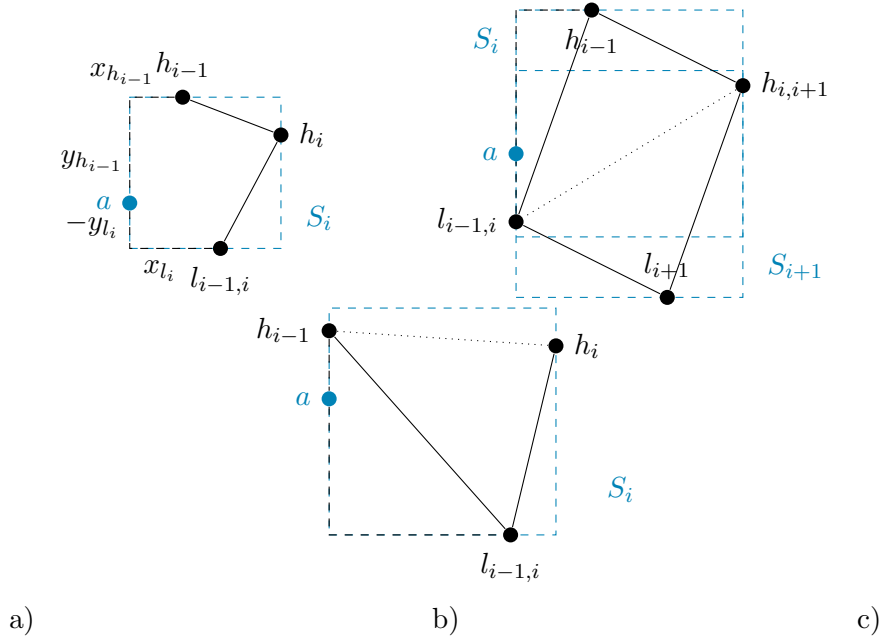


Figure 5: The 3 cases of the proof of Lemma 11. The dotted edges are edges that are in the  $L_\infty$ -Delaunay triangulation but not in its  $\text{Yao}_4^\infty$  subgraph.

To obtain the desired bound on the sum of the lengths of the two paths from  $a$  to  $h_i$ , it is thus sufficient to bound the second part with  $2(1 + \sqrt{2})(x_{h_i} - x_{a'}) + 2y_{h_i}$ . Note that this reduction in analysis works as long as the sum of the first part is bounded by  $2(1 + \sqrt{2})x_{a'}$ . In order to make the computation of the second part easier, we can assume, for analysis purposes only, that  $a = a'$ , i.e.  $a$  is on the W side of  $S_i$ .

Let us now detail the three different cases.

**Case 1:**  $(h_{i-1}, h_i) \in \text{Yao}_4^\infty$  and  $[l_i h_i]$  is steep.

Since  $[l_i h_i]$  is steep and  $h_i$  is promising,  $l_i$  must be on the S side of  $S_i$  and  $(l_i, h_i)$  is a SE edge and thus in  $\text{Yao}_4^\infty$ . Note that  $h_{i-1}$  can be on the W or N side of  $S_i$ .

The first path from  $a$  to  $h_i$  we consider consists of the maximal high sequence from  $a$  to  $h_{i-1}$  and the edge  $(h_{i-1}, h_i)$  (see Figure 5 a)). Considering this path we get

$$d_Y(a, h_i) \leq y_{h_{i-1}} + x_{h_{i-1}} + d_2(h_{i-1}, h_i) \quad (1)$$

The second path consists of the maximal low sequence from  $a$  to  $l_{i-1}$  and edge  $(l_{i-1}, h_i)$ . From this we get

$$d_Y(a, h_i) \leq x_{l_i} - y_{l_i} + d_2(l_i, h_i) \quad (2)$$

The sum of the upper bounds in inequalities (1) and (2) is at most the perimeter of square  $S_i$ , whose side length is  $x_{h_i}$ . Therefore the sum is bounded by  $4x_{h_i} \leq 2(1 + \sqrt{2})x_{h_i} + 2y_{h_i}$ , which completes the proof for this case.

**Case 2:**  $(h_{i-1}, h_i) \in \text{Yao}_4^\infty$  and  $[l_i h_i]$  is gentle.

Since  $h_i$  is not inductive and  $[l_i h_i]$  is gentle, it must mean that  $(l_i, h_i) \notin \text{Yao}_4^\infty$ ,  $[l_i h_{i-1}]$ ,  $[h_{i-1} h_i]$ ,  $[l_i l_{i+1}]$ , and  $[l_{i+1} h_i]$  are steep, gentle, steep, and gentle, respectively, and edges  $(l_{i-1} = l_i, h_{i-1})$ ,  $(h_{i-1}, h_i = h_{i+1})$ ,  $(l_{i-1} = l_i, l_{i+1})$ , and  $(l_{i+1}, h_i = h_{i+1})$  are all in  $\text{Yao}_4^\infty$ .

Finally, since  $(l_i, h_i)$  is gentle and not in  $\text{Yao}_4^\infty$ ,  $h_i = h_{i+1}$  must be also the promising point of  $S_{i+1}$  as shown in Figure 5 b).

The first path we consider is composed of the maximal high sequence to  $h_{i-1}$  plus the edge  $(h_{i-1}, h_i)$ . Since  $[h_{i-1}h_i]$  is gentle, using this path we get:

$$d_Y(a, h_i) \leq y_{h_{i-1}} + x_{h_{i-1}} + \sqrt{2}(x_{h_i} - x_{h_{i-1}})$$

The second path is composed of the maximal low sequence to  $l_i$  and edges  $(l_i, l_{i+1})$  and  $(l_{i+1}, h_i)$ . Since  $[l_i l_{i+1}]$  is gentle, using this path we get:

$$d_Y(a, h_i) \leq -y_{l_i} + \sqrt{2}x_{l_{i+1}} + (x_{h_i} - x_{l_{i+1}}) + (y_{h_i} - y_{l_{i+1}})$$

Because  $y_{h_{i-1}} - y_{l_i} \leq x_{h_i}$ ,  $y_{h_{i+1}} - y_{l_{i+1}} \leq x_{h_i}$ ,  $x_{h_{i-1}} + \sqrt{2}(x_{h_i} - x_{h_{i-1}}) \leq \sqrt{2}x_{h_i}$ , and  $(x_{h_i} - x_{l_{i+1}}) + \sqrt{2}x_{l_{i+1}} \leq \sqrt{2}x_{h_i}$ , the sum of the two bounds is at most  $2(1 + \sqrt{2})x_{h_i}$  which completes the proof for this case.

**Case 3:  $(h_{i-1}, h_i)$  is not an edge of  $\text{Yao}_4^\infty$ .** By Lemmas 2 and 6,  $(h_{i-1}, h_i)$  must be a WE edge in  $S_i$ ,  $l_i$  lies on the S side of  $S_i$  and edges  $(h_{i-1}, l_i)$  and  $(l_i, h_i)$  are in  $\text{Yao}_4^\infty$ . Since  $h_i$  is not inductive, line segment  $[l_i h_i]$  must be steep. Figure 5 c) illustrates this case.

The first path we consider consists of the maximal high sequence from  $a$  to  $h_{i-1}$  followed by edges  $(h_{i-1}, l_i)$  and  $(l_i, h_i)$ . Using this path, we get

$$d_Y(a, h_i) \leq y_{h_{i-1}} + d_2(h_{i-1}, l_i) + d_2(l_i, h_i).$$

The bound is largest when  $h_{i-1}$  is the NW vertex of square  $S_i$  and  $l_i$  is the SE vertex. The second path uses the maximal low sequence from  $a$  to  $l_i$  followed by edge  $(l_i, h_i)$ . It gives us the bound

$$d_Y(a, h_i) \leq -y_{l_i} + x_{l_i} + d_2(l_i, h_i).$$

The length of this path is largest when  $l_i$  is the SE vertex as well. If  $h_{i-1}$  and  $l_i$  are, respectively, the NW and SE vertex of  $S_i$ , then the minimum of  $y_{h_{i-1}} + d_2(h_{i-1}, l_i)$  and  $-y_{l_i} + x_{l_i}$  can be as high as  $(1 + \sqrt{2}/2)x_{h_i}$  achieved when  $|y_{l_i}| = (\sqrt{2}/2)x_{h_i}$  and  $y_{h_{i-1}} = (1 - \sqrt{2}/2)x_{h_i}$ . Then  $d_Y(a, h_i) \leq (1 + \sqrt{2})x_{h_i} + y_{h_i}$ .

**Induction step.** Let  $S_i$  be promising but not inductive square. Assume that the lemma holds for the promising squares in the sequence  $S_1, \dots, S_{i-1}$  and that this sequence is free of inductive squares. Let  $h_j, \dots, h_{i-1}$  be the maximal high sequence ending at  $h_{i-1}$  and  $l_g, \dots, l_i$  be the maximal low sequence ending at  $l_{i-1} = l_i$ . One of  $j$  or  $g$  could be 0. By induction, or because  $h_j = h_0 = a$ ,  $d_Y(a, h_j) \leq (1 + \sqrt{2})x_{h_j} + y_{h_j}$ ; similarly,  $d_Y(a, l_g) \leq (1 + \sqrt{2})x_{l_g} - y_{l_g}$ .

Using the bound on the length of a maximal high sequence path, we get  $d_Y(a, h_{i-1}) \leq (1 + \sqrt{2})x_{h_j} + (x_{h_{i-1}} - x_{h_j}) + y_{h_{i-1}}$  and a similar sum for  $d_Y(a, l_i)$ . By again splitting the analysis just as in the three induction cases, we can assume that  $a$  lies on the W side of  $S_i$  and that  $d_Y(a, h_{i-1}) \leq x_{h_{i-1}} + y_{h_{i-1}}$ . The bound for the path using the maximal low sequence is similarly obtained:  $d_Y(a, l_i) \leq -y_{l_i} + x_{l_i}$ . The analysis of the three cases thus reduces to the same geometric arguments as in the induction step.  $\square$

We can finally prove Lemma 8.

**Proof of Lemma 8.** As already noted, Lemma 11 implies that if no square  $S_1, \dots, S_k$  is inductive  $d_Y(a, b) \leq (1 + \sqrt{2})x + y$ . Let  $S_j$  be the first inductive square and let  $h_j$  be the

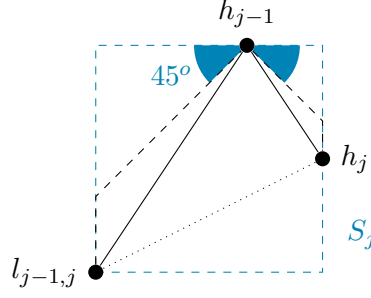


Figure 6: If  $P_1 = d_2(l_j, h_{j-1}) + d_2(h_{j-1}, h_j)$  and  $P_2 = \sqrt{2}(x_{h_j} - x_{l_j}) + (x_{h_j} - x_{l_j}) - (y_{h_j} - y_{l_j})$ , the difference  $P_2 - P_1$  is minimized when  $l_j$  is (essentially) the SW vertex of  $S_j$ . In that case,  $P_2$  is exactly the length of the shown dashed path. By the triangular inequality,  $P_1 \leq P_2$ .

inductive point in  $S_j$ ; the proof when  $l_j$  is the inductive point is similar. The definition of inductive point leads us to consider two cases.

**Case 1:**  $(l_j, h_j) \in \text{Yao}_4^\infty$ . Since  $[l_j h_j]$  has a gentle slope, it follows that  $d_2(l_j, h_j) \leq \sqrt{2}(x_{h_j} - x_{l_j})$ . Let  $l_i, l_{i+1}, \dots, l_{j-1} = l_j$  be the maximal low path ending at  $l_j$ . Note that  $d_Y(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$ . Either  $l_i = l_0 = a$  or  $l_i$  is a promising point in promising square  $S_i$ ; by Lemma 11 we have that  $d_Y(a, l_i) \leq (1 + \sqrt{2})x_{l_i} - y_{l_i}$ . Putting all this together, we get

$$\begin{aligned} d_Y(a, h_j) + (y_{h_j} - y) &\leq d_Y(a, l_i) + d_Y(l_i, l_j) + d_2(l_j, h_j) + y_{h_j} \\ &\leq (1 + \sqrt{2})x_{l_i} - y_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + \sqrt{2}(x_{h_j} - x_{l_j}) + y_{h_j} \\ &\leq \sqrt{2}x_{h_j} + x_{l_j} + y_{h_j} - y_{l_j} \leq (1 + \sqrt{2})x_{h_j} \end{aligned}$$

where the last inequality follows  $x_{l_j} + y_{h_j} - y_{l_j} \leq x_{h_j}$ , i.e. from the assumption that segment  $[l_j h_j]$  has a gentle slope.

**Case 2:**  $(l_j, h_j) \notin \text{Yao}_4^\infty$ . Then, by Lemma 2, edges  $(l_{j-1} = l_j, h_{j-1})$ ,  $(h_{j-1}, h_j = h_{j+1})$ ,  $(l_{j-1} = l_j, l_{j+1})$ , and  $(l_{j+1}, h_j = h_{j+1})$  must exist. Furthermore, by definition of inductive point, either  $[l_j h_{j-1}]$  and  $[h_{j-1} h_j]$  are both gentle, or  $[l_j l_{j+1}]$  and  $[l_{j+1} h_j]$  are both gentle, or  $[h_{j-1} h_j]$  is steep, or  $[l_{j+1} h_j]$  is steep. In the first two cases,  $d_Y(l_j, h_j) \leq \sqrt{2}(x_{h_j} - x_{l_j})$  and the Case 1 argument applies. Of the last two cases, we assume the former as the last case can be handled using a symmetric argument.

As Figure 6 illustrates, because  $[h_{j-1} h_j]$  is steep,

$$d_2(l_j, h_{j-1}) + d_2(h_{j-1}, h_j) \leq \sqrt{2}(x_{h_j} - x_{l_j}) + (x_{h_j} - x_{l_j}) - (y_{h_j} - y_{l_j}).$$

We can now repeat the argument from Case 1, except that we use path  $l_j, h_{j-1}, h_j$  and the above bound on its length instead of edge  $(l_j, h_j)$  and its length bound  $\sqrt{2}(x_{h_j} - x_{l_j})$ .  $\square$

## 4 Conclusion and perspectives

In this paper we determine the stretch factor of  $L_1$ -, and  $L_\infty$ -Delaunay triangulations, and also of the  $\text{Yao}_4^\infty$ -graph. We believe that techniques developed in this paper will help provide

tighter bounds for Delaunay triangulations defined by other regular polygons and, ultimately, tighter bounds for classical Delaunay triangulations.

From routing perspectives, it is interesting to construct routes in geometric graphs that can be determined *locally* from neighbor's coordinates only [BCD09]. Unfortunately, the route that appears implicitly in our proof is built using non-local decisions. It will be interesting to know whether in the  $L_1$ -Delaunay triangulation or in the Yao $_4^\infty$ -graph a route with stretch  $\sqrt{4 + 2\sqrt{2}}$  can be constructed using a local routing algorithm. For TD-Delaunay triangulations, [BFvRV12] showed that there is no local routing algorithm that achieves a stretch that is less than  $5/\sqrt{3} \approx 2.88$ , whereas the stretch factor is actually 2. We leave open the questions regarding the gap between the stretch factor of  $L_1$ -Delaunay triangulations and Yao $_4^\infty$ -graphs and the stretch that is possible using local routing.

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