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# RIGHT-INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE DIFFEOMORPHISM GROUP OF THE CIRCLE

JOACHIM ESCHER AND BORIS KOLEV

ABSTRACT. In this paper, we study the geodesic flow of a right-invariant metric induced by a general Fourier multiplier on the diffeomorphism group of the circle and on some of its homogeneous spaces. This study covers in particular right-invariant metrics induced by Sobolev norms of fractional order. We show that, under a certain condition on the symbol of the inertia operator (which is satisfied for the fractional Sobolev norm  $H^s$  for  $s \geq 1/2$ ), the corresponding initial value problem is well-posed in the smooth category and that the Riemannian exponential map is a smooth local diffeomorphism. Paradigmatic examples of our general setting cover, besides all traditional Euler equations induced by a local inertia operator, the Constantin-Lax-Majda equation, and the Euler-Weil-Petersson equation.

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## 1. INTRODUCTION

The interest for geodesic flows on diffeomorphism groups goes back to Arnold [1]. He recast the Euler equations of hydrodynamics of an ideal fluid as the geodesic flow for the  $L^2$  right invariant Riemannian metric on the volume preserving diffeomorphism group. Arnold's paper was, somehow, rather formal from the analytical point of view. The well-posedness of the geodesic flow was established, subsequently, by Ebin and Marsden in [9]. To do so, they introduced Hilbert manifolds of diffeomorphisms of class  $H^q$ , and used them to approximate the *Fréchet manifold* of smooth diffeomorphisms. This framework was extended thereafter to other equations of physical relevance [7, 20, 23, 25, 31, 37, 11]. Among these studies, right invariant metrics induced by  $H^k$  Sobolev norms ( $k \in \mathbb{N}$ ) on the diffeomorphism group of the circle,  $\text{Diff}^\infty(\mathbb{S}^1)$ , have been extensively investigated [7, 23, 25, 37]. In [11], well-posedness of the geodesic flow for the homogeneous  $H^{1/2}$  right-invariant metric on the homogeneous space  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$  was established. The homogeneous  $H^{3/2}$  right-invariant metric on  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$  was considered in [15].

It is the aim of the present paper to study well-posedness of geodesic flows and their corresponding Euler equation for  $H^s$  Sobolev norms on  $\text{Diff}^\infty(\mathbb{S}^1)$ , where  $s \in \mathbb{R}^+$ . One of the main difficulty which arise immediately, in this context, is that the inertia operator is *non-local*. More precisely, such a *right-invariant* metric on  $\text{Diff}^\infty(\mathbb{S}^1)$  is induced by an inner product

$$\langle u, v \rangle = \int_{\mathbb{S}^1} (Au)v \, dx,$$

on  $\text{Vect}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)$ , where  $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  is a (non-local) *Fourier multiplier*. To be able to make use of the framework proposed by Ebin and Marsden, one needs to extend the metric *smoothly* on  $\mathcal{D}^q(\mathbb{S}^1)$ , the Hilbert manifold of diffeomorphisms of Sobolev class  $H^q$ . When  $A$  is of finite order  $r \geq 0$ , it extends to a bounded linear operator from  $H^q(\mathbb{S}^1)$  to  $H^{q-r}(\mathbb{S}^1)$  for  $q$  large enough, and the smoothness of the metric is reduced to the following question, where

$$R_\varphi : v \mapsto v \circ \varphi, \quad \varphi \in \mathcal{D}^q(\mathbb{S}^1), v \in H^q(\mathbb{S}^1).$$

**Problem.** *Given a Fourier multiplier  $A$  of order  $r \geq 0$ , under which conditions is the mapping*

$$(1) \quad \varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

*smooth?*

Note that the problem is not trivial in general, because the mapping

$$(\varphi, v) \mapsto R_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is *not differentiable* (see [9] for instance), and the mapping  $\varphi \mapsto R_\varphi$  is not even continuous (see remark B.4). It is however a well-known fact that when  $A$  is a differential operator of order  $r$ ,  $A_\varphi$  is a linear differential operator whose coefficients are polynomial expressions of  $1/\varphi_x$  and the derivatives of  $\varphi$  up to order  $r$  (e.g.  $D_\varphi = (1/\varphi_x)D$ ), see [9, 10] for instance. In that case,  $\varphi \mapsto A_\varphi$  is smooth (in fact real analytic) for  $q \geq r$ .

However, for a general Fourier multiplier, we are not aware of any results in this direction. In theorem 3.7, however, we give a *sufficient condition on the symbol* of  $A$  which ensures that the mapping (1) is smooth. This answers a question raised in [9, Appendix A], at least in the case of the diffeomorphism group of the circle. Up to the authors knowledge, these results are new.

*Remark 1.1.* Of course, there are Fourier multiplication operators  $A$ , of order less than 1, for which the mapping

$$\varphi \mapsto A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth. However, the present proof of theorem 3.7 works only for  $r \geq 1$ . So far, the authors have not been able to exhibit a counter-example which would show that the conclusion of theorem 3.7 is false for  $0 \leq r < 1$ . They are not aware either of an example of a Fourier multiplier for which the conclusion of theorem 3.7 fails for all  $q \geq 0$ .

Theorem 3.7 applies, in particular, to the inertia operator  $\Lambda^{2s}$  of the Sobolev metric  $H^s$  on  $\text{Diff}^\infty(\mathbb{S}^1)$  for  $s \in \mathbb{R}$  and  $s \geq 1/2$  (corollary 3.9). This allows us to prove that the corresponding *weak Riemannian* metric and its geodesic spray can be smoothly extended to the Hilbert manifold approximation  $\mathcal{D}^q(\mathbb{S}^1)$  for sufficiently large  $q \in \mathbb{R}$ . As a corollary, we are able to prove local existence and uniqueness of geodesics on  $\mathcal{D}^q(\mathbb{S}^1)$  and  $\text{Diff}^\infty(\mathbb{S}^1)$  (theorem 4.3), as well as the well-posedness of the corresponding *Euler equation* (corollary 4.4).

It is a well known result that the *group exponential map* on  $\text{Diff}^\infty(\mathbb{S}^1)$  is not locally surjective [30]. In [7], it was shown, moreover, that the *Riemannian exponential map* for the  $L^2$  metric on  $\text{Diff}^\infty(\mathbb{S}^1)$  was not a local diffeomorphism. However, due to the fact that the spray of the  $H^s$  metric is smooth for  $s \geq 1/2$  (theorem 3.10), we are able to prove that the exponential map on  $\text{Diff}^\infty(\mathbb{S}^1)$  is a local diffeomorphism, in that case (theorem 5.1). From this fact, we can deduce that given two nearby diffeomorphisms, there is a unique geodesic which joins them and that this geodesic is a *local minimum* of both the arc-length and the energy functionals. For  $s = 1/2$ , this local minimum is, however, not a global minimum [2]. This exhibits a surprising difference with finite dimensional Riemannian geometry.

We close our study by extending our results to Euler equations on some homogeneous spaces of  $\text{Diff}^\infty(\mathbb{S}^1)$ , namely  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ , where  $\text{Rot}(\mathbb{S}^1)$  is the subgroup of all rigid rotations of the circle  $\mathbb{S}^1$  and  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$ , where  $\text{PSL}(2, \mathbb{R})$  is the subgroup of all rigid Möbius transformations which preserve the circle  $\mathbb{S}^1$ . The first case includes the Constantin-Lax-Majda equation [11, 12]. The second case includes the Euler-Weil-Petersson equation, which is related to the Weil-Petersson metric on the universal Teichmüller space  $T(1)$  [33, 38].

The plan of the paper is as follows. In Section 2, we recall basic materials on right-invariant metrics on the diffeomorphism group. Section 3 is devoted to the study of the smoothness of the extended metric and its spray on the Hilbert manifolds  $\mathcal{D}^q(\mathbb{S}^1)$ . In Section 4, we prove local existence and uniqueness of the initial value problem for the geodesics of the right-invariant  $H^s$  metric on  $\text{Diff}^\infty(\mathbb{S}^1)$  and well-posedness of the corresponding

Euler equation. In Section 5 we deal with the Riemannian exponential map and discuss the problem of minimization of the arc-length and the energy. In Section 6 we extend our study to the homogeneous spaces  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$  and  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$ . We prove well-posedness for the corresponding Euler equations. In Appendix A, we prove some lemmas on Fourier multipliers, while in Appendix B, we provide estimates and local boundedness properties for the right translation operator  $R_\varphi$ .

## 2. GEOMETRIC FRAMEWORK

Let  $\text{Diff}^\infty(\mathbb{S}^1)$  be the group of all smooth and orientation preserving diffeomorphisms on the circle. This group is equipped with a *Fréchet manifold* structure, modelled on the *Fréchet vector space*  $C^\infty(\mathbb{S}^1)$  (see Guieu and Roger [18]). Since, moreover, composition and inversion are smooth for this structure, we say that  $\text{Diff}^\infty(\mathbb{S}^1)$  is a *Fréchet-Lie group*, cf. [19]. Its Lie algebra,  $\text{Vect}(\mathbb{S}^1)$ , is the space of smooth vector fields on the circle. It is isomorphic to  $C^\infty(\mathbb{S}^1)$  with the Lie bracket given by

$$[u, w] = u_x w - u w_x.$$

Let  $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  be a  $L^2$ -symmetric, positive definite, continuous linear operator on  $C^\infty(\mathbb{S}^1)$ , we define the following inner product on the Lie algebra  $\text{Vect}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)$

$$\langle u, w \rangle := \int_{\mathbb{S}^1} (Au)w \, dx = \int_{\mathbb{S}^1} u(Aw) \, dx.$$

Translating this inner product on each tangent space, we get one on each tangent space  $T_\varphi \text{Diff}^\infty(\mathbb{S}^1)$ , given by

$$(2) \quad \langle v_1, v_2 \rangle_\varphi = \langle v_1 \circ \varphi^{-1}, v_2 \circ \varphi^{-1} \rangle_{\text{id}} = \int_{\mathbb{S}^1} v_1(A_\varphi v_2) \varphi_x \, dx,$$

where  $v_1, v_2 \in T_\varphi \text{Diff}^\infty(\mathbb{S}^1)$ ,  $A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}$ , and  $R_\varphi(v) := v \circ \varphi$ . One generates this way a *smooth, weak Riemannian metric* on  $\text{Diff}^\infty(\mathbb{S}^1)$ . For historical reasons going back to Euler [13],  $A$  is called the *inertia operator* of the right-invariant metric.

A *covariant derivative* on a Fréchet manifold is a way of differentiating vector fields along paths. In general, a torsion-free, covariant derivative, compatible with a *weak Riemannian metric* does not exist but if it does, it is unique. According to Arnold, a necessary and sufficient for the existence of a covariant derivative compatible with a right-invariant metric on  $\text{Diff}^\infty(\mathbb{S}^1)$  is the existence of the *Arnold bilinear operator*

$$B(u, v) = \frac{1}{2} \left( \text{ad}_u^\top v + \text{ad}_v^\top u \right),$$

where  $u, v \in C^\infty(\mathbb{S}^1)$  and  $\text{ad}_u^\top$  is the adjoint of the operator  $\text{ad}_u$ , with respect to  $A$  (see [1] for instance). If  $\varphi(t)$  is any path in  $\text{Diff}^\infty(\mathbb{S}^1)$  and  $\xi(t)$  is a field of tangent vectors along the path, we define the covariant derivative along the path to be

$$D_t \xi = R_\varphi \left( w_t + \frac{1}{2} [u, w] + B(u, w) \right),$$

where  $u(t) := \varphi_t \circ \varphi^{-1}$  and  $w(t) := \xi(t) \circ \varphi^{-1}$ . One can check that this covariant derivative is torsion-free and metric-compatible.

**Lemma 2.1.** *If  $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  is invertible and commutes with  $d/dx$ , then, the map  $\text{ad}_u^\top$  is well defined and given by*

$$\text{ad}_u^\top w = A^{-1} [2(Aw)u_x + (Aw)_x u],$$

for  $u, w \in C^\infty(\mathbb{S}^1)$ .

*Proof.* We have

$$\langle \text{ad}_u v, w \rangle = \int_{\mathbb{S}^1} (Aw)(u_x v - uv_x) dx = \int_{\mathbb{S}^1} [2(Aw)u_x + (Aw)_x u] v dx$$

where  $u, v, w \in C^\infty(\mathbb{S}^1)$ . But since  $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  is invertible, we get, finally

$$\text{ad}_u^\top w = A^{-1} [2(Aw)u_x + (Aw)_x u].$$

□

A *geodesic* is a path  $\varphi(t)$  in  $\text{Diff}^\infty(\mathbb{S}^1)$ , which is an extremal curve of the energy functional

$$\mathcal{E} := \frac{1}{2} \int_0^1 \langle u(t), u(t) \rangle dt,$$

where  $u(t) = \varphi_t \circ \varphi^{-1}$ . The corresponding Euler-Lagrange equation

$$D_t \varphi_t = 0$$

is equivalent to the following first order equation

$$(3) \quad u_t = -B(u, u) = -A^{-1} \{ (Au)_x u + 2(Au)u_x \},$$

called the *Euler equation*.

*Example.* For the  $L^2$ -metric ( $A = I$ ), the corresponding Euler equation (3) is the *inviscid Burgers equation*

$$u_t + 3uu_x = 0.$$

*Example.* For the  $H^1$ -metric ( $A = I - d^2/dx^2$ ), the corresponding Euler equation (3) is the *Camassa-Holm equation*

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0,$$

a model in the theory of shallow water waves [5, 14].

Let  $\varphi$  be the flow of the time dependent vector field  $u$  i.e.,  $\varphi_t = u \circ \varphi$  and let  $v = \varphi_t$ . Then  $u$  solves the Euler equation (3), if and only if,  $(\varphi, v)$  is a solution of

$$(4) \quad \begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases}$$

where

$$S_\varphi(v) := (R_\varphi \circ S \circ R_{\varphi^{-1}})(v),$$

and

$$S(u) := A^{-1} \{ [A, u]u_x - 2(Au)u_x \}.$$

The *second order vector field* on  $\text{Diff}^\infty(\mathbb{S}^1)$ , defined by

$$(5) \quad F : (\varphi, v) \mapsto (\varphi, v, v, S_\varphi(v))$$

is called the *geodesic spray*, following Lang [26].

Suppose now that  $A$  is a differential operator of order  $r$ . Then, the right hand side of the Euler equation is of *order 1*. It is however quite surprising that, in Lagrangian coordinates, the propagator of evolution equation of the geodesic flow has better mapping properties, provided that the *order  $r$*  of  $A$  is not less than 1. Indeed, in that case, the quadratic operator

$$S(u) := A^{-1} \{[A, u]u_x - 2(Au)u_x\}$$

is of order 0 because the commutator  $[A, u]$  is of order less than  $\leq r - 1$ . One might expect, that for a larger class of operators  $A$ , the quadratic operator  $S$  to be of order 0 and the second order system (4) to be the local expression of an ODE on some suitable Banach manifold.

This observation is at the root of a strategy proposed in the 70' by Ebin and Marsden [9] to study well-posedness of the Euler equation. Following their approach, if we can prove *local existence and uniqueness of geodesics* (ODE) on diffeomorphism groups then the PDE (Euler equation) is *well-posed*. To do so, it is necessary to introduce an approximation of the Fréchet–Lie group  $\text{Diff}^\infty(\mathbb{S}^1)$  by *Hilbert manifolds*. Let  $H^q(\mathbb{S}^1)$  be the completion of  $C^\infty(\mathbb{S}^1)$  for the norm

$$\|u\|_{H^q} := \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^q |\hat{u}_k|^2 \right)^{1/2},$$

where  $q \in \mathbb{R}, q \geq 0$ . We recall that  $H^q(\mathbb{S}^1)$  is a multiplicative algebra for  $q > 1/2$  (cf. [40, Theorem 2.8.3]). This means that

$$\|uv\|_{H^q(\mathbb{S}^1)} \lesssim \|u\|_{H^q(\mathbb{S}^1)} \|v\|_{H^q(\mathbb{S}^1)}, \quad u, v \in H^q(\mathbb{S}^1).$$

**Definition 2.2.** We say that a  $C^1$  diffeomorphism  $\varphi$  of  $\mathbb{S}^1$  is of class  $H^q$  if for any of its lifts to  $\mathbb{R}$ ,  $\tilde{\varphi}$ , we have

$$\tilde{\varphi} - \text{id} \in H^q(\mathbb{S}^1).$$

For  $q > 3/2$ , the set  $\mathcal{D}^q(\mathbb{S}^1)$  of  $C^1$ -diffeomorphisms of the circle which are of class  $H^q$  has the structure of a *Hilbert manifold*, modelled on  $H^q(\mathbb{S}^1)$  (see [9]). The manifold  $\mathcal{D}^q(\mathbb{S}^1)$  is also a *topological group* but *not a Lie group* (composition and inversion in  $\mathcal{D}^q(\mathbb{S}^1)$  are continuous but *not differentiable*). Note however, that, given  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ ,

$$u \mapsto R_\varphi(u) := u \circ \varphi, \quad H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is a *smooth map*, and that

$$(u, \varphi) \mapsto u \circ \varphi, \quad H^{q+k}(\mathbb{S}^1) \times \mathcal{D}^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is of class  $C^k$ .

The Fréchet Lie group  $\text{Diff}^\infty(\mathbb{S}^1)$  may be viewed as an inverse limit of *Hilbert manifolds* (ILH)

$$\text{Diff}^\infty(\mathbb{S}^1) = \bigcap_{q > \frac{3}{2}} \mathcal{D}^q(\mathbb{S}^1),$$

and we call the scales of manifolds  $\mathcal{D}^q(\mathbb{S}^1)_{q > 3/2}$ , a Hilbert manifold approximation of  $\text{Diff}^\infty(\mathbb{S}^1)$ .

*Remark 2.3.* Note that the tangent bundle of the Hilbert manifold  $\mathcal{D}^q(\mathbb{S}^1)$  is trivial. Indeed, let  $\mathfrak{t} : T\mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{R}$  be a trivialisation of the tangent bundle of the circle. Then

$$\Psi : T\mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), \quad \xi \mapsto \mathfrak{t} \circ \xi$$

defines a smooth vector bundle isomorphism because  $\mathfrak{t}$  is smooth (see [9, Page 107]).

Within this framework, the case where the inertia operator  $A$  is a *differential* operator with constant coefficients has been extensively studied in the literature (see for instance [6, 7, 10]). It is the aim of the present paper to extend these results when  $A$  is a general *Fourier multiplier*, that is, a continuous linear operator on  $C^\infty(\mathbb{S}^1)$ , which commutes with  $D := d/dx$ . In that case, we get

$$(Au)(x) = \sum_{k \in \mathbb{Z}} a(k) \hat{u}(k) \exp(2i\pi kx),$$

where  $\hat{u}(k)$  is the  $k$ -th Fourier coefficients of  $u$  (see lemma A.1). The sequence  $a : \mathbb{Z} \rightarrow \mathbb{C}$  is called the *symbol* of  $A$  and we shall use the notation  $A = \mathbf{op}(a(k))$ . When  $a(k) = \mathcal{O}(|k|^r)$ , the Fourier multiplier  $A = \mathbf{op}(a(k))$  extends, for each  $q \geq r$ , to a bounded linear operator in  $\mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ . It is said to be of order  $r$ .

*Example.* The inertia operator for the  $H^s$  Sobolev metric ( $s \geq 1/2$ ), defined by

$$\Lambda_{2s} := \mathbf{op}((1 + k^2)^s)$$

is of order  $2s \geq 1$ .

### 3. SMOOTHNESS OF THE EXTENDED METRIC AND SPRAY

The fact that a right-invariant metric, defined by (2), and its geodesic spray  $F$ , defined by (5) are smooth on  $\text{Diff}^\infty(\mathbb{S}^1)$  is unfortunately useless to establish the well-posedness of the geodesic flow. What we need to do is to study under which conditions, the metric and its spray *can be extended smoothly* to the Hilbert approximation manifolds  $\mathcal{D}^q(\mathbb{S}^1)$ . In this section, we provide a criteria on the inertia operator  $A$  (satisfied by almost all known examples) which ensures the smoothness of the metric on the extended manifolds  $\mathcal{D}^q(\mathbb{S}^1)$ , for  $q$  large enough.

For general materials on Banach manifolds, we refer to [26]. Let  $X$  be a Banach manifold modelled over a Banach space  $E$ . We recall that a Riemannian metric  $g$  on  $X$  is a smooth, symmetric, positive definite, covariant 2-tensor field on  $X$ . In other words, we have for each  $x \in X$  a symmetric, positive definite, bounded, bilinear form  $g(x)$  on  $T_x X$  and, in any local chart  $U$ , the mapping

$$x \rightarrow g(x), \quad U \rightarrow \mathcal{L}_{\text{sym}}^2(E, \mathbb{R})$$

is smooth. Given any  $x \in X$ , we can then consider the bounded, linear operator

$$h_x : T_x X \rightarrow T_x^* X,$$



called the *flat map* and defined by  $h_x(\xi_x) = g(x)(\xi_x, \cdot)$ . The metric is *strong* if  $h_x$  is a topological linear isomorphism for all  $x \in X$ , whereas it is *weak* if  $h_x$  is only injective for all  $x \in X$ .

Given  $A \in \text{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ , it induces a *bounded inner product* on each tangent space,  $T_\varphi \mathcal{D}^q(\mathbb{S}^1)$  for each  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ , given by

$$\langle v_1, v_2 \rangle_\varphi = \int_{\mathbb{S}^1} v_1(A_\varphi v_2) \varphi_x dx,$$

where  $A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}$ . To conclude, however, that the family  $\langle \cdot, \cdot \rangle_\varphi$  defines a (weak) Riemannian metric on the Banach manifold  $\mathcal{D}^q(\mathbb{S}^1)$ , we need to show that the mapping

$$\varphi \mapsto \varphi_x A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}^2(H^q(\mathbb{S}^1), H^{-q}(\mathbb{S}^1)),$$

is smooth.

*Remark 3.1.* Note that even when the flat map

$$\tilde{A} : (\varphi, v) \mapsto (\varphi, \varphi_x A_\varphi v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^{-q}(\mathbb{S}^1)$$

is smooth and defines an injective vector bundle morphism, its image

$$\mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$$

is not a subbundle of  $T^*\mathcal{D}^q(\mathbb{S}^1)$  in the sense of [26, III.3], because  $H^{q-r}(\mathbb{S}^1)$  is *not a closed subspace* of  $H^{-q}(\mathbb{S}^1)$ .

For  $q > 3/2$ , the mappings  $\varphi \mapsto \varphi_x$  and  $\varphi \mapsto 1/\varphi_x$  are smooth from  $\mathcal{D}^q(\mathbb{S}^1) \rightarrow H^{q-1}(\mathbb{S}^1)$ . Thus, for  $r \geq 1$  and  $q - r \geq 0$ , lemma B.1 shows that the metric is smooth if and only if

$$\varphi \mapsto A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth. If this holds, we can compute, for each  $n \geq 1$ , the  $n$ -th Fréchet differential<sup>1</sup>

$$\partial_\varphi^n A_\varphi \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)).$$

which is itself smooth.

**Lemma 3.2.** *We have*

$$(6) \quad \partial_\varphi^n A_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi A_n R_\varphi^{-1}(v, \delta\varphi_1, \dots, \delta\varphi_n),$$

where

$$A_n := \partial_{\text{id}} A_\varphi \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is the  $(n+1)$ -linear operator defined inductively by  $A_0 = A$  and

$$(7) \quad A_{n+1}(u_0, u_1, \dots, u_{n+1}) = [\nabla_{u_{n+1}}, A_n(\cdot, u_1, \dots, u_n)]u_0 \\ - \sum_{k=1}^n A_n(u_0, u_1, \dots, \nabla_{u_{n+1}} u_k, \dots, u_n),$$

where  $\nabla$  is the canonical connection on the Lie group  $\mathbb{S}^1$ .

<sup>1</sup>We have chosen to denote by  $\partial$  the Fréchet differential to avoid the confusion with the already used notation  $D = d/dx$ .

*Remark 3.3.* For  $n = 1$ , we get

$$A_1(u_0, u_1) = [\nabla_{u_1}, A]u_0,$$

and for  $n = 2$ , we get

$$A_2(u_0, u_1, u_2) = ([\nabla_{u_2}, [\nabla_{u_1}, A]] - [\nabla_{\nabla_{u_2}u_1}, A])u_0.$$

*Proof.* We will make the computations for smooth functions. The general result follows from a density argument and the fact that the expressions are continuous. Formula (6) is trivially true for  $n = 0$ . Now suppose it is true for some  $n \in \mathbb{N}$ , that is

$$\partial_\varphi^n A_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi A_n R_\varphi^{-1}(v, \delta\varphi_1, \dots, \delta\varphi_n),$$

Let  $\varphi(s)$  be a smooth path in  $\text{Diff}^\infty(\mathbb{S}^1)$  such that

$$\varphi(0) = \varphi, \quad \partial_s \varphi(s)|_{s=0} = \delta\varphi_{n+1}.$$

Set  $u_k = \delta\varphi_k \circ \varphi^{-1}$ , for  $1 \leq k \leq n+1$  and  $u_0 = v \circ \varphi^{-1}$ . We compute first

$$\partial_s \{R_{\varphi(s)} w\}_{s=0} = R_\varphi(u_{n+1} w_x),$$

for  $w \in C^\infty(\mathbb{S}^1)$ , and

$$\partial_s \{R_{\varphi(s)}^{-1} w\}_{s=0} = -u_{n+1} (R_\varphi^{-1} w)_x,$$

for  $w \in C^\infty(\mathbb{S}^1)$ . Hence

$$\begin{aligned} \partial_s \{R_\varphi A_n R_\varphi^{-1}(v, \delta\varphi_1, \dots, \delta\varphi_n)\}_{s=0} = \\ R_\varphi \{u_{n+1} (A_n(u_0, \dots, u_n))_x\} - \sum_{k=0}^n R_\varphi A_n(u_0, \dots, u_{n+1}(u_k)_x, \dots, u_n), \end{aligned}$$

which gives the recurrence relation (7), since

$$\begin{aligned} u_{n+1} (A_n(u_0, \dots, u_n))_x - A_n(u_{n+1}(u_0)_x, u_1, \dots, u_n) = \\ [\nabla_{u_{n+1}}, A_n(\cdot, u_1, \dots, u_n)]u_0, \end{aligned}$$

□

We shall prove now the following *necessary and sufficient condition* for smoothness.

**Theorem 3.4** (Smoothness Theorem). *Let  $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$  be a continuous linear operator of order  $r \geq 1$ . Let  $q > 3/2$  with  $q - r \geq 0$ . Then*

$$\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

*is smooth, if and only if, each  $A_n$  extends to a bounded  $(n+1)$ -linear operator in  $\mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ .*

The idea of the proof of theorem 3.4, which is inductive, is the following. First, we show that if  $A_n$  is bounded, then the mapping

$$\varphi \mapsto A_{n,\varphi} := R_\varphi A_n R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is locally bounded. Then, we prove that if  $A_{n+1,\varphi}$  is locally bounded, then  $A_{n,\varphi}$  is locally Lipschitz. Finally we show that if  $A_{n+1,\varphi}$  is locally Lipschitz, then  $A_{n,\varphi}$  is  $C^1$ . The full detail proof, given below, requires the following two elementary lemmas, which will be stated without proof.

**Lemma 3.5.** *Let  $X$  be a topological space and  $E$  a Banach space. Let  $f : [0, 1] \times X \rightarrow E$  be a continuous mapping. Then the mapping*

$$g(x) := \int_0^1 f(t, x) dt$$

*is continuous.*

**Lemma 3.6.** *Let  $E, F$  be Banach spaces and  $U$  a convex, open set in  $E$ . Let  $\alpha : U \rightarrow \mathcal{L}(E, F)$  be a continuous mapping and  $f : U \rightarrow F$  a mapping such that*

$$f(y) - f(x) = \int_0^1 \alpha(ty + (1-t)x)(y-x) dt,$$

*for all  $x, y \in U$ . Then  $f$  is  $C^1$  on  $U$  and  $df = \alpha$ .*

*Proof of theorem 3.4.* Note first that for  $q > 3/2$  and  $q-r \geq 0$ , the mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^{q-r}(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is locally bounded (lemma B.2) and that the mapping

$$(\varphi, v) \mapsto v \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is continuous (corollary B.3). Since, moreover, the mapping

$$\varphi \mapsto \varphi^{-1}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1)$$

is continuous for  $q > 3/2$  (see [9, 22]), we conclude that

$$\varphi \mapsto R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^q(\mathbb{S}^1))$$

is locally bounded and that

$$(\varphi, v) \mapsto v \circ \varphi^{-1}, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is continuous.

The fact that the boundedness of the  $A_n$  is a necessary condition results from lemma 3.2. Conversely, suppose that each  $A_n$  is bounded in  $\mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ , then

$$A_{n,\varphi} := R_\varphi A_n R_{\varphi^{-1}} \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)),$$

is a bounded  $(n+1)$ -linear operator, with

$$\begin{aligned} \|A_{n,\varphi}\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} &\leq \\ &\|R_\varphi\|_{\mathcal{L}(H^{q-r}, H^{q-r})} \|A_n\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} \|R_{\varphi^{-1}}\|_{\mathcal{L}(H^q, H^q)}^{n+1}, \end{aligned}$$

and we conclude, thus, that

$$\varphi \mapsto A_{n,\varphi}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is locally bounded, for each  $n \in \mathbb{N}$ .

(a) We will first show that  $\varphi \mapsto A_{n,\varphi}$  is *locally Lipschitz continuous*<sup>2</sup>, for each  $n \in \mathbb{N}$ . Let  $\psi \in \mathcal{D}^q(\mathbb{S}^1)$  be given. Because  $\varphi \mapsto A_{n+1,\varphi}$  is *locally*

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<sup>2</sup>On  $\mathcal{D}^q(\mathbb{S}^1)$ , we did not introduce any distance compatible with the topology. The concept of a *locally Lipschitz mapping*  $f : M \rightarrow E$ , from a Banach manifold  $M$  to a Banach vector space  $E$  does not require such an additional structure. It is defined using a local chart, and then shown to be independent of the choice of the particular chart.

bounded, it is possible to find a neighbourhood  $U$  of  $\psi$  and a positive constant  $K$  such that

$$\|A_{n+1,\varphi}\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} \leq K, \quad \forall \varphi \in U.$$

We can further assume (using a local chart) that  $U$  is a ball in  $H^q(\mathbb{S}^1)$ . Pick now  $\varphi_0$  and  $\varphi_1$  in  $\text{Diff}^\infty(\mathbb{S}^1) \cap U$  and set  $\varphi(t) := (1-t)\varphi_0 + t\varphi_1$  for  $t \in [0, 1]$ . Choosing  $v_0, \dots, v_n \in C^\infty(\mathbb{S}^1)$  with  $\|v_j\|_{H^q} \leq 1$ , we obtain from lemma 3.2 that

$$A_{n,\varphi_1}(v_0, \dots, v_n) - A_{n,\varphi_0}(v_0, \dots, v_n) = \int_0^1 A_{n+1,\varphi(t)}(v_0, \dots, v_n, \varphi_1 - \varphi_0) dt.$$

This implies

$$\|A_{n,\varphi_1}(v_0, \dots, v_n) - A_{n,\varphi_0}(v_0, \dots, v_n)\|_{H^{q-r}} \leq K \|\varphi_1 - \varphi_0\|_{H^q},$$

for all  $v_0, \dots, v_n \in C^\infty(\mathbb{S}^1)$  with  $\|v_j\|_{H^q} \leq 1$ . The assertion that  $A_{n,\varphi}$  is Lipschitz continuous follows from the density of the embedding  $C^\infty(\mathbb{S}^1) \hookrightarrow H^q(\mathbb{S}^1)$ , and continuity of the mapping

$$(\varphi, v_0, \dots, v_n) \mapsto A_{n,\varphi}(v_0, \dots, v_n), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)^{n+1} \rightarrow H^{q-r}(\mathbb{S}^1).$$

(b) We will now show by induction, that  $\varphi \mapsto A_\varphi$  is of class  $C^n$  for all  $n \in \mathbb{N}$ , and that its  $n$ -th Fréchet derivative is  $A_{n,\varphi}$ . For each  $n \geq 1$ , let

$$\alpha_n : \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}\left(H^q(\mathbb{S}^1), \mathcal{L}^n(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))\right),$$

be the *Lipschitz continuous* mapping defined by

$$\alpha_n(\varphi) := [\delta\varphi_n \mapsto A_{n,\varphi}(\cdot, \dots, \cdot, \delta\varphi_n)].$$

Let  $U$  be a local chart in  $\mathcal{D}^q(\mathbb{S}^1)$ , that we choose to be a convex open subset of  $H^q(\mathbb{S}^1)$ . By its very definition, we have

$$A_{\varphi_1}(v) - A_{\varphi_0}(v) = \int_0^1 A_{1,t\varphi_1+(1-t)\varphi_0}(v, \varphi_1 - \varphi_0) dt,$$

for all  $\varphi_0, \varphi_1 \in U \cap C^\infty(\mathbb{S}^1)$  and  $v \in C^\infty(\mathbb{S}^1)$ . But, the continuity of the mapping  $\varphi \mapsto A_{1,\varphi}$ , together with lemma 3.5, and the density of the embedding  $C^\infty(\mathbb{S}^1) \hookrightarrow H^q(\mathbb{S}^1)$ , permit to conclude that this formula is still true for all  $\varphi_0, \varphi_1 \in U$  and  $v \in H^q(\mathbb{S}^1)$ . Therefore, we can write in  $\mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$

$$A_{\varphi_1} - A_{\varphi_0} = \int_0^1 \alpha_1(t\varphi_1 + (1-t)\varphi_0)(\varphi_1 - \varphi_0) dt,$$

and, by virtue of lemma 3.6, we conclude that  $\varphi \mapsto A_\varphi$  is  $C^1$  and that  $DA_\varphi = \alpha_1$ . A similar argument shows that, for each  $n \geq 1$ , we have

$$\alpha_n(\varphi_1) - \alpha_n(\varphi_0) = \int_0^1 \alpha_{n+1}(t\varphi_1 + (1-t)\varphi_0)(\varphi_1 - \varphi_0) dt,$$

and hence that  $\alpha_n$  is  $C^1$  with  $D\alpha_n = \alpha_{n+1}$ . This completes the proof.  $\square$

When  $A$  is a Fourier multiplier, a criteria on the symbol  $a$  of  $A$  which ensures that all the  $A_n$  are bounded and thus that the metric is smooth is given below. It's proof is a direct consequence of lemma A.4, lemma A.6 and corollary A.7 in Appendix A.

**Theorem 3.7.** *Let  $A = \mathbf{op}(a(k))$  be a Fourier multiplier of order  $r \geq 1$ . Suppose that its symbol  $a$  extends to  $\mathbb{R}$  and that for each  $n \geq 1$ , the function*

$$f_n(\xi) := \xi^{n-1}a(\xi)$$

*is of class  $C^{n-1}$ , that  $f_n^{(n-1)}$  is absolutely continuous and that there exists  $C_n > 0$  such that*

$$(8) \quad \left| f_n^{(n)}(\xi) \right| \leq C_n(1 + \xi^2)^{(r-1)/2},$$

*almost everywhere. Then,*

$$\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

*is smooth for  $q > 3/2$  and  $q - r \geq 0$ .*

*Remark 3.8.* This criteria is always satisfied when the symbol  $A = \mathbf{op}(a(k))$  belongs to the class  $\mathcal{S}^r$ , that is when  $a$  can be extended to a smooth function on  $\mathbb{R}$  such that

$$(9) \quad a^{(k)}(\xi) = \mathcal{O}(|\xi|^{r-k}), \quad \forall k \in \mathbb{N}.$$

This applies in particular to the inertia operator  $\Lambda^{2s}$  of the Sobolev metric  $H^s$ , when  $s \geq 1/2$ . Indeed, let  $a_s(\xi) := (1 + \xi^2)^s$  be the symbol of  $\Lambda^{2s}$ . One can check that

$$a_s^{(k)}(\xi) = \frac{p_k(\xi)}{(1 + \xi^2)^k} a_s(\xi),$$

for  $k \geq 1$ , where  $p_k$  is a polynomial function with  $d(p_k) \leq k$ . Thus, (9) is true for  $a_s$ , and we have the following result.

**Corollary 3.9.** *Let  $s \in \mathbb{R}$  and  $\Lambda^{2s} := \mathbf{op}((1 + n^2)^s)$ . If  $s \geq 1/2$  then the mapping*

$$\varphi \mapsto R_\varphi \circ \Lambda^{2s} \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-2s}(\mathbb{S}^1))$$

*is smooth for  $q > 3/2$  and  $q - 2s \geq 0$ .*

On a finite dimensional manifold, as soon as the metric is  $C^k$ , the geodesic spray is  $C^{k-1}$ , because the components of the spray, in any local chart, involve the *Christoffel symbols* which depend on the first derivatives of the metric. We might, therefore, expect some kind of analog results to hold for a weak Riemannian metric on a Banach manifold, as soon as the spray exists.

**Theorem 3.10.** *Let  $A$  be a Fourier multiplier of order  $r \geq 1$  and let  $q > 3/2$ , with  $q - r \geq 0$ . Suppose, moreover, that*

$$\varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \text{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

*is smooth. Then the geodesic spray*

$$(\varphi, v) \mapsto S_\varphi(v) = R_\varphi \circ S \circ R_{\varphi^{-1}}(v),$$

*where*

$$S(u) = A^{-1} \{ [A, u]u_x - 2(Au)u_x \}.$$

*extends smoothly to  $T\mathcal{D}^q(\mathbb{S}^1) = \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ .*

*Proof.* Let  $P(u) := (Au)u_x$  and  $Q(u) := [A, u]u_x$ . Then,

$$S_\varphi(v) = A_\varphi^{-1} \{Q_\varphi(v) - 2P_\varphi(v)\}.$$

The proof reduces to establish, using the chain rule, that the three mappings

$$(\varphi, v) \mapsto P_\varphi(v), \quad (\varphi, v) \mapsto Q_\varphi(v), \quad (\varphi, w) \mapsto A_\varphi^{-1}(w)$$

are smooth.

(a) We have  $P_\varphi(v) = (A_\varphi(v))(D_\varphi(v))$ . But

$$(\varphi, v) \mapsto A_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth by hypothesis, whereas

$$(\varphi, v) \mapsto D_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-1}(\mathbb{S}^1)$$

is smooth since  $D_\varphi(v) = v_x/\varphi_x$  and  $H^{q-1}(\mathbb{S}^1)$  is a multiplicative algebra for  $q > 3/2$ . To conclude that

$$(\varphi, v) \mapsto P_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth, we use the fact that pointwise multiplication extends to a bounded bilinear mapping

$$H^{q-r}(\mathbb{S}^1) \times H^{q-1}(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1),$$

if  $q-1 > 1/2$  and  $0 \leq q-r \leq q-1$  (c.f. lemma B.1).

(b) By virtue of lemma 3.2, we have

$$\partial_\varphi A_\varphi(v, v) = A_{1, \varphi}(v, v) = -Q_\varphi(v),$$

and therefore

$$(\varphi, v) \mapsto Q_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth.

(c) The set  $\text{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$  is open in  $\mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$  and the mapping

$$P \mapsto P^{-1}, \quad \text{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)) \rightarrow \text{Isom}(H^{q-r}(\mathbb{S}^1), H^q(\mathbb{S}^1))$$

is smooth (it is even real analytic). Besides  $A_\varphi \in \text{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ , for all  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ , and the mapping

$$\varphi \mapsto A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \text{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth. Thus

$$(\varphi, w) \mapsto A_\varphi^{-1}(w), \quad H^{q-r}(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is smooth.  $\square$

We have, in particular, the following corollary.

**Corollary 3.11.** (*Smoothness of the  $H^s$  metric and its spray*) *Let  $s \geq 1/2$  and assume that  $q > 3/2$ ,  $q - 2s \geq 0$ . Then the right-invariant, weak Riemannian metric defined on  $\text{Diff}^\infty(\mathbb{S}^1)$  by the inertia operator  $A = \Lambda^{2s}$  extends to a smooth weak Riemannian metric on the Banach manifold  $\mathcal{D}^q(\mathbb{S}^1)$  with a smooth geodesic spray.*

*Remark 3.12.* When  $A$  is a *differential operator*, theorem 3.10 can be sharpened; in that case, we can conclude that the spray is smooth when  $q > 3/2$  and  $q \geq r - 1$ . Indeed, we have

$$\int_{\mathbb{S}^1} \{[A, u]u_x - 2(Au)u_x\} dx = 0,$$

so that  $[A, u]u_x - 2(Au)u_x = DB(u)$  where  $B$  is a quadratic, differential operator of order  $r - 1$  (see [34, Chapter 4]). Hence

$$(\varphi, v) \mapsto B_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r+1}(\mathbb{S}^1)$$

is smooth for  $q \geq r - 1$ . Moreover, the symbol  $a$  of  $A$  is a real, even polynomial (because  $A$  is  $L^2$ -symmetric) with no real roots (because  $A$  is invertible). Therefore,  $A$  can be written as

$$A = A'(\alpha - iD)(\bar{\alpha} - iD),$$

where  $\alpha \in \mathbb{C} \setminus \mathbb{R}$  and  $A'$  is an invertible, differential operator of degree  $r - 2$ . We have thus

$$A^{-1}D = (A')^{-1} \frac{1}{2} \left( \frac{\alpha}{\operatorname{Im} \alpha} (\alpha - iD)^{-1} - \frac{\bar{\alpha}}{\operatorname{Im} \alpha} (\bar{\alpha} - iD)^{-1} \right).$$

The conclusion follows now from the fact that

$$\varphi \mapsto A'_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \operatorname{Isom}(H^q(\mathbb{S}^1), H^{q-r+2}(\mathbb{S}^1))$$

is smooth if  $q > 3/2$  and  $q \geq r - 2$  and that

$$\varphi \mapsto (\alpha - iD)_\varphi, (\bar{\alpha} - iD)_\varphi \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \operatorname{Isom}(H^q(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))$$

are smooth if  $q > 3/2$ . This applies in particular for the Camassa-Holm equation where the spray is smooth for  $q > \frac{3}{2}$  (in [32], it was proved that the spray is of class  $C^1$  for  $q > \frac{3}{2}$ ).

#### 4. WELL-POSEDNESS

In this section, we will prove local existence and uniqueness of the initial value problem for the geodesics of the right-invariant  $H^s$  metric on the Fréchet-Lie group  $\operatorname{Diff}^\infty(\mathbb{S}^1)$ , and more generally for any right-invariant weak Riemannian metric for which the inertia operator  $A$  is such that

$$(10) \quad \varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \operatorname{Isom}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth, for  $q > 3/2$  and  $q - r \geq 0$ . Under these assumptions, the metric admits a smooth spray  $F_q$  defined on  $T\mathcal{D}^q(\mathbb{S}^1)$  (c.f. theorem 3.10) and we can apply the Picard-Lindelöf theorem. For each  $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$ , there exists a *unique non-extendable* solution

$$(\varphi, v) \in C^\infty(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{S}^1)),$$

of the Cauchy problem

$$(11) \quad \begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases}$$

with  $\varphi(0) = \varphi_0$  and  $v(0) = v_0$ , defined on some *maximal interval of existence*  $J_q(\varphi_0, v_0)$ , which is open and contains 0. Note that in general  $J_q(\varphi_0, v_0) \neq \mathbb{R}$ , meaning that the solutions are not *global*.

To prove well-posedness of the Cauchy problem (11) on the smooth manifold  $T\text{Diff}^\infty(\mathbb{S}^1)$ , we need precise regularity properties of solutions to (11) on each Hilbert approximation manifold  $T\mathcal{D}^q(\mathbb{S}^1)$ . More precisely, assume that  $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ . Then, we may solve (11) in  $T\mathcal{D}^q(\mathbb{S}^1)$  and in  $T\mathcal{D}^{q+1}(\mathbb{S}^1)$ . Since solutions on each level are non-extendable, we clearly have

$$(12) \quad J_{q+1}(\varphi_0, v_0) \subset J_q(\varphi_0, v_0),$$

which could lead to

$$\bigcap_q J_q(\varphi_0, v_0) = \{0\}.$$

The remarkable observation that the maximal interval of existence is independent of the parameter  $q$ , due to the right-invariance of the spray (cf. lemma 4.1) was pointed out in [9, Theorem 12.1]. This makes it possible to avoid Nash-Moser type schemes to prove local existence of smooth geodesics.

**Lemma 4.1** (No loss, nor Gain). *Given  $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ , we have*

$$J_{q+1}(\varphi_0, v_0) = J_q(\varphi_0, v_0),$$

for  $q > (3/2)$  and  $q - r \geq 0$ .

*Proof.* Let  $\Phi_q$  be the flow of the spray  $F_q$  and  $R_s$  be the (right) action of the rotation group  $\mathbb{S}^1$  on  $\mathcal{D}^q(\mathbb{S}^1)$ , defined by

$$(R_s \cdot \varphi)(x) := \varphi(x + s), \quad \varphi \in \mathcal{D}^q(\mathbb{S}^1), \quad x \in \mathbb{S}^1.$$

This action induces an action on  $T\mathcal{D}^q(\mathbb{S}^1)$  given by

$$(R_s \cdot (\varphi, v))(x) := (\varphi(x + s), v(x + s)), \quad (\varphi, v) \in T\mathcal{D}^q(\mathbb{S}^1), \quad x \in \mathbb{S}^1.$$

Note that if  $(\varphi, v) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ , then<sup>3</sup>

$$s \mapsto R_s \cdot (\varphi, v), \quad \mathbb{S}^1 \rightarrow T\mathcal{D}^q(\mathbb{S}^1)$$

is a  $C^1$  map, and that

$$\frac{d}{ds} R_s \cdot (\varphi, v) = (\varphi_x, v_x).$$

Therefore, if  $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ , we get

$$(13) \quad \left. \frac{d}{ds} \right|_{s=0} \Phi_q(t, R_s \cdot (\varphi_0, v_0)) = \partial_{(\varphi, v)} \Phi_q(t, (\varphi_0, v_0)) \cdot (\varphi_{0,x}, v_{0,x}).$$

On the other hand, the spray  $F_q$  is invariant under each right-invariant translation  $R_\varphi$  where  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ . The same is true for its flow  $\Phi_q$ , and hence

$$\Phi_q(t, R_s \cdot (\varphi_0, v_0)) = R_s \cdot \Phi_q(t, (\varphi_0, v_0)) \quad \text{for all } t \in J_q(\varphi_0, v_0), \quad s \in \mathbb{R}.$$

We get thus

$$\partial_{(\varphi, v)} \Phi_q(t, (\varphi_0, v_0)) \cdot (\varphi_{0,x}, v_{0,x}) = (\varphi_x(t), v_x(t)).$$

But  $\partial_{(\varphi, v)} \Phi_q(t, (\varphi_0, v_0)) \cdot (\varphi_{0,x}, v_{0,x})$  belongs to  $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ , and hence

$$(\varphi(t), v(t)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1) \quad \text{for all } t \in J_q(\varphi_0, v_0).$$

---

<sup>3</sup>We will avoid to write  $TR_s, T(TR_s), \dots$  and simply keep the notation  $R_s$ .



We conclude therefore that

$$J_q(\varphi_0, v_0) = J_{q+1}(\varphi_0, v_0),$$

which completes the proof.  $\square$

*Remark 4.2.* Lemma 4.1 states that there is no loss of spatial regularity during the evolution of (11). By reversing the time direction, it follows from the unique solvability that there is also no gain of regularity in the following sense: Let  $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$  be given and assume that  $(\varphi(t_1), v(t_1)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$  for some  $t_1 \in J_q(\varphi_0, v_0)$ . Then  $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ .

We get therefore the following local existence result.

**Theorem 4.3.** *Let (10) be satisfied and consider the geodesic flow on the tangent bundle  $T\text{Diff}^\infty(\mathbb{S}^1)$  induced by the inertia operator  $A$ . Then, given any  $(\varphi_0, v_0) \in T\text{Diff}^\infty(\mathbb{S}^1)$ , there exists a unique non-extendable solution*

$$(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$$

of (11) on the maximal interval of existence  $J$ , which is open and contains 0.

And we obtain well-posedness of the Euler equation.

**Corollary 4.4.** *The corresponding Euler equation has for any initial data  $u_0 \in C^\infty(\mathbb{S}^1)$  a unique non-extendable smooth solution*

$$u \in C^\infty(J, C^\infty(\mathbb{S}^1)).$$

The maximal interval of existence  $J$  is open and contains 0.

It is known that the Euler equation induced by the inertia operator

$$A = \mathbf{op}(1 + k^2)$$

leads to the classical periodic Camassa-Holm equation

$$(14) \quad u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{S}^1,$$

cf. [7]. It may be interesting to briefly discuss another possible option for  $A$ , namely

$$A = \mathbf{op}(|k|^r + \delta_0(k)).$$

Observe that Theorem 3.7 is applicable provided  $r \geq 1$ . In that case, the mapping

$$\varphi \mapsto R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for  $q > (3/2)$  and  $q-r \geq 0$ . Since in addition,  $A$  is a topological linear isomorphism from  $H^q(\mathbb{S}^1)$  onto  $H^{q-r}(\mathbb{S}^1)$ , the operator  $A$  satisfies clearly assumption (10) and thus Theorem 4.3 guarantees the well-posedness, in the smooth category, of the corresponding Euler equation

$$(15) \quad m_t + um_x + 2u_x m = 0, \quad m = \mu(u) + (-\Delta)^{r/2} u$$

where  $(-\Delta)^{r/2} := \mathbf{op}(|k|^r)$  and  $\mu(u) := \int_{\mathbb{S}^1} u$ . Note that  $\int_{\mathbb{S}^1} m \, dx$  is a conserved quantity for the evolution under (15), since  $\int_{\mathbb{S}^1} m_t \, dx = 0$ . Equation (15) is of particular interest for the values  $r = 2$  and  $r = 1$ , respectively. In the first case we get the so-called  $\mu$ -Hunter-Saxton equation, cf. [28, 12]

$$(16) \quad u_{txx} + uu_{xxx} + 2u_x u_{xx} - 2\mu(u)u_x = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

In the case  $r = 1$  we get the so-called *generalized CLM equation*, cf. [12]

$$(17) \quad Hu_{tx} + uHu_{xx} + 2\mu(u)u_x + 2u_xHu_x = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

where  $H = \mathbf{op}(i \operatorname{sgn}(k))$  denotes the Hilbert transform, acting on the spatial variable  $x \in \mathbb{S}^1$ . Note that  $\mathbf{op}(|k|) = H \circ D = (-\Delta)^{1/2}$ .

## 5. EXPONENTIAL MAP AND MINIMISATION PROBLEMS

The geodesic flow  $\Phi_q$  on the Hilbert manifold  $T\mathcal{D}^q(\mathbb{S}^1)$  satisfies the following remarkable property

$$\Phi_q(t, \varphi, \sigma v) = \Phi_q(\sigma t, \varphi, v), \quad \sigma > 0,$$

which is a consequence of the quadratic nature of the geodesic spray [26, Chapter 4]. Therefore, the time one map of the flow is defined on some open set  $W_q$  of  $T\mathcal{D}^q(\mathbb{S}^1)$ . The *exponential map*  $\mathbf{exp}_q$  is defined as

$$\mathbf{exp}_q : (\varphi, v) \mapsto \pi \circ \Phi_q(1, \varphi, v), \quad W_q \rightarrow \mathcal{D}^q(\mathbb{S}^1),$$

where  $\pi : T\mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1)$  is the canonical projection. For each  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ , we denote by  $\mathbf{exp}_{q,\varphi}$ , the restriction of  $\mathbf{exp}_q$  to the tangent space  $T_\varphi\mathcal{D}^q(\mathbb{S}^1)$ . Thus

$$\mathbf{exp}_{q,\varphi} : T_\varphi\mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1).$$

If the spray  $F_q$  is smooth, then  $\mathbf{exp}_{q,\text{id}}$  is a local diffeomorphism from a neighbourhood  $V_q$  of the  $0 \in T_{\text{id}}\mathcal{D}^q(\mathbb{S}^1)$  onto a neighbourhood  $U_q$  of  $\text{id} \in \mathcal{D}^q(\mathbb{S}^1)$  [26, Theorem 4.1].

This last assertion is in general no longer true on a *Fréchet manifold* and in particular on  $\text{Diff}^\infty(\mathbb{S}^1)$ . One may find useful to recall on this occasion that the *group exponential* of  $\text{Diff}^\infty(\mathbb{S}^1)$  is not a local diffeomorphism [30]. Moreover, the Riemannian exponential map for the  $L^2$  metric (Burgers equation) on  $\text{Diff}^\infty(\mathbb{S}^1)$  is not a local  $C^1$ -diffeomorphism near the origin [6]. Nevertheless, it has been established in [6], that for the Camassa-Holm equation – which corresponds to the Euler equation of the  $H^1$  metric on  $\text{Diff}^\infty(\mathbb{S}^1)$  – and more generally for  $H^k$  metrics ( $k \geq 1$ ) (see [7]), the Riemannian exponential map was in fact a *smooth local diffeomorphism*. This result is still true for  $H^s$  right-invariant metrics on  $\text{Diff}^\infty(\mathbb{S}^1)$  provided  $s \in [1/2, +\infty)$ .

**Theorem 5.1.** *The exponential mapping  $\mathbf{exp}_{\text{id}}$  for the  $H^s$ -metric on  $\text{Diff}^\infty(\mathbb{S}^1)$  is a smooth local diffeomorphism from a neighbourhood  $V$  of 0 in  $\text{Vect}(\mathbb{S}^1)$  onto a neighbourhood  $U$  of  $\text{id}$  in  $\text{Diff}^\infty(\mathbb{S}^1)$  for each  $s \geq 1/2$ .*

The proof of theorem 5.1 relies mainly on a linearized version of the *no loss, no gain lemma* 4.1, and is stated below. The full proof is similar to the one given for [10, Theorem 14] and will be omitted.

**Lemma 5.2.** *Let  $v \in V_q \cap H^{q+1}(\mathbb{S}^1)$ , we have*

$$T_v\mathbf{exp}_{q,\text{id}}(H^{q+1}(\mathbb{S}^1)) = H^{q+1}(\mathbb{S}^1).$$

*Proof.* Let  $G_q : W_q \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times \mathcal{D}^q(\mathbb{S}^1)$  be the smooth mapping defined by

$$G_q(\varphi, v) := (\varphi, \mathbf{exp}_q(\varphi, v)).$$

From the invariance of the flow  $\Phi_q^t$  under the right action of  $\text{Diff}^\infty(\mathbb{S}^1)$ , we deduce that

$$G_q(R_s \cdot (\varphi, v)) = R_s \cdot G_q(\varphi, v)$$

where  $R_s$  denotes the natural action on derived spaces, induced by the action of the rotation group  $\mathbb{S}^1$  on  $\mathcal{D}^q(\mathbb{S}^1)$

$$(R_s \cdot \varphi)(x) := \varphi(x + s).$$

Therefore, we have

$$(18) \quad TG_q.(R_s \cdot (\varphi, v, \delta\varphi, \delta v)) = R_s \cdot TG_q.(\varphi, v, \delta\varphi, \delta v).$$

Now, if  $(\varphi, v, \delta\varphi, \delta v) \in T^2\mathcal{D}^{q+1}(\mathbb{S}^1)$ , then

$$s \mapsto R_s \cdot (\varphi, v, \delta\varphi, \delta v), \quad \mathbb{S}^1 \rightarrow T^2\mathcal{D}^q(\mathbb{S}^1)$$

is a  $C^1$  mapping and

$$\frac{d}{ds} R_s \cdot (\varphi, v, \delta\varphi, \delta v) = (\varphi_x, v_x, \delta\varphi_x, \delta v_x).$$

Therefore, taking derivatives in  $s$ , at  $s = 0$ , in equation (18), we get

$$\begin{aligned} & T^2G_q.(\varphi, v, \delta\varphi, \delta v, \varphi_x, v_x, \delta\varphi_x, \delta v_x) \\ &= (\varphi, v, \delta\varphi, \delta v, \varphi_x, (\mathbf{exp}_q(\varphi, v))_x, \delta\varphi_x, (T\mathbf{exp}_q.(\varphi, v, \delta\varphi, \delta v))_x). \end{aligned}$$

Since the left hand-side of the preceding equation lies in  $T^2(\mathcal{D}^q(\mathbb{S}^1) \times \mathcal{D}^q(\mathbb{S}^1))$ , we deduce that

$$T_v\mathbf{exp}_{q,\varphi}(H^{q+1}(\mathbb{S}^1)) \subset H^{q+1}(\mathbb{S}^1)$$

as soon as  $(\varphi, v) \in \mathcal{D}^{q+1}(\mathbb{S}^1) \times H^{q+1}(\mathbb{S}^1)$ . Since a similar statement can be made for  $(T_v\mathbf{exp}_{q,\varphi})^{-1}$ , the proof of lemma 5.2 is complete.  $\square$

*Remark 5.3.* Let  $U$  and  $V$  be the neighbourhoods introduced in theorem 5.1. We define

$$\mathcal{V} := \bigcup_{\varphi \in \text{Diff}^\infty(\mathbb{S}^1)} R_\varphi V,$$

which is an open neighbourhood of the *zero section* in  $T\text{Diff}^\infty(\mathbb{S}^1)$ , and

$$\mathcal{U} := \{(\varphi, \psi) \in \text{Diff}^\infty(\mathbb{S}^1) \times \text{Diff}^\infty(\mathbb{S}^1); \psi \circ \varphi^{-1} \in U\},$$

which is an open neighbourhood of the *diagonal* in  $\text{Diff}^\infty(\mathbb{S}^1) \times \text{Diff}^\infty(\mathbb{S}^1)$ . One can deduce, from theorem 5.1, that the mapping

$$G : \mathcal{V} \rightarrow \mathcal{U}, \quad (\varphi, v) \mapsto (\varphi, \mathbf{exp}_\varphi(v))$$

is a smooth diffeomorphism.

The restriction of  $\mathbf{exp}_{q,\varphi}$  to  $V_q$  defines a local chart around  $\text{id}$  in  $\mathcal{D}^q(\mathbb{S}^1)$ . On this chart, called a *normal chart*, we have *local polar coordinates*, defined as follows. Given  $\varphi \in U_q - \{\text{id}\}$ , there is a  $v \in V_q \setminus \{0\}$  such that  $\varphi = \mathbf{exp}_{q,\varphi}(v)$ . Letting now

$$w := v / \|v\|_{H^s}, \quad \rho := \|v\|_{H^s},$$

we have that  $\varphi = \mathbf{exp}(\rho w)$  and  $(\rho, w)$  are called the *polar coordinates* of  $\varphi \in U - \{\text{id}\}$ . Note that  $(\rho, w)$  depend smoothly of  $\varphi$  and that  $\rho(\varphi) \rightarrow 0$  as  $\varphi \rightarrow \text{id}$ .

As can be checked in [26], the following result is valid not only for a strong Riemannian metric but also for a *weak* Riemannian metric, *provided* there exists a compatible, symmetric covariant derivative.

**Lemma 5.4.** *For a piecewise  $C^1$  curve  $\varphi : [a, b] \rightarrow U_q - \{\text{id}\}$ , we have the inequality*

$$(19) \quad L_s(\gamma) \geq |\rho(b) - \rho(a)|,$$

where

$$L_s(\gamma) := \int_a^b \|R_{\varphi^{-1}}\varphi_t\|_{H^s} dt.$$

A consequence of lemma 5.4 is that the length of any path which *lies inside* the normal neighbourhood is bounded below by  $r := |\rho(b) - \rho(a)|$ . Note also that a path, of constant velocity norm, which minimizes locally the arc-length minimizes also the *energy* defined as

$$E_s(\gamma) := \frac{1}{2} \int_a^b \|R_{\varphi^{-1}}\varphi_t\|_{H^s}^2 dt.$$

We get therefore the following theorem.

**Theorem 5.5.** *Let  $s \geq 1/2$ . Given two nearby diffeomorphisms  $\varphi, \psi \in \text{Diff}^\infty(\mathbb{S}^1)$ , there exists a unique geodesic for the right-invariant  $H^s$  metric on  $\text{Diff}^\infty(\mathbb{S}^1)$ , joining them and which minimizes locally the arc-length and the energy.*

On a *strong Riemannian manifold*, given two nearby points, there exists a unique geodesic, joining these two points, which minimizes *globally* the arc-length and the energy. Note however, that for a *weak metric*, this might not be true. Indeed, in lemma 5.4, the bound (19) might not be true for a path which leaves the normal neighbourhood *before* leaving the (weak ball) of radius  $r$  defined as

$$B_s(\text{id}, r) := \{\varphi \in U; \rho(\varphi) \leq r\}.$$

This happens, in particular, for the critical exponent  $s = 1/2$  as it follows from [2]. Note also that, for the  $L^2$  metric, the situation is even worse since the energy has no local minimum [4]. The problem whether the geodesic joining two nearby points is a *global minimum* for  $s > 1/2$  seems to be still an open problem.

To make this clear we close this section by a remark concerning the geodesic semi-distance  $d_s$  induced by the  $H^s$  metric and defined as the greatest lower bound of path-lengths  $L_s(\varphi)$ , for piecewise  $C^1$  paths  $\varphi(t)$  in  $\text{Diff}^\infty(\mathbb{S}^1)$  joining  $\varphi_0$  and  $\varphi_1$ . It was first shown in [29], that this semi-distance vanishes identically for the  $L^2$  right-invariant metric on the diffeomorphism group of any compact manifold. More recently, it was shown in [2] that  $d_s$  vanishes identically on  $\text{Diff}^\infty(\mathbb{S}^1)$  if  $s \in [0, 1/2]$ , whereas  $d_s$  is a distance for  $s > 1/2$

$$\forall \varphi_0, \varphi_1 \in \text{Diff}^\infty(\mathbb{S}^1), \quad \varphi_0 \neq \varphi_1 \Rightarrow d_s(\varphi_0, \varphi_1) > 0.$$

Anyway, it should be noted that lemma 5.4 does not imply that the geodesic semi-distance is in fact a distance.

## 6. EULER EQUATIONS ON HOMOGENEOUS SPACES

The theory of Euler equations on a homogeneous space  $G/K$  has been developed in [24]. In that case, the geodesic flow for a right-invariant Riemannian metric on the homogeneous space  $G/K$ , can be reduced to the so called *Euler-Poincaré* equation

$$(20) \quad m_t = \text{ad}_u^* m, \quad m \in \mathfrak{g}^*,$$

on the dual space  $\mathfrak{g}^*$  of the Lie algebra of  $G$  (see [24] or Poincaré's original paper [36]). Unfortunately, there is no natural *contravariant* formulation of equation (20), leading to an *Euler equation*, as it is the case for a Lie group. In that case, the Eulerian velocity (defined using a lift  $g(t)$  in  $G$  of a path  $x(t)$  in  $G/K$ ) is only defined *up to a path* in  $K$  and the relation between  $u$  and  $m$  is not one-to-one (see [39] for a recent survey on this subject). Another way to treat the problem is to introduce *sub-Riemannian geometry* on  $G$  (see [16, 17] for a deep study of this approach for  $\text{Diff}^\infty(\mathbb{S}^1)$ ).

These difficulties clear away if  $K$  is a normal subgroup. Indeed, in that case, the coset manifold  $G/K$  is a Lie group equipped with a right-invariant Riemannian metric. But this special case is not very useful for our study, since  $\text{Diff}^\infty(\mathbb{S}^1)$  is a *simple* group: it has no nontrivial normal subgroups (see [18]).

Hopefully, there is another situation where a right-invariant Riemannian metric on a homogeneous space can be reduced to the ordinary theory of the Euler equation on a Lie group, namely when there exists a section of the projection map  $\pi : G \rightarrow G/K$  onto a subgroup  $H$  of  $G$ . This situation occurs for  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ , in which case  $H := \text{Diff}_1^\infty(\mathbb{S}^1)$ , the subgroup of diffeomorphisms which fix one point, and for  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$  in which case  $H := \text{Diff}_3^\infty(\mathbb{S}^1)$ , the subgroup diffeomorphisms which fix three points. More precisely, we have the following.

**Lemma 6.1.** *Let  $G$  be a group and  $H, K$  some subgroups of  $G$ . Suppose that*

- (1) *The restriction to  $H$  of the projection map  $\pi : G \rightarrow G/K$  is surjective,*
- (2)  *$H \cap K = \{e\}$ .*

*Then  $H$  acts simply and transitively on  $G/K$ .*

*Remark 6.2.* As a result, if the hypothesis of lemma 6.1 are satisfied, then  $G/K$  inherits a group structure. Note, however that the restriction of the projection  $\pi : H \rightarrow G/K$  is a group morphism, if and only if,  $K$  is a normal subgroup of  $G$ .

*Proof.* By definition, the projection map  $\pi$  sends an element  $g \in G$  to the coset  $Kg$ . To show that the (right) action of  $H$  on  $G/K$  is transitive, it suffices to show that for any coset  $Kg$  we can find  $h \in H$  such that  $Kh = Kg$ . But this means precisely that  $\pi : H \rightarrow G/K$  is surjective. Hence the transitivity of the action is equivalent to the surjectivity of  $\pi$ . To prove that the action is simple, it is enough to show that the only element  $h \in H$  which fixes the coset  $K$  is  $h = e$ , the unit element. But this means  $Kh = K$ , and thus  $h \in K \cap H$ , which leads to  $h = e$  by condition (2). Note that this implies that  $\pi : H \rightarrow G/K$  is injective.  $\square$

The mentioned scenario is summarized in the following proposition.

**Proposition 6.3.** *Suppose that  $H$  and  $K$  are closed subgroups of a Lie group  $G$  such that  $H \cap K = \{e\}$  and such that  $\pi : H \rightarrow G/K$  is surjective. Let  $\mathfrak{g}$ ,  $\mathfrak{h}$ , and  $\mathfrak{k}$  denote the Lie algebras of  $G$ ,  $H$  and  $K$ , respectively. Let  $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$  be the inertia operator of a (degenerate) non-negative inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ , which satisfies the following conditions:*

$$(21) \quad \ker A = \mathfrak{k},$$

and

$$(22) \quad \langle \text{Ad}_k u, \text{Ad}_k v \rangle = \langle u, v \rangle, \quad \forall k \in K, \forall u, v \in \mathfrak{g},$$

where  $\text{Ad}$  is the adjoint action of  $G$  on its Lie algebra  $\mathfrak{g}$ . Then,  $A$  induces a right-invariant Riemannian metric  $\gamma$  on  $G/K$  and  $\pi : H \rightarrow G/K$  is a Riemannian isometry between  $H$ , endowed with the right-invariant metric induced by  $A$ , and  $(G/K, \gamma)$ .

*Remark 6.4.* In the situation described, we have  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$  and  $\mathfrak{h}^*$  can be identified with

$$\mathfrak{k}^0 = \{m \in \mathfrak{g}^*; (m, w) = 0, \forall w \in \mathfrak{k}\}.$$

Now, condition (22) implies<sup>4</sup>

$$(23) \quad \langle \text{ad}_w^* u, v \rangle = -\langle u, \text{ad}_w v \rangle, \quad \forall w \in \mathfrak{k}, \forall u, v \in \mathfrak{g}.$$

Therefore, we have

$$(\text{ad}_u^* A(v), w) = -(\text{ad}_v^* A(u), w), \quad \forall w \in \mathfrak{k}, \forall u, v \in \mathfrak{g},$$

where  $\text{ad}^*$  is the coadjoint action of  $\mathfrak{g}$  on  $\mathfrak{g}^*$ , defined by

$$(\text{ad}_u^* m, v) = -(m, [u, v]), \quad u, v \in \mathfrak{g}, m \in \mathfrak{g}^*.$$

Hence

$$\text{ad}_u^* A(v) + \text{ad}_v^* A(u) \in \mathfrak{k}^0 = \text{im } A$$

and Arnold's bilinear operator

$$B(u, v) = \frac{1}{2} A^{-1} \left[ \text{ad}_u^* A(v) + \text{ad}_v^* A(u) \right]$$

is well-defined as a mapping from  $\mathfrak{h} \times \mathfrak{h}$  to  $\mathfrak{h}$ . The Euler equation on  $\mathfrak{h}$  is given by

$$(24) \quad u_t = -B(u, u) = -A^{-1} \left[ \text{ad}_u^* A(u) \right].$$

In the next subsections, we will extend our main theorems to some geodesic equations on  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$  and  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$ . Since the proofs are very similar to what has been done so far, we will not exhibit all the details but only point out crucial changes.

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<sup>4</sup>If the subgroup  $K$  is connected, (22) and (23) are equivalent.

**6.1. Euler equations on the coadjoint orbit**  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ . Let  $\text{Rot}(\mathbb{S}^1)$  denotes the subgroup of all rigid rotations of  $\mathbb{S}^1$  and

$$\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1),$$

be the corresponding homogeneous space of right cosets. Let  $\text{Diff}_1^\infty(\mathbb{S}^1)$  be the subgroup of  $\text{Diff}^\infty(\mathbb{S}^1)$  consisting of all diffeomorphisms of  $\mathbb{S}^1$  which fix one arbitrarily point (say  $x_0$ ). It is easy to check that the conditions of lemma 6.1 are satisfied and hence that the canonical projection

$$\text{Diff}_1^\infty(\mathbb{S}^1) \rightarrow \text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$$

is a bijection. The group  $\text{Diff}_1^\infty(\mathbb{S}^1)$  is a Fréchet Lie group and we can use the Fréchet manifold structure of  $\text{Diff}_1^\infty(\mathbb{S}^1)$  to endow the quotient space  $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$  with a Fréchet manifold structure, so that the canonical projection becomes a diffeomorphism. The Lie algebras of  $\text{Diff}_1^\infty(\mathbb{S}^1)$  and  $\text{Rot}(\mathbb{S}^1)$  are given by

$$C_1^\infty(\mathbb{S}^1) := \{u \in C^\infty(\mathbb{S}^1); u(x_0) = 0\} \quad \text{and} \quad \mathbb{R} \cdot w_0,$$

respectively, where  $w_0$  stands for the constant function with value 1.

$\text{Diff}_1^\infty(\mathbb{S}^1)$  is an ILH space; a Hilbert approximation being given by the Hilbert manifolds

$$\mathcal{D}_1^q(\mathbb{S}^1) := \{\varphi \in \mathcal{D}^q(\mathbb{S}^1); \varphi(x_0) = x_0\},$$

modelled on the Hilbert spaces

$$H_1^q(\mathbb{S}^1) := \{u \in H^q(\mathbb{S}^1); u(x_0) = 0\}.$$

Note that  $\mathcal{D}_1^q(\mathbb{S}^1)$  is a closed submanifold of the Hilbert manifold  $\mathcal{D}^q(\mathbb{S}^1)$  and a closed topological subgroup of  $\mathcal{D}^q(\mathbb{S}^1)$ , for  $q > 3/2$ .

Let  $A = \mathbf{op}(p(k))$  be a  $L^2$ -symmetric, Fourier multiplier on  $C^\infty(\mathbb{S}^1)$  and assume that its symbol satisfies

$$p(k) = 0 \quad \text{iff} \quad k = 0,$$

which is equivalent to  $\ker A = \mathbb{R} \cdot w_0$ . We have  $\text{ad}_{w_0} = -D$ , so that condition (23) is satisfied. Since  $\text{Rot}(\mathbb{S}^1)$  is connected, hypotheses of proposition 6.3 are fulfilled. If  $A$  is of order  $r \geq 1$ , then  $A$  extends to  $H_1^q(\mathbb{S}^1)$ , for all  $q > 3/2$ , and

$$A \in \text{Isom}(H_1^q(\mathbb{S}^1), \hat{H}_1^{q-r}(\mathbb{S}^1)),$$

where

$$\hat{H}_1^{q-r}(\mathbb{S}^1) := \{m \in H^{q-r}(\mathbb{S}^1); \hat{m}(0) = 0\}.$$

Then, for each  $\varphi \in \mathcal{D}_1^q(\mathbb{S}^1)$ ,  $A$  induces a positive inner product on each tangent space,  $T_\varphi \mathcal{D}_1^q(\mathbb{S}^1)$ , with flat map

$$\tilde{A}_\varphi = \varphi_x A_\varphi \in \text{Isom}(H_1^q(\mathbb{S}^1), \hat{H}_1^{q-r}(\mathbb{S}^1)).$$

Note that  $\tilde{A}(T\mathcal{D}_1^q(\mathbb{S}^1)) = \mathcal{D}_1^q(\mathbb{S}^1) \times \hat{H}_1^{q-r}(\mathbb{S}^1)$  and the proof of theorem 6.5 is similar to that of theorem 3.10 and will be omitted.

**Theorem 6.5.** *Let  $A = \mathbf{op}(p(k))$  be a  $L^2$ -symmetric, non negative, Fourier multiplier of order  $r \geq 1$ , satisfying*

$$(25) \quad p(k) = 0 \iff k = 0.$$

Assume, in addition, that

$$\varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for  $q > 3/2$  and  $q - r \geq 0$ . Then, the induced right-invariant metric on  $\mathcal{D}_1^q(\mathbb{S}^1)$  is smooth and has a smooth spray. Moreover, given any  $(\varphi_0, v_0) \in T\text{Diff}_1^\infty(\mathbb{S}^1)$ , there exists a unique non-extendable solution

$$(\varphi, v) \in C^\infty(J, T\text{Diff}_1^\infty(\mathbb{S}^1))$$

of the Cauchy problem for the associated geodesic spray on the maximal interval of existence  $J$ .

We briefly discuss two special instances, namely

$$A = \mathbf{op}(k^2) \quad \text{and} \quad A = \mathbf{op}(|k|).$$

In the first case,  $A = \mathbf{op}(k^2)$ , we get the periodic *Hunter-Saxton equation* (HS), see [21, 42, 3, 27],

$$(26) \quad u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0.$$

When  $A = \mathbf{op}(|k|)$ , we get the *Constantin-Lax-Majda equation* (CLM), see [8, 41, 11],

$$(27) \quad \partial_t(Hu_x) + uHu_{xx} + 2u_xHu_x = 0,$$

where  $H = \mathbf{op}(i \operatorname{sgn}(k))$  denotes the Hilbert transform, acting on the spatial variable  $x \in \mathbb{S}^1$ .

Clearly, both symbols  $(k^2)_{k \in \mathbb{Z}}$  and  $(|k|)_{k \in \mathbb{Z}}$  satisfy (25). Moreover, they also fulfill the hypotheses of theorem 3.7, so that theorem 6.5 is applicable to both (26) and (27).

**6.2. Euler equations on the coadjoint orbit**  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$ . Let  $\text{PSL}(2, \mathbb{R})$  denotes the subgroup of all rigid Möbius transformations which preserves the circle  $\mathbb{S}^1$  and let

$$\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R}),$$

be the corresponding homogeneous space of right cosets. Let  $\text{Diff}_3^\infty(\mathbb{S}^1)$  be the subgroup of  $\text{Diff}^\infty(\mathbb{S}^1)$  consisting of all diffeomorphisms of  $\mathbb{S}^1$  which fix 3 arbitrary distinct points (say  $x_0, x_1, x_2$ ). Then  $\text{PSL}(2, \mathbb{R}) \cap \text{Diff}_3^\infty(\mathbb{S}^1) = \{e\}$  and the canonical projection

$$\text{Diff}_3^\infty(\mathbb{S}^1) \rightarrow \text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$$

is a bijection. The group  $\text{Diff}_3^\infty(\mathbb{S}^1)$  is a Fréchet Lie group and we can use this Fréchet structure to endow the quotient space  $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$  with a Fréchet manifold structure. In that way, the canonical projection becomes a diffeomorphism. The Lie algebras of  $\text{Diff}_3^\infty(\mathbb{S}^1)$  is given by

$$\mathcal{C}_3^\infty(\mathbb{S}^1) := \{u \in C^\infty(\mathbb{S}^1); u(x_0) = 0, u(x_1) = 0, u(x_2) = 0\},$$

whereas the Lie algebra  $\mathfrak{sl}(2, \mathbb{R}) \subset C^\infty(\mathbb{S}^1)$  of  $\text{PSL}(2, \mathbb{R})$  is the 3-dimensional subalgebra of  $C^\infty(\mathbb{S}^1)$ , generated by

$$w_0(x) := 1, \quad w_1(x) := \cos(x), \quad w_{-1}(x) := \sin(x).$$

An ILH structure on  $\text{Diff}_3^\infty(\mathbb{S}^1)$  is given by the Hilbert manifolds

$$\mathcal{D}_3^q(\mathbb{S}^1) := \{\varphi \in \mathcal{D}^q(\mathbb{S}^1); \varphi(x_0) = x_0, \varphi(x_1) = x_1, \varphi(x_2) = x_2\},$$



modelled on the Hilbert spaces

$$H_3^q(\mathbb{S}^1) := \{u \in H^q(\mathbb{S}^1); u(x_0) = 0, u(x_1) = 0, u(x_2) = 0\}.$$

Note that  $\mathcal{D}_3^q(\mathbb{S}^1)$  is a closed submanifold of the Hilbert manifold  $\mathcal{D}^q(\mathbb{S}^1)$  and a topological subgroup of  $\mathcal{D}^q(\mathbb{S}^1)$ , for  $q > 3/2$ .

**Theorem 6.6.** *Let  $A = \mathbf{op}(p(k))$  be a  $L^2$ -symmetric, non negative, Fourier multiplier of order  $r \geq 1$ , satisfying*

$$(28) \quad p(k) = 0 \iff k \in \{-1, 0, 1\},$$

and

$$\langle \text{ad}_w u, v \rangle = -\langle u, \text{ad}_w v \rangle, \quad \forall w \in \mathfrak{sl}(2, \mathbb{R}), \forall u, v \in C^\infty(\mathbb{S}^1).$$

Assume, in addition, that

$$\varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for  $q > 3/2$  and  $q - r \geq 0$ . Then, the induced right-invariant metric on  $\mathcal{D}_3^q(\mathbb{S}^1)$  is smooth and has a smooth spray. Moreover, given any  $(\varphi_0, v_0) \in \text{TDiff}_3^\infty(\mathbb{S}^1)$ , there exists a unique non-extendable solution

$$(\varphi, v) \in C^\infty(J, \text{TDiff}_3^\infty(\mathbb{S}^1))$$

of the Cauchy problem for the associated geodesic spray on the maximal interval of existence  $J$ .

*Proof.* The proof is similar to that of theorem 3.10, except for point (c). Note that  $A$  extends to  $H_3^q(\mathbb{S}^1)$ , for all  $q > 3/2$ , and that

$$A \in \text{Isom}(H_1^q(\mathbb{S}^1), \hat{H}_3^{q-r}(\mathbb{S}^1)),$$

where

$$\hat{H}_3^{q-r}(\mathbb{S}^1) := \{m \in H^{q-r}(\mathbb{S}^1); \hat{m}(0) = 0, \hat{m}(1) = 0, \hat{m}(-1) = 0\}.$$

Let

$$\tilde{A} : \mathcal{D}_3^q(\mathbb{S}^1) \times H_3^q(\mathbb{S}^1) \rightarrow \mathcal{D}_3^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1), \quad (\varphi, v) \rightarrow (\varphi, \varphi_x A_\varphi v).$$

Given  $\varphi \in \mathcal{D}_3^q(\mathbb{S}^1)$ , we have

$$\tilde{A}_\varphi(H_3^q(\mathbb{S}^1)) = \left\{ m \in H^{q-r}(\mathbb{S}^1); \varphi_x(m \circ \varphi) \in \hat{H}_3^{q-r}(\mathbb{S}^1) \right\},$$

so we cannot conclude immediately that  $\tilde{A}(\text{T}\mathcal{D}_3^q(\mathbb{S}^1))$  is a trivial bundle, as this was the case for  $\tilde{A}(\text{T}\mathcal{D}^q(\mathbb{S}^1))$  and  $\tilde{A}(\text{T}\mathcal{D}_1^q(\mathbb{S}^1))$ . To overcome this difficulty, we first remark that  $\tilde{A}$  is a vector bundle morphism. Moreover, the continuous linear map  $\tilde{A}_\varphi : H_3^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$  is injective and *splits*<sup>5</sup> because  $\tilde{A}_\varphi(H_3^q(\mathbb{S}^1))$  is a closed subspace of  $H^{q-r}(\mathbb{S}^1)$ , for every  $\varphi \in \mathcal{D}_3^q(\mathbb{S}^1)$ . Then according to [26, Proposition 3.1],  $\tilde{A}(\text{T}\mathcal{D}_3^q(\mathbb{S}^1))$  is a subbundle of  $\mathcal{D}_3^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$  which is isomorphic to  $\mathcal{D}_3^q(\mathbb{S}^1) \times \hat{H}_3^{q-r}(\mathbb{S}^1)$ . Finally, an argument similar to the one given in point (c) of the proof of theorem 3.10 does apply and achieves the proof.  $\square$

<sup>5</sup>If  $E, F$  are Banach spaces and  $\Lambda : E \rightarrow F$  is a continuous linear map, which is injective, then we say that  $\Lambda$  *splits* if  $\Lambda(E)$  is *closed* and *complemented* in  $F$  (i.e there exists a closed subspace  $G$  of  $F$  such that  $F = \Lambda(E) \oplus G$ ). Note that if  $F$  is a Hilbert space, then every injective, continuous linear map with closed range, splits.

An important application of Theorem 6.6 is the Euler-Weil-Petersson equation, which corresponds to the inertia operator

$$A := HD(D^2 + 1) = \mathbf{op}(|k|(k^2 - 1)).$$

This equation has been related with the Weil-Petersson metric on the universal Teichmüller space  $T(1)$  in [33, 38]. The corresponding geodesic flow has been extensively studied in [15]. One of the main results of this paper is that the inertia operator  $A$  defines on a suitable replacement<sup>6</sup> for the “diffeomorphism group of class  $H^{3/2}$ ”, a right-invariant *strong Riemannian structure* which is, moreover, geodesically complete.

Our point of view is completely different because we are interested in the geodesic flow for the right-invariant metric on the Fréchet Lie group  $\text{Diff}_3^\infty(\mathbb{S}^1)$  and its Hilbert approximation  $\mathcal{D}^s(\mathbb{S}^1)$  for  $s > 3/2$ . The price to pay is the fact that the metric only defines a *weak Riemannian structure*. Nevertheless, theorem 6.6 applies in this case. Indeed,  $A$  satisfies the hypothesis of theorem 3.7 and all conditions of theorem 6.6. This proves short time existence of geodesics on  $\text{Diff}_3^\infty(\mathbb{S}^1)$ , which doesn't seem to be a consequence of the results in [15].

#### APPENDIX A. FOURIER MULTIPLIERS

In this Appendix, we recall and establish some basic results about *Fourier multipliers*. Here and in the following, we use the notation

$$\mathbf{e}_k(x) = \exp(2\pi i k x),$$

for  $k \in \mathbb{Z}$  and  $x \in \mathbb{S}^1$ .

**Lemma A.1.** *Let  $A$  be a continuous linear operator on the Fréchet space  $C^\infty(\mathbb{S}^1, \mathbb{C})$ . Then the following three conditions are equivalent:*

- (1)  $A$  commutes with all rotations  $R_s$ .
- (2)  $[A, D] = 0$ , where  $D = d/dx$ .
- (3) For each  $k \in \mathbb{Z}$ , there is a  $a(k) \in \mathbb{C}$  such that  $A\mathbf{e}_k = a(k)\mathbf{e}_k$ .

In that case, we say that  $A$  is a Fourier multiplier.

Since every smooth function on the unit circle  $\mathbb{S}^1$  can be represented by its Fourier series, we get that

$$(29) \quad (Au)(x) = \sum_{k \in \mathbb{Z}} a(k) \hat{u}(k) \mathbf{e}_k(x),$$

for every Fourier multiplier  $A$  and every  $u \in C^\infty(\mathbb{S}^1)$ , where

$$\hat{u}(k) := \int_{\mathbb{S}^1} u(x) \overline{\mathbf{e}_k(x)} dx,$$

stands for the  $k$ -th Fourier coefficients of  $u$ . The sequence  $a : \mathbb{Z} \rightarrow \mathbb{C}$  is called the *symbol* of  $A$ . We use also the notation  $A := \mathbf{op}(a(k))$  for the Fourier multiplier induced by the sequence  $a$ .

---

<sup>6</sup> $\mathcal{D}^s(\mathbb{S}^1)$ , the space of homeomorphisms of class  $H^s$  as well as their inverse is a topological group, if and only if,  $s > 3/2$ .

*Proof.* Given  $s \in \mathbb{R}$  and  $u \in C^\infty(\mathbb{S}^1)$ , let  $u_s(x) := u(x + s)$ . If  $A$  commutes with translations we have

$$(Au)_s(x) = (Au_s)(x).$$

Taking the derivative of both sides of this equation with respect to  $s$  at 0 and using the continuity of  $A$ , we get  $DAu = ADu$  which proves the implication (1)  $\Rightarrow$  (2).

If  $[A, D] = 0$ , then both  $A\mathbf{e}_k$  and  $\mathbf{e}_k$  are solutions of the linear differential equation  $u' = (-2\pi ik)u$  and are therefore equal, up to a multiplicative constant  $a(k)$ . This proves that (2)  $\Rightarrow$  (3).

If  $A\mathbf{e}_k = a(k)\mathbf{e}_k$ , for each  $k \in \mathbb{Z}$  and  $A$  is continuous, then we have representation (29). Therefore

$$\begin{aligned} (Au)_s(x) &= \sum_{k \in \mathbb{Z}} a(k) \hat{u}(k) \mathbf{e}_k(x + s) \\ &= \sum_{k \in \mathbb{Z}} a(k) \hat{u}_s(k) \mathbf{e}_k(x) = (Au_s)(x), \end{aligned}$$

which proves that (3)  $\Rightarrow$  (1).  $\square$

*Remark A.2.* The space of Fourier multipliers is a *commutative subalgebra* of the algebra of linear operators on  $C^\infty(\mathbb{S}^1, \mathbb{C})$ . It contains all linear differential operators with constant coefficients. Note that a Fourier multiplier  $A$  is  $L^2$ -symmetric iff its symbol  $a$  is real.

Let  $I_n := \{1, \dots, n\}$ . Given a function  $f$  and  $n \geq 1$ , we introduce

$$S_{f,n}(m_0, m_1, \dots, m_n) := \sum_{p=0}^n (-1)^p \sum_{\substack{J \subset I_n, \\ |J|=p}} f_n(m_0 + \sum_{j \in J} m_j),$$

where  $f_n(\xi) := \xi^{n-1} f(\xi)$  and  $m_j \in \mathbb{R}$ , for  $0 \leq j \leq n$ .

**Lemma A.3.** *For each  $n \geq 1$ , we have*

$$(30) \quad \begin{aligned} S_{f,n+1}(m_0, m_1, \dots, m_{n+1}) &= (m_0 + \dots + m_n) S_{f,n}(m_0, \dots, m_n) \\ &\quad - \sum_{k=1}^n m_k S_{f,n}(m_0, \dots, m_k + m_{n+1}, \dots, m_n) \end{aligned}$$

*Proof.* Given  $J \subset I_n$ , the expression

$$\begin{aligned} (m_0 + \dots + m_n) f_n(m_0 + \sum_{j \in J} m_j) &- (m_0 + m_{n+1}) f_n(m_0 + m_{n+1} + \sum_{j \in J} m_j) \\ &- \sum_{k=1}^n m_k f_n(m_0 + \sum_{j \in J} m_j + \delta_J(k) m_{n+1}), \end{aligned}$$

can be recast as

$$\begin{aligned} (m_0 + \sum_{j \in J} m_j) f_n(m_0 + \sum_{j \in J} m_j) \\ - (m_0 + m_{n+1} + \sum_{j \in J} m_j) f_n(m_0 + m_{n+1} + \sum_{j \in J} m_j), \end{aligned}$$

which is equal to

$$f_{n+1}\left(m_0 + \sum_{j \in J} m_j\right) + f_{n+1}\left(m_0 + m_{n+1} + \sum_{j \in J} m_j\right),$$

since  $f_{n+1}(\xi) = \xi f_n(\xi)$ . Therefore, the right hand-side of (30) is equal to

$$\begin{aligned} \sum_{p=0}^n (-1)^p \sum_{\substack{J \subset I_{n+1}, \\ |J|=p, n+1 \notin J}} f_{n+1}\left(m_0 + \sum_{j \in J} m_j\right) \\ + \sum_{p=0}^n (-1)^{p+1} \sum_{\substack{J \subset I_{n+1}, \\ |J|=p+1, n+1 \in J}} f_{n+1}\left(m_0 + \sum_{j \in J} m_j\right), \end{aligned}$$

which is exactly

$$S_{f, n+1}(m_0, m_1, \dots, m_{n+1}).$$

□

**Lemma A.4.** *Let  $A = \mathbf{op}(a(k))$  be a Fourier multiplier on  $C^\infty(\mathbb{S}^1)$ , and let  $(A_n)$  be the sequence of multilinear operators defined inductively in lemma 3.2. Then, for each  $n \geq 1$ , we have*

$$(31) \quad A_n(\mathbf{e}_{m_0}, \dots, \mathbf{e}_{m_n}) = a_n(m_0, m_1, \dots, m_n) \mathbf{e}_{m_0 + m_1 + \dots + m_n},$$

where

$$(32) \quad a_n(m_0, m_1, \dots, m_n) = (2\pi i)^n m_0 S_{a, n}(m_0, m_1, \dots, m_n).$$

*Remark A.5.* For  $n = 1$ , we have

$$a_1(m_0, m_1) = (2\pi i) m_0 \left[ a(m_0) - a(m_0 + m_1) \right]$$

and for  $n = 2$ , we get

$$\begin{aligned} a_2(m_0, m_1, m_2) = (2\pi i)^2 m_0 \left[ (m_0 + m_1 + m_2) a(m_0 + m_1 + m_2) \right. \\ \left. - (m_0 + m_1) a(m_0 + m_1) - (m_0 + m_2) a(m_0 + m_2) + m_0 a(m_0) \right]. \end{aligned}$$

*Proof.* From relation (7), we get

$$\begin{aligned} a_{n+1}(m_0, \dots, m_{n+1}) = (2\pi i) \left[ (m_0 + \dots + m_n) a_n(m_0, \dots, m_n) \right. \\ \left. - \sum_{k=0}^n m_k a_n(m_0, \dots, m_k + m_{n+1}, \dots, m_n) \right]. \end{aligned}$$

For  $n = 1$ , relation (32) is clear. Now, suppose inductively that this relation holds for some  $n \geq 1$ . Then,  $a_{n+1}(m_0, \dots, m_{n+1})$  can be expressed as

$$\begin{aligned} (2\pi i) \left[ (m_0 + \dots + m_n) (2\pi i)^n m_0 S_{a, n}(m_0, \dots, m_n) \right. \\ \left. - \sum_{k=0}^n m_k (2\pi i)^n m_0 S_{a, n}(m_0, \dots, m_k + m_{n+1}, \dots, m_n) \right], \end{aligned}$$

which is equal to

$$(2\pi i)^{n+1} m_0 S_{a, n+1}(m_0, m_1, \dots, m_{n+1}),$$

by virtue of lemma A.3. This achieves the proof.  $\square$

Recall that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *absolutely continuous* if  $f$  has a derivative almost everywhere, the derivative is locally Lebesgue integrable and

$$f(b) = f(a) + \int_a^b f'(\tau) d\tau,$$

for all  $a, b \in \mathbb{R}$ .

**Lemma A.6.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $n \geq 1$ . Suppose that  $f_n(\xi) := \xi^{n-1}f(\xi)$  is of class  $C^{n-1}$ , that  $f_n^{(n-1)}$  is absolutely continuous and that there exists  $C_n > 0$  and  $r \geq 1$  such that*

$$(33) \quad \left| f_n^{(n)}(\xi) \right| \leq C_n(1 + \xi^2)^{(r-1)/2},$$

almost everywhere. Then

$$|S_{f,n}(m_0, m_1, \dots, m_n)| \leq C_n \prod_{j=1}^n |m_j| \sum_{J \subset I_n} \left( 1 + \left( m_0 + \sum_{j \in J} m_j \right)^2 \right)^{(r-1)/2},$$

for all  $m_0, m_1, \dots, m_n \in \mathbb{R}$ .

*Proof.* Fix  $n \geq 1$  and  $m_0, m_1, \dots, m_n \in \mathbb{R}$ . Let  $g_k$  be the sequence of functions defined inductively by

$$g_0(\xi) = f_n(\xi), \quad g_{k+1}(\xi) = g_k(\xi) - g_k(\xi + m_{n-k}),$$

for  $k = 0, \dots, n-1$ . We have in particular

$$g_n(m_0) = S_{f,n}(m_0, m_1, \dots, m_n).$$

Let  $K_0 = \{m_0\}$  and for  $p = 1, \dots, n$ , let  $K_p$  be the convex set generated by  $K_{p-1}$  and  $K_{p-1} + m_p$ . Note that  $K_n$  is the convex hull of the points  $m_0 + \sum_{j \in J} m_j$ , for all subset  $J$  of  $\{1, \dots, n\}$ . Let

$$M := \max_{\xi \in K_n} (1 + \xi^2)^{(r-1)/2}.$$

By hypothesis, we have  $\left| g_0^{(n)}(\xi) \right| \leq C_n M$  almost everywhere on  $K_n$ , and using the mean value theorem, we get inductively

$$\left| g_k^{(n-k)}(\xi) \right| \leq C_n M |m_n| \cdots |m_{n-k+1}|, \quad \forall \xi \in K_{n-k},$$

for  $k = 1, \dots, n$ . In particular, we have

$$|g_n(m_0)| \leq C_n M \prod_{j=1}^n |m_j|.$$

Let's now estimate  $M$ . For  $r \geq 1$ , the function  $\xi \mapsto (1 + \xi^2)^{(r-1)/2}$  has no local maximum on  $\mathbb{R}$  (or is constant). Thus, it attains its maximum on the

compact, convex set  $K_n$  at some extremal point  $m_0 + \sum_{j \in J_0} m_j$  and we get

$$\begin{aligned} M &= \left( 1 + \left( m_0 + \sum_{j \in J_0} m_j \right)^2 \right)^{(r-1)/2} \\ &\leq \sum_{J \subset I_n} \left( 1 + \left( m_0 + \sum_{j \in J} m_j \right)^2 \right)^{(r-1)/2}, \end{aligned}$$

which achieves the proof.  $\square$

**Corollary A.7.** *Let  $A = \mathbf{op}(a(k))$  be a Fourier multiplier of order  $r \geq 1$ . Suppose that  $a$  satisfies the hypothesis of lemma A.6. Then, each  $A_n$  extends to a bounded multilinear operator*

$$A_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

when  $q > 3/2$  and  $q - r \geq 0$ .

*Proof.* Given smooth functions  $u_0, \dots, u_n$ , we have

$$\|A_n(u_0, \dots, u_n)\|_{H^{q-r}}^2 = \sum_{k \in \mathbb{Z}} (1 + k^2)^{q-r} \left| \hat{A}_n(k) \right|^2,$$

where

$$\begin{aligned} \hat{A}_n(k) &:= \langle A_n(u_0, \dots, u_n), e_k \rangle_{L^2} \\ &= \sum_{m_0 + \dots + m_n = k} a_n(m_0, \dots, m_n) \widehat{u}_0(m_0) \cdots \widehat{u}_n(m_n). \end{aligned}$$

If we assume, moreover, that  $a$  satisfies the hypothesis of lemma A.6, we get

$$\left| \hat{A}_n(k) \right|^2 \lesssim \sum_{J \subset I_n} \left( \hat{A}_{n,J}(k) \right)^2$$

where

$$\begin{aligned} \hat{A}_{n,J}(k) &:= \sum_{m_0 + \dots + m_n = k} \left( 1 + \left( m_0 + \sum_{j \in J} m_j \right)^2 \right)^{(r-1)/2} \\ &\quad \left| \widehat{u}'_0(m_0) \right| \cdots \left| \widehat{u}'_n(m_n) \right|. \end{aligned}$$

Recall now, that, for smooth functions on  $C^\infty(\mathbb{S}^1)$ , we have

$$\widehat{v_0 \cdots v_n}(k) = \sum_{m_0 + \dots + m_n = k} \widehat{v}_0(m_0) \cdots \widehat{v}_n(m_n).$$

Therefore,  $\hat{A}_{n,J}(k)$  is the  $k$ -th Fourier coefficient of

$$\Lambda^{r-1} \left( \tilde{u}_0 \prod_{j \in J} \tilde{u}_j \right) \prod_{j \in I_n \setminus J} \tilde{u}_j,$$

where  $\Lambda^s := \mathbf{op}((1 + k^2)^{s/2})$  and  $\tilde{u}_j$  is the smooth function with Fourier coefficients

$$\widehat{\tilde{u}}_j(k) = \left| \widehat{u}'_j(k) \right|.$$

Thus

$$\|A_n(u_0, \dots, u_n)\|_{H^{q-r}}^2 \lesssim \sum_{J \subset I_n} \left\| \Lambda^{r-1} \left( \tilde{u}_0 \prod_{j \in J} \tilde{u}_j \right) \prod_{j \in I_n \setminus J} \tilde{u}_j \right\|_{H^{q-r}}^2.$$

Now, because  $q-1 > 1/2$  and  $0 \leq q-r \leq q-1$ , we deduce from lemma B.1 that

$$\begin{aligned} \|A_n(u_0, \dots, u_n)\|_{H^{q-r}}^2 &\lesssim \sum_{J \subset I_n} \left\| \Lambda^{r-1} \left( \tilde{u}_0 \prod_{j \in J} \tilde{u}_j \right) \right\|_{H^{q-r}}^2 \left\| \prod_{j \in I_n \setminus J} \tilde{u}_j \right\|_{H^{q-1}}^2 \\ &\lesssim \sum_{J \subset I_n} \left\| \tilde{u}_0 \prod_{j \in J} \tilde{u}_j \right\|_{H^{q-1}}^2 \left\| \prod_{j \in I_n \setminus J} \tilde{u}_j \right\|_{H^{q-1}}^2 \\ &\lesssim \|\tilde{u}_0\|_{H^{q-1}}^2 \cdots \|\tilde{u}_n\|_{H^{q-1}}^2, \end{aligned}$$

because  $H^{q-1}(\mathbb{S}^1)$  is a multiplicative algebra. Since, moreover,  $u'_j$  and  $\tilde{u}_j$  have the same  $H^{q-1}$  norm, we obtain finally

$$\|A_n(u_0, \dots, u_n)\|_{H^{q-r}} \lesssim \|u_0\|_{H^q} \cdots \|u_n\|_{H^q},$$

which achieves the proof.  $\square$

#### APPENDIX B. BOUNDEDNESS PROPERTIES OF RIGHT TRANSLATIONS

In this Appendix, we provide some explicit estimates for the right representation of  $\mathcal{D}^q(\mathbb{S}^1)$  on  $\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))$ , when  $q > 3/2$  and  $0 \leq \rho \leq q$ . These results are certainly not new but we give here very precise estimates.

Let us recall first the Slobodeckij spaces  $W^{s,p}(\mathbb{S}^1)$ . For  $m \in \mathbb{N}$  and  $1 \leq p \leq \infty$ , the  $W^{m,p}$ -norm of a measurable functions on  $\mathbb{S}^1$  is defined by

$$\|u\|_{W^{m,p}} := \sum_{j=0}^m \|u^{(j)}\|_{L^p},$$

where  $u^{(j)}$  denotes the derivative of order  $j$ . When  $s > 0$  is not an integer and  $p < \infty$ , the  $W^{s,p}$ -norm is defined by

$$\|u\|_{W^{s,p}} := \|u\|_{W^{m,p}} + \mathfrak{p}_{\sigma,p}(u^{(m)})$$

where

$$m = [s] \quad \text{and} \quad s = m + \sigma,$$

and the semi-norm  $\mathfrak{p}_{\sigma,p}$  ( $0 < \sigma < 1$ ) is defined by

$$\mathfrak{p}_{\sigma,p}(w) = \left( \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|w(x) - w(y)|^p}{|x - y|^{1+p\sigma}} dx dy \right)^{1/p}.$$

In the following, we write  $\mathfrak{p}_{\sigma,2} = \mathfrak{p}_\sigma$ , when there is no ambiguity. The Banach space  $W^{s,p}(\mathbb{S}^1)$  is by definition the completion of  $C^\infty(\mathbb{S}^1)$  with respect to the  $W^{s,p}$ -norm and when  $p = 2$ , we get the Hilbert space

$$W^{s,2}(\mathbb{S}^1) = H^s(\mathbb{S}^1).$$

Recall that for  $1 \leq p, q < \infty$  and  $r, s \in \mathbb{R}$  such that

$$r \geq s, \quad \text{and} \quad r - \frac{1}{p} \geq s - \frac{1}{q},$$

we have the continuous *Sobolev embeddings* (see [40])

$$W^{r,p}(\mathbb{S}^1) \subset W^{s,q}(\mathbb{S}^1),$$

and

$$H^s(\mathbb{S}^1) \subset W^{m,\infty}(\mathbb{S}^1),$$

for  $s = m + \sigma$ ,  $m \in \mathbb{N}$  and  $\sigma > 1/2$ .

Recall that the space  $H^s(\mathbb{S}^1)$  is a multiplicative algebra for  $s > 1/2$ . We have, moreover, the following result which is a discrete version of [22, Lemma 2.3].

**Lemma B.1.** *Let  $q > 1/2$  and  $0 \leq \rho \leq q$ . Then, pointwise multiplication in  $C^\infty(\mathbb{S}^1)$  extends to a continuous bilinear mapping*

$$H^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1).$$

*Proof.* Given  $u, v \in C^\infty(\mathbb{S}^1)$ , we have

$$\|uv\|_{H^\rho}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^\rho |\widehat{uv}(n)|^2.$$

Let us introduce the sequences

$$\tilde{u}(k) := (1 + k^2)^{q/2} |\hat{u}(k)|, \quad \text{and} \quad \tilde{v}(l) := (1 + l^2)^{\rho/2} |\hat{v}(l)|,$$

so that

$$\|\tilde{u}\|_{l^2} = \|u\|_{H^q}, \quad \text{and} \quad \|\tilde{v}\|_{l^2} = \|v\|_{H^\rho}.$$

We have thus

$$\begin{aligned} (1 + n^2)^{\rho/2} |\widehat{uv}(n)| &= (1 + n^2)^{\rho/2} \left| \sum_{k+l=n} \hat{u}(k) \hat{v}(l) \right| \\ &\lesssim \sum_{\substack{k+l=n \\ |k| \leq |l|}} \frac{1}{(1 + k^2)^{q/2}} \tilde{u}(k) \tilde{v}(l) + \sum_{\substack{k+l=n \\ |k| > |l|}} \frac{1}{(1 + l^2)^{q/2}} \tilde{u}(k) \tilde{v}(l) \\ &\lesssim (\lambda^q \tilde{u} * \tilde{v})(n) + (\tilde{u} * \lambda^q \tilde{v})(n), \end{aligned}$$

where  $\lambda^q$  is the sequence defined by

$$\lambda^q(k) := \frac{1}{(1 + k^2)^{q/2}}$$

and  $*$  stand for the convolution of sequences. By virtue of the Young inequality

$$\|a * b\|_{l^2} \lesssim \|a\|_{l^1} \|b\|_{l^2},$$

which is valid for any complex sequences  $a, b$ , we get

$$\|\lambda^q \tilde{u} * \tilde{v}\|_{l^2} \lesssim \|\lambda^q \tilde{u}\|_{l^1} \|\tilde{v}\|_{l^2}, \quad \|\tilde{u} * \lambda^q \tilde{v}\|_{l^2} \lesssim \|\lambda^q \tilde{v}\|_{l^1} \|\tilde{u}\|_{l^2},$$

and by Cauchy-Schwarz, we obtain finally

$$\|uv\|_{H^\rho} \lesssim \|\lambda^q\|_{l^2} \|\tilde{u}\|_{l^2} \|\tilde{v}\|_{l^2} \lesssim \|u\|_{H^q} \|v\|_{H^\rho},$$

which achieves the proof.  $\square$

The main estimates which have been used in this paper concerning the right representation of  $\mathcal{D}^q(\mathbb{S}^1)$  on  $\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))$  are given below. Note that the cases overlap.

**Lemma B.2.** *Let  $q > 3/2$ ,  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$  and  $0 \leq \rho \leq q$ .*



**Case 1:** For  $0 \leq \rho \leq 1$ , we have

$$(34) \quad \|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))} \leq C_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}).$$

**Case 2:** For  $0 \leq \rho \leq 2$ , we have

$$(35) \quad \|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))} \leq C_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}}).$$

**Case 3:** For  $3/2 < \rho \leq 3$ , we have

$$(36) \quad \|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))} \leq C_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|\varphi_x\|_{H^{\rho-1}}.$$

**Case 4:** For  $\rho > 5/2$ , we have

$$(37) \quad \|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))} \leq C_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{\rho-2}}) \|\varphi_x\|_{H^{\rho-1}}.$$

In each case,  $C_\rho$  is a positive, continuous function on  $(\mathbb{R}^+)^2$ , depending on  $\rho$ .

*Proof.* Let  $q > 3/2$  and fix  $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ . Since the estimates involve only linear expressions of  $u$  and  $C^\infty(\mathbb{S}^1)$  is dense in  $H^\rho(\mathbb{S}^1)$ , it is enough to establish them for  $u \in C^\infty(\mathbb{S}^1)$ . A change of variables leads to

$$\|u \circ \varphi\|_{L^2} \leq \|1/\varphi_x\|_{L^\infty}^{1/2} \|u\|_{L^2},$$

whereas

$$\mathfrak{p}_\sigma(u \circ \varphi) \leq \|1/\varphi_x\|_{L^\infty} \|\varphi_x\|_{L^\infty}^{(1+2\sigma)/2} \mathfrak{p}_\sigma(u),$$

for  $\sigma \in (0, 1)$ . Moreover, we have

$$\left\| (u \circ \varphi)^{(1)} \right\|_{L^2} \lesssim \|\varphi_x\|_{L^\infty} \|1/\varphi_x\|_{L^\infty}^{1/2} \|u_x\|_{L^2},$$

which proves the *first case*. Note that this proves also (35) for  $0 \leq \rho \leq 1$ , because  $\|\varphi_x\|_{L^\infty} \leq \|\varphi_x\|_{H^{q-1}}$ . Suppose now that  $\rho = 1 + \sigma$ , where  $0 < \sigma \leq 1$ . In this case, we have

$$\begin{aligned} \|u \circ \varphi\|_{H^\rho} &\lesssim \|u \circ \varphi\|_{H^1} + \|(u_x \circ \varphi)\varphi_x\|_{H^\sigma} \\ &\lesssim C_1 (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}}) \|u\|_{H^1} \\ &\quad + C_\sigma (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}}) \|u_x\|_{H^\sigma} \|\varphi_x\|_{H^{q-1}}, \end{aligned}$$

where we have used lemma B.1 and (35) for  $0 \leq \rho \leq 1$ . This completes the proof of the *second case*. If  $\rho = 1 + \sigma$  with  $1/2 < \sigma \leq 1$ , we have using (34) and the fact that  $H^\sigma(\mathbb{S}^1)$  is a multiplicative algebra

$$\begin{aligned} \|u \circ \varphi\|_{H^\rho} &\lesssim \|u \circ \varphi\|_{H^1} + \|u_x \circ \varphi\|_{H^\sigma} \|\varphi_x\|_{H^\sigma} \\ &\lesssim C_1 (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|u\|_{H^1} \\ &\quad + C_\sigma (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|u_x\|_{H^\sigma} \|\varphi_x\|_{H^\sigma}. \end{aligned}$$

Now, noting that

$$1 \leq \|1/\varphi_x\|_{L^\infty} \|\varphi_x\|_{L^\infty} \leq \|1/\varphi_x\|_{L^\infty} \|\varphi_x\|_{H^\sigma},$$

we get

$$\|u \circ \varphi\|_{H^\rho} \lesssim C_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|\varphi_x\|_{H^{\rho-1}} \|u\|_{H^\rho},$$

where  $C_\rho$  is a positive, continuous function on  $(\mathbb{R}^+)^2$ . If  $\rho = 2 + \sigma$  where  $0 < \sigma \leq 1$ , we have by virtue of lemma B.1 and (34)

$$\begin{aligned} \left\| (u \circ \varphi)^{(2)} \right\|_{H^\sigma} &\lesssim \|u_{xx} \circ \varphi\|_{H^\sigma} \|(\varphi_x)^2\|_{H^1} + \|u_x \circ \varphi\|_{H^1} \|\varphi_{xx}\|_{H^\sigma} \\ &\lesssim C_\sigma (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|u_{xx}\|_{H^\sigma} \|\varphi_x\|_{L^\infty} \|\varphi_x\|_{H^1} \\ &\quad + C_1 (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|u_x\|_{H^1} \|\varphi_{xx}\|_{H^\sigma} \end{aligned}$$

because  $\|(\varphi_x)^2\|_{H^1} \lesssim \|\varphi_x\|_{L^\infty} \|\varphi_x\|_{H^1}$ . Therefore

$$\begin{aligned} \|u \circ \varphi\|_{H^\rho} &\lesssim \|u \circ \varphi\|_{H^1} + \left\| (u \circ \varphi)^{(2)} \right\|_{H^\sigma} \\ &\lesssim C_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{L^\infty}) \|\varphi_x\|_{H^{\rho-1}} \|u\|_{H^\rho}, \end{aligned}$$

where  $C_\rho$  is a positive, continuous function. This proves *the third case*. If  $\rho = 2 + \sigma$  and  $1/2 < \sigma \leq 1$  we get immediately (37) from the preceding computation because then  $\|\varphi_x\|_{L^\infty} \lesssim \|\varphi_x\|_{H^\sigma}$ . Suppose now that  $\rho = m + \sigma$ , where  $m \geq 3$  and  $\sigma \in [0, 1)$ . Given an integer  $n$ , we have

$$(u \circ \varphi)^{(n+1)} = \sum_{k=0}^n \left( u_x^{(k)} \circ \varphi \right) W_{n,k}(\varphi),$$

where  $W_{n,k}(\varphi)$  is a homogeneous polynomial of degree  $k+1$  in the variables  $\varphi_x, \dots, \varphi_x^{(n-k)}$ . The sequence  $W_{n,k}$  is defined by

$$W_{n,n}(\varphi) = \varphi_x^{n+1}, \quad W_{n,0}(\varphi) = \varphi_x^{(n)},$$

and

$$W_{n+1,k}(\varphi) = W_{n,k-1}(\varphi)\varphi_x + W_{n,k}(\varphi)', \quad 1 \leq k \leq n.$$

In particular, for  $n \geq 2$ , we have

$$W_{n,1}(\varphi) = (n+1)\varphi_x\varphi_x^{(n-1)} + P_n(\varphi_x, \dots, \varphi_x^{(n-2)}),$$

where  $P_n$  is a homogeneous polynomial of degree 2. Thus, for  $m \geq 2$ , we get

$$\begin{aligned} \left\| (u \circ \varphi)^{(m+1)} \right\|_{L^2} &\lesssim \|u_x\|_{L^\infty} \left\| \varphi_x^{(m)} \right\|_{L^2} + \|u_{xx} \circ \varphi\|_{L^2} \|W_{m,1}(\varphi)\|_{L^\infty} \\ &\quad + \sum_{k=2}^m \left\| u_x^{(k)} \circ \varphi \right\|_{L^2} \|W_{m,k}(\varphi)\|_{L^\infty} \\ (38) \quad &\lesssim \left[ 1 + \|1/\varphi_x\|_{L^\infty} \left( \|\varphi_x\|_{L^\infty} + \sum_{k=1}^m \|\varphi_x\|_{H^{m-1}}^k \right) \right] \\ &\quad \times \|\varphi_x\|_{H^{m-1}} \|u\|_{H^m}, \end{aligned}$$

because

$$\|W_{m,1}(\varphi)\|_{L^\infty} \lesssim \|\varphi_x\|_{L^\infty} \|\varphi_x\|_{H^m} + \|\varphi_x\|_{H^{m-1}}^2,$$

and

$$\|W_{m,k}(\varphi)\|_{L^\infty} \lesssim \|\varphi_x\|_{H^{m-1}}^{k+1}, \quad 2 \leq k \leq m.$$

Starting, with  $m = 2$ , we get first (37) for  $\rho = 3$  and the case when  $\rho$  is an integer  $m \geq 3$  is obtained by an inductive argument, using (38). Now, using lemma B.1, we have for  $m \geq 3$  and  $0 < \sigma < 1$

$$\left\| (u \circ \varphi)^{(m)} \right\|_{H^\sigma} \lesssim \|u_x \circ \varphi\|_{H^1} \left\| \varphi_x^{(m-1)} \right\|_{H^\sigma} + \sum_{k=1}^{m-1} \left\| u_x^{(k)} \circ \varphi \right\|_{H^\sigma} \|W_{m-1,k}(\varphi)\|_{H^1}.$$

But, for  $2 \leq k \leq m-1$  we have

$$\|W_{m-1,k}(\varphi)\|_{H^1} \lesssim \|\varphi_x\|_{H^{m-2}}^{k+1}.$$

and for  $k=1$  and  $m-1 \geq 2$ , we have

$$\|W_{m-1,1}(\varphi)\|_{H^1} \lesssim \|\varphi_x\|_{H^{m-2}} \|\varphi_x\|_{H^{m-1}}.$$

We get therefore

$$\begin{aligned} \|(u \circ \varphi)^{(m)}\|_{H^\sigma} &\lesssim \left( \|1/\varphi_x\|_{L^\infty}^{1/2} + \|1/\varphi_x\|_{L^\infty}^{3/2} + \|1/\varphi_x\|_{L^\infty}^{1/2} \sum_{k=1}^{m-1} \|\varphi_x\|_{H^{m-2}}^k \right) \\ &\quad \times \|\varphi_x\|_{H^{\rho-1}} \|u\|_{H^\rho} \end{aligned}$$

which shows that estimate (37) is also true when  $\rho > 5/2$  is not necessary an integer. This proves the *fourth case* and achieves the proof.  $\square$

**Corollary B.3.** *Let  $q > 3/2$  and  $0 \leq \rho \leq q$ . Then, the mapping*

$$(\varphi, u) \mapsto u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1)$$

*is continuous.*

*Proof.* Lemma B.2 shows that the mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))$$

is locally bounded for  $q > 3/2$  and  $0 \leq \rho \leq q$ . To establish the continuity of the mapping

$$(\varphi, u) \mapsto u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1),$$

it is thus sufficient to prove that

$$\varphi \mapsto u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1).$$

is continuous for each fixed  $u \in H^\rho(\mathbb{S}^1)$ . Let  $\varphi_0 \in \mathcal{D}^q(\mathbb{S}^1)$  and  $\varepsilon > 0$ . Choose a neighbourhood  $V$  of  $\varphi_0$ , that we may suppose to be a ball in a local chart of the Banach manifold  $\mathcal{D}^q(\mathbb{S}^1)$ , and on which

$$\|R_\varphi\|_{\mathcal{L}(H^\rho(\mathbb{S}^1), H^\rho(\mathbb{S}^1))} < K,$$

for some positive constant  $K$ . Since  $C^\infty(\mathbb{S}^1)$  is dense in  $H^\rho(\mathbb{S}^1)$ , we can find  $w \in C^\infty(\mathbb{S}^1)$  such that

$$\|u - w\|_{H^\rho} < \varepsilon/K.$$

We have thus

$$\|u \circ \varphi - u \circ \varphi_0\|_{H^\rho} < \|w \circ \varphi - w \circ \varphi_0\|_{H^\rho} + 2\varepsilon.$$

Now, let  $\varphi(t) := t\varphi + (1-t)\varphi_0$ . We have, pointwise

$$w \circ \varphi - w \circ \varphi_0 = \int_0^1 (w_x \circ \varphi(t))(\varphi - \varphi_0) dt,$$

and we deduce from lemma B.1, that

$$\|w \circ \varphi - w \circ \varphi_0\|_{H^\rho} \leq C \int_0^1 \|w_x \circ \varphi(t)\|_{H^\rho} \|\varphi - \varphi_0\|_{H^q} dt,$$

for some positive constant  $C$ , which depends only on  $q$  and  $\rho$ . Thus

$$\|w \circ \varphi - w \circ \varphi_0\|_{H^\rho} \leq CK \|w_x\|_{H^\rho} \|\varphi - \varphi_0\|_{H^q},$$

and therefore, if  $\|\varphi - \varphi_0\|_{H^q}$  is small enough, we get that

$$\|u \circ \varphi - u \circ \varphi_0\|_{H^p} < 3\varepsilon,$$

which achieves the proof.  $\square$

*Remark B.4.* The mapping

$$(\varphi, u) \mapsto R_\varphi(u) := u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1).$$

is continuous for  $q > 3/2$  (see [9]), but that does not imply that the mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1))$$

is continuous with respect to the operator norm on  $\mathcal{L}(H^q(\mathbb{S}^1))$  (*norm continuity*). *Norm continuity* obviously implies *continuity* but the converse is false. Indeed, a general result in the theory of semigroups of linear operators states that a semigroup on a Banach space  $E$  is norm continuous at 0, if and only if, its infinitesimal generator is bounded on  $E$ , cf. [35, Theorem 1.2]. Let now  $q > 3/2$  and let  $\tau_s$  be the rotation by the angle  $s$  on  $\mathbb{S}^1$ . Then the representation of the group  $\{R_{\tau_s}; s \in \mathbb{R}\}$  is continuous on  $H^q(\mathbb{S}^1)$ . But it cannot be norm continuous, since its infinitesimal generator  $D$  is not bounded on  $H^q(\mathbb{S}^1)$ . A direct argument, which shows that  $\|R_{\tau_s} - \text{Id}\|_{\mathcal{L}(H^q(\mathbb{S}^1))}$  is bounded away from 0 for all  $s$  near 0 is runs as follows: Let  $s \in (-1/2, 1/2)$  and  $u_s$  be a periodic, bump function with support in  $(k - s/2, k + s/2)$  ( $k \in \mathbb{Z}$ ) with  $\|u_s\|_{L^2} = 1$ . We have then

$$\|R_{\tau_s} u_s - u_s\|_{H^q(\mathbb{S}^1)}^2 = 2 \|u_s\|_{H^q(\mathbb{S}^1)}^2,$$

because  $u_s$  and  $R_{\tau_s} u$  are  $H^q(\mathbb{S}^1)$ -orthogonal and  $R_{\tau_s}$  is an  $H^q(\mathbb{S}^1)$ -isometry. Hence

$$\|R_{\tau_s} - \text{Id}\|_{\mathcal{L}(H^q(\mathbb{S}^1))} \geq \sqrt{2} \quad \text{for} \quad \frac{-1}{2} < s < \frac{1}{2},$$

which proves that the representation  $\varphi \mapsto R_\varphi$  is not *norm continuous*.

Nevertheless, we have the following result.

**Corollary B.5.** *Let  $q > 3/2$ . Then, the mappings*

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))$$

and

$$\varphi \mapsto R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))$$

are continuous.

*Proof.* Note first that since  $\varphi \mapsto \varphi^{-1}$  from  $\mathcal{D}^q(\mathbb{S}^1)$  to  $\mathcal{D}^q(\mathbb{S}^1)$  is continuous (see for instance [22]), it is enough to show that

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^{q-1}(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))$$

is continuous. Let  $\varphi_0 \in \mathcal{D}^q(\mathbb{S}^1)$ . Choose a neighbourhood  $V$  of  $\varphi_0$ , that we may suppose to be a ball in a local chart of the Banach manifold  $\mathcal{D}^q(\mathbb{S}^1)$ , and on which

$$\|R_\varphi\|_{\mathcal{L}(H^{q-1}(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))} < K,$$

for some positive constant  $K$ . Now, let  $\varphi(t) := t\varphi + (1-t)\varphi_0$  and  $v \in H^q(\mathbb{S}^1)$ . We have, pointwise

$$v \circ \varphi - v \circ \varphi_0 = \int_0^1 (v_x \circ \varphi(t))(\varphi - \varphi_0) dt.$$

Hence

$$\|v \circ \varphi - v \circ \varphi_0\|_{H^{q-1}} \leq C \int_0^1 \|w_x \circ \varphi(t)\|_{H^{q-1}} \|\varphi - \varphi_0\|_{H^{q-1}} dt,$$

for some positive constant  $C$  and from which we deduce that

$$\|R_\varphi - R_{\varphi_0}\|_{\mathcal{L}(H^q(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))} \leq CK \|\varphi - \varphi_0\|_{H^q}.$$

This shows that

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^{q-1}(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))$$

is locally Lipschitz continuous and completes the proof.  $\square$

Another interpreting consequence of estimates B.2 is the following results, which extends [22, Theorem 1.2] beyond the critical exponent.

**Corollary B.6.** *Let  $q > 3/2$ . Then, the mappings*

$$(\varphi, v) \mapsto v \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-1}(\mathbb{S}^1)$$

and

$$(\varphi, v) \mapsto v \circ \varphi^{-1}, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-1}(\mathbb{S}^1)$$

are  $C^1$ .

*Proof.* We are going to show that

$$(\varphi, v) \mapsto v \circ \varphi^{-1}, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-1}(\mathbb{S}^1)$$

is  $C^1$ . The proof that the first mapping is  $C^1$  is similar and easier. Observe first that if  $\varphi(s)$  and  $v(s)$  are smooth paths in  $\text{Diff}^\infty(\mathbb{S}^1)$  and  $C^\infty(\mathbb{S}^1)$  respectively with  $\varphi(0) = \varphi$  and  $v(0) = v$ , we have

$$\partial_s [v(s) \circ \varphi(s)^{-1}]_{s=0} = (\delta v \circ \varphi^{-1}) - \left( \frac{v_x \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \right) (\delta \varphi \circ \varphi^{-1}),$$

where  $\delta \varphi = \partial_s \varphi(s)|_{s=0}$  and  $\delta v = \partial_s v(s)|_{s=0}$ . Let  $U$  be a local chart in  $\mathcal{D}^q(\mathbb{S}^1)$ , that we assume to be a ball in  $H^q(\mathbb{S}^1)$ . Given  $\varphi_0, \varphi_1$  in  $\text{Diff}^\infty(\mathbb{S}^1) \cap U$  and  $v_0, v_1 \in C^\infty(\mathbb{S}^1)$ , we set

$$\varphi(t) := (1-t)\varphi_0 + t\varphi_1, \quad \text{and} \quad v(t) := (1-t)v_0 + tv_1,$$

for  $t \in [0, 1]$ . We have therefore

$$(39) \quad v_1 \circ \varphi_1^{-1} - v_0 \circ \varphi_0^{-1} = \int_0^1 ((v_1 - v_0) \circ \varphi(t)^{-1}) dt \\ - \int_0^1 \left( \frac{v_x(t) \circ \varphi(t)^{-1}}{\varphi_x(t) \circ \varphi(t)^{-1}} \right) ((\varphi_1 - \varphi_0) \circ \varphi(t)^{-1}) dt.$$

Now, using lemma 3.5, we observe that both sides of (39) are continuous expressions of  $\varphi_k, v_k$  ( $k = 0, 1$ ). Using a density argument, we conclude therefore, that the equality is still true in  $H^{q-1}(\mathbb{S}^1)$  if we take  $\varphi_0, \varphi_1 \in \mathcal{D}^q(\mathbb{S}^1)$  and  $v_0, v_1 \in \mathcal{D}^q(\mathbb{S}^1)$ . Furthermore, the mapping

$$(\varphi, v) \mapsto \left[ (\delta \varphi, \delta v) \mapsto R_{\varphi^{-1}} \delta v - \left( \frac{v_x \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \right) R_{\varphi^{-1}} \delta \varphi \right]$$

from

$$\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \quad \text{to} \quad \mathcal{L}(H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), H^{q-1}(\mathbb{S}^1))$$

is continuous by corollary B.5 and the fact that  $H^{q-1}(\mathbb{S}^1)$  is a multiplicative algebra for  $q > 3/2$ . We conclude the proof using lemma 3.6.  $\square$

To conclude this Appendix, we provide an estimate for the norm of  $(\varphi^{-1})_x$  which might be useful, on its own.

**Lemma B.7.** *Let  $q > 3/2$ . Then*

$$\|(\varphi^{-1})_x\|_{H^{q-1}} \lesssim C_q(\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}})$$

where  $C_q$  is a positive, continuous function on  $(\mathbb{R}^+)^2$ .

*Proof.* Given  $\sigma \in (0, 1)$ , a change of variables leads to the estimate

$$\|w \circ \varphi^{-1}\|_{H^\sigma} \lesssim \left( \|\varphi_x\|_{L^\infty}^{1/2} + \|\varphi_x\|_{L^\infty} \|1/\varphi_x\|_{L^\infty}^{(1+2\sigma)/2} \right) \|w\|_{H^\sigma},$$

for any  $w \in H^\sigma(\mathbb{S}^1)$  and any  $C^1$  diffeomorphism  $\varphi$ , whereas

$$\|w \circ \varphi^{-1}\|_{H^1} \lesssim \left( \|\varphi_x\|_{L^\infty}^{1/2} + \|\varphi_x\|_{L^\infty}^{3/2} \right) \|1/\varphi_x\|_{L^\infty} \|w\|_{H^1},$$

for any  $w \in H^1(\mathbb{S}^1)$  and any  $C^1$  diffeomorphism  $\varphi$ .

1) Suppose first that  $q = 1 + \sigma$  and thus  $\sigma > 1/2$ . We get

$$\begin{aligned} \|(\varphi^{-1})_x\|_{H^\sigma} &= \left\| \frac{1}{\varphi_x} \circ \varphi^{-1} \right\|_{H^\sigma} \\ &\leq \left( \|\varphi_x\|_{L^\infty}^{1/2} + \|\varphi_x\|_{L^\infty} \|1/\varphi_x\|_{L^\infty}^{(1+2\sigma)/2} \right) \|1/\varphi_x\|_{H^\sigma}. \end{aligned}$$

But a direct computation shows that

$$\|1/\varphi_x\|_{H^\sigma} \lesssim \|1/\varphi_x\|_{L^\infty}^2 \|\varphi_x\|_{H^\sigma},$$

which finishes the proof of the lemma for  $q < 2$ , since  $\|\varphi_x\|_{L^\infty} \lesssim \|\varphi_x\|_{H^\sigma}$ .

2) Suppose now that  $q = m + \sigma$ , where  $m \geq 2$  and  $\sigma \in [0, 1)$ . Given an integer  $n$ , we have

$$\left( \frac{1}{\varphi_x \circ \varphi^{-1}} \right)^{(n)} = \frac{1}{(\varphi_x \circ \varphi^{-1})^{2n+1}} P_n(\varphi_x \circ \varphi^{-1}, \dots, \varphi_x^{(n)} \circ \varphi^{-1}),$$

where  $P_n$  is a homogeneous polynomial of degree  $n$ , which is of partial degree at most one in  $\varphi_x^{(n)} \circ \varphi^{-1}$ . We have therefore

$$\begin{aligned} \left\| \frac{1}{\varphi_x \circ \varphi^{-1}} \right\|_{H^{m-2}} &\lesssim \sum_{k=0}^{m-2} \|1/\varphi_x\|_{L^\infty}^{2k+1} \|P_k\|_{L^\infty} \\ &\lesssim \sum_{k=0}^{m-2} \|1/\varphi_x\|_{L^\infty}^{2k+1} \|\varphi_x\|_{H^{m-1}}^k, \end{aligned}$$

whereas

$$\left\| \left( \frac{1}{\varphi_x \circ \varphi^{-1}} \right)^{(m-1)} \right\|_{H^\sigma} \lesssim \|1/\varphi_x\|_{H^1}^{2m-1} \|P_{m-1}\|_{H^\sigma},$$

by virtue of lemma B.1. But

$$P_{m-1} = a_0(\varphi) + a_1(\varphi) \left( \varphi_x^{(m-1)} \circ \varphi^{-1} \right),$$

where  $a_j(\varphi)$  is a homogeneous polynomial of degree  $m-1-j$  in the variables  $\varphi_x \circ \varphi^{-1}, \dots, \varphi_x^{(m-2)} \circ \varphi^{-1}$ . Therefore, we have

$$\|P_{m-1}\|_{H^\sigma} \lesssim \|a_0(\varphi)\|_{H^1} + \|a_1(\varphi)\|_{H^1} \left\| \varphi_x^{(m-1)} \circ \varphi^{-1} \right\|_{H^\sigma},$$

where

$$\|a_j(\varphi)\|_{H^1} \lesssim \left( \|\varphi_x\|_{L^\infty}^{1/2} + \|\varphi_x\|_{L^\infty}^{3/2} \right)^{m-1-j} \|1/\varphi_x\|_{L^\infty}^{m-1-j} \|\varphi_x\|_{H^{m-1}}^{m-1-j},$$

and

$$\left\| \varphi_x^{(m-1)} \circ \varphi^{-1} \right\|_{H^\sigma} \lesssim \left( \|\varphi_x\|_{L^\infty}^{1/2} + \|\varphi_x\|_{L^\infty} \|1/\varphi_x\|_{L^\infty}^{(1+2\sigma)/2} \right) \|\varphi_x\|_{H^{q-1}}$$

which ends the proof.  $\square$

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