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RIGHT-INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE DIFFEOMORPHISMS GROUP OF THE CIRCLE

JOACHIM ESCHER AND BORIS KOLEV

ABSTRACT. In this paper we study the geodesic flow of a right-invariant metric induced by a general Fourier multiplier on the diffeomorphisms group of the circle and on some of its homogeneous spaces. This study covers in particular right-invariant metrics induced by Sobolev norms of fractional order. We show that, under a certain condition on the symbol of the inertia operator (which is satisfied for the fractional Sobolev norm H^s for $s \geq 1/2$), the corresponding initial value problem is well-posed in the smooth category and that the Riemannian exponential map is a smooth local diffeomorphism. Paradigmatic examples of our general setting cover, besides all traditional Euler equations induced by a local inertia operator, the Constantin-Lax-Majda equation, and the Euler-Weil-Petersson equation.

CONTENTS

1. Introduction	2
2. Geometric Background	4
2.1. Euler equation on a Lie group	4
2.2. The diffeomorphisms group of the circle	5
3. Fourier multipliers and proof of the main theorem	7
3.1. Recursive formulas and multi-symbols	9
3.2. Extension to Sobolev spaces	12
4. Smoothness of the metric and the spray	17
5. Existence results for geodesics in the smooth category	20
6. Exponential map and geodesic distance	23
7. Euler equations on homogeneous spaces	25
7.1. Euler equation on a homogeneous space	25
7.2. Euler equations on the coadjoint orbit $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$	27
7.3. Euler equations on the coadjoint orbit $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$	29
Appendix A. Boundedness properties of right translations	31
Acknowledgments	35
References	35

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1. INTRODUCTION

It is a fundamental observation due to Arnold [2] that the Euler equations of hydrodynamics of an ideal fluid on a compact Riemannian manifold M can be recast as the geodesic flow for a right invariant Riemannian metric on the group of volume preserving smooth diffeomorphisms of M . In this general picture, there is a great latitude in the choice of the inertia operator (i.e the induced inner product on the Lie algebra of the group) which generates the metric. In the classical papers [2, 11], L^2 inner products have been used. This interpretation was extended thereafter to other equations of physical relevance [6, 20, 22, 24, 30, 34, 13]. Among these studies, the particular case of right invariant metrics induced by H^k Sobolev norms with $k \in \mathbb{N}$ and $k \geq 1$ have been extensively investigated [6, 22, 24, 34]. In [13] a (non local) inertia operator on $\text{Diff}^\infty(\mathbb{S}^1)$, the diffeomorphisms group of the circle, of the form HD , where H denotes the Hilbert transform and D the spatial derivative, has been considered. In [15] the third order operator $A = HD(D^2 + 1)$ has been studied as an inertia operator on a suitable approximation of the homogeneous space $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$.

It is the aim of the present paper to extend to *non-local* inertia operators on $\text{Diff}^\infty(\mathbb{S}^1)$, some earlier results obtained for differential operators. The essence of the method is to use a Lagrangian formalism for the geodesic flow that leads to evolution equations on the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ with a *tame* propagator. More precisely, consider a *right-invariant* metric on $\text{Diff}^\infty(\mathbb{S}^1)$, induced by an inner product

$$\langle u, v \rangle = \int_{\mathbb{S}^1} (Au)v \, dx,$$

on $\text{Vect}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)$, where $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$ is a *Fourier multiplier*¹. The corresponding *sharp map* $T\text{Diff}^\infty(\mathbb{S}^1) \rightarrow T\text{Diff}^\infty(\mathbb{S}^1)^*$ is given by $(\varphi, v) \mapsto (\varphi, \varphi_x A_\varphi)$, where $A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}$ and $R_\varphi(v) := v \circ \varphi$. Does this metric extends *smoothly* on $\mathcal{D}^q(\mathbb{S}^1)$, the Hilbert manifold and topological group of diffeomorphisms of orientation preserving diffeomorphisms of Sobolev class H^q of the circle? When A is of finite order $r \geq 0$, it extends to a bounded linear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$ for q large enough and we are lead to the following natural question.

Problem. *Given a Fourier multiplier P of order $r \geq 0$, under which conditions is the mapping*

$$\varphi \mapsto P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

smooth?

Notice that the problem is not trivial in general, because the mapping

$$(\varphi, v) \mapsto R_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is *not differentiable* (see [11] for instance). It is however a well-known fact that when P is a differential operator of order r , P_φ is a linear differential operator whose coefficients are polynomial expressions of $1/\varphi_x$ and the derivatives of φ up to order r (see [11, 12] for instance). In that case, $\varphi \mapsto P_\varphi$

¹Standard definitions and basic facts on Fourier multipliers are collected in Section 3.

is smooth for $q > r + 3/2$. However for a general Fourier multiplier we are not aware of any results in this direction. In our main theorem below, we give a *sufficient condition on the symbol* of P which ensures that this map is smooth. This answers a question raised in [11, Appendix A], at least in the case of the diffeomorphisms group of the circle. Up to the authors knowledge, these results are new.

Theorem 1.1. *Let $P = \mathbf{op}(p(k))$ be a Fourier multiplier of order $r \geq 1$. Suppose that its symbol p extends to \mathbb{R} and that for each $n \geq 1$, the function*

$$f_n(\xi) := \xi^{n-1}p(\xi)$$

is of class C^{n-1} , that $f_n^{(n-1)}$ is absolutely continuous and that there exists $C_n > 0$ such that

$$(1.1) \quad \left| f_n^{(n)}(\xi) \right| \leq C_n(1 + \xi^2)^{(r-1)/2},$$

almost everywhere. Then the map

$$\varphi \mapsto P_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for each $q \in (\frac{3}{2} + r, \infty)$.

The proof of Theorem 1.1 is given in Section 3.

Remark 1.2. The hypothesis of Theorem 1.1 are always satisfied when p can be extended to a smooth function on \mathbb{R} such that $p^{(k)}(\xi) = O(|\xi|^{r-k})$ at infinity, for all $k \in \mathbb{N}$. This applies in particular when P is a differential operator (i.e. when p is a polynomial).

Remark 1.3. Of course there are Fourier multiplication operators of order less than 1 for which the mapping

$$\varphi \mapsto P_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth. However, the present proof of Theorem 1.1 works only for $r \geq 1$. So far, the authors have not been able to exhibit a counter-example which would show that the conclusion of Theorem 1.1 is false for $0 \leq r < 1$. They are not aware either of an example of a Fourier multiplier for which the conclusion of Theorem 1.1 fails for all $q \geq 0$.

Theorem 1.1 applies in particular to the inertia operator Λ^{2s} of the Sobolev metric H^s on $\text{Diff}^\infty(\mathbb{S}^1)$ for $s \in \mathbb{R}$ and $s \geq 1/2$.

Corollary 1.4. *Let $s \in \mathbb{R}$ and $\Lambda^{2s} := \mathbf{op}((1 + n^2)^s)$. If $s \geq 1/2$ then the mapping*

$$\varphi \mapsto R_\varphi \circ \Lambda^{2s} \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-2s}(\mathbb{S}^1))$$

is smooth for any $q \in (\frac{3}{2} + 2s, \infty)$.

Proof of Corollary 1.4. We have to check that the symbol of Λ^{2s} satisfies the hypothesis of Theorem 1.1. In view of Remark 1.2, it is sufficient to show that $g_s^{(k)}(\xi) = O(|\xi|^{2s-k})$ where $g_s(\xi) := (1 + \xi^2)^s$, for all $k \in \mathbb{N}$. This can be checked easily using the fact that

$$g_s^{(k)}(\xi) = \frac{p_k(\xi)}{(1 + \xi^2)^k} g_s(\xi),$$

for $k \geq 1$, where p_k is a polynomial function with $d(p_k) \leq k$. \square

Corollary 1.4 allows us to prove that the corresponding *weak Riemannian* metric and its geodesic spray can be smoothly extended to the Hilbert manifold approximation $\mathcal{D}^q(\mathbb{S}^1)$ for sufficiently large $q \in \mathbb{R}$. As a corollary, we are able to prove local existence and uniqueness of geodesics on $\text{Diff}^\infty(\mathbb{S}^1)$, cf. Theorem 5.3. A particular case of Theorem 5.3 is the following result:

Corollary 1.5. *The geodesic flow on the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ induced by the Sobolev norm H^s is locally well-posed in the smooth category, provided $s \geq 1/2$. This means that given any $(\varphi_0, v_0) \in T\text{Diff}^\infty(\mathbb{S}^1)$, there exists a unique geodesic*

$$(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$$

with respect to the H^s -metric emerging from (φ_0, v_0) .

It is known that for weak Riemannian metrics the exponential mapping fails in general to be a local diffeomorphism, cf. [6]. We clarify the picture to some extent by proving the following result in Section 6.

Theorem 1.6. *The exponential mapping \exp at the unit element id for the H^s -metric on $\text{Diff}^\infty(\mathbb{S}^1)$ is a smooth local diffeomorphism from a neighbourhood of zero in $\text{Vect}(\mathbb{S}^1)$ to a neighbourhood of id on $\text{Diff}^\infty(\mathbb{S}^1)$ for each $s \geq 1/2$.*

We close our study by extending our results to Euler equations on some homogeneous spaces of $\text{Diff}^\infty(\mathbb{S}^1)$, namely $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$, where $\text{Rot}(\mathbb{S}^1)$ is the subgroup of all rigid rotations of the circle \mathbb{S}^1 and $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$, where $\text{PSL}(2, \mathbb{R})$ is the subgroup of all rigid Möbius transformations which preserve the circle \mathbb{S}^1 . The first case includes the Hunter-Saxton equation [26] and the Constantin-Lax-Majda equation [13, 14]. The second case includes the Euler-Weil-Petersson equation, which is related to the Weil-Petersson metric on the universal Teichmüller space $T(1)$, cf. [31, 35].

The plan of the paper is as follows. In Section 2 we recall some well-known facts on the geometry of the Euler equations and some basic material on weak Riemannian metrics. In Section 3 we provide the proofs of our main results: Theorem 1.1 and Corollary 1.4. Section 4 is devoted to the study of the smoothness of the metric and the geodesic spray on the extended Hilbert manifolds $\mathcal{D}^q(\mathbb{S}^1)$. In Section 5 we prove local existence and uniqueness of the initial value problem for the geodesics of the right-invariant H^s metric on $\text{Diff}^\infty(\mathbb{S}^1)$, while in Section 6 we deal with the Riemannian exponential mapping and discuss the problem of geodesic distance. In Section 7 we extend our study to the homogeneous spaces $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ and $\text{Diff}^\infty(\mathbb{S}^1)/\text{PSL}(2, \mathbb{R})$. We prove local existence result for the corresponding Euler equations. Technical lemmas on local boundedness of right translations are collected in Appendix A.

2. GEOMETRIC BACKGROUND

2.1. Euler equation on a Lie group. A *right-invariant* Riemannian metric on a Lie group G is defined by its value at the unit element, that is by an inner product on the Lie algebra \mathfrak{g} of the group. If this inner product is represented by an invertible operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$, for historical reasons, going back to the work of Euler on the motion of the rigid body, this inner

product is called the *inertia operator*. The *Levi-Civita connection* of such a Riemannian metric is itself *right-invariant* and given by

$$(2.1) \quad \nabla_{\xi_u} \xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v),$$

where ξ_v is the right-invariant vector field on G generated by $v \in \mathfrak{g}$ and B is the right-invariant tensor field on G , generated by the bilinear operator

$$(2.2) \quad B(u, v) = \frac{1}{2} \left[\text{ad}_u^\top v + \text{ad}_v^\top u \right]$$

where $v, w \in \mathfrak{g}$ and ad_v^\top is the *adjoint* (relatively to the inertia operator A) of the natural action of the Lie algebra on itself given by

$$\text{ad}_v : w \mapsto [v, w].$$

Remark 2.1. Notice that

$$\text{ad}_v^\top = A^{-1} \text{ad}_v^* A$$

where ad^* is the coadjoint action of \mathfrak{g} on itself, defined by

$$(\text{ad}_v^* m, w) := -(m, \text{ad}_v w)$$

for $m \in \mathfrak{g}^*$ and $v, w \in \mathfrak{g}$.

Given a smooth path $g(t)$ in G , we define its *Eulerian velocity*, which lies in the Lie algebra \mathfrak{g} , by

$$u(t) = R_{g^{-1}(t)} \dot{g}(t)$$

where R_g stands for the right translation² in G . It can then be shown, see e.g. [12] that $g(t)$ is a *geodesic* if and only if its Eulerian velocity u satisfies the first order equation

$$(2.3) \quad u_t = -B(u, u).$$

This equation for the velocities is known as the *Euler equation*.

2.2. The diffeomorphisms group of the circle. Let $\text{Diff}^\infty(\mathbb{S}^1)$ be the group of all smooth and orientation preserving diffeomorphisms on the circle. This group is naturally equipped with a *Fréchet manifold* structure; it can be covered by charts taking values in the *Fréchet vector space* $C^\infty(\mathbb{S}^1)$ and in such a way that the change of charts are smooth maps ([11] for more details). Since the composition and the inverse map are smooth for this structure we say that $\text{Diff}^\infty(\mathbb{S}^1)$ is a *Fréchet-Lie group*, cf. [19]. Its Lie algebra $T_{id}\text{Diff}^\infty(\mathbb{S}^1) = \text{Vect}(\mathbb{S}^1)$ is isomorphic to $C^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[u, v] = u_x v - u v_x.$$

From a topological point of view, $\text{Diff}^\infty(\mathbb{S}^1)$ may be viewed as an inverse limit of *Hilbert manifolds*; an ILH (*inverse limit Hilbert*) Lie group. More precisely, let $q \in \mathbb{R}$ with $q > 3/2$ be given and let $\mathcal{D}^q(\mathbb{S}^1)$ denote the set of all orientation preserving diffeomorphisms φ of the circle \mathbb{S}^1 , such that both φ and φ^{-1} belong to the fractional Sobolev space $H^q(\mathbb{S}^1)$. Then $\mathcal{D}^q(\mathbb{S}^1)$ is

²We use the same notation R_φ for this diffeomorphism as well as for its tangent map.

a Hilbert manifold and a topological group [11] but not a *Lie group*³. We have

$$\text{Diff}^\infty(\mathbb{S}^1) = \bigcap_{q > \frac{3}{2}} \mathcal{D}^q(\mathbb{S}^1),$$

and we call the scales of manifolds $\mathcal{D}^q(\mathbb{S}^1)_{q > 3/2}$, a Hilbert manifold approximation of $\text{Diff}^\infty(\mathbb{S}^1)$.

Like any Lie group, $\text{Diff}^\infty(\mathbb{S}^1)$ is a parallelizable manifold

$$T\text{Diff}^\infty(\mathbb{S}^1) \sim \text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1).$$

Notice that it is also the case of the Hilbert manifold (and topological group) $\mathcal{D}^q(\mathbb{S}^1)$ but for different reasons. Indeed, let $\mathfrak{t} : T\mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{R}$ be a trivialization of the tangent bundle of the circle. Then

$$\Psi : T\mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1), \quad \xi_\varphi \mapsto (\mathfrak{t}_1 \circ \xi_\varphi, \mathfrak{t}_2 \circ \xi_\varphi)$$

defines a smooth vector bundle isomorphism because \mathfrak{t} is smooth.

A *right-invariant* metric on $\text{Diff}^\infty(\mathbb{S}^1)$ is defined by an inner product on the Lie algebra $\text{Vect}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)$. In the present paper, we assume that this inner product is given by

$$\langle u, v \rangle = \int_{\mathbb{S}^1} (Au)v \, dx,$$

where $A : C^\infty(\mathbb{S}^1) \rightarrow C^\infty(\mathbb{S}^1)$ is an invertible *Fourier multiplier* (see section 3 for precise definitions). By translating the above inner product, we obtain an inner product on each tangent space $T_\varphi\text{Diff}^\infty(\mathbb{S}^1)$

$$(2.4) \quad \langle \eta, \xi \rangle_\varphi = \langle \eta \circ \varphi^{-1}, \xi \circ \varphi^{-1} \rangle_{id} = \int_{\mathbb{S}^1} \eta(A_\varphi \xi) \varphi_x \, dx,$$

where $\eta, \xi \in T_\varphi\text{Diff}^\infty(\mathbb{S}^1)$ and $A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}$. This family of pre-Hilbertian structures, indexed by $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$, is smooth because composition and inversion are smooth on the Fréchet Lie group $\text{Diff}^\infty(\mathbb{S}^1)$. This way we obtain a right-invariant, *weak Riemannian* metric on $\text{Diff}^\infty(\mathbb{S}^1)$.

On a Fréchet manifold, only covariant derivatives along curves are meaningful and in general, the existence of a symmetric, covariant derivative, compatible with a weak Riemannian metric, that is

$$\frac{d}{dt} \langle \xi, \eta \rangle_\varphi = \left\langle \frac{D\xi}{Dt}, \eta \right\rangle_\varphi + \left\langle \xi, \frac{D\eta}{Dt} \right\rangle_\varphi,$$

is far from being granted. Nevertheless, in the situation we consider, the map ad_u^\top is well defined and given by

$$\text{ad}_u^\top v = A^{-1} \left(2(Av)u_x + (Av)_x u \right)$$

for $u, v \in C^\infty(\mathbb{S}^1)$. Hence, one can define

$$B(u, v) = \frac{1}{2} A^{-1} \left[2(Av)u_x + (Av)_x u + 2(Au)v_x + (Au)_x v \right]$$

³Indeed, on $\mathcal{D}^q(\mathbb{S}^1)$, right translation $R_\varphi : \psi \mapsto \psi \circ \varphi$ is linear, continuous and hence smooth (for fixed φ); whereas left translation $L_\varphi : \psi \mapsto \varphi \circ \psi$ is continuous but not differentiable in general [11, 19].

and check that the expression

$$\frac{D\xi(t)}{Dt} = \left(\varphi, w_t + \frac{1}{2}[u, w] + B(u, w) \right),$$

where $u = \varphi_t \circ \varphi^{-1}$, and $\xi(t) = (\varphi(t), w(t))$ in $\text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$ is a vector field defined along the curve $\varphi(t) \in \text{Diff}^\infty(\mathbb{S}^1)$, defines a right-invariant, symmetric covariant derivative on $\text{Diff}^\infty(\mathbb{S}^1)$ which is compatible with the metric induced by A . The corresponding Euler equation on $\text{Diff}^\infty(\mathbb{S}^1)$ is given by

$$(2.5) \quad u_t = -A^{-1} \{ (Au)_x u + 2(Au)u_x \}.$$

The right hand side of the Euler equation is of *order 1* because if $u \in H^q(\mathbb{S}^1)$ then $A^{-1}[u(Au_x)] \in H^{q-1}(\mathbb{S}^1)$. Hence the Euler equation cannot be realized as an ODE on the Hilbert space $H^q(\mathbb{S}^1)$. It is however quite surprising that in Lagrangian coordinates the propagator of evolution equation of the geodesic flow possesses better mapping properties, provided that the *order* of A is not less than 1. In fact, let φ be the flow of the time dependent vector field u and let $v = \varphi_t$. Then $v_t = (u_t + uu_x) \circ \varphi$ and u solves the Euler equation (2.5) if and only if (φ, v) is a solution of

$$(2.6) \quad \begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases}$$

where

$$S_\varphi(v) := (R_\varphi \circ S \circ R_{\varphi^{-1}})(v),$$

and

$$S(u) := A^{-1} \{ [A, u]u_x - 2(Au)u_x \}.$$

The *second order vector field* on $\text{Diff}^\infty(\mathbb{S}^1)$, defined in a local chart by

$$F : (\varphi, v) \mapsto (v, S_\varphi(v))$$

is called the *geodesic spray* of the metric, cf. [25].

The main observation is that if A is a *differential operator* of order $r \geq 1$ then the quadratic operator

$$S(u) := A^{-1} \{ [A, u]u_x - 2(Au)u_x \}$$

is of order 0 because the commutator $[A, u]$ is of order less than $\leq r - 1$. One might expect, that for a larger class of operators A , the quadratic operator S to be of order 0 and the second order system (2.6) to be the local expression of an ODE on the Hilbert manifold $T\mathcal{D}^q(\mathbb{S}^1)$. The special case where A is a *differential operator* with constant coefficients has been extensively studied in [5, 6, 12]. It is the aim of the present paper to extend these results for a general Fourier multiplier, under certain conditions on its symbol.

3. FOURIER MULTIPLIERS AND PROOF OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.1. To do so, we will introduce some techniques to study the differentiability of *the conjugates of a Fourier multiplier*. Here and in the following, we use the notation

$$\mathbf{e}_n(x) = \exp(2\pi i n x),$$

for $n \in \mathbb{Z}$ and $x \in \mathbb{S}^1$.

Lemma 3.1. *Let P be a continuous linear operator on the Fréchet space $C^\infty(\mathbb{S}^1, \mathbb{C})$. Then the following three conditions are equivalent:*

- (1) P commutes with all rotations R_s .
- (2) $[P, D] = 0$, where $D = d/dx$.
- (3) For each $n \in \mathbb{Z}$, there is a $p(n) \in \mathbb{C}$ such that $P\mathbf{e}_n = p(n)\mathbf{e}_n$.

In that case, we say that P is a Fourier multiplier.

Since every smooth function on the unit circle \mathbb{S}^1 can be represented by its Fourier series, we get that

$$(3.1) \quad (Pu)(x) = \sum_{k \in \mathbb{Z}} p(k) \hat{u}(k) \mathbf{e}_k(x),$$

for every Fourier multiplier P and every $u \in C^\infty(\mathbb{S}^1)$, where

$$\hat{u}(k) := \int_{\mathbb{S}^1} u(x) \mathbf{e}_{-k}(x) dx,$$

stands for the k -th Fourier coefficients of u . The sequence $p : \mathbb{Z} \rightarrow \mathbb{C}$ is called the *symbol* of P . We use also the notation $P := \mathbf{op}(p(k))$ for the Fourier multiplier induced by the sequence p .

Proof. Given $s \in \mathbb{R}$ and $u \in C^\infty(\mathbb{S}^1)$, let $u_s(x) := u(x + s)$. If P commutes with translations we have

$$(Pu)_s(x) = (Pu_s)(x).$$

Taking the derivative of both sides of this equation with respect to s at 0 and using the continuity of P , we get $DPu = PDu$ which proves the implication (1) \Rightarrow (2).

If $[P, D] = 0$, then both $P\mathbf{e}_n$ and \mathbf{e}_n are solutions of the linear differential equation

$$u' = (-2\pi in)u$$

and, are therefore equal, up to a multiplicative constant $p(n)$. This proves that (2) \Rightarrow (3).

If $P\mathbf{e}_n = p(n)\mathbf{e}_n$, for each $n \in \mathbb{Z}$ and P is continuous, then we have representation (3.1). Therefore

$$\begin{aligned} (Pu)_s(x) &= \sum_{k \in \mathbb{Z}} p(k) \hat{u}(k) \mathbf{e}_k(x + s) \\ &= \sum_{k \in \mathbb{Z}} p(k) \hat{u}_s(k) \mathbf{e}_k(x) = (Pu_s)(x), \end{aligned}$$

which proves that (3) \Rightarrow (1). □

Remark 3.2. The space of Fourier multipliers is a *commutative subalgebra* of the algebra of linear operators on $C^\infty(\mathbb{S}^1, \mathbb{C})$. It contains all linear differential operators with constant coefficients. Notice that a Fourier multiplier P is L^2 -symmetric iff its symbol p is real.

3.1. Recursive formulas and multi-symbols. Let $(\varphi, v) \mapsto P_\varphi(v)$ be a smooth mapping on $\text{Diff}^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$, where P is linear in v . The partial Gâteaux derivative of P in the first variable φ and in the direction $\delta\varphi_1 \in C^\infty(\mathbb{S}^1)$ is a smooth map which is linear both in v and $\delta\varphi_1$ and that we will denote by

$$(3.2) \quad \partial_\varphi P_\varphi(v, \delta\varphi_1).$$

Therefore, the partial Gâteaux derivative of P in the variable φ is a mapping of three independent variables : $\varphi, v, \delta\varphi_1$. The second partial derivative of P in directions $\delta\varphi_1, \delta\varphi_2 \in C^\infty(\mathbb{S}^1)$ is the partial Gâteaux derivative of (3.2) in the variable φ and in the direction $\delta\varphi_2$. We will denote it by

$$\partial_\varphi^2 P_\varphi(v, \delta\varphi_1, \delta\varphi_2).$$

It can be checked that this expression is symmetric in $\delta\varphi_1, \delta\varphi_2$ (see [19]). Inductively, we define this way the n -th partial derivative of P in directions $\delta\varphi_1, \dots, \delta\varphi_n$ and we write it as

$$\partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n).$$

The space of linear operators on a Fréchet space is a locally convex topological vector space, but in general is not a Fréchet space (see [19]). For this reason, we will avoid taking limits and derivatives of linear operators. In the sequel, if such equalities appear for notational simplicity, it just means equality of mappings.

Let P denote a general Fourier multiplier on $C^\infty(\mathbb{S}^1)$. We will now study conjugation

$$P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}}$$

of P with right translations R_φ , where $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$. We will derive a recursion formula for the n -th derivative with respect to φ of such operators and introduce multi-symbols for multilinear operators.

Lemma 3.3. *Let P be a continuous, linear operator on $C^\infty(\mathbb{S}^1)$ and let*

$$P_\varphi = R_\varphi P R_\varphi^{-1},$$

where $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$. Then, given $n \in \mathbb{N}$, we have

$$(3.3) \quad \partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi P_n R_\varphi^{-1}(v, \delta\varphi_1, \dots, \delta\varphi_n),$$

where P_n is the $(n+1)$ -linear operator defined inductively by $P_0 = P$ and

$$(3.4) \quad P_{n+1}(u_0, u_1, \dots, u_{n+1}) = [u_{n+1}D, P_n(\cdot, u_1, \dots, u_n)]u_0, \\ - \sum_{k=1}^n P_n(u_0, u_1, \dots, u_{n+1}Du_k, \dots, u_n).$$

Remark 3.4. For a Fourier multiplier, that is, if $[P, D] = 0$, we have

$$P_1(u_0, u_1) = ([u_1, P]D)u_0,$$

and

$$P_2(u_0, u_1, u_2) = ([u_1, [u_2, P]]D^2 + [u_1, P][u_2, D]D + [u_2, P][u_1, D]D)u_0.$$

Proof. Formula (3.3) is trivially true for $n = 0$. Now suppose it is true for some $n \in \mathbb{N}$, that is

$$\partial_\varphi^n P_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi P_n R_\varphi^{-1}(v, \delta\varphi_1, \dots, \delta\varphi_n),$$

Let $\varphi(s)$ be a smooth path in $\text{Diff}^\infty(\mathbb{S}^1)$ such that

$$\varphi(0) = \varphi, \quad \partial_s \varphi(s)|_{s=0} = \delta\varphi_{n+1},$$

and set $u_k = \delta\varphi_k \circ \varphi^{-1}$, for $0 \leq k \leq n+1$. We compute first

$$\dot{R}_\varphi w := \partial_s R_{\varphi(s)} w|_{s=0} = R_\varphi u_{n+1} D w,$$

for $w \in C^\infty(\mathbb{S}^1)$, and

$$\dot{u}_k := \partial_s (\delta\varphi_k \circ \varphi(s)^{-1})|_{s=0} = -u_{n+1} D u_k,$$

for $0 \leq k \leq n$. Therefore

$$\begin{aligned} \partial_s R_\varphi P_n R_\varphi^{-1}(v, \delta\varphi_1, \dots, \delta\varphi_n)|_{s=0} = \\ \dot{R}_\varphi P_n(u_0, \dots, u_n) + \sum_{k=0}^n R_\varphi P_n(u_0, \dots, \dot{u}_k, \dots, u_n), \end{aligned}$$

which gives the recurrence relation (3.4), since

$$\dot{R}_\varphi P_n(u_0, \dots, u_n) + R_\varphi P_n(\dot{u}_0, \dots, \dot{u}_n) = R_\varphi [u_{n+1} D, P_n(\cdot, u_1, \dots, u_n)] u_0,$$

and

$$P_n(u_0, u_1, \dots, \dot{u}_k, \dots, u_n) = -P_n(u_0, u_1, \dots, u_{n+1} D u_k, \dots, u_n),$$

for $1 \leq k \leq n$. □

Lemma 3.5. *Let P be a Fourier multiplier on $C^\infty(\mathbb{S}^1)$, and let P_n be the multilinear operator defined in Lemma 3.3 for some $n \in \mathbb{N}$. Then we have*

$$(3.5) \quad P_n(\mathbf{e}_{m_0}, \dots, \mathbf{e}_{m_n}) = p_n(m_0, m_1, \dots, m_n) \mathbf{e}_{m_0+m_1+\dots+m_n},$$

where the sequence p_n is defined inductively by $p_0 = p$ (the symbol of P) and

$$(3.6) \quad p_{n+1}(m_0, \dots, m_{n+1}) = (2\pi i) \left[(m_0 + \dots + m_n) p_n(m_0, \dots, m_n) - \sum_{k=0}^n m_k p_n(m_0, \dots, m_k + m_{n+1}, \dots, m_n) \right],$$

where $m_k \in \mathbb{Z}$, for $k = 0, \dots, n$.

Remark 3.6. For $n = 1$, we have

$$(3.7) \quad p_1(m_0, m_1) = (2\pi i) m_0 (p_0(m_0) - p_0(m_0 + m_1))$$

and for $n = 2$, we get

$$(3.8) \quad p_2(m_0, m_1, m_2) = (2\pi i)^2 m_0 \left((m_0 + m_1 + m_2) p_0(m_0 + m_1 + m_2) - (m_0 + m_1) p_0(m_0 + m_1) - (m_0 + m_2) p_0(m_0 + m_2) + m_0 p_0(m_0) \right).$$

Proof. Invoking Lemma 3.1, the case $n = 0$ is clear. Suppose that equation (3.5) is true for some $n \geq 0$. Then, using recurrence relation (3.4), we have

$$P_{n+1}(\mathbf{e}_{m_0}, \dots, \mathbf{e}_{m_{n+1}}) = \mathbf{e}_{m_{n+1}} D(P_n(\mathbf{e}_{m_0}, \dots, \mathbf{e}_{m_n})) \\ - \sum_{k=0}^n P_n(\mathbf{e}_{m_0}, \dots, \mathbf{e}_{m_{n+1}} D\mathbf{e}_{m_k}, \dots, \mathbf{e}_{m_n}),$$

which is equal to

$$(2\pi i) \left\{ (m_0 + \dots + m_n) p_n(m_0, \dots, m_n) \right. \\ \left. - \sum_{k=0}^n m_k p_n(m_0, \dots, m_k + m_{n+1}, \dots, m_n) \right\} \mathbf{e}_{m_0 + \dots + m_{n+1}}.$$

This shows that equation (3.5) is true for $n + 1$ with

$$p_{n+1}(m_0, \dots, m_{n+1}) = (2\pi i) \left[(m_0 + \dots + m_n) p_n(m_0, \dots, m_n) \right. \\ \left. - \sum_{k=0}^n m_k p_n(m_0, \dots, m_k + m_{n+1}, \dots, m_n) \right]$$

and achieves the proof. \square

Corollary 3.7. *Under the notations of Lemma (3.5), we have*

$$(3.9) \quad p_n(m_0, m_1, \dots, m_n) = \\ (2\pi i)^n m_0 \left[\sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} f_n(m_0 + \sum_{j \in I} m_j) \right],$$

for each $n \geq 1$, where $f_n(k) = k^{n-1} p_0(k)$, $k \in \mathbb{Z}$.

Proof. For $n = 1$, we have

$$p_1(m_0, m_1) = (2\pi i) m_0 (p_0(m_0) - p_0(m_0 + m_1))$$

so equation (3.9) is true for $n = 1$. Now, suppose inductively that this equation is valid for some $n \geq 1$. Using the recurrence relation (3.6), we get

$$p_{n+1}(m_0, \dots, m_{n+1}) = (2\pi i)^{n+1} m_0 \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} \left\{ \right. \\ (m_0 + \dots + m_n) f_n(m_0 + \sum_{j \in I} m_j) \\ \left. - \sum_{k=1}^n m_k f_n(m_0 + \sum_{j \in I} m_j + \delta_I(k) m_{n+1}) \right. \\ \left. - (m_0 + m_{n+1}) f_n(m_0 + \sum_{j \in I} m_j + m_{n+1}) \right\},$$

which can be rewritten as

$$(2\pi i)^{n+1} m_0 \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} \left\{ (m_0 + \sum_{j \in I} m_j) f_n(m_0 + \sum_{j \in I} m_j) \right. \\ \left. - (m_0 + \sum_{j \in I} m_j + m_{n+1}) f_n(m_0 + \sum_{j \in I} m_j + m_{n+1}) \right\}.$$

using the fact that $f_{n+1}(t) = t f_n(t)$, we have therefore

$$p_{n+1}(m_0, m_1, \dots, m_{n+1}) = (2\pi i)^{n+1} m_0 \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} \left\{ \right. \\ \left. f_{n+1}(m_0 + \sum_{j \in I} m_j) - f_{n+1}(m_0 + \sum_{j \in I} m_j + m_{n+1}) \right\},$$

which is equal to

$$(2\pi i)^{n+1} m_0 \left\{ \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n+1\}, \\ |I|=p, n+1 \notin I}} f_{n+1}(m_0 + \sum_{j \in I} m_j) \right. \\ \left. + \sum_{p=0}^n (-1)^{p+1} \sum_{\substack{I \subset \{1, \dots, n+1\}, \\ |I|=p+1, n+1 \in I}} f_{n+1}(m_0 + \sum_{j \in I} m_j) \right\}.$$

But this last expression is exactly

$$(2\pi i)^{n+1} m_0 \sum_{p=0}^{n+1} (-1)^p \sum_{\substack{I \subset \{1, \dots, n+1\}, \\ |I|=p}} f_{n+1}(m_0 + \sum_{j \in I} m_j),$$

which achieves the proof. \square

3.2. Extension to Sobolev spaces. In this subsection, we provide a sufficient criterion on the symbol of the original operator P , which ensures that the operators P_n extend to suitable Sobolev spaces.

Given two Banach spaces E and F , we denote by $\mathcal{L}(E, F)$, the Banach spaces of continuous, linear mappings from E to F . Given Banach spaces E_1, \dots, E_m, F , recall that a m -multilinear mapping U from the m -fold Cartesian product $E_1 \times \dots \times E_m$ into F is continuous (*bounded*) iff there is a constant $c > 0$ such that

$$\|U(e_1, e_2, \dots, e_m)\|_F \leq c \|e_1\|_{E_1} \|e_2\|_{E_2} \cdots \|e_m\|_{E_m}.$$

The space $\mathcal{L}^m(E_1, \dots, E_m; F)$ of continuous, m -multilinear mapping from $E_1 \times \dots \times E_m$ into F , endowed with the *operator norm*

$$\|U\|_{\mathcal{L}^m} := \sup \left\{ \frac{\|U(e_1, \dots, e_m)\|_F}{\|e_1\|_{E_1} \cdots \|e_m\|_{E_m}}; e_1, \dots, e_m \neq 0 \right\}$$

is a Banach space. We recall further that the canonical map

$$\mathcal{L}(E_1, \mathcal{L}(E_2, \dots, \mathcal{L}(E_r, F) \dots)) \rightarrow \mathcal{L}^m(E_1, \dots, E_m; F)$$

is a topological linear isomorphism, which is norm-preserving, and when

$$E_1 = \cdots = E_m = E,$$

we shall write

$$\mathcal{L}^m(E_1, \dots, E_r; F) = \mathcal{L}^m(E, F).$$

We define the Sobolev space $H^q(\mathbb{S}^1)$ as the completion of $C^\infty(\mathbb{S}^1)$ for the norm

$$\|u\|_{H^q(\mathbb{S}^1)} := \left(\sum_{k \in \mathbb{Z}} (1 + k^2)^q |\hat{u}_k|^2 \right)^{1/2}.$$

We recall that $H^q(\mathbb{S}^1)$ is a multiplicative algebra for $q > 1/2$ (cf. [37, Theorem 2.8.3]). This means that there exists a positive constant C_q such that

$$\|uv\|_{H^q(\mathbb{S}^1)} \leq C_q \|u\|_{H^q(\mathbb{S}^1)} \|v\|_{H^q(\mathbb{S}^1)}, \quad u, v \in H^q(\mathbb{S}^1).$$

A Fourier multiplier $P = \mathbf{op}(p(k))$ with symbol p is said to be of order $r \in \mathbb{R}$ if there exists a constant $C > 0$ such that

$$|p(k)| \leq C (1 + k^2)^{r/2},$$

for every $k \in \mathbb{Z}$. In that case, for each $q \geq r$, the operator P extends to a bounded linear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$. We express this fact by the notation $P \in \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$.

Lemma 3.8. *Let P be a Fourier multiplier of order $r \geq 0$ and P_n be the $(n + 1)$ -multilinear operator defined by the recurrence relation (3.4) with $P_0 := P$. Suppose that there exists a constant $C_n > 0$, such that*

$$(3.10) \quad |p_n(m_0, \dots, m_n)| \leq C_n (1 + m_0^2)^{r/2} \cdots (1 + m_n^2)^{r/2}$$

for all $m_j \in \mathbb{Z}$. Then P_n extends to a bounded multilinear operator

$$P_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

for all $q > r + 1/2$.

Proof. Assume that (3.10) holds true. By virtue of Proposition 3.5, we have

$$\|P_n(u_0, \dots, u_n)\|_{H^{q-r}}^2 = \sum_{l \in \mathbb{Z}} (1 + l^2)^{q-r} \left| \sum_{m_0 + \dots + m_n = l} \hat{u}_0(m_0) \cdots \hat{u}_n(m_n) p_n(m_0, \dots, m_n) \right|^2,$$

for any choice of smooth functions u_0, u_1, \dots, u_n , since $(\mathbf{e}_l)_{l \in \mathbb{Z}}$ is an orthogonal system in $H^{q-r}(\mathbb{S}^1)$ and $\|\mathbf{e}_l\|_{H^{q-r}}^2 = (1 + l^2)^{q-r}$. Therefore, the inequality (3.10) implies that

$$\|P_n(u_0, \dots, u_n)\|_{H^{q-r}}^2 \leq C_n^2 \sum_{l \in \mathbb{Z}} (1 + l^2)^{q-r} \left(\sum_{m_0 + \dots + m_n = l} \prod_{j=0}^n (1 + m_j^2)^{r/2} |\hat{u}_j(m_j)| \right)^2.$$

Observe now that, given smooth functions v_0, v_1, \dots, v_n , we have

$$\widehat{v_0 \cdots v_n}(l) = \sum_{m_0 + \cdots + m_n = l} \hat{v}_0(m_0) \cdots \hat{v}_n(m_n).$$

In addition $H^{q-r}(\mathbb{S}^1)$ is a multiplicative algebra for $q - r > 1/2$. Thus, we can find a constant $C'_{n,q-r}$ such that

$$(3.11) \quad \sum_{l \in \mathbb{Z}} (1 + l^2)^{q-r} \left| \sum_{m_0 + \cdots + m_n = l} \hat{v}_0(m_0) \cdots \hat{v}_n(m_n) \right|^2 \leq C'^2_{n,q-r} \|v_0\|_{H^{q-r}}^2 \cdots \|v_n\|_{H^{q-r}}^2,$$

for all smooth functions v_0, v_1, \dots, v_n . Putting now

$$\hat{v}_j(m_j) := (1 + m_j^2)^{r/2} |\hat{u}_j(m_j)|, \quad j = 0, \dots, n$$

in (3.11) and using the fact that the functions with Fourier coefficient $\hat{v}(m)$ and $|\hat{v}(m)|$ have the same H^{q-r} norm, we obtain

$$\|P_n(u_0, \dots, u_n)\|_{H^{q-r}} \leq C_n C'_{n,q-r} \|u_0\|_{H^q} \cdots \|u_n\|_{H^q},$$

which implies the assertion. \square

Finally, we will need to define a condition on the symbol of the Fourier multiplier P in order that the operators P_n are bounded. For this purpose, the following lemma will be useful. Recall that a real function f is said to be *absolutely continuous* on \mathbb{R} if f has a derivative almost everywhere, the derivative is locally Lebesgue integrable and

$$f(b) = f(a) + \int_a^b f'(\tau) d\tau,$$

for all $a, b \in \mathbb{R}$.

Lemma 3.9. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of class C^{n-1} with $n \geq 1$. Suppose that $f^{(n-1)}$ is absolutely continuous and that there exists $C > 0$ and $r \geq 1$ such that*

$$\left| f^{(n)}(\xi) \right| \leq C(1 + \xi^2)^{(r-1)/2},$$

almost everywhere. Then

$$\left| \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} f\left(m_0 + \sum_{j \in I} m_j\right) \right| \leq 2^{n(r-1)/2} C(1 + m_0^2)^{(r-1)/2} \prod_{j=1}^n (1 + m_j^2)^{r/2},$$

for all $m_0, m_1, \dots, m_n \in \mathbb{Z}$.

Proof. Let g_k be the sequence of functions defined inductively by

$$g_0(\xi) = f(\xi), \quad g_{k+1}(\xi) = g_k(\xi + m_{n-k}) - g_k(\xi),$$

for $k = 0, \dots, n-1$. We have in particular

$$g_n(\xi) = (-1)^n \sum_{p=0}^n (-1)^p \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=p}} f\left(\xi + \sum_{j \in I} m_j\right).$$

Let $K_0 = \{m_0\}$ and for $p = 1, \dots, n$, let K_p be the convex set generated by K_{p-1} and $K_{p-1} + m_p$. Notice that K_n is the convex hull of the points $m_0 + \sum_{j \in I} m_j$, for all subset I of $\{1, \dots, n\}$. Let

$$M := \max_{\xi \in K_n} (1 + \xi^2)^{(r-1)/2}.$$

By hypothesis, we have $|g_0^{(n)}(\xi)| \leq CM$ almost everywhere on K_n , and using the mean value theorem, we get inductively

$$\left|g_k^{(n-k)}(\xi)\right| \leq CM |m_n| \cdots |m_{n-k+1}|, \quad \forall \xi \in K_{n-k},$$

for $k = 1, \dots, n$. In particular, we have

$$|g_n(m_0)| \leq CM \prod_{j=1}^n |m_j|.$$

Let's now estimate the constant M . The function $\xi \mapsto 1 + \xi^2$ attains its maximum on K_n at some extremal point $m_0 + \sum_{j \in I_0} m_j$ and since the m_j are integers, we have

$$\max_{\xi \in K_n} (1 + \xi^2) = 1 + \left(m_0 + \sum_{j \in I_0} m_j\right)^2 \leq 2^n \prod_{j=0}^n (1 + m_j^2).$$

Moreover, if $r-1 \geq 0$, we get

$$M = \max_{\xi \in K_n} (1 + \xi^2)^{(r-1)/2} \leq 2^{n(r-1)/2} \prod_{j=0}^n (1 + m_j^2)^{(r-1)/2},$$

and since $|m_j| \leq \sqrt{1 + m_j^2}$, this achieves the proof. \square

So far, the condition on the symbol of P , in Theorem 1.1, ensures that

$$P_n \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

for $q \in (\frac{1}{2} + r, \infty)$ and $n \in \mathbb{N}$. To achieve the proof of Theorem 1.1, we will now show that if each P_n extends to a bounded multilinear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$, then the mapping

$$\varphi \mapsto P_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for $q \in (\frac{3}{2} + r, \infty)$.

Proposition 3.10. *Let P be a Fourier multiplier of order $r \geq 1$ and assume that $q \in (\frac{3}{2} + r, \infty)$. Suppose further that the operators P_n , defined in Lemma 3.3, belong to $\mathcal{L}^{(n+1)}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$ for each $n \in \mathbb{N}$. Then the mapping*

$$\varphi \mapsto P_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth with $D^n P_\varphi = P_{n,\varphi} := R_\varphi P_n R_{\varphi^{-1}}$, for $n \geq 1$.

Before giving the proof of this proposition, it is worth to recall two elementary lemmas that we will state without proof.

Lemma 3.11. *Let X be a topological space and E a Banach space. Let $f : X \times [0, 1] \rightarrow E$ be a continuous map. Then the map*

$$g(x) := \int_0^1 f(t, x) dt$$

is continuous.

Lemma 3.12. *Let E, F be Banach spaces and U a convex, open set in E . Let $\alpha : U \rightarrow \mathcal{L}(E, F)$ be a continuous map and $f : U \rightarrow F$ a map such that*

$$f(y) - f(x) = \int_0^1 \alpha(ty + (1-t)x)(y-x) dt,$$

for all $x, y \in U$. Then f is C^1 on U and $df = \alpha$.

Proof of Proposition 3.10. Notice first that, for each $n \in \mathbb{N}$, we have

$$P_{n,\varphi} \in \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1)),$$

with

$$\begin{aligned} \|P_{n,\varphi}\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} &\leq \\ &\|R_\varphi\|_{\mathcal{L}(H^{q-r}, H^{q-r})} \|P_n\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} \|R_{\varphi^{-1}}\|_{\mathcal{L}(H^q, H^q)}^{n+1}. \end{aligned}$$

Moreover, the mapping

$$(\varphi, v_0, \dots, v_n) \mapsto P_{n,\varphi}(v_0, \dots, v_n), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)^{n+1} \rightarrow H^{q-r}(\mathbb{S}^1)$$

is continuous, whereas

$$\varphi \mapsto P_{n,\varphi}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is locally bounded (Proposition A.2).

(a) We will first show that $\varphi \mapsto P_{n,\varphi}$ is *locally Lipschitz continuous*⁴. Let $\psi \in \mathcal{D}^q(\mathbb{S}^1)$ be given. Because $\varphi \mapsto P_{n+1,\varphi}$ is *locally bounded*, it is possible to find a neighbourhood U of ψ and a positive constant K such that

$$\|P_{n+1,\varphi}\|_{\mathcal{L}^{n+1}(H^q, H^{q-r})} \leq K, \quad \forall \varphi \in U.$$

We can further assume (using a local chart) that U is a ball in $H^q(\mathbb{S}^1)$. Pick now φ_0 and φ_1 in $\text{Diff}^\infty(\mathbb{S}^1) \cap U$ and set $\varphi(t) := (1-t)\varphi_0 + t\varphi_1$ for $t \in [0, 1]$. Choosing $v_0, \dots, v_n \in C^\infty(\mathbb{S}^1)$ with $\|v_j\|_{H^q} \leq 1$, we obtain from Lemma 3.3 that

$$\begin{aligned} P_{n,\varphi_1}(v_0, \dots, v_n) - P_{n,\varphi_0}(v_0, \dots, v_n) &= \\ &\int_0^1 P_{n+1,\varphi(t)}(v_0, \dots, v_n, \varphi_1 - \varphi_0) dt. \end{aligned}$$

This implies

$$\|P_{n,\varphi_1}(v_0, \dots, v_n) - P_{n,\varphi_0}(v_0, \dots, v_n)\|_{H^{q-r}} \leq K \|\varphi_1 - \varphi_0\|_{H^q},$$

⁴On $\mathcal{D}^q(\mathbb{S}^1)$, we did not introduce any distance compatible with the topology. The concept of a *locally Lipschitz map* $f : M \rightarrow E$, from a Banach manifold M to a Banach vector space E does not require such an additional structure. It is defined using a local chart, and then shown to be independent of the choice of the particular chart.

for all $v_0, \dots, v_n \in C^\infty(\mathbb{S}^1)$ with $\|v_j\|_{H^q} \leq 1$. The assertion that $P_{n,\varphi}$ is Lipschitz continuous follows from the density of the embedding $C^\infty(\mathbb{S}^1) \hookrightarrow H^q(\mathbb{S}^1)$.

(b) We will now show by induction, that $\varphi \mapsto P_\varphi$ is of class C^n for all $n \in \mathbb{N}$, and that its n -th Fréchet derivative is $P_{n,\varphi}$. For each $n \geq 1$, let

$$\alpha_n : \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}\left(H^q(\mathbb{S}^1), \mathcal{L}^n(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))\right),$$

be the *Lipschitz continuous* mapping defined by

$$\alpha_n(\varphi) := [\delta\varphi_n \mapsto P_{n,\varphi}(\cdot, \dots, \cdot, \delta\varphi_n)].$$

Let U be a local chart in $\mathcal{D}^q(\mathbb{S}^1)$, that we choose to be a convex open subset of $H^q(\mathbb{S}^1)$. By its very definition, we have

$$P_{\varphi_1}(v) - P_{\varphi_0}(v) = \int_0^1 P_{1,t\varphi_1+(1-t)\varphi_0}(v, \varphi_1 - \varphi_0) dt,$$

for all $\varphi_0, \varphi_1 \in U \cap C^\infty(\mathbb{S}^1)$ and $v \in C^\infty(\mathbb{S}^1)$. But, the continuity of the map $\varphi \mapsto P_{1,\varphi}$, together with Lemma 3.11, and the density of the embedding $C^\infty(\mathbb{S}^1) \hookrightarrow H^q(\mathbb{S}^1)$, permit to conclude that this formula is still true for all $\varphi_0, \varphi_1 \in U$ and $v \in H^q(\mathbb{S}^1)$. Therefore, we can write in $\mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$

$$P_{\varphi_1} - P_{\varphi_0} = \int_0^1 \alpha_1(t\varphi_1 + (1-t)\varphi_0)(\varphi_1 - \varphi_0) dt,$$

and, by virtue of Lemma 3.12, we conclude that $\varphi \mapsto P_\varphi$ is C^1 and that $DP_\varphi = \alpha_1$. A similar argument shows that, for each $n \geq 1$, we have

$$\alpha_n(\varphi_1) - \alpha_n(\varphi_0) = \int_0^1 \alpha_{n+1}(t\varphi_1 + (1-t)\varphi_0)(\varphi_1 - \varphi_0) dt,$$

and hence that α_n is C^1 with $D\alpha_n = \alpha_{n+1}$. This completes the proof. \square

4. SMOOTHNESS OF THE METRIC AND THE SPRAY

For general materials on Banach manifolds, we refer to [25]. Let X be a Banach manifold modelled over a Banach space E . A Riemannian metric g on X is a smooth, symmetric, positive definite, covariant 2-tensor field on X . In other words, we have for each $x \in X$ a symmetric, positive definite, bounded, bilinear form $g(x)$ on $T_x X$ and, in any local chart U , the mapping

$$x \rightarrow g(x), \quad U \rightarrow \mathcal{L}_{\text{sym}}^2(E, \mathbb{R})$$

is smooth. Given any $x \in X$, we can then consider the bounded, linear operator

$$h_x : T_x X \rightarrow T_x^* X,$$

called the *flat map* and defined by $h_x(\xi_x) = g(x)(\xi_x, \cdot)$. The mapping $h : TX \rightarrow T^*X$ is a vector bundle morphism. The metric is *strong* if h_x is a topological linear isomorphism for all $x \in X$, whereas it is *weak* if h_x is only injective for all $x \in X$.

Given a *strong* Riemannian metric on X , there exists always a unique symmetric covariant derivative compatible with the metric (see [25]). This covariant derivative is given in a local chart U by the formula

$$\nabla_u v := d_x v \cdot u + \Gamma_x(u, v)$$

where $v \in C^\infty(U, E)$, $u \in E$ and Γ_x is the *Christoffel map*, determined by

$$(4.1) \quad 2g(x)(\Gamma_x(u, v), w) = d_x g \cdot u(v, w) + d_x g \cdot v(u, w) - d_x g \cdot w(u, v)$$

where $u, v, w \in E$.

Geodesics are defined as extremal curves of the energy functional

$$\mathcal{K} := \frac{1}{2} \int_0^1 g(x)(\dot{x}, \dot{x}) dt.$$

The equations for the *geodesics* can be obtained as follows. The pullback $h^*\Omega$ on TX of the canonical (strong) symplectic structure Ω on T^*X is itself a (strong) symplectic structure on TX . The Hamiltonian vector field F on TX associated to the energy function

$$K : TX \rightarrow \mathbb{R}, \quad \xi_x \mapsto \frac{1}{2}g(x)(\xi_x, \xi_x),$$

and defined by $dK = -h^*\Omega(F, \cdot)$ is a smooth quadratic, second order vector field, called the *geodesic spray*. Its integral curves are the geodesics. In a local chart $U \times E$ of TX , the Hamiltonian vector field F is given by

$$F(x, v) := (v, S_x(v)), \quad S_x(v) = -\Gamma_x(v, v),$$

where $x \in U$ and $v \in E$.

Conversely, given a smooth spray (i.e a quadratic second order vector field) on the Banach manifold X , it induces a symmetric covariant derivative on X , which however may *not be metric*, in the sense that it may not be compatible with any Riemannian metric on X (cf. [25, Chapter VIII]).

The preceding construction is no longer true for a *weak* Riemannian metric on X , in general. The standard proof of the existence of a symmetric covariant derivative, compatible with the metric, requires the invertibility of the maps h_x , which is not available for a weak metric. However, if such a covariant derivative exists, it is unique. It can be shown moreover, that a weak Riemannian metric which admits a geodesic spray has a compatible, symmetric covariant derivative (cf. [25, Chapter VIII]).

Consider now a L^2 -symmetric and positive definite Fourier multiplier A on $C^\infty(\mathbb{S}^1)$. Suppose moreover, that A extends, for all q large enough, to a bounded, linear isomorphism

$$A : H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

for some fixed $r \geq 1$. In terms of the symbol a of A , this is equivalent to assume that a does not vanish, and that

$$a(n) = O(|n|^r), \quad \frac{1}{a(n)} = O(|n|^{-r}).$$

Let $H^{-q}(\mathbb{S}^1)$ denote the topological dual of $H^q(\mathbb{S}^1)$ considered as a Banach space (and, of course, isomorphic to $H^q(\mathbb{S}^1)$). Then any function in $L^2(\mathbb{S}^1)$ belongs to $H^{-q}(\mathbb{S}^1)$. More precisely, there is a topological embedding

$$L^2(\mathbb{S}^1) \hookrightarrow H^{-q}(\mathbb{S}^1).$$

In particular, A induces a continuous, injective linear operator from $H^q(\mathbb{S}^1)$ to $H^{-q}(\mathbb{S}^1)$, whose range is $H^{q-r}(\mathbb{S}^1)$.

In that case, the right-invariant metric induced by A on $\text{Diff}^\infty(\mathbb{S}^1)$ extends and provides a continuous family of positive inner products on each tangent space

$$\langle v_1, v_2 \rangle_\varphi = \int_{\mathbb{S}^1} v_1(A_\varphi v_2) \varphi_x dx.$$

If we suppose moreover that A fulfils the hypothesis of Theorem 1.1, then

$$\varphi \mapsto \varphi_x A_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth and A induces a *weak* Riemannian metric on $\mathcal{D}^q(\mathbb{S}^1)$. The corresponding flat map, is given by

$$h : (\varphi, v) \mapsto (\varphi, \varphi_x A_\varphi v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^{-q}(\mathbb{S}^1).$$

It is an injective vector bundle morphism. Its image $\text{im } h$ is the vector bundle $\mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$ which maps continuously and one-to-one in the bundle $\mathcal{D}^q(\mathbb{S}^1) \times H^{-q}(\mathbb{S}^1)$. Notice however that $\mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$ is not a subbundle of $\mathcal{D}^q(\mathbb{S}^1) \times H^{-q}(\mathbb{S}^1)$ in the sense of [25, III.3].

Proposition 4.1. *Let A be a Fourier multiplier of order $r \geq 1$. Suppose that A induces an isomorphism from $H^q(\mathbb{S}^1)$ onto $H^{q-r}(\mathbb{S}^1)$ and that*

$$\varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for $q \in (\frac{3}{2} + r, \infty)$. Then the geodesic spray, given by

$$(\varphi, v) \mapsto S_\varphi(v) = R_\varphi \circ S \circ R_{\varphi^{-1}}(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1),$$

where

$$S(u) = A^{-1} \{ [A, u]u_x - 2(Au)u_x \}.$$

is well defined and smooth,

Proof. Given an operator K , we introduce the following notation

$$\tilde{K}(\varphi, v) := (\varphi, K_\varphi(v)),$$

where $K_\varphi(v) = R_\varphi \circ K \circ R_{\varphi^{-1}}(v)$. Let $P(u) := (Au)u_x$ and $Q(u) := [A, u]u_x$. We have

$$S_\varphi(v) = A_\varphi^{-1} \{ Q_\varphi(v) - 2P_\varphi(v) \}.$$

Although P and Q are smooth operators, we cannot conclude directly that these results carry over when conjugated with translation in $\mathcal{D}^q(\mathbb{S}^1)$ since for $q > 3/2$ these sets only form topological groups: neither composition nor inversion are differentiable.

(a) We have $P_\varphi(v) = (A_\varphi(v))(D_\varphi(v))$. But

$$(\varphi, v) \mapsto A_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is smooth by hypothesis, whereas

$$(\varphi, v) \mapsto D_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^{q-1}(\mathbb{S}^1)$$

is smooth since $D_\varphi(v) = v_x/\varphi_x$, and $H^{q-1}(\mathbb{S}^1)$ is a multiplicative algebra. Since $q - r > 1/2$ and $r \geq 1$, $P_\varphi(v) \in H^{q-r}(\mathbb{S}^1)$ and we can conclude that

$$\tilde{P} : \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1),$$

is smooth.

(b) By virtue of Proposition 3.10 and Lemma 3.3, we have

$$\partial_\varphi A_\varphi(v, v) = A_{1, \varphi}(v, v) = -Q_\varphi(v).$$

and therefore

$$\tilde{Q} : \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1),$$

is smooth.

(c) Since

$$D_{(\varphi, v)} \tilde{A}(\delta\varphi, \delta v) = \begin{pmatrix} \text{id} & 0 \\ * & A_\varphi \end{pmatrix}$$

is a bounded, linear, invertible operator from $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$, we conclude, using the inverse mapping theorem for Banach spaces, that

$$\tilde{A}^{-1} : \mathcal{D}^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1) \rightarrow \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$$

is smooth.

The assertion now follows from the chain rule. \square

As an application of the preceding proposition and Corollary 1.4, we obtain in particular the following result.

Corollary 4.2. (*Smoothness of the H^s metric and its spray*) *Let $s \geq 1/2$ be given and assume that $q \in (\frac{3}{2} + 2s, \infty)$. Then the right-invariant, weak Riemannian metric defined on $\text{Diff}^\infty(\mathbb{S}^1)$ by the inertia operator $A = \Lambda^{2s}$ extends to a smooth weak Riemannian metric on the Banach manifold $\mathcal{D}^q(\mathbb{S}^1)$ with a smooth geodesic spray.*

5. EXISTENCE RESULTS FOR GEODESICS IN THE SMOOTH CATEGORY

In this section, we will prove local existence and uniqueness of the initial value problem for the geodesics of the right-invariant H^s metric on the Fréchet-Lie group $\text{Diff}^\infty(\mathbb{S}^1)$, and more generally for any right-invariant weak Riemannian metric for which the inertia operator satisfies the hypothesis of Theorem 1.1.

For this we shall use the Hilbert manifold approximation $\{\mathcal{D}^q(\mathbb{S}^1)\}_{q>3/2}$ of $\text{Diff}^\infty(\mathbb{S}^1)$ and the corresponding results of the previous section. The remarkable observation that the maximal interval of existence is independent of the parameter q , due to the right-invariance of the spray (cf. Lemma 5.1) was pointed out in [11, Theorem 12.1]. This makes it possible to avoid Nash-Moser type schemes to prove Theorem 5.3.

In what follows, we start with a right-invariant metric on $\text{Diff}^\infty(\mathbb{S}^1)$, which inertia operator A is a Fourier multiplier of order $r \geq 1$. We suppose further that, for all $q \in (\frac{3}{2} + r, \infty)$, A induces an isomorphism from $H^q(\mathbb{S}^1)$ onto $H^{q-r}(\mathbb{S}^1)$ and that the mapping

$$(5.1) \quad \varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth. Under these assumptions, the metric admits a smooth spray $F_q : T\mathcal{D}^q(\mathbb{S}^1) \rightarrow T\mathcal{D}^q(\mathbb{S}^1)$ (cf. Proposition 4.1) and we can apply the Picard-Lindelöf theorem. For each $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$, there exists a *unique non-extendable* solution

$$(\varphi, v) \in C^\infty(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{S}^1)),$$

of the Cauchy problem

$$(5.2) \quad \begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases}$$

with $\varphi(0) = \varphi_0$ and $v(0) = v_0$, defined on some *maximal interval of existence* $J_q(\varphi_0, v_0)$, which is open and contains 0. Note that in general $J_q(\varphi_0, v_0) \neq \mathbb{R}$, meaning that the solutions are not *global*. Furthermore, letting

$$\text{dom}^{(q)} := \bigcup_{(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)} J_q(\varphi_0, v_0) \times \{(\varphi_0, v_0)\}$$

and

$$\Phi_q(t, (\varphi_0, v_0)) := (\varphi(t), v(t)), \quad t \in J_q(\varphi_0, v_0),$$

we know that $\text{dom}^{(q)}$ is open in $\mathbb{R} \times T\mathcal{D}^q(\mathbb{S}^1)$ and that

$$(5.3) \quad \Phi_q \in C^\infty(\text{dom}^{(q)}, \mathbb{R} \times T\mathcal{D}^q(\mathbb{S}^1)),$$

cf. [1, Section 10]. The mapping Φ_q is called the *flow* on $T\mathcal{D}^q(\mathbb{S}^1)$, induced by the vector field $(v, S_\varphi(v))$ and $\text{dom}^{(q)}$ is its *maximal domain of definition*.

To prove well-posedness of the Cauchy problem (5.2) on the smooth manifold $T\text{Diff}^\infty(\mathbb{S}^1)$, we need precise regularity properties of solutions to (5.2) on each Hilbert approximation manifold $T\mathcal{D}^q(\mathbb{S}^1)$, with $q > (3/2) + r$. More precisely, assume that $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$. Then we may solve (5.2) in $T\mathcal{D}^q(\mathbb{S}^1)$ and in $T\mathcal{D}^{q+1}(\mathbb{S}^1)$. Since solutions on each level are non-extendable, we clearly have

$$(5.4) \quad J_{q+1}(\varphi_0, v_0) \subset J_q(\varphi_0, v_0).$$

The fact that $J_{q+1}(\varphi_0, v_0)$ is not a proper subset of $J_q(\varphi_0, v_0)$, which could lead to $\cap_q J_q(\varphi_0, v_0) = \{0\}$, is ruled out by the following result.

Lemma 5.1 (Ebin-Marsden, 1970). *Given $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$, we have*

$$J_{q+1}(\varphi_0, v_0) = J_q(\varphi_0, v_0).$$

Before giving the proof of Lemma 5.1, recall that the translation group $\{T(s); s \in \mathbb{R}\}$ in $H^\sigma(\mathbb{S}^1)$ ($\sigma \geq 0$), defined by

$$T(s)v(x) := v(x+s), \quad v \in H^\sigma(\mathbb{S}^1), \quad x \in \mathbb{S}^1,$$

is a *strongly continuous group* in $\mathcal{L}(H^\sigma(\mathbb{S}^1))$, with infinitesimal generator given by D and domain of definition $H^{\sigma+1}(\mathbb{S}^1)$. This means in particular that, given $v_0 \in H^{\sigma+1}(\mathbb{S}^1)$, we have that

$$[s \mapsto T(s)v_0] \in C^1(\mathbb{R}, H^\sigma(\mathbb{S}^1))$$

with

$$\frac{d}{ds}T(s)v_0 = T(s)(Dv_0), \quad s \in \mathbb{R}.$$

There is also a one parameter group of right translations in $\mathcal{D}^q(\mathbb{S}^1)$ for which we use the same notation $T(s)\varphi_0(x) := \varphi_0(x+s)$ for $\varphi_0 \in \mathcal{D}^\sigma(\mathbb{S}^1)$ and $x \in \mathbb{S}^1$.

Proof. Let

$$\Phi_q(\cdot, (\varphi_0, v_0)) = (\varphi, v) \in C^\infty(J_q(\varphi_0, v_0), T\mathcal{D}^q(\mathbb{S}^1))$$

be the solution to (5.2) with initial datum (φ_0, v_0) . From (5.3) we easily deduce that

$$(5.5) \quad \frac{d}{ds}\Phi_q(t, T(s)(\varphi_0, v_0))|_{s=0} = D_{(\varphi, v)}\Phi_q(t, (\varphi_0, v_0))(\varphi_{0,x}, v_{0,x}).$$

On the other hand, the spray F_q is $\mathcal{D}^q(\mathbb{S}^1)$ -equivariant, we get in particular that

$$\Phi_q(t, T(s)(\varphi_0, v_0)) = T(s)\Phi_q(t, (\varphi_0, v_0)) \quad \text{for all } t \in J_q(\varphi_0, v_0), s \in \mathbb{R}.$$

Thus the left hand side of (5.5) equals

$$\frac{d}{ds}T(s)\Phi_q(t, (\varphi_0, v_0))|_{s=0} = \partial_x\Phi_q(t, (\varphi_0, v_0)) = (\varphi_x(t), v_x(t)).$$

Combining these observations, we get

$$D_{(\varphi, v)}\Phi_q(t, (\varphi_0, v_0))(\varphi_{0,x}, v_{0,x}) = (\varphi_x(t), v_x(t)).$$

But (5.3) reveals that the left hand side of the latter identity belongs to $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$, which in turn implies that

$$(\varphi(t), v(t)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1) \quad \text{for all } t \in J_q(\varphi_0, v_0).$$

By the unique solvability of (5.2), we conclude that

$$J_q(\varphi_0, v_0) \subset J_{q+1}(\varphi_0, v_0).$$

Invoking (5.4), the proof is completed. \square

Remark 5.2. Lemma 5.1 states that there is no loss of spatial regularity during the evolution of (5.2). By reversing the time direction, it follows from the unique solvability that there is also no gain of regularity in the following sense: Let $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$ be given and assume that $(\varphi(t_1), v(t_1)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ for some $t_1 \in J_q(\varphi_0, v_0)$. Then $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$.

Theorem 5.3. *Let (5.1) be satisfied and consider the geodesic flow on the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ induced by the inertia operator A . Then, given any $(\varphi_0, v_0) \in T\text{Diff}^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution*

$$(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$$

of (5.2) on the maximal interval of existence J , which is open and contains 0.

Proof. The result follows from (5.3), Lemma 5.1 and [12, Lemma 8], cf. the proof of Theorem 12 in [12]. \square

Corollary 5.4. *Let $s \geq 1/2$ be given and consider the right-invariant Sobolev H^s -metric on $\text{Diff}^\infty(\mathbb{S}^1)$. Then the corresponding geodesic equation has for any initial data in the tangent bundle $T\text{Diff}^\infty(\mathbb{S}^1)$ a unique non-extendable smooth solution $(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(\mathbb{S}^1))$. The maximal interval of existence J is open and contains 0.*

Proof. Let $s \geq 1/2$ be given. Then Corollary 1.4 ensures that the smoothness assertion in (5.1) is satisfied for $\mathbf{op}((1+k^2)^s)$. The hypothesis on the invertibility in (5.1) is obvious in this case. Thus the result follows from Theorem 5.3. \square

It is known that the Euler equation induced by the inertia operator

$$A = \mathbf{op}(1+k^2)$$

leads to the classical periodic Camassa-Holm equation

$$(5.6) \quad u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{S}^1,$$

cf. [6]. It may be interesting to briefly discuss another possible option for A , namely

$$A = \mathbf{op}(|k|^r + \delta_0(k)).$$

Observe that Theorem 1.1 is applicable provided $r \geq 1$. In that case, the mapping

$$\varphi \mapsto R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for $q > (3/2) + r$. Since in addition, A is a topological linear isomorphism from $H^q(\mathbb{S}^1)$ onto $H^{q-r}(\mathbb{S}^1)$, the operator A satisfies clearly assumption (5.1) and thus Theorem 5.3 guarantees the well-posedness, in the smooth category, of the corresponding Euler equation

$$(5.7) \quad m_t + um_x + 2u_x m = 0, \quad m = \mu(u) + (-\Delta)^{r/2}u$$

where $(-\Delta)^{r/2} := \mathbf{op}(|k|^r)$ and $\mu(u) := \int_{\mathbb{S}^1} u$. Note that $\int_{\mathbb{S}^1} m dx$ is a conserved quantity for the evolution under (5.7), since $\int_{\mathbb{S}^1} m_t dx = 0$. Equation (5.7) is of particular interest for the values $r = 2$ and $r = 1$, respectively. In the first case we get the so-called μ -Hunter-Saxton equation, cf. [27, 14]

$$(5.8) \quad u_{txx} + uu_{xxx} + 2u_x u_{xx} - 2\mu(u)u_x = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

In the case $r = 1$ we get the so-called *generalized CLM equation*, cf. [14]

$$(5.9) \quad Hu_{tx} + uHu_{xx} + 2\mu(u)u_x + 2u_x Hu_x = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

where $H = \mathbf{op}(i \operatorname{sgn}(k))$ denotes the Hilbert transform, acting on the spatial variable $x \in \mathbb{S}^1$. Note that $\mathbf{op}(|k|) = H \circ D = (-\Delta)^{1/2}$.

6. EXPONENTIAL MAP AND GEODESIC DISTANCE

The geodesic flow of a smooth spray on a Banach manifold X satisfies the following remarkable property

$$\varphi(t, x_0, \sigma u_0) = \varphi(\sigma t, x_0, u_0), \quad \sigma > 0,$$

which is a consequence of the quadratic nature of the geodesic equation [25]. Therefore, given $x_0 \in X$, the Riemannian *exponential mapping* \exp_{x_0} , defined as the time one of the flow is well defined in a neighbourhood of 0 in $T_{x_0}X$ for each point x_0 . It is moreover a local diffeomorphism from a neighbourhood V of 0 in $T_{x_0}X$ onto a neighbourhood $U(x_0)$ of x_0 in X [25]. This last assertion is in general no longer true on a *Fréchet manifold* and in particular on $\operatorname{Diff}^\infty(\mathbb{S}^1)$. One may find useful to recall on this occasion that the *group exponential* of $\operatorname{Diff}^\infty(\mathbb{S}^1)$ is not a local diffeomorphism [29]. Moreover, the Riemannian exponential map for the L^2 metric (Burgers equation) on

$\text{Diff}^\infty(\mathbb{S}^1)$ is not a local C^1 -diffeomorphism near the origin [5]. Nevertheless, it has been established in [5], that for the Camassa-Holm equation – which corresponds to the Euler equation of the H^1 metric on $\text{Diff}^\infty(\mathbb{S}^1)$ – and more generally for H^k metrics ($k \geq 1$) (see [6]), the Riemannian exponential map was in fact a smooth local diffeomorphism. This result is still true for H^s right-invariant metrics on $\text{Diff}^\infty(\mathbb{S}^1)$ provided $s \in [1/2, +\infty)$. The proof of Theorem 1.6 is similar to the one given in [12] and will be omitted. It requires only the smoothness of the spray on $T\mathcal{D}^q(\mathbb{S}^1)$ for all q large enough.

On a *strong Riemannian manifold*, given two nearby points, there exists a unique geodesic, joining these two points, which minimizes (globally) the *arc-length*. This is a consequence of the existence of normal neighbourhoods. This is no longer true for a *weak Riemannian metric* (pre-Hilbertian structure) in general.

To make this clear we close this section by a remark concerning the geodesic semi-distance d_s induced by the H^s metric and defined as the greatest lower bound of path-lengths $L_s(\varphi)$, for piecewise C^1 paths $\varphi(t)$ in $\text{Diff}^\infty(\mathbb{S}^1)$ joining φ_0 and φ_1 . It was first shown in [28], that this semi-distance vanishes identically for the L^2 right-invariant metric on the diffeomorphisms group of any compact manifold. More recently, it was shown in [3] that d_s vanishes identically on $\text{Diff}^\infty(\mathbb{S}^1)$ if $s \in [0, 1/2]$, whereas d_s is a distance for $s > 1/2$

$$\forall \varphi_0, \varphi_1 \in \text{Diff}^\infty(\mathbb{S}^1), \quad \varphi_0 \neq \varphi_1 \Rightarrow d_s(\varphi_0, \varphi_1) > 0.$$

Since the geodesic spray of the weak H^s right-invariant Riemannian metric on $\mathcal{D}^q(\mathbb{S}^1)$ is smooth for $q > (3/2) + 2s$ and $s \geq 1/2$, its exponential mapping on $\mathcal{D}^q(\mathbb{S}^1)$ is a diffeomorphism from a neighbourhood V of 0 in $H^q(\mathbb{S}^1)$ to a neighbourhood U of the identity in $\mathcal{D}^q(\mathbb{S}^1)$. This leads to the existence of *local polar coordinates* in the *normal chart* U . These coordinates are defined as follows. Given $\varphi \in U - \{\text{id}\}$, there is a $v \in V \setminus \{0\}$ such that $\varphi = \text{exp}(v)$. Letting now

$$w := v / \|v\|_{H^s}, \quad \rho := \|v\|_{H^s},$$

we have that $\varphi = \text{exp}(\rho w)$ and (ρ, w) are called the *polar coordinates* of $\varphi \in U - \{\text{id}\}$. Notice that (ρ, w) depend smoothly of φ and that $\rho(\varphi) \rightarrow 0$ as $\varphi \rightarrow \text{id}$.

As can be checked in [25], the following result is valid not only for a strong Riemannian metric but also for a *weak* Riemannian metric, *provided* there exists a compatible, symmetric covariant derivative.

Lemma 6.1. *For a piecewise C^1 curve $\gamma : [a, b] \rightarrow U(\varphi_0) - \{\varphi_0\}$, we have the inequality*

$$L_s(\gamma) \geq |\rho(b) - \rho(a)|.$$

However, it should be noticed that Lemma 6.1 does not imply that the geodesic semi-distance is in fact a distance. What Lemma 6.1 says, is that the length of any path which *lies inside* the normal neighbourhood is bounded below by $r := |\rho(b) - \rho(a)|$. However for a path which leaves the normal neighbourhood, this might not be true. Such a path could leave the normal neighbourhood *before* leaving the (weak ball) of radius r defined as

$$B_s(\text{id}, r) := \{\varphi \in U; \rho(\varphi) \leq r\}.$$

In fact this happens for the critical exponent $s = 1/2$ as it follows from [3].

7. EULER EQUATIONS ON HOMOGENEOUS SPACES

In this section, we will extend our main theorems to some geodesic equations on homogeneous spaces of $\text{Diff}^\infty(\mathbb{S}^1)$. Since the proof are very similar to what has been done so far, we will not give all the details but only point out the crucial new ingredients.

7.1. Euler equation on a homogeneous space. Consider now a non-negative, *degenerate* inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{g} of some Lie group G and let

$$A : \mathfrak{g} \rightarrow \mathfrak{g}^*$$

be the corresponding inertia operator. Suppose moreover that $\ker A = \mathfrak{k}$, where \mathfrak{k} is the Lie algebra of a closed subgroup K . By (right) translating this inner product over all tangent space $T_g G$, we obtain a right-invariant, symmetric, two-tensor field γ on G , such that

$$(7.1) \quad \gamma(g)(R_g w, \xi_g) = 0,$$

for all $w \in \mathfrak{k}$ and $\xi_g \in T_g G$. If moreover, γ is invariant under the left action of K on G , that is

$$(7.2) \quad L_k^* \gamma = \gamma$$

for all $k \in K$, then γ goes down on the quotient space and induces a right G -invariant Riemannian metric on the space G/K of right cosets ($Kg, g \in G$). The study of a degenerate inertia operator may therefore be reduced to the study of an invariant metric on a homogeneous space.

Remark 7.1. Condition (7.2) is equivalent to the following condition

$$(7.3) \quad \langle \text{Ad}_k u, \text{Ad}_k v \rangle = \langle u, v \rangle,$$

for all $k \in K$ and $u, v \in \mathfrak{g}$. Now, condition (7.3) implies

$$(7.4) \quad \langle \text{ad}_w u, v \rangle = -\langle u, \text{ad}_w v \rangle,$$

for all $w \in \mathfrak{k}$ and all $u, v \in \mathfrak{g}$ and if the subgroup K is connected, (7.3) and (7.4) are equivalent. Finally, condition (7.4) can also be rewritten as

$$(7.5) \quad \text{ad}_w^* \circ A = A \circ \text{ad}_w,$$

for all $w \in \mathfrak{k}$.

The theory of Euler equations on a homogeneous space G/K has been developed in [23]. In that case, the geodesic flow of a right-invariant Riemannian metric on the homogeneous space G/K , can be reduced to the so called *Euler-Poincaré* equation

$$(7.6) \quad m_t = \text{ad}_u^* m, \quad m \in \mathfrak{g}^*,$$

using a Hamiltonian reduction process (see [23] or the original paper of Poincaré [33]). This equation is obtained as follows. Let

$$L : v_x \mapsto \frac{1}{2} \langle v_x, v_x \rangle_x, \quad TM \rightarrow \mathbb{R}$$

be the energy function on M and $\pi : G \rightarrow M$ be the canonical projection. We define $\bar{L} : TG \rightarrow TM$ as the pullback of L by $T\pi$, the tangent map of

π . The derivative of \bar{L} along the fiber (*fiber derivative*), or Legendre map defines a smooth map $P : TG \rightarrow TG^*$ and we can introduce right and left momentum by

$$m^R : TG \rightarrow \mathfrak{g}^*, \quad \xi_g \mapsto P(\xi_g) \circ R_g,$$

and

$$m^L : TG \rightarrow \mathfrak{g}^*, \quad \xi_g \mapsto P(\xi_g) \circ L_g.$$

Since \bar{L} is right invariant, $m^L(\dot{g})$ is constant along the lift $g(t)$ of any geodesic on M (Noether's theorem) and the Euler-Poincaré equation (7.6) results from the observation that $m^R(\dot{g}) = \text{Ad}_g^* m^L(\dot{g})$, and where we define $u(t) := R_{g^{-1}}\dot{g}$.

Unfortunately, there is no useful *contravariant* formulation of this equation similar to the “genuine Euler equation on a Lie group” (2.3). Indeed, in this case, the Eulerian velocity (defined using a lift $g(t)$ in G of a path $x(t)$ in M) is only defined *up to a path* in K and the relation between u and m is not simple (see [36] for a recent survey on this subject). Another way to treat the problem is to introduce *sub-Riemannian geometry* on G (see [16, 17] for a deep study of this approach for $\text{Diff}^\infty(\mathbb{S}^1)$).

These difficulties clear away if K is a normal subgroup. Indeed, in that case the coset manifold G/K is a Lie group equipped with a right-invariant Riemannian metric. But this special case is not very useful for our study, since $\text{Diff}^\infty(\mathbb{S}^1)$ is a *simple* group: it has no nontrivial normal subgroups (see [18]). However, there is another situation where the study of a right-invariant Riemannian metric on a homogeneous space can be reduced to the ordinary theory of the Euler equation on a Lie group and this situation applies to the Hunter-Saxton equation and the Weil-Petersson equation. It is described by the lemma below.

Lemma 7.2. *Let G be a group and H, K some subgroups of G . Suppose that*

- (1) *The restriction to H of the projection map $\pi : G \rightarrow G/K$ is surjective,*
- (2) *$H \cap K = \{e\}$.*

Then H acts simply and transitively on G/K .

Remark 7.3. As a result, if the hypothesis of Lemma 7.2 are satisfied, then G/K inherits a group structure. Notice, however that the restriction of the projection $\pi : H \rightarrow G/K$ is a group morphism if and only if K is a normal subgroup of G .

Proof. By definition, the projection map π sends an element $g \in G$ to the coset Kg . To show that the (right) action of H on G/K is transitive, it suffices to show that for any coset Kg we can find $h \in H$ such that $Kh = Kg$. But this means precisely that $\pi : H \rightarrow G/K$ is surjective. Hence the transitivity of the action is equivalent to the surjectivity of π .

To prove that the action is simple, it is enough to show that the only element $h \in H$ which fixes the coset K is $h = e$, the unit element. But this means $Kh = K$, and thus $h \in K \cap H$, which leads to $h = e$ by condition (2). Notice that this implies that $\pi : H \rightarrow G/K$ is injective. \square

We will summarize all the preceding considerations in the following proposition.

Proposition 7.4. *Suppose that H and K are closed subgroups of a Lie group G such that $H \cap K = \{e\}$ and such that $\pi : H \rightarrow G/K$ is surjective. Let \mathfrak{g} , \mathfrak{h} , and \mathfrak{k} denote the Lie algebras of G , H , and K , respectively. Finally, let $\langle \cdot, \cdot \rangle$ be a non-negative inner product on \mathfrak{g} whose inertia operator $A : \mathfrak{g} \rightarrow \mathfrak{g}^*$ satisfies the following conditions:*

- (i) $\ker A = \mathfrak{k}$,
- (ii) $\text{Ad}_k^* \circ A = A \circ \text{Ad}_k$, for all $k \in K$.

Then, A induces a right-invariant, Riemannian metric γ on the homogeneous space $M := G/K$. Moreover, π is a Riemannian isometry between (M, γ) and the Riemannian space defined by the Lie group H endowed with the right-invariant metric induced by the inner product on \mathfrak{h} .

Remark 7.5. In the situation described, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$ and \mathfrak{h}^* can be identified with

$$\mathfrak{k}^0 = \{m \in \mathfrak{g}^*; \langle m, w \rangle = 0, \forall w \in \mathfrak{k}\}.$$

Moreover, the inertia operator A induces an isomorphism $A : \mathfrak{h} \rightarrow \mathfrak{k}^0 \simeq \mathfrak{h}^*$. Now condition (ii) (and the symmetry of A) leads to

$$\langle \text{ad}_u^* A(v), w \rangle = -\langle \text{ad}_v^* A(u), w \rangle$$

for all $u, v \in \mathfrak{g}$ and all $w \in \mathfrak{k}$. Hence,

$$\text{ad}_u^* A(v) + \text{ad}_v^* A(u) \in \mathfrak{k}^0 = \text{im } A,$$

and the bilinear operator

$$B(u, v) = \frac{1}{2} A^{-1} \left[\text{ad}_u^* A(v) + \text{ad}_v^* A(u) \right]$$

is well-defined on $\mathfrak{g} \times \mathfrak{g}$. The Euler equation on \mathfrak{h} is given by

$$(7.7) \quad u_t = -B(u, u) = -A^{-1} \left[\text{ad}_u^* A(u) \right].$$

7.2. Euler equations on the coadjoint orbit $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$. Let $\text{Rot}(\mathbb{S}^1)$ denotes the subgroup of all rigid rotations of \mathbb{S}^1 and

$$\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1),$$

be the corresponding homogeneous space of right cosets. Let $\text{Diff}_1^\infty(\mathbb{S}^1)$ be the subgroup of $\text{Diff}^\infty(\mathbb{S}^1)$ consisting of all diffeomorphisms of \mathbb{S}^1 which fix one arbitrarily point (say x_0). It is easy to check that the conditions of Lemma 7.2 are satisfied and hence that the canonical projection

$$\text{Diff}_1^\infty(\mathbb{S}^1) \rightarrow \text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$$

is a bijection. The group $\text{Diff}_1^\infty(\mathbb{S}^1)$ is a Fréchet Lie group and we can use the Fréchet manifold structure of $\text{Diff}_1^\infty(\mathbb{S}^1)$ to endow the quotient space $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ with a Fréchet manifold structure, so that the canonical projection becomes a diffeomorphism.

The Lie algebras of $\text{Diff}_1^\infty(\mathbb{S}^1)$ and $\text{Rot}(\mathbb{S}^1)$ are given by

$$\mathcal{C}_1^\infty(\mathbb{S}^1) := \{u \in C^\infty(\mathbb{S}^1); u(x_0) = 0\} \quad \text{and} \quad \mathbb{R} \cdot w_0,$$

respectively, where w_0 stands for the constant function with value 1.

$\text{Diff}_1^\infty(\mathbb{S}^1)$ is an ILH space; a Hilbert approximation being given by the Hilbert manifolds

$$\mathcal{D}_1^q(\mathbb{S}^1) := \{\varphi \in \mathcal{D}^q(\mathbb{S}^1); \varphi(x_0) = x_0\},$$

modelled on the Hilbert spaces

$$H_1^q(\mathbb{S}^1) := \{u \in H^q(\mathbb{S}^1); u(x_0) = 0\}.$$

Notice that $\mathcal{D}_1^q(\mathbb{S}^1)$ is a closed submanifold of the Hilbert manifold $\mathcal{D}^q(\mathbb{S}^1)$ and a topological subgroup of $\mathcal{D}^q(\mathbb{S}^1)$, for $q > 3/2$.

Let $A = \mathbf{op}(p(k))$ be a L^2 -symmetric, Fourier multiplier on $C^\infty(\mathbb{S}^1)$ and assume that its symbol satisfies

$$p(k) = 0 \quad \text{iff } k = 0,$$

which is equivalent to $\ker A = \mathbb{R} \cdot w_0$. Since $\text{ad}_{w_0} = -D$, $\text{ad}_{w_0}^* = -D$ and A commutes with D , hypothesis (ii) of Proposition 7.4 is satisfied. Therefore, A induces a weak right-invariant Riemannian metric on $\text{Diff}^\infty(\mathbb{S}^1)/\text{Rot}(\mathbb{S}^1)$ and the corresponding Euler equation on $C_1^\infty(\mathbb{S}^1)$ is given by

$$u_t = -A^{-1} \{(Au)_x u + 2(Au)u_x\}.$$

If A is of order $r \geq 1$, then A extends to $H_1^q(\mathbb{S}^1)$, for all $q > 3/2$, and

$$A \in \text{Isom}(H_1^q(\mathbb{S}^1), \hat{H}_1^{q-r}(\mathbb{S}^1)),$$

where

$$\hat{H}_1^{q-r}(\mathbb{S}^1) := \{m \in H^{q-r}(\mathbb{S}^1); \hat{m}(0) = 0\}.$$

Then, A induces a positive inner product on each tangent space, $T_\varphi \mathcal{D}_1^q(\mathbb{S}^1)$, with a continuous flat map

$$\tilde{A}(\varphi, v) := (\varphi, \varphi_x A_\varphi(v)),$$

defined on $T\mathcal{D}_1^q(\mathbb{S}^1) = \mathcal{D}_1^q(\mathbb{S}^1) \times H_1^q(\mathbb{S}^1)$. Notice that

$$\tilde{A}_\varphi \in \text{Isom}(H_1^q(\mathbb{S}^1), \hat{H}_1^{q-r}(\mathbb{S}^1)),$$

for each $\varphi \in \mathcal{D}_1^q(\mathbb{S}^1)$ and that $\tilde{A}(T\mathcal{D}_1^q(\mathbb{S}^1)) = \mathcal{D}_1^q(\mathbb{S}^1) \times \hat{H}_1^{q-r}(\mathbb{S}^1)$. The proof of the proposition below is similar to that of proposition 4.1 and the proof will be omitted.

Theorem 7.6. *Let $A = \mathbf{op}(p(k))$ be a L^2 -symmetric, non negative, Fourier multiplier of order $r \geq 1$, satisfying*

$$(7.8) \quad p(k) = 0 \iff k = 0.$$

Assume that in addition that

$$\varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for $q > 3/2$. Then, the induced right-invariant metric on $\mathcal{D}_1^q(\mathbb{S}^1)$ is smooth and has a smooth spray. Moreover, given any $(\varphi_0, v_0) \in T\text{Diff}_1^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution

$$(\varphi, v) \in C^\infty(J, T\text{Diff}_1^\infty(\mathbb{S}^1))$$

of the Cauchy problem for the associated geodesic spray

$$(7.9) \quad \begin{cases} \varphi_t = v, & \varphi(0) = \varphi_0 \\ v_t = S_\varphi(v), & v(0) = v_0. \end{cases}$$

on the maximal interval of existence J .

We briefly discuss two special instances, namely $A = \mathbf{op}(k^2)$ and $A = \mathbf{op}(|k|)$. In the first case $A = \mathbf{op}(k^2)$ the Euler equation reads as

$$(7.10) \quad u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

and is known as the periodic *Hunter-Saxton* equation, cf. [21, 39, 4, 26].

For the inertia operator $A = \mathbf{op}(|k|)$ we get the so called CLM equation, cf. [7, 38, 13].

$$(7.11) \quad \partial_t(Hu_x) + uHu_{xx} + 2u_xHu_x = 0, \quad t > 0, \quad x \in \mathbb{S}^1,$$

where $H = \mathbf{op}(i \operatorname{sgn}(k))$ denotes the Hilbert transform, acting on the spatial variable $x \in \mathbb{S}^1$.

Clearly, both symbols $(k^2)_{k \in \mathbb{Z}}$ and $(|k|)_{k \in \mathbb{Z}}$ satisfy (7.8). Moreover they also fulfill the hypotheses of Theorem 1.1, so that Theorem 7.6 is applicable to (7.10) and (7.11).

Remark 7.7. Recall that $\mu(u)$ is conserved under the evolution of (5.7). Hence if we choose an initial condition $u_0 \in C^\infty(\mathbb{S}^1)$ with $\mu(u_0) = 0$, the corresponding solution to the μ -Hunter-Saxton and the generalized (CLM) correlates to the solution of (7.10) and (7.11), respectively. Note that $\mu(u_0) = 0$ implies that u_0 has a zero. After rotation, we may assume that $u_0(x_0) = 0$, meaning that the Eulerian velocity $u(t)$ satisfies $u = \varphi_t \circ \varphi^{-1}$, where (φ, v) is the solution to (7.9) with initial data (id, u_0) .

7.3. Euler equations on the coadjoint orbit $\operatorname{Diff}^\infty(\mathbb{S}^1)/\operatorname{PSL}(2, \mathbb{R})$. Let $\operatorname{PSL}(2, \mathbb{R})$ denotes the subgroup of all rigid Möbius transformations which preserves the circle \mathbb{S}^1 and let

$$\operatorname{Diff}^\infty(\mathbb{S}^1)/\operatorname{PSL}(2, \mathbb{R}),$$

be the corresponding homogeneous space of right cosets. Let $\operatorname{Diff}_3^\infty(\mathbb{S}^1)$ be the subgroup of $\operatorname{Diff}^\infty(\mathbb{S}^1)$ consisting of all diffeomorphisms of \mathbb{S}^1 which fix 3 arbitrary distinct points (say x_0, x_1, x_2). Then $\operatorname{PSL}(2, \mathbb{R}) \cap \operatorname{Diff}_3^\infty(\mathbb{S}^1) = \{e\}$ and the canonical projection

$$\operatorname{Diff}_3^\infty(\mathbb{S}^1) \rightarrow \operatorname{Diff}^\infty(\mathbb{S}^1)/\operatorname{PSL}(2, \mathbb{R})$$

is a bijection. The group $\operatorname{Diff}_3^\infty(\mathbb{S}^1)$ is a Fréchet Lie group and we can use this Fréchet structure to endow the quotient space $\operatorname{Diff}^\infty(\mathbb{S}^1)/\operatorname{PSL}(2, \mathbb{R})$ with a Fréchet manifold structure. In that way, the canonical projection becomes a diffeomorphism.

The Lie algebras of $\operatorname{Diff}_3^\infty(\mathbb{S}^1)$ is given by

$$C_3^\infty(\mathbb{S}^1) := \{u \in C^\infty(\mathbb{S}^1); u(x_0) = 0, u(x_1) = 0, u(x_2) = 0\},$$

whereas the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \subset C^\infty(\mathbb{S}^1)$ of $\operatorname{PSL}(2, \mathbb{R})$ is the 3-dimensional subalgebra of $C^\infty(\mathbb{S}^1)$, generated by

$$w_0(x) := 1, \quad w_1(x) := \cos(x), \quad w_{-1}(x) := \sin(x).$$

An ILH structure on $\operatorname{Diff}_3^\infty(\mathbb{S}^1)$ is given by the Hilbert manifolds

$$\mathcal{D}_3^q(\mathbb{S}^1) := \{\varphi \in \mathcal{D}^q(\mathbb{S}^1); \varphi(x_0) = x_0, \varphi(x_1) = x_1, \varphi(x_2) = x_2\},$$

modelled on the Hilbert spaces

$$H_3^q(\mathbb{S}^1) := \{u \in H^q(\mathbb{S}^1); u(x_0) = 0, u(x_1) = 0, u(x_2) = 0\}.$$

Notice that $\mathcal{D}_3^q(\mathbb{S}^1)$ is a closed submanifold of the Hilbert manifold $\mathcal{D}^q(\mathbb{S}^1)$ and a topological subgroup of $\mathcal{D}^q(\mathbb{S}^1)$, for $q > 3/2$.

Theorem 7.8. *Let $A = \mathbf{op}(p(k))$ be a L^2 -symmetric, non negative, Fourier multiplier of order $r \geq 1$, satisfying*

$$(7.12) \quad p(k) = 0 \iff k \in \{-1, 0, 1\},$$

and

$$\mathrm{ad}_w^* \circ A = A \circ \mathrm{ad}_w,$$

for all $w \in \mathfrak{sl}(2, \mathbb{R})$. Assume that in addition that

$$\varphi \mapsto A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for $q > 3/2$. Then, the induced right-invariant metric on $\mathcal{D}_3^q(\mathbb{S}^1)$ is smooth and has a smooth spray. Moreover, given any $(\varphi_0, v_0) \in \mathrm{TDiff}_3^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution

$$(\varphi, v) \in C^\infty(J, \mathrm{TDiff}_3^\infty(\mathbb{S}^1))$$

of the Cauchy problem for the associated geodesic spray

$$\begin{cases} \varphi' = v, & \varphi(0) = \varphi_0 \\ v' = S_\varphi(v), & v(0) = v_0. \end{cases}$$

on the maximal interval of existence J .

Proof. The proof is similar to that of Proposition 4.1, except for point (c). In fact, notice that A extends to $H_3^q(\mathbb{S}^1)$, for all $q > 3/2$, and that

$$A \in \mathrm{Isom}(H_1^q(\mathbb{S}^1), \hat{H}_3^{q-r}(\mathbb{S}^1)),$$

where

$$\hat{H}_3^{q-r}(\mathbb{S}^1) := \{m \in H^{q-r}(\mathbb{S}^1); \hat{m}(0) = 0, \hat{m}(1) = 0, \hat{m}(-1) = 0\}.$$

Let

$$\tilde{A}(\varphi, v) := (\varphi, \varphi_x A_\varphi(v)),$$

be the flat map defined on $\mathrm{T}\mathcal{D}_3^q(\mathbb{S}^1) = \mathcal{D}_3^q(\mathbb{S}^1) \times H_3^q(\mathbb{S}^1)$. It takes values in $\mathcal{D}_3^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$, but given $\varphi \in \mathcal{D}_3^q(\mathbb{S}^1)$, we have

$$\tilde{A}_\varphi(H_3^q(\mathbb{S}^1)) = \left\{ m \in H^{q-r}(\mathbb{S}^1); \varphi_x m \circ \varphi \in \hat{H}_3^{q-r}(\mathbb{S}^1) \right\},$$

so we cannot conclude immediately that $\tilde{A}(\mathrm{T}\mathcal{D}_3^q(\mathbb{S}^1))$ is a trivial bundle as this was the case for $\tilde{A}(\mathrm{T}\mathcal{D}^q(\mathbb{S}^1))$ and $\tilde{A}(\mathrm{T}\mathcal{D}_1^q(\mathbb{S}^1))$. To overcome this difficulty, we first remark that

$$\tilde{A} : \mathcal{D}_3^q(\mathbb{S}^1) \times H_3^q(\mathbb{S}^1) \rightarrow \mathcal{D}_3^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$$

is a vector bundle morphism. Moreover, the continuous linear map

$$\tilde{A}_\varphi : H_3^q(\mathbb{S}^1) \rightarrow H^{q-r}(\mathbb{S}^1)$$

is injective and *splits*⁵ because, $\tilde{A}_\varphi(H_3^q(\mathbb{S}^1))$ is a closed subspace of $H^{q-r}(\mathbb{S}^1)$, for every $\varphi \in \mathcal{D}_3^q(\mathbb{S}^1)$. Then according to Proposition 3.1 in [25, Chapter 3], $\tilde{A}(T\mathcal{D}_3^q(\mathbb{S}^1))$ is a subbundle of $\mathcal{D}_3^q(\mathbb{S}^1) \times H^{q-r}(\mathbb{S}^1)$ which is isomorphic to $\mathcal{D}_3^q(\mathbb{S}^1) \times H_3^{q-r}(\mathbb{S}^1)$. Then, an argument similar to the one given in the point (c) of the proof of Proposition 4.1 does apply and achieves the proof. \square

An important application of Theorem 7.8 is the Euler-Weil-Petersson equation, which corresponds to the inertia operator

$$A := HD(D^2 + 1) = \mathbf{op}(|k|(k^2 - 1)).$$

This equation has been related with the Weil-Petersson metric on the universal Teichmüller space $T(1)$ in [31, 35]. The corresponding geodesic flow has been extensively studied in [15]. Recall first that $\mathcal{D}^s(\mathbb{S}^1)$, the space of homeomorphisms of class H^s as well as their inverse is a topological group only for $s > 3/2$ and that $3/2$ is therefore a critical exponent. One of the main results in [15] is that, the inertia operator A defines on a suitable replacement for the “ $H^{3/2}$ diffeomorphisms group”, a right-invariant *strong Riemannian structure* which is moreover geodesically complete (i.e., geodesics are defined for all times).

Our point of view here is completely different in the sense that we want to study the corresponding right-invariant metric on the Fréchet Lie group $\text{Diff}_3^\infty(\mathbb{S}^1)$ and its Hilbert approximations $\mathcal{D}^s(\mathbb{S}^1)$ for $s > 3/2$. The price to pay is the fact that the metric only defines a *weak Riemannian structure*. Nevertheless, theorem 7.8 applies in this case. Indeed, A satisfies the hypothesis of theorem 1.1 and all conditions of theorem 7.8. This shows the local existence of geodesics on $\text{Diff}_3^\infty(\mathbb{S}^1)$, which doesn't seem to be a consequence of the results in [15].

APPENDIX A. BOUNDEDNESS PROPERTIES OF RIGHT TRANSLATIONS

In this section we provide some local boundedness properties for the right representation of $\mathcal{D}^q(\mathbb{S}^1)$ on $H^q(\mathbb{S}^1)$. It was proven in [8] (see also [9, 10, 11]) that the diffeomorphisms group $\mathcal{D}^q(\mathbb{S}^1)$ is a topological group under composition if and only if q is strictly bigger than the critical exponent $3/2$. It was also established there that the mapping

$$(A.1) \quad (\varphi, u) \mapsto R_\varphi(u) := u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \rightarrow H^q(\mathbb{S}^1)$$

is continuous (*continuity* of the right representation of $\mathcal{D}^q(\mathbb{S}^1)$ on $H^q(\mathbb{S}^1)$). Notice however that this does not imply that the mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^q(\mathbb{S}^1))$$

is continuous with respect to the operator norm on $\mathcal{L}(H^q(\mathbb{S}^1))$ (*norm continuity*).

Remark A.1. *Norm continuity* obviously implies *continuity* but the converse is false. Indeed, a general result in the theory of semigroups of linear operators states that a semigroup on a Banach space E is norm continuous at

⁵If E, F are Banach spaces and $\Lambda : E \rightarrow F$ is a continuous linear map, which is injective, then we say that Λ *splits* if $\Lambda(E)$ is *closed* and *complemented* in F (i.e there exists a closed subspace G of F such that $F = \Lambda(E) \oplus G$). Notice that if F is a Hilbert space, then every injective, continuous linear map with closed range, splits.

0 if and only if its infinitesimal generator is bounded on E , cf. [32, Theorem 1.2]. Let now $q > 3/2$ and let τ_s be the rotation by the angle s on \mathbb{S}^1 . Then the representation of the group $\{R_{\tau_s}; s \in \mathbb{R}\}$ is continuous on $H^q(\mathbb{S}^1)$. But it cannot be norm continuous, since its infinitesimal generator D is not bounded on $H^q(\mathbb{S}^1)$. A direct argument, which shows that $\|R_{\tau_s} - \text{Id}\|_{\mathcal{L}(H^q(\mathbb{S}^1))}$ is bounded away from 0 for all s near 0 is runs as follows: Let $s \in (-1/2, 1/2)$ and u_s be a periodic, bump function with support in $(k - s/2, k + s/2)$ ($k \in \mathbb{Z}$) with $\|u_s\|_{L^2} = 1$. We have then

$$\|R_{\tau_s} u_s - u_s\|_{H^q(\mathbb{S}^1)}^2 = 2 \|u_s\|_{H^q(\mathbb{S}^1)}^2,$$

because u_s and $R_{\tau_s} u$ are $H^q(\mathbb{S}^1)$ -orthogonal and R_{τ_s} is an $H^q(\mathbb{S}^1)$ -isometry. Hence

$$\|R_{\tau_s} - \text{Id}\|_{\mathcal{L}(H^q(\mathbb{S}^1))} \geq \sqrt{2} \quad \text{for} \quad \frac{-1}{2} < s < \frac{1}{2},$$

which proves that the representation $\varphi \mapsto R_\varphi$ is not *norm continuous*.

Proposition A.2. *Given $\rho, q \in \mathbb{R}$ such that $3/2 < \rho \leq q$, the mapping*

$$(\varphi, u) \mapsto u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1)$$

is continuous. Moreover

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^\rho(\mathbb{S}^1))$$

and

$$\varphi \mapsto R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^\rho(\mathbb{S}^1))$$

are locally bounded.

Before entering into the details of the proof of Lemma A.2, let us recall some notations. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, the $W^{m,p}$ -norm of a measurable functions on \mathbb{S}^1 is defined by

$$\|u\|_{W^{m,p}} := \sum_{j=0}^m \left\| u^{(j)} \right\|_{L^p},$$

where $u^{(j)}$ denotes the derivative of order j . When $s > 0$ is not an integer and $p < \infty$, the $W^{s,p}$ -norm is defined by

$$\|u\|_{W^{s,p}} := \|u\|_{W^{m,p}} + \mathfrak{p}_{\sigma,p}(u^{(m)})$$

where

$$m = [s] \quad \text{and} \quad s = m + \sigma,$$

and the semi-norm $\mathfrak{p}_{\sigma,p}$ ($0 < \sigma < 1$) is defined by

$$\mathfrak{p}_{\sigma,p}(w) = \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|w(x) - w(y)|^p}{|x - y|^{1+p\sigma}} dx dy \right)^{1/p}.$$

In the following, we write $\mathfrak{p}_{\sigma,2} = \mathfrak{p}_\sigma$, when there is no ambiguity. The Banach space $W^{s,p}(\mathbb{S}^1)$ is by definition the completion of $C^\infty(\mathbb{S}^1)$ with respect to the $W^{s,p}$ -norm and when $p = 2$, we get the Hilbert space

$$W^{s,2}(\mathbb{S}^1) = H^s(\mathbb{S}^1).$$

Recall that for $1 \leq p, q < \infty$ and $r, s \in \mathbb{R}$ such that

$$r \geq s, \quad \text{and} \quad r - \frac{1}{p} \geq s - \frac{1}{q},$$

we have the continuous *Sobolev embeddings* (see [37])

$$W^{r,p}(\mathbb{S}^1) \subset W^{s,q}(\mathbb{S}^1),$$

and

$$H^s(\mathbb{S}^1) \subset W^{m,\infty}(\mathbb{S}^1),$$

for $s = m + \sigma$, $m \in \mathbb{N}$ and $\sigma > 1/2$. Moreover, $H^s(\mathbb{S}^1)$ is a multiplicative algebra for $s > 1/2$.

Lemma A.3. *Let $\sigma \in (0, 1)$. Then, pointwise multiplication in $C^\infty(\mathbb{S}^1)$ extends to a continuous bilinear mapping*

$$H^\sigma(\mathbb{S}^1) \times H^1(\mathbb{S}^1) \rightarrow H^\sigma(\mathbb{S}^1).$$

Proof. For $\sigma > 1/2$, $H^\sigma(\mathbb{S}^1)$ is a multiplicative algebra and the lemma is obvious. So we may suppose $\sigma \leq 1/2$. We have first

$$\|uv\|_{L^2} \leq \|u\|_{L^2} \|v\|_{L^\infty},$$

for $(u, v) \in H^\sigma(\mathbb{S}^1) \times H^1(\mathbb{S}^1)$. Next, we have

$$\begin{aligned} \mathfrak{p}_\sigma(uv)^2 &= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|(u(x) - u(y))v(x) + u(y)(v(x) - v(y))|^2}{|x - y|^{1+2\sigma}} dx dy \\ &\leq 2\|v\|_{L^\infty}^2 \mathfrak{p}_\sigma(u)^2 + 2 \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|u(y)|^2 |v(x) - v(y)|^2}{|x - y|^{1+2\sigma}} dx dy. \end{aligned}$$

But Hölder's inequality leads to

$$\begin{aligned} &\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|u(y)|^2 |v(x) - v(y)|^2}{|x - y|^{1+2\sigma}} dx dy \\ &\leq \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} |u(y)|^{2\alpha} dx dy \right)^{1/\alpha} \left(\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|v(x) - v(y)|^{2\alpha'}}{|x - y|^{(1+2\sigma)\alpha'}} dx dy \right)^{1/\alpha'} \end{aligned}$$

where $1 < \alpha, \alpha' < \infty$ and $1/\alpha + 1/\alpha' = 1$. Therefore we get

$$\mathfrak{p}_\sigma(uv)^2 \leq 2\|v\|_{L^\infty}^2 \|u\|_{H^\sigma}^2 + 2\|u\|_{L^{2\alpha}}^2 \|v\|_{W^{\sigma+1/2\alpha, 2\alpha'}}^2.$$

Now, by virtue of Sobolev's embeddings theorem, there exists positive constants C_1, C_2, C_3 such that

$$\begin{aligned} \|v\|_{L^\infty} &\leq C_1 \|v\|_{H^1}, \\ \|v\|_{W^{\sigma+1/2\alpha, 2\alpha'}} &\leq C_2 \|v\|_{H^1}, \\ \|u\|_{L^{2\alpha}} &\leq C_3 \|u\|_{H^\sigma}, \end{aligned}$$

provided we choose $1/2\sigma \leq \alpha' \leq 1/\sigma$, which achieves the proof. \square

Proof of proposition A.2. The continuity of the mapping

$$(\varphi, u) \mapsto u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1)$$

results from the continuity of (A.1), since $3/2 < \rho \leq q$. Moreover, since $\varphi \mapsto \varphi^{-1}$ is a homeomorphism of $\mathcal{D}^q(\mathbb{S}^1)$ for $q > 3/2$ (see [11] and references therein), it is sufficient to show that the mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \rightarrow \mathcal{L}(H^\rho(\mathbb{S}^1))$$

is locally bounded.

First, notice that if φ is a C^1 -diffeomorphism and $\sigma \in (0, 1)$, a change of variables leads to the estimate

$$\|w \circ \varphi\|_{H^\sigma} \leq \left(\|1/\varphi_x\|_{L^\infty}^{1/2} + \|1/\varphi_x\|_{L^\infty} \|\varphi_x\|_{L^\infty}^{(1+2\sigma)/2} \right) \|w\|_{H^\sigma},$$

for any $w \in H^\sigma(\mathbb{S}^1)$.

Suppose now that $\rho = 1 + \sigma$, and thus $\sigma > 1/2$. Given $\varphi \in \text{Diff}^\infty(\mathbb{S}^1)$ and $v \in C^\infty(\mathbb{S}^1)$, we have

$$\|v \circ \varphi\|_{L^2}^2 = \int_{\mathbb{S}^1} |v(x)|^2 \frac{1}{\varphi_x \circ \varphi^{-1}(x)} dx \leq \|1/\varphi_x\|_{L^\infty} \|v\|_{L^2}^2,$$

and

$$\begin{aligned} \|(v \circ \varphi)_x\|_{H^\sigma} &\leq C \|v_x \circ \varphi\|_{H^\sigma} \|\varphi_x\|_{H^\sigma} \\ &\leq C \left(\|1/\varphi_x\|_{L^\infty}^{1/2} + \|1/\varphi_x\|_{L^\infty} \|\varphi_x\|_{L^\infty}^{(1+2\sigma)/2} \right) \|\varphi_x\|_{H^\sigma} \|v_x\|_{H^\sigma}. \end{aligned}$$

Therefore

$$\begin{aligned} \|v \circ \varphi\|_{H^\rho} &\leq \|v \circ \varphi\|_{L^2} + \|(v \circ \varphi)_x\|_{H^\sigma} \\ &\leq \beta_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}}) \|v\|_{H^\rho}, \end{aligned}$$

where β_ρ is a positive, continuous function on $(\mathbb{R}^+)^2$.

Suppose then that $\rho = m + \sigma$, where $m \geq 2$ and $\sigma \in [0, 1)$, we have

$$\|(v \circ \varphi)_x\|_{L^2}^2 = \int_{\mathbb{S}^1} |v_x(x)|^2 \varphi_x \circ \varphi^{-1}(x) dx \leq \|\varphi_x\|_{L^\infty} \|v\|_{H^1}^2,$$

and more generally

$$\left\| (v \circ \varphi)^{(j)} \right\|_{L^2}^2 \leq p_j (\|\varphi_x\|_{H^{q-1}}) \|v\|_{H^j}^2, \quad 1 \leq j \leq m,$$

where p_j is a positive, polynomial function. Besides

$$(v \circ \varphi)^{(m)} = (v_x \circ \varphi) \varphi_x^{(m-1)} + \sum_{j=2}^m \left(v^{(j)} \circ \varphi \right) W_j$$

where W_j is a monomial in the variables $\varphi_x, \varphi_x^{(1)}, \dots, \varphi_x^{(m-2)}$. Hence, making use of Lemma A.3, we get

$$\left\| (v \circ \varphi)^{(m)} \right\|_{H^\sigma} \leq C \left(\|v_x \circ \varphi\|_{H^1} \left\| \varphi_x^{(m-1)} \right\|_{H^\sigma} + \sum_{j=2}^m \left\| v^{(j)} \circ \varphi \right\|_{H^\sigma} \|W_j\|_{H^1} \right),$$

for some positive constant C . Therefore, we get that

$$\|v \circ \varphi\|_{H^\rho} \leq \beta_\rho (\|1/\varphi_x\|_{L^\infty}, \|\varphi_x\|_{H^{q-1}}) \|v\|_{H^\rho}$$

where β_ρ is a positive, continuous function on $(\mathbb{R}^+)^2$. By a density argument and the continuity of the mapping

$$(\varphi, u) \mapsto u \circ \varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \times H^\rho(\mathbb{S}^1) \rightarrow H^\rho(\mathbb{S}^1)$$

this estimate is still true when $\varphi \in \mathcal{D}^q(\mathbb{S}^1)$ and $v \in H^\rho(\mathbb{S}^1)$. This achieves the proof. \square

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