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On quasi-inverse monoids (and premorphisms)

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# Quasi-inverse monoids (and premorphisms)

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## Abstract

In this paper, we define and study a class of monoids ordered by a two sided version of Nambooripad order. These monoids, quasi-inverse in some sense, are equipped with a variation of McAlister and Reilly premorphisms. The resulting category is shown to be a super-category of the category of monoids equipped with morphisms.

An expansion construction, à la Birget and Rhodes, that lifts every monoid morphism to a quasi-inverse monoid premorphism, is provided. This shows that, in some sense, the classical notion of language recognizability by monoid can be generalized to a richer notion of language quasi-recognizability by quasi-inverse monoid.

While former studies of inverse monoids in the context of language theory rather led to negative results as the class of definable languages by means of inverse monoids collapses, the new framework proposed here reopens the way towards the integration of the mathematical richness of inverse monoid theory in algebraic studies of formal language theory.

*Nota : this research report is unpublished and will probably remain as such. Ongoing developments, within language theory, of the notion of quasi-recognizability tell us that definitions proposed in this report are yet not as potentially stable as one could expect.*

## 1 Introduction

Monoids (or semigroups) theory plays a central rôle in formal language theory. Via the notion of algebraic recognizability and, especially, the associated

notion of syntactic monoid, a considerable amount of decision results have been obtained.

In view of the mathematical richness of inverse semigroups theory, many attempts have thus been conducted to incorporate that richness within formal language theory. However, aside the inherent mathematical interests of these studies, most of these approaches have been unsatisfactory in terms of expressive power. Languages of words recognizable by inverse monoids turned out to be rather poor [10] and recognizable subsets of free inverse monoids, i.e. languages of birooted trees, also turned out to be considerably limited [14] compared to the notion of regular tree languages. A recent study [6] of recognizable languages of tiles - one-dimensional unidirectional birooted trees - even shows that these subsets are essentially characterized by means of special covers of periodic bi-infinite words.

In all cases, such a weak expressiveness can be explained by the fact that morphisms, in that they preserve products, convey far too much structure. The class of associated automata, necessarily inversible in some sense, is thus limited. We thereby seek to identify a *relaxation* of the notion of morphism itself.

For languages of tiles, equivalently subsets of McAlister inverse monoid [9], we already proposed [5] a notion of *quasi-recognizability*: recognizability by means of *premorphisms* instead of morphisms. Defined on *ordered monoids*, premorphisms are monotonic mappings that are only required to be submultiplicative w.r.t. the monoid product [11], i.e.  $\varphi(xy) \leq \varphi(x)\varphi(y)$  instead of  $\varphi(xy) = \varphi(x)\varphi(y)$ . The resulting quasi-recognizable subsets are then shown [5] to capture *essentially* the expressive power of Monadic Second Order Logic (MSO) definability : a typical yardstick of expressive power.

However, the notion quasi-recognizability briefly sketched above is incomplete for it does not imply MSO definability. Indeed, one need to restrict both the class of (ordered) monoids and the class of premorphisms so that the resulting inverse images are definable in MSO. Setting up an adequate notion of quasi-recognizability becomes thus a rather delicate task seeking for the right balance between gaining in expressiveness by relaxing classical hypothesis, and preserving MSO definability by imposing some restriction.

Our goal in this paper is to make such a proposal and study the abstract properties of these adequate ordered monoids and premorphisms. The associated methods, ideas and results that induce a generalization of the classical notion of recognizability is potentially applicable in many other context than the one previously studied. We aim thus at providing here mathematical foundations, within ordered semigroup and inverse semigroup theory, of this potentially new bridge between semigroup theory in algebra and formal lan-

guage theory in computer science.

## Outline

The first stage of our work amounts to define the class of admissible monoids upon which quasi-recognizability is well-behaved. It occurs that these monoids are tightly related with the idea that

We give two equivalent axiomatic definition of it.

The first axiomatic definition deals with monoids equipped with two unary projection  $x \mapsto x_L$  and  $x \mapsto x_R$  that maps every element to (some notion of) canonical left and right local units assumed to be commuting idempotents. A number of axioms defining the properties of these projections are then given that eventually capture the class of Lawson's  $U$ -semiadequate monoids [7]. In that approach,  $U$  is defined to be the image of  $S$  via both projections.

An additional axiom is provided to restrict to these monoids were the induced two-sided natural order is stable with respect to the product.

Following this point of view, the resulting class of monoids can be seen as a class of *quasi-inverse* monoids in the sense that, even in the absence of inverses themselves,  $x_L$  *essentially* behaves like  $x^{-1}x$  and  $x_R$  *essentially* behaves like  $xx^{-1}$ . Such a *metaphor* considerably helps developing intuition on the way these canonical local units behave.

As such, the class of quasi-inverse monoids generalizes the class of inverse, regular, ample or adequate monoids as well as even larger classes such as (two sided) restriction monoids [4, 1]. In fact, such a study of classes of monoids (or semigroups) that *essentially* behave like inverse monoid without the inverses themselves already started with Fountains' notion of ample and adequate monoids [2, 3]. Our proposal, which justification comes from a language theoretical point of view, is just another one. It inherits from the many studies and proposal already conducted since Fountains' pioneered work.

The second axiomatic definition is based on partially ordered monoid. More precisely, we consider these monoids which subunits, elements smaller than the unit, are commuting idempotents and where the order relation is the two-sided Nambooripad natural order [12] induced by these subunits. Requiring moreover that the left (and the right) stabilizer of every element contain a least subunits, we eventually capture the class of Lawson's  $U$ -semiadequate monoids [7], where  $U$  is the set of subunits.

In the case the order is stable, commutation of idempotent subunits can be dropped (as it is enforced by stability hypothesis) and this leads to a characterization of stable  $U$ -semiadequate monoids.

In both general or stable case, the fact we are dealing with monoids instead of semigroups also make a somehow subtle difference with Lawson's definition [7]. Indeed, it is striking that in a  $U$ -semiadequate monoid  $M$  ordered by the natural order, set  $U$  equals the set of subunits of  $M$ , i.e. those elements that are smaller than or equal to 1. It follows that our quasi-inverse monoids, already known as  $U$ -semi adequate monoids, can just be called semiadequate ordered monoids or, following the new naming convention of the *York school* founded by Fountains, *weakly adequate ordered monoids*.

Various examples of quasi-inverse monoids are provided in order to illustrate these definitions. Of course, a major source of examples comes from taking submonoids of inverse monoids and closing them under left projection mapping  $x \mapsto x_L = x^{-1}x$  and right projection mapping  $x \mapsto x_R = xx^{-1}$ . These are stable inverse monoids.

Another example, rather unexpected, is the relation monoid  $\mathcal{P}(Q \times Q)$  where an order, coarser than inclusion order, turns it into a non stable quasi-inverse monoid with partial bijections defining all subunits.

The second stage of our proposal amounts to define the adequate premorphism that behaves nicely on quasi-inverse monoids. The main concern is that, aiming at defining languages by means of premorphism inverse images, we need to ensure these inverse images remains simple.

Does there exist any non trivial admissible monoids and premorphisms? An expansion, in the sense of Birget and Rhodes [13], is provided. It uniformly maps every monoid  $M$  (resp. morphism  $\varphi$ ) to a non trivial admissible quasi-inverse monoid  $\mathcal{Q}(M)$  (resp. non trivial admissible premorphism  $\mathcal{Q}(\varphi)$ ). This last result constitute the corner stone of our proposal.

## More related works

Although  $U$ -semiadequate are defined in [7] it appears that there and in following studies, the emphasis is put on  $U$ -semiadequate semigroups that moreover satisfies the congruence property, i.e.  $U$ -semiadequate semigroups where both  $\tilde{\mathcal{L}}$  and  $\tilde{\mathcal{R}}$  are congruences.

In our study, we are rather concerned with stable quasi-inverse monoids, i.e.  $U$ -semiadequate monoids where the induced (two-sided) natural order is stable under product. It follows that we do not need the congruence property. Moreover, our main construction  $\mathcal{Q}(M)$  leads to quasi-inverse monoids

that do not satisfy that congruence property. It follows that, despite a tight relationship, our work still diverges from previous studies.

Our expansion construction has many similarities with constructions *à la* McAlister, studied and developed further for instance in Cornock PhD [1]. However, the product in an expansion just seems unrelated with the products defined there. Despite its simplicity and robustness, despite the richness of the underlying field, it seems (so far) that our construction is still original. Anyhow, as  $\mathcal{Q}(S)$  still looks like a sort of a double semi-direct product between  $S$  and the two lattices  $\mathcal{U}_s(S)$  and  $\mathcal{U}_p(S)$  of left and right ideals of  $S$  with intersection as product, a potential link with formerly defined constructions need to be further investigated.

## Notations

In the remainder of the text, given a monoid  $S$  with neutral written 1 and (possibly) absorbant element written 0, we write  $\leq_p$  for the prefix preorder and  $\leq_s$  for the suffix preorder, defined, for all  $x$  and  $y \in S$  by  $x \leq_p y$  when  $xz = y$  for some  $z \in S$  and  $x \leq_s y$  when  $zx = y$  for some  $z \in S$ .

Observe that, under both prefix and suffix preorder, 1 is the least element of  $S$  and 0 (if ever) is the greatest.

We also write  $x^{-1}(y) = \{z \in S : xz = y\}$  and  $(y)x^{-1} = \{z \in S : zx = y\}$ . This notation extend to sets as follows. For all  $x \in S$  and  $Y \subseteq S$  we write  $x^{-1}(Y) = \{z \in S : xz \in Y\}$  and  $(Y)x^{-1} = \{z \in S : zx \in Y\}$ .

## 2 Quasi-inverse monoids

In this section, we define quasi-inverse monoid that can be seen as almost inverse monoids where we only require, for all element  $x$ , the existence of a left context  $x_L$  behaving “like”  $x^{-1}x$  and a right context  $x_R$  behaving “like”  $xx^{-1}$ .

### 2.1 Axiomatic definition

We give here the axiomatic definition of quasi-inverse monoid and study some of its immediate properties.

**Definition.** A *quasi-inverse monoid* is a monoid  $S$  equipped with two mappings  $x \mapsto x_L$  and  $x \mapsto x_R$ , called the left and right context mappings, such for all  $x$  and  $y \in S$ :

- (Q0)  $(x_L y_L)_L = (x_L y_L)_R = x_L y_L$  and  $(x_R y_R)_L = (x_R y_R)_R = x_R y_R$ ,
- (Q1)  $x_L x_L = x_L$  and  $x_R x_R = x_R$ ,
- (Q2)  $x x_L = x$  and  $x_R x = x$ ,
- (Q3)  $(xy)_L y_L = (xy)_L$  and  $x_R (xy)_R = (xy)_R$ ,
- (Q4)  $x_L y_L = y_L x_L$  and  $x_R y_R = y_R x_R$ ,

**Lemma 1** *In a quasi-inverse monoid  $S$ , we have  $1_L = 1_R = 1$ .*

*Proof.* By axiom (Q2) we have  $1 \cdot 1_L = 1$  hence  $1_L = 1$  since 1 is neutral. By symmetry we also have  $1_R = 1$ .  $\square$

**Lemma 2** *Let  $S$  be a quasi-inverse monoid, let  $C_L(S) = \{x_L \in S : x \in S\}$  and let  $C_R(S) = \{x_R \in S : x \in S\}$ . We have  $C_L(S) = C_R(S)$  from now on written  $C(S)$ .*

*Moreover,  $C(S)$  is a commutative submonoid of idempotents of  $S$  with, for all  $x \in C(S)$ ,  $x_L = x_R = x$ , i.e. left and right context operators are identities over context elements.*

*Proof.* Assume  $S$  is a quasi-inverse monoid. By Lemma 1 we have  $1_R = 1_L = 1$ . It follows that, by (Q0) taking  $y = 1$ , for all  $x \in C_R(S)$  (resp.  $x \in C_L(S)$ ) one has  $x_R = x_L = x$  hence  $x \in C_L(S)$  (resp.  $x \in C_R(S)$ ). Then closure of  $C(S)$  follows from (Q0) again. Idempotency is provided by (Q1). Commutativity is provided by (Q4).  $\square$

The importance of axiom (Q3) becomes clear in Lemma 14 below.

**Lemma 3** *Let  $S$  be a quasi-inverse monoid. If  $0 \in S$  then  $0_R = 0_L$  from now on written  $\perp$  and, for all  $x \in C(S)$ ,  $\perp x = x \perp = \perp$ .*

*Proof.* Assume  $0 \in S$ . By (Q3) for all  $y \in S$ , we have  $(0y)_L y_L = (0y)_L$  hence, for all  $y \in S$ ,  $0_L y_L = 0_L$ . By symmetry, for all  $y \in S$  we also have  $y_R 0_R = 0_R$ . Now by Lemma 2, taking  $y = 0_L$  or  $y = 0_R$  we deduce that  $0_R = 0_L$  with the announced property within context element.  $\square$

**Remark.** We observe first that  $S = \{1, 0\}$  with  $0_L = 0_R = 0$  is a quasi-inverse monoid. We observe also that  $S = \{1, 0, \perp\}$  with  $0_L = 0_R = \perp$  is also a quasi-inverse monoid. This suggests that taking  $\perp = 0$  or  $\perp \neq 0$  leads to two types of quasi-inverse monoids with zero. This remark is however not developed further.



**Lemma 4** *Let  $S$  be a quasi-inversive monoid  $S$  and let  $x$  and  $y \in S$ . If  $x \leq_s y$  (resp.  $x \leq_p y$ ) then  $yx_L = y$  (resp.  $x_R y = y$ ).*

*If both  $x \leq_s y$  and  $y \leq_s x$ , i.e.  $x$  and  $y$  are  $\mathcal{L}$ -equivalent, (resp. both  $x \leq_p y$  and  $y \leq_p x$ , i.e.  $x$  and  $y$  are  $\mathcal{R}$ -equivalent) then  $x_L = y_L$  (resp.  $x_R = y_R$ ).*

*Proof.* Let  $x$  and  $y$  as above. By symmetry, we only prove the suffix case. Assume  $x \leq_s y$  henceforth  $y = xz$ . By (Q2) on  $y$  we have  $yx_L = yy_L x_L$ , hence by (Q3)  $yx_L = yy_L$  hence by (Q2) again,  $yx_L = y$ .

If both  $x \leq_s y$  and  $y \leq_s x$  we have, by (Q3), both  $y_L x_L = y_L$  and  $x_L y_L = x_L$  hence  $x_L = y_L$  since, by (Q4), context elements commute.  $\square$

## 2.2 Trivial and non trivial Quasi-inverse monoids

One may ask if, when  $S$  is a quasi-inverse monoid, the left and right context mappings are uniquely determined. The answer is no in general as shown by the following definition.

**Definition.** Let  $S$  be an arbitrary monoid. Let  $S^0 = S + 0$  be the *trivial extension* of monoid  $S$  with a *new* zero element and the mappings  $x \mapsto x_L$  and  $x \mapsto x_R$  defined by  $0_L = 0_R = 0$  and, for all  $x \in S$ ,  $x_L = x_R = 1$ .

**Lemma 5** *The trivial extension  $S^0$  of a monoid  $S$  is a quasi-inverse monoid.*

*Proof.* Straightforward.  $\square$

Before developing the notion of non trivial quasi-inverse monoid, we must say that, as proved in Lemma 10, as soon as  $C(S)$  is known, then both mappings  $x \mapsto x_L$  and  $x \mapsto x_R$  are uniquely determined.

The notion of trivial extension of a monoid and the fact that, in particular, inverse monoid are quasi-inverse monoid leads us to the following definition.

**Definition.** A quasi-inverse monoid  $S$  is said *non trivial* when, for every  $x \in S$ , if  $x_L = 1$  (resp. if  $x_R = 1$ ) then there is  $y \in S$  such that  $yx = 1$  (resp.  $xy = 1$ ), i.e.  $x$  has a left (resp. right) group inverse.

**Remark.** We observe that, in particular, in a non trivial monoid with zero, one must have  $0_L = 0_R = \perp < 1$ .

**Theorem 6** *An inverse monoid is a non trivial quasi-inverse monoid.*

*Proof.* Let  $S$  be an universe monoid. Recall that this means  $S$  is a monoid such that for all  $x \in S$  there is a unique elements  $x^{-1} \in S$ , called the pseudo inverse of  $x$ , such that  $xx^{-1}x = x$  and  $x^{-1}xx^{-1} = x^{-1}$ .

In particular, observe that for all  $x \in S$  both  $x^{-1}x$  and  $xx^{-1}$  are idempotent. Denoting by  $E(S)$  the set of idempotent of  $S$ , it is well-known that  $E(S)$  is a commutative monoid. Even more, it can be shown [8] that idempotence commutation hypothesis is actually equivalent with the unicity of pseudo inverses.

Let us show that  $S$  is quasi-inverse. In order to do so, for every  $x \in S$ , let us take  $x_L = x^{-1}x$  and  $x_R = xx^{-1}$ . Since for all  $x$  and  $y \in S$  one has  $(xy)^{-1} = y^{-1}x^{-1}$  all axioms of quasi-inverse monoid are immediate.

For instance, for axiom (Q3), given  $x$  and  $y \in S$ , one has  $(xy)_L y_L = y^{-1}x^{-1}xyy^{-1}y$  hence by commutation  $(xy)_L y_L = y^{-1}yy^{-1}x^{-1}xy$  hence because  $y^{-1}yy^{-1} = y^{-1}$   $(xy)_L y_L = (xy)_L$ .

The fact it is non trivial then immediately follows from the definition since, for all  $x$ , if  $x_L = 1$  then  $x^{-1}x = 1$  and thus  $x^{-1}$  is a left group inverse of  $x$  and if  $x_R = 1$  then  $xx^{-1} = 1$  and thus  $x^{-1}$  is a right group inverse of  $x$ .  $\square$

**Remark.** Observe that when  $S$  is an inverse monoid with left and right context operators defined as above, for all  $x$  and  $y \in S$ , if  $x_L y = y$  (resp.  $x_R y = y$ ) then  $x \leq_s y$  (resp.  $x \leq_p y$ ). In other words, an inverse monoid also satisfies the converse of the property stated in Lemma 4. For quasi-inverse monoids, even non trivial ones, this is no longer true as illustrated, in particular, by quasi-inverse extensions  $\mathcal{Q}(S)$  of arbitrary monoids  $S$  defined in Section 5.

The following theorem shows how inverse monoids generate many quasi-inverse monoids.

**Theorem 7** *Let  $S$  be an inverse monoid, let  $X \subseteq S$  be a subset of  $S$  and let  $\langle X \rangle_{\mathcal{Q}}$  be the submonoid of  $S$  induced by  $X$  that is moreover closed under both left and right projection mapping  $x \mapsto x_L = x^{-1}x$  and  $x \mapsto x_R = xx^{-1}$ . Then  $\langle X \rangle_{\mathcal{Q}}$  is a quasi-inverse monoid.*

*Proof.* The proof is essentially the same as the proof of Theorem 6. This illustrates the fact that, indeed, inverses themselves are not needed in quasi-inverse monoids.  $\square$

### 2.3 $U$ -semiadequate monoids

We relate here quasi-inverse monoids with  $U$ -semiadequate semigroups as defined in [7].

**Definition.** Let  $S$  be a semigroup and let  $U \subseteq E(S)$  be a subset of idempotents of  $S$ . Extended Green relations  $\tilde{\mathcal{L}}$  and  $\tilde{\mathcal{R}}$  are defined from  $U$ , for all  $x$  and  $y \in S$ , by:

$$x\tilde{\mathcal{L}}y \text{ if for all } e \in U, xe = x \Leftrightarrow ye = y,$$

$$x\tilde{\mathcal{R}}y \text{ if for all } e \in U, ex = x \Leftrightarrow ey = y.$$

One can easily check that relation  $\tilde{\mathcal{L}}$  (resp.  $\tilde{\mathcal{R}}$ ) is a left (resp. right) congruence relation that generalizes Green's relation  $\mathcal{L}$  (resp.  $\mathcal{R}$ ). Moreover, for all  $x \in S$  and  $y \in U$ , if  $y\tilde{\mathcal{L}}x$  (resp.  $y\tilde{\mathcal{R}}x$ ) then, since  $y$  is idempotent,  $yy = y$  hence, by definition,  $xy = x$  (resp.  $yx = x$ ).

**Definition.** A semigroup  $S$  is a  $U$ -semiabundant when for all  $x \in S$  there is at least one  $x^+ \in U$  such that  $x^+\tilde{\mathcal{R}}x$  and one  $x^* \in U$  such that  $x^*\tilde{\mathcal{L}}x$ .

A semigroup  $S$  is  $U$ -semiadequate when it is  $U$ -semiabundant and  $U$  is a commutative submonoid of  $S$ .

**Lemma 8** *A quasi-inverse monoid  $S$  is a  $C(S)$ -semiadequate monoid.*

*Proof.* Let  $S$  be a quasi-inverse monoid. Let  $x \in S$ .

Let  $y \in C(S)$ . Since  $xx_L = x$ , if  $x_Ly = x_L$  then  $xy = xx_Ly = xx_L = x$ . Conversely, if  $xy = x$  then, by axiom (Q3),  $(xy)_Ly_L = (xy)_L$  with, by axiom (Q0),  $y_L = y$ , hence  $x_Ly = x_L$ .

In other words, we have  $x\tilde{\mathcal{L}}x_L$  and, by symmetrical arguments,  $x\tilde{\mathcal{R}}x_R$ .

Assume now there is some  $y \in C(S)$  such that  $y\tilde{\mathcal{L}}x$ . This means in particular, since, by axiom (Q2),  $xx_L = x$ , that  $yx_L = y$  hence  $y \leq x_L$ . But, as observed above, since  $y$  is idempotent, we have  $yy = y$  hence, by definition of  $\tilde{\mathcal{L}}$ ,  $xy = x$ .  $\square$

The following properties is proved in [7]:

**Lemma 9** *In a  $U$ -semiadequate semigroup each  $\tilde{\mathcal{L}}$ -class (resp.  $\tilde{\mathcal{R}}$ -class) has a unique element of  $U$ .*

*Proof.* Follows from the facts (see [7] Lemma1.3) that, in a  $U$ -semiadequate semigroup Green relation  $\mathcal{L}$  (resp.  $\mathcal{R}$ ) and extended Green  $\tilde{\mathcal{L}}$  (resp.  $\tilde{\mathcal{R}}$ ) are equal over  $U$  and the fact that, over commuting idempotents,  $\mathcal{L}$  and  $\mathcal{R}$  relations are just equality.  $\square$

**Lemma 10** *A  $U$ -semiadequate monoid  $S$  is a quasi-inverse monoid taking, for all  $x \in S$ ,  $x_L = x^+$ ,  $x_R = x^*$ .*

*Proof.* Let  $S$  be a  $U$ -semiadequate monoid. For all  $x \in S$ , we put  $x_L = x^+$  and  $x_R = x^*$ .

Observe first, since  $1 \in S$ , that  $1^* = 1^+ = 1$  henceforth  $1 \in U$ . Indeed, given  $1^* \in U$  we have  $1^*1 = 1$  hence  $1^* = 1$  since 1 is neutral. By symmetrical argument, we have  $1^+ = 1$ .

By hypothesis,  $U$  is a commutative subsemigroup (hence a submonoid) of idempotents with, by Lemma 9, for all  $x \in U$ ,  $x^* = x^+ = x$ , hence axioms (Q0), (Q1) and (Q4) of quasi-inverse monoids are satisfied.

As we have already observed that axiom (Q2) is satisfied it remains now to prove axiom (Q3). Let  $x$  and  $y \in S$ . Since  $yy^+ = y$  we also have  $(xy)y^+ = (xy)$  hence  $(xy)^+y^+ = (xy)^+$  since  $(xy)^+\tilde{\mathcal{L}}xy$ . A symmetrical argument shows that  $x^*(xy)^* = (xy)^*$ .  $\square$

**Corollary 11** *Quasi-inverse monoids and  $U$ -semiadequate monoids are equivalent notions.*

## 2.4 Examples of quasi-inverse monoids

Let  $Q$  be a set and let  $\mathcal{P}(Q \times Q)$  be the relation monoid with product defined, for every  $X$  and  $Y \subseteq Q \times Q$  by  $XY = \{(p, q) \in Q \times Q : \exists r \in Q, (p, r) \in X, (r, q) \in Y\}$ . For every  $X \subseteq Q \times Q$ , let also  $X_L = \{(q, q) \in Q \times Q : \exists p \in Q, (p, q) \in X\}$  and  $X_R = \{(p, p) \in Q \times Q : \exists q \in Q, (p, q) \in X\}$ .

**Theorem 12** *Monoid  $\mathcal{P}(Q \times Q)$  equipped with left and right projection defined above is a quasi-inverse monoid.*

*Proof.* Let  $U = \{X \subseteq Q \times Q : X \subseteq I_Q\}$  with  $I_Q = \{(q, q) \in Q \times Q : q \in Q\}$  the identity relation. It shall be clear that  $U = \{X_L : X \subseteq Q \times Q\} = \{X_R : X \subseteq Q \times Q\}$  and, for every  $X$  and  $Y \in U$ ,  $XY = X \cap Y$ . This proves that axioms (Q0), (Q1) and (Q4) are satisfied. Axiom (Q2) is an immediate consequence of definition. Last, axiom (Q3) follows from the fact that for every relation  $X$  and  $Y \subseteq Q \times Q$ , we obviously have  $(XY)_L \subseteq Y_L$  and  $(XY)_R \subseteq X_R$ .  $\square$

**Remark.** Axiom (Q5) presented below that is equivalent to the fact the natural order is stable is not satisfied by  $\mathcal{P}(Q \times Q)$  as shown by the following example.

Let  $Q = \{1, 2, 3\}$ , let  $X = \{(1, 1), (2, 2), (1, 3)\}$ , let  $Y = \{(1, 1), (2, 2), (3, 2)\}$  and let  $Z = \{(1, 1), (2, 2)\}$ . By construction  $XY = \{(1, 1), (2, 2), (1, 2)\}$ ,  $XZY = \{(1, 1), (2, 2)\} = Z$  and  $(XZY)_L = (XZY)_R = Z$ . Now, since  $Z(XY)Z = XY$  it follows that  $(XZY)_LXY(XZY)_R = XY \neq XZY$ .

### 3 Naturally ordered quasi-inverse monoids

We give here the definition of the natural order in a quasi-inverse monoid and review some properties of that order.

#### 3.1 Natural order

We define here, over quasi-inverse monoids, the analogous of the natural order in inverse monoid.

**Definition.** In a quasi-inverse monoid  $S$ , let  $\leq$  be the natural relation defined, for every  $x$  and  $y \in S$  by  $x \leq y$  when  $x = x_R y x_L$ .

**Lemma 13** *The relation  $\leq$  in a quasi-inverse monoid  $S$  is an order relation. When  $S$  is an inverse monoid, it coincides with Nambooripad natural order in  $S$ .*

*Proof.* Relation  $\leq$  is obviously a preorder, i.e. reflexive and transitive. Let us show it is anti-symmetric. Assume  $x \leq y$ . By definition,  $x = x_R y x_L$ . By left product with  $y_R$  and right product with  $y_L$  we have  $y_R x y_L = y_R x_R y x_L y_L$  hence by context element commutation  $y_R x y_L = x_R (y_R y y_L) x_L$  hence  $y_R x y_L = x_R y x_L$ . Assume now  $y \leq x$ , this means  $y = y_R x y_L$  hence  $x = y$ .

When  $S$  is inverse, we now that  $x = x_R y x_L$ ,  $x = x_R y$ ,  $x = y x_L$ ,  $x = e y$  for some  $e \in E(S)$  and  $x = y f$  for some  $f \in E(S)$  are all equivalent properties, the last two being Nambooripad [12] natural orders definitions.  $\square$

**Remark.** In general, it is not true that, in a quasi-inverse monoid, if  $x \leq y$  then for all  $z \in S$ ,  $xz \leq yz$  and  $zx \leq zy$ , i.e. the natural order is not necessarily stable under product. The missing axiom (Q5) is provided in Section 4.1.

**Lemma 14** *Let  $S$  be a quasi-inverse monoid. The set  $C(S)$  of context elements ordered by natural order is a meet semi-lattice with 1 as maximum element,  $\perp = 0_R = 0_L$  as minimum element in the case  $0 \in S$ , and for all  $x$  and  $y \in C(S)$ ,  $x \wedge y = xy$ .*

*Proof.* Let  $x \in C(S)$ . By Lemma 2 one has  $x_L = x_R = x$  and, by (Q1),  $xx = x$  hence  $x = x_R 1 x_L$  hence  $x \leq 1$ .

Assume  $0 \in S$ . By Lemma 3, let  $\perp = 0_L = 0_R$ . For all  $x \in C(S)$  we have, again by Lemma 3,  $\perp = 0_R x 0_L$  hence  $\perp \leq x$ .

Let us prove  $C(S)$  is a meet semi-lattice with product as meet operator. In order to do so, let  $x$  and  $y \in C(S)$ . Observe first that, by commutation and idempotence of context elements we do have  $xy \leq x$  and  $xy \leq y$ . Conversely, let  $z \in C(S)$  such that (a)  $z \leq x$  and (b)  $z \leq y$ . By (a) we have  $zxy = zy$  hence by (b)  $zxy = z$  hence, by commutation of contexts and idempotence of  $z$ ,  $z = zxyz$  hence  $z \leq xy$ . Altogether, this means that  $x \wedge y$  is defined with  $x \wedge y = xy$ .  $\square$

**Remark.** In particular, for all  $x$  and  $y \in C(S)$ ,  $x \leq y$  if and only if  $xy = x$ .

**Lemma 15** *When  $S$  is a quasi-inverse monoid the set  $C(S)$  of context elements of  $S$  equals the set  $U(S) = \{x \in S : x \leq 1\}$  of subunits of  $S$  ordered by the (quasi-inverse) natural order.*

*Proof.* Lemma 14 above already tells us that  $C(S) \subseteq U(S)$ . Let thus  $x \in S$  such that  $x \leq 1$ . By definition of the quasi-inverse natural order this means that  $x = x_R x_L$  hence, since  $C(S)$  is a submonoid of  $S$  (Lemma 2),  $x \in C(S)$ .  $\square$

**Lemma 16** *When  $S$  is a quasi-inverse monoid, for all  $x$  and  $y \in S$ , if  $x \leq y$  then  $x_R \leq (yx_L)_R \leq y_R$  and  $x_L \leq (x_R y)_L \leq y_L$ ,*

*Proof.* Let  $x$  and  $y \in S$  with  $S$  quasi-inverse.

Assume  $x \leq y$ . By symmetry, we only need to prove  $x_R \leq (yx_L)_R \leq y_R$ .

Since  $x \leq y$  this means that  $x = x_R y x_L$  hence  $y x_L \leq_s x$  hence, by (Q3),  $x_R (y x_L)_R = x_R$  that is to say  $x_R \leq (y x_L)_R$ . But we also have  $y_R (y x_L) = y x_L$  hence  $y_R \leq_p y x_L$  hence, by (Q3) again,  $(y x_L)_R y_R = (y x_L)_R$ .  $\square$

**Remark.** One may ask about the converse. Assuming both  $x_L \leq y_L$  and  $x_R \leq y_R$  could we have  $x \leq y$  for any  $x$  and  $y \in S$ ? By Lemma 4 this would mean that whenever  $x$  and  $y \in S$  are  $\mathcal{H}$ -equivalent, i.e. both  $\mathcal{R}$  and  $\mathcal{L}$ -equivalent, then they are equal. This will not be true in general as already illustrated by inverse monoids.

### 3.2 Natural order revisited

We show here that the natural order definition can be further simplified as a two-sided variation of Nambooripad natural order [12] restricted to idempotents of  $C(S)$ .

**Lemma 17** *When  $S$  is a quasi-inverse monoid, for every  $e$  and  $f \in C(S)$  and  $x \in S$ ,  $exf \leq x$ .*

*Proof.* Let  $S$  be a quasi-inverse monoid. Let  $e$  and  $f \in C(S)$  and  $x \in S$ . Observe first that  $(exf)_{Re} = (exf)_R$ . In fact, we have  $e \leq_p exf$  hence, by property (Q3) of quasi-inverse monoid,  $(exf)_R(e)_R = (exf)_R$ . But, by Lemma 2 we have  $(e)_R = e$  hence the claim. Symmetrically, we have  $f(exf)_L = (exf)_L$ . Altogether,  $(exf)_{Rx}(exf)_L = (exf)_R exf (exf)_R$  and thus  $exf \leq x$ .  $\square$

It follows that:

**Corollary 18** *In a quasi-inverse monoid  $S$ , for all  $x$  and  $y \in S$ ,  $x \leq y$  if and only if there exists two context elements  $e$  and  $f \in C(S)$  such that  $x = eyf$ .*

**Remark.** We can also define two other orders in a quasi-inverse monoid by saying  $x \leq_L y$  (resp.  $x \leq_R y$ ) when there exists some  $e \in C(S)$  such that  $x = ye$  (resp.  $x = ey$ ).

It occurs however that these orders are both weaker than the natural order defined above, i.e.  $(\leq_L \cup \leq_R) \subseteq \leq$  and the inclusion can be strict as shown by  $\mathcal{Q}(S)$  defined below in Section 5.

As already observed in [7] for  $U$ -semiadequate semigroups, we have:

**Lemma 19** *In a quasi-inverse monoid  $\leq = (\leq_L \circ \leq_R) = (\leq_R \circ \leq_L)$  where  $\circ$  denotes relation composition.*

*Proof.* We obviously have  $\leq \subseteq (\leq_L \circ \leq_R)$  since whenever  $x = eyf$  then  $x = zf$  with  $z = ey$ . Conversely, both  $\leq_L \subseteq \leq$  and  $\leq_R \subseteq \leq$  as particular instance of  $\leq$  definition, hence, by transitivity  $(\leq_L \circ \leq_R) \subseteq \leq$ .  $\square$

## 4 Stable quasi-inverse monoid

Quasi-inverse monoids for which the natural is a stable order are characterized by means of an additional axiom (Q5). We also provide another complete axiomatization of stable quasi-inverse monoids when seen as ordered monoids.

## 4.1 The missing axiom

We provide here the missing axiom for natural order on quasi-inverse monoid to be stable by product.

**Definition.** A quasi-inverse monoid is called *stable* when for all  $x, y \in S$ , the following property is satisfied:

$$(Q5) \text{ for all } z \in C(S), (xzy)_Rxy(xzy)_L = xzy.$$

**Lemma 20** *A quasi-inverse monoid is stable (as a quasi-inverse monoid) if and only if, ordered by the (quasi-inverse) natural order, it is a (stable) ordered monoid.*

*Proof.* Let  $S$  be a quasi-inverse monoid. Observe that axiom (Q5) essentially says that for all  $x$  and  $y \in S$ , for all  $z \in C(S)$ ,  $xzy \leq xy$ .

If  $S$  ordered by the natural order is a (stable) ordered monoid then axiom (Q5) is obviously satisfied since  $z \leq 1$ .

Conversely assume (Q5) holds in  $S$ . Let  $x, x' \in S$  such that  $x \leq x'$  and let  $y \in S$ .

Since  $x = x_Rx'_L$  we have  $xy = x_Rx'_Ly$ . By Lemma 17 this implies that  $xy \leq x'_Ly$  hence by axiom (Q5),  $xy \leq x'y$ .

By symmetrical arguments, we can prove that  $yx \leq yx'$ .

□

## 4.2 Well-behaved ordered monoid

The following definition is adapted from [6]. It occurs that when the natural order of a quasi-inverse monoid is stable then it enjoys a rather simple axiomatization.

**Definition.** A monoid  $S$  is a *well-behaved ordered monoid* when it is equipped with an order relation  $\leq$  such that:

$$(W0) \text{ for all } x, y \text{ and } z \in S, \text{ if } x \leq y \text{ then } xz \leq yz \text{ and } zx \leq zy, \\ \text{i.e. the order relation } \leq \text{ is } \textit{stable} \text{ under product,}$$

and given the set  $U(S) = \{x \in S : x \leq 1\}$  of *subunits* of  $S$ :

$$(W1) \text{ for all } x \in U(S), xx = 1, \text{ i.e. subunit elements are idempotents,}$$

$$(W2) \text{ for all } x \text{ and } y \in S, \text{ if } x \leq y \text{ then there is } e \text{ and } f \in U(S) \\ \text{such that } x = e y f, \text{ i.e. the order relation is a natural order,}$$



(W3) for all  $x \in S$ , both sets  $L_x = \{e \in U(S) : xe = x\}$  and  $R_x = \{e \in U(S) : ex = x\}$  have least element,

**Lemma 21** *Let  $S$  be a stable quasi-inverse monoid with natural order  $\leq$ . Monoid  $S$  ordered by  $\leq$  is a well-behaved ordered monoid.*

*Proof.* Axiom (W0) follows from axiom (Q5) and Lemma 20.

Axiom (W1). Assume there is some  $x \in S$  such that  $x \leq 1$ . This means, by definition of the natural order, that  $x = x_L x_R$  hence by Lemma 2,  $x_L = x$  (and  $x_R = x$ ) hence by axiom (Q1)  $x$  is idempotent.

Axiom (W2) follows from the definition of the natural order and the fact that  $C(S) = U(S)$ .

Axiom (W3). By symmetry, we only prove the left case. Let  $y \in U(S)$  such that  $xy = x$ . We have to show that  $x_L \leq y_R$  or, equivalently by Lemma 14, that  $x_L y_L = x_L$ . But, by (Q3) we have  $(xy)_L y_L = (xy)_L$  with  $(xy)_L = x_L$  since  $xy = x$  hence  $x_L y_L = x_L$ .  $\square$

**Lemma 22** *Let  $S$  be a well-behaved ordered monoid  $S$  with order  $\leq$  and subunits  $U(S)$ . Defining, for all  $x \in S$ ,  $x_L = \bigwedge \{y \in U(S) : xy = x\}$  and  $x_R = \bigwedge \{y \in U(S) : yx = x\}$  turns  $S$  into a stable quasi-inverse monoid.*

*Proof.* Axiom (Q4) follows from stability (W0) and idempotence of subunits (W1).

Axioms (Q0) to (Q2) then immediately follows from axiom (W1) and (W3).

Axiom (Q3) follows from (W'') and the observation that for all  $x$  and  $y \in S$  and for all  $z \in U(S)$ , if  $zx = x$  (resp.  $yz = y$ ) then  $zxy = xy$  (resp.  $xyz = xy$ ).

Last, axioms (Q5) follows from Lemma20.  $\square$

In other words, quasi-inverse monoid and well-ordered monoid are equivalent algebraic structures.

### 4.3 Stable $F^*$ -property

We consider a stronger version of the  $F^*$ -property (see [5]) where we assume moreover that maximal elements of a  $F^*$ -quasi-inverse monoids form a submonoid.

**Definition.** An quasi-inverse monoid is said *stably  $F^*$*  when:

- (1) any non zero element  $x \in S$  lies beneath a unique maximal element  $\hat{x} \in S$ ,

- (2) for all non zero  $x$  and  $y \in S$ ,  $\widehat{xy} \neq 0$  and  $\widehat{xy} = \widehat{x}\widehat{y}$ , i.e. the set  $\widehat{S}$  of maximal elements of  $S$  is a submonoid of  $S$ .

**Remark.** Observe that the trivial extension  $S^0$  of an arbitrary monoid  $S$  is a stably  $F^*$ -quasi-inverse monoid. In Section 5 we will prove that this is true as well for the non trivial extension  $\mathcal{Q}(S)$  of  $S$ .

Applied to inverse monoid, the stably  $F^*$ -property induces a rather strong additional property.

**Lemma 23** *Let  $S$  be an inverse monoid ordered by natural order. If  $S$  is stably  $F^*$  then  $\widehat{S}$  is a group.*

*Proof.* Let  $x \in \widehat{S}$ . We have  $x^{-1} \in \widehat{S}$ . Otherwise, since  $x^{-1} \neq 0$ , by  $F^*$  property,  $x^{-1} < \widehat{x^{-1}}$  and thus, as the natural order is stable by inverse mapping,  $x < (\widehat{x^{-1}})^{-1}$  which contradicts the fact that  $x \in \widehat{S}$ .

We conclude then by stability assumption of  $\widehat{S}$ . Indeed, this means that both  $xx^{-1}$  and  $x^{-1}x \in \widehat{S}$  hence, since both  $x^{-1}x \leq 1$  and  $xx^{-1} \leq 1$ , by  $F^*$ -assumption,  $xx^{-1} = x^{-1}x = 1$ .  $\square$

**Remark.** Though fairly simple, it seems that this fact was left unnoticed. This illustrates again the well-known fact that semigroup theories and related monoid theories sometimes slightly differs and the study presented here is concerned with monoids rather than semigroups.

## 5 A non trivial quasi-inverse extension

In this section, we define from arbitrary monoid  $S$ , sort of a monoid of positive tiles  $\mathcal{Q}(S)$  much in the same way the monoid  $T_A$  of positive tiles [6] over the alphabet  $A$  is been built upon  $A^*$  with a canonical injection from  $S$  to  $\mathcal{Q}(S)$ . It turns out that  $\mathcal{Q}(S)$  is a non trivial quasi-inverse monoid.

### 5.1 Prefix and suffix upper sets

Let  $S$  be a monoid. Let  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) defined to be the set of upward closed subsets of  $S$  preordered by  $\leq_p$  (resp.  $\leq_s$ ).

More precisely, as  $S$  is a monoid hence with  $1 \in S$ ,  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) is the set  $U \subseteq S$  such that  $US = U$  (resp.  $SU = U$ ).

For both  $x = p$  or  $x = s$ , elements of  $\mathcal{U}_x(S)$  are from now on called  $x$ -upper set. The set  $\mathcal{U}_x(S)$  is turned into a monoid by taking  $\cap$  as product. One can check that, indeed, the intersection of two  $x$ -upper sets is indeed a

upper set. The neutral (or maximal) element is then  $S$  itself the absorbant is the empty set  $\emptyset$ .

**Remark.** In semigroup theory non empty elements of  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) are sometimes called right ideals (resp. left ideals) of  $S$ . As ideals in order theory must satisfy some extra condition we prefer to stick to the notion of upper sets.

**Lemma 24** *Let  $S$  be some monoid and let  $x \in S$ . One have:*

(0)  $xS$  is a  $p$ -upper set (resp.  $Sx$  a  $s$ -upper set).

Moreover, for every  $p$ -upper set (resp.  $s$ -upper set)  $U \subseteq S$ :

(1) if  $x \in U$  then  $x^{-1}(U) = S$  (resp.  $(U)x^{-1} = S$ ),

(2)  $xU$  is a  $p$ -upper set (resp.  $Ux$  is a  $s$ -upper set),

(3)  $x^{-1}(U)$  is a  $p$ -upper set (resp.  $(U)x^{-1}$  is a  $s$ -upper sets),

(4)  $xx^{-1}(U) \subseteq U \subseteq x^{-1}(xU)$  (resp.  $(U)x^{-1}x \subseteq U \subseteq (Ux)x^{-1}$ ),

*Proof.* Straightforward. □

## 5.2 The extension

**Definition.** Let  $S$  be a monoid. The quasi-inverse extension of  $S$ , written  $\mathcal{Q}(S)$  is defined to be

$$\mathcal{Q}(S) = (\mathcal{U}_s(S) - \emptyset) \times S \times (\mathcal{U}_p(S) - \emptyset) + 0$$

with product defined by

$$(L_1, x_1, R_1) \cdot (L_2, x_2, R_2) = (L_1 \cap (L_2)x_1^{-1}, x_1x_2, R_2 \cap x_2^{-1}(R_1))$$

when both  $L_1 \cap (L_2)x_1^{-1} \neq \emptyset$  and  $R_2 \cap x_2^{-1}(R_1) \neq \emptyset$ , and the product equals 0 otherwise. This product is extended to 0 as expected 0 being the absorbant elements.

Before proving in Theorem 27 that the monoid  $\mathcal{Q}(S)$  is a quasi-invers monoid, let us examine some properties of the idempotent elements of  $S$ .

**Lemma 25** *A non zero triple  $(L, x, R) \in \mathcal{Q}(S)$  is idempotent if and only if  $x \in S$  is idempotent,  $L \subseteq (L)x^{-1}$  and  $R \subseteq x^{-1}(R)$ .*

*In particular any element of the form  $(L, 1, R) \in \mathcal{Q}(S)$  is idempotent.*

*Proof.* Let  $(L, x, R)$  be an idempotent of  $\mathcal{Q}(S)$ . By definition of the product in  $\mathcal{Q}(S)$  this means that  $xx = x$  hence  $x$  is idempotent in  $S$ ,  $L \cap (L)x^{-1} = L$  hence  $L \subseteq (L)x^{-1}$  and  $R \cap x^{-1}(R) = R$  hence  $R \subseteq x^{-1}R$ .

Conversely, let  $(L, x, R)$  be a non zero element of  $Qq(S)$  with  $x$  idempotent,  $L \subseteq (L)x^{-1}$  and  $R \subseteq x^{-1}(R)$ . We have  $(L, x, R).(L, x, R) = (L \cap (L)x^{-1}, xx, R \cap x^{-1}(R))$ . By idempotence,  $xx = x$ . The fact that  $L = (L \cap (L)x^{-1})$  (resp.  $R = R \cap x^{-1}(R)$ ) immediately follows from the fact that  $L \subseteq (L)x^{-1}$  (resp.  $R \subseteq x^{-1}(R)$ ).  $\square$

**Lemma 26** *The set  $C(\mathcal{Q}(S))$  of elements of the form  $(L, 1, R)$  in  $\mathcal{Q}(S)$  is a commutative monoid.*

*Proof.* Let  $(L, 1, R)$  and  $(M, 1, N)$  be two elements in  $\mathcal{Q}(S)$  with the desired form.

Assume first the product  $(L, 1, R).(M, 1, N)$  is non zero. By definition of the product, this means that  $(L, 1, R).(M, 1, N) = (L \cap M, 1, R \cap N)$  hence, by symmetrical arguments,  $(L, 1, R).(M, 1, N) = (M, 1, N).(L, 1, R)$ .

In the case  $(L, x, R).(M, y, N) = 0$  similar arguments show that we also have  $(M, y, N).(L, x, R) = 0$  since either  $L \cap M = \emptyset$  or  $R \cap N = \emptyset$ .  $\square$

**Theorem 27** *When  $S$  be a monoid,  $\mathcal{Q}(S)$  with left and right context mappings defined, for every non zero element  $(L, x, R)$  by:*

$$(L, x, R)_L = (Lx, 1, R) \text{ and } (L, x, R)_R = (L, 1, xR)$$

*is a quasi-inverse monoid.*

*Moreover, the mapping  $i : S \rightarrow Qq(S)$  defined  $i(x) = (S, x, S)$  is a one-to-one morphism.*

*Proof.* Let  $S$  and  $\mathcal{Q}(S)$  defined as above.

Observe first that Lemma 24 ensures us that the left and right context mappings defined as above are indeed well-defined, i.e. for all  $q = (L, x, R) \in \mathcal{Q}(S)$ ,  $q_L = (Lx, 1, R) \in \mathcal{Q}(S)$  and  $q_R = (L, 1, xR) \in \mathcal{Q}(S)$ .

Axiom (Q0) and (Q4) follow from Lemma 26 above.

Axiom (Q1) follows from Lemma 25

Let us prove axiom (Q2). Let  $q = (L, x, R)$ . We need to prove that  $qq_L = q$ . By definition of the product  $qq_L = (L \cap (Lx)x^{-1}, x.1, x^{-1}(xR) \cap 1^{-1}(R))$  hence the result since, by Lemma 24 we have  $L \subseteq (Lx)x^{-1}$  and  $R \subseteq x^{-1}(xR)$ , and we obviously have  $x1 = x$ , and  $1^{-1}(R) = R$ .

Let us prove property (Q3). Let  $q = (L, x, R)$  and  $p = (M, y, N)$  be some non zero element of  $\mathcal{Q}(S)$ . Assume  $q \leq_s p$ . We need to prove that  $pLqL = pL$ .

Since  $q \leq_s p$ , this means there is some  $r = (O, z, P)$  such that  $p = rq$ , equivalently (a)  $M = O \cap (L)z^{-1}$ , (b)  $y = zx$ , and (c)  $N = R \cap x^{-1}(P)$ .

Now, we have  $pqL = (M \cap (Lx)y^{-1}, y, R \cap N)$ . We want to prove it equals  $(M, y, N)$ .

Observe first that, by (a) and (b),  $M \subseteq (Lx)y^{-1}$  hence  $M \cap (Lx)y^{-1} = M$ . In fact, let  $u \in M$ . By (a),  $u \in (L)z^{-1}$  hence  $uz \in L$  hence  $uzx \in Lx$ . But by (b)  $uzx = uy$  hence  $uy \in Lx$  hence  $u \in (Lx)y^{-1}$ .

By (c), we have  $N \subseteq R$  hence  $R \cap N = N$ .

The proof of the last statement is straightforward since, by Lemma 24,  $x^{-1}S = S = Sx^{-1}$  for arbitrary  $x \in S$  hence, over images by  $i$  of elements of  $S$ , the product in  $\mathcal{Q}(S)$  just mimics the product in  $S$ .  $\square$

The following theorem says that our construction above essentially extend to arbitrary monoid the construction of the monoid of positive tiles [6] from the free monoid  $A^*$ .

### 5.3 Natural order in quasi-inverse extensions

Let us now review some additional properties of the natural order in  $\mathcal{Q}(S)$ .

**Lemma 28** *In  $\mathcal{Q}(S)$ , for every elements  $(L_1, x_1, R_1)$  and  $(L_2, x_2, R_2)$  we have  $(L_1, x_1, R_1) \leq (L_2, x_2, R_2)$  if and only if  $L_1 \subseteq L_2$ ,  $x_1 = x_2$  and  $R_1 \subseteq R_2$ .*

*Proof.* Let  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$ .

We have  $u_1 \leq u_2$  when, by definition,  $u_1 = (u_1)R u_2 (u_1)L$  or, stated explicitly:  $(L_1, x_1, R_1) = (L_1, 1, x_1 R_1) \cdot (L_2, x_2, R_2) \cdot (L_1 x_1, 1, R_1)$ .

It follows that  $u_1 \leq u_2$  if and only if (1)  $x_1 = x_2 = x$ , (2)  $L_1 = L_1 \cap L_2 \cap (L_1 x)x^{-1}$ , (3)  $R_1 = R_1 \cap R_2 \cap x^{-1}(x R_1)$ .

However, by Lemma 24, we have  $L_1 \subseteq (L_1 x)x^{-1}$  and  $R_2 \subseteq x^{-1}(x R_2)$  hence (2) can be rewritten  $L_1 = L_1 \cap L_2$  or equivalently  $L_1 \subseteq L_2$  and (3) can be rewritten  $R_1 = R_1 \cap R_2$  henceforth  $R_1 \subseteq R_2$ .  $\square$

**Theorem 29** *Quasi-inverse monoid  $\mathcal{Q}(S)$  is a stable, i.e. for every  $u, v$  and  $w \in \mathcal{Q}(S)$ , if  $u \leq v$  then  $uw \leq vw$  and  $wu \leq wv$ .*

*Proof.* Let  $u = (L_1, x_1, R_1)$ ,  $v = (L_2, x_2, R_2)$  and  $w = (M, y, N)$ . Assume  $u \leq v$ . By Lemma 28 above, we have  $x_1 = x_2$  from now on denoted by  $x$ ,  $L_1 \subseteq L_2$  and  $R_1 \subseteq R_2$ .

By definition of the product we have  $uw = (L_1 \cap (M)x^{-1}, xy, N \cap y^{-1}(R_1))$  and  $vw = (L_2 \cap (M_2)x^{-1}, xy, N \cap y^{-1}(R_2))$ . Hence  $uw \leq vw$  by applying Lemma 28 and stability of the inclusion order by intersection and residual.

Symmetrical arguments show that  $wu \leq wv$ .  $\square$

**Remark.** By applying Lemma 20 we could have proven instead that the quasi-inverse extension  $\mathcal{Q}(S)$  of  $S$  satisfies axiom (Q5)

Lemma 28 has more consequences that are listed below.

**Lemma 30** *The extension  $\mathcal{Q}(S)$  of monoid  $S$  is a meet semi-lattice. It is moreover (upward) bounded complete and (upward) directed complete.*

*Proof.* By Lemma 28 we easily check that  $\mathcal{Q}(S)$  satisfies the  $F^*$ -property, i.e. any non zero element  $(L, x, R) \in \mathcal{Q}(S)$  lies beneath a unique maximal element  $(S, x, S)$ . The meet of two elements  $(L_1, x_1, R_1)$  and  $(L_2, x_2, R_2)$  is then just defined as 0 when  $x_1 \neq x_2$  and  $(L_1 \cap L_2, x_1, R_1 \cap R_2)$  otherwise.

Let  $X \subseteq \mathcal{Q}(S)$  a non empty either directed or bounded subset of  $\mathcal{Q}(S)$ . We have to show that  $X$  admits a greatest lower bound  $\bigvee X$ .

If  $X = \{0\}$  then  $\bigvee X = 0$  and we are done. Otherwise, in both directed and bounded case, by Lemma 28, there is some  $s \in S$  such that every  $u \in X$  is of the form  $(L_u, x, R_u)$ . It follows that  $\bigvee X = (\bigcup_{u \in X} L_u, s, \bigcup_{u \in X} R_u)$ .  $\square$

**Lemma 31** *Monoid  $\mathcal{Q}(Q)$  is a stably  $F^*$ -quasi-inverse monoid.*

*Proof.* By Lemma 28 maximal elements are all elements of the form  $(S, x, S)$  with  $x \in S$  that forms a submonoid of  $\mathcal{Q}(S)$ . Moreover, again by Lemma 28 any non zero element, hence of the form  $(L, x, R)$  lies beneath the unique maximal elements  $(S, x, S)$ .  $\square$

Last, the following Lemma tells how extended Green relations behave on maximal elements of  $\mathcal{Q}(S)$ .

**Lemma 32** *In  $\mathcal{Q}(S)$ , restricted to maximal elements, relation  $\tilde{\mathcal{L}}$  equals  $\mathcal{L}$ -equivalence and relation  $\tilde{\mathcal{R}}$  equals  $\mathcal{R}$ -equivalence.*

*Proof.* Let  $u = (L_1, x_1, R_1)$ ,  $v = (L_2, x_2, R_3)$ . Assume  $u \tilde{\mathcal{R}} v$ . This means that for all  $e = (L, 1, R)$  one has  $eu = u$  if and only if  $ev = v$ .

On the left, this implies that for all  $L \in \mathcal{U}_s(S)$ , we have  $L_1 \subseteq L$  if and only if  $L_2 \subseteq L$ . As this must be true for  $L = L_1$  or  $L = L_2$  this implies that  $L_1 = L_2$

On the right, this implies that for all  $R \in \mathcal{U}_p(S)$ , we have  $R_1 \subseteq x_1^{-1}(R)$  if and only if  $R_2 \subseteq x_2^{-1}(R)$ .

Assume now  $R_1 = R_2 = S$ .

Taking  $R = x_2S$  this implies  $1 \in x_1^{-1}(x_2S)$  hence  $x_2 \leq_p x_1$ . Taking  $R = x_1S$  this implies  $1 \in x_2^{-1}(x_1S)$  hence  $x_1 \leq_p x_2$ . It follows that  $x_1$  and  $x_2$  are  $\mathcal{R}$ -equivalent and thus, so are  $u$  and  $v$ .  $\square$

## 6 Quasi-inverse monoids and prehomomorphisms

In this section we show that our extension construction  $\mathcal{Q}(S)$  induces an expansion in the sense of Birget and Rhodes [13] of the category of trivial quasi-inverse monoid with (well-behaved) prehomomorphisms into the category of stable quasi-inverse monoids with well-behaved prehomomorphisms.

### 6.1 Prehomomorphisms

The following definition is adapted from McAlister and Reilly [11].

**Definition.** Let  $S$  and  $T$  be two ordered monoids with zeros. A mapping  $\varphi : S \rightarrow T$  is a *prehomomorphism* when  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , for all  $x$  and  $y \in S$ , if  $x \leq y$  then  $\varphi(x) \leq \varphi(y)$  and, for all  $x$  and  $y \in S$ ,  $\varphi(xy) \leq \varphi(x)\varphi(y)$

A prehomomorphism  $\varphi$  such that  $\varphi(xy) < \varphi(x)\varphi(y)$  if and only if  $xy = 0$  is called a *trivial prehomomorphism*.

Observe that given a prehomomorphism  $\varphi : S \rightarrow T$ ,  $(\varphi(S))^*$  is a submonoid of  $T$  while, in general,  $\varphi(S)$  may not be closed under product.

**Lemma 33** *For every prehomomorphism  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow U$ , the mapping  $\varphi\psi : S \rightarrow U$  defined for all  $x \in S$  by  $\varphi\psi(x) = \psi(\varphi(x))$  is a prehomomorphism.*

In other words, ordered monoids and prehomomorphisms forms a category.

### 6.2 Well-behaved prehomomorphisms

**Definition.** Let  $S$  and  $T$  be two stably  $F^*$ -quasi-inverse monoid. A prehomomorphism  $\varphi : S \rightarrow T$  is a well-behaved prehomomorphism when the following condition are satisfied:

- (P1) for all  $x$  and  $y \in \widehat{S}$ ,  $\varphi(xy) \in \widehat{T}$ ,
- (P2) for all  $x, y$  and  $z \in \widehat{S}$ ,  $\varphi(x_L y z_R) = (\varphi(x))_L \varphi(y) (\varphi(z))_R$ ,

where  $\widehat{S}$  (resp.  $\widehat{T}$ ) denotes the set of maximal elements of  $S$  (resp.  $T$ ).

In particular, as an immediate consequence of (P1) and (P2), for all non zero element  $x \in S$ , if  $\varphi(x) \neq 0$ , we have  $\widehat{\varphi(x)} = \varphi(\widehat{x})$  and, if  $x \in \widehat{S}$ ,  $\varphi(x_L) = (\varphi(x))_L$  and  $\varphi(x_R) = (\varphi(x))_R$ .

**Lemma 34** *Stably  $F^*$ -quasi-inverse monoids with well-behaved prehomomorphisms defined a category that contains the category of monoid and morphism as a (isomorphic) subcategory.*

*Proof.* Observe first that any monoid morphism  $\varphi : S \rightarrow T$  can be lifted to a trivial prehomomorphism  $\varphi : S^0 \rightarrow T^0$  by taking additionally  $\varphi(0) = 0$ . As both  $S^0$  and  $T^0$  are stably  $F^*$ -quasi-inverse monoids,  $\varphi$  extended in such a way is moreover a well-behaved prehomomorphism.

Let  $S$ ,  $T$  and  $U$  be three stably  $F^*$ -quasi-inverse monoids. Let then  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow U$  be two well-behaved prehomomorphisms. By Lemma 33,  $\psi \circ \varphi : S \rightarrow U$  is a prehomomorphism.

The fact it is well-behaved is immediate from the definitions and Lemma 34 above.  $\square$

### 6.3 Quasi-inverse extension and prehomomorphisms

We have already shown that for every monoid  $S$  there is a one-to-one morphism  $i : S \rightarrow \mathcal{Q}(S)$ , the canonical injection of  $S$  into  $\mathcal{Q}(S)$ . It can be extended to a trivial one-to-one prehomomorphism  $i : S^0 \rightarrow \mathcal{Q}(S)$  by taking  $i(0) = 0$ . Remember that  $S^0$  is the trivial extension of  $S$  with an *additional* new zero. What about a canonical (inverse) surjection from  $\mathcal{Q}(S)$  onto  $S^0$  ?

**Definition.** For all monoid  $S$ , let define the mapping  $\sigma_S : \mathcal{Q}(S) \rightarrow S^0$  that maps 0 to 0 and any non zero triple  $(L, x, R) \in \mathcal{Q}(S)$  to  $x$ .

**Lemma 35** *For all monoid  $S$ , the mapping  $\sigma_S : \mathcal{Q}(S) \rightarrow S^0$  is a (trivial hence well-behaved) onto prehomomorphism.*

*Proof.* Mapping  $\sigma_S$  is obviously onto and monotonic. It remains to prove it is sub-multiplicative. Let  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$  two non zero elements of  $\mathcal{Q}(S)$ . If  $u_1 u_2 = 0$  we have  $\varphi(u_1 u_2) = 0 < \varphi(u_1) \varphi(u_2) = x_1 x_2$ . If  $u_1 u_2 \neq 0$  then  $\varphi(u_1 u_2) = x_1 x_2$  henceforth  $\varphi(u_1 u_2) = \varphi(u_1) \varphi(u_2)$ . In other words,  $\varphi$  is a trivial prehomomorphism.  $\square$



## 6.4 Expansion prehomomorphism

We provide here the last piece that show that our extension construction is actually an expansion in the sense of Birget and Rhodes in the category of ordered monoids and prehomomorphisms.

**Definition.** Let  $S$  and  $T$  be two monoids. Let  $\varphi : S \rightarrow T$  be a monoid morphism and let still denote by  $\varphi$  the lift of  $\varphi$  to  $\mathcal{P}(S)$  defined by as  $\varphi : \mathcal{P}(S) \rightarrow \mathcal{P}(T)$  by taking, for every  $X \subseteq S$ ,  $\varphi(X) = \{\varphi(x) : x \in X \cap S\}$ .

Let then  $\mathcal{Q}(\varphi)$  be the expansion mapping  $\mathcal{Q}(\varphi) : \mathcal{Q}(S) \rightarrow \mathcal{Q}(T)$  defined by  $\mathcal{Q}(\varphi)(0) = 0$  and for all non zero element  $u = (L, x, R) \in \mathcal{Q}(S)$ , by  $\mathcal{Q}(\varphi)(u) = (T\varphi(L), \varphi(x), \varphi(R)T)$ .

**Lemma 36** *For all monoids  $S$  and  $T$ , for all (monoid) morphism  $\varphi : S \rightarrow T$ , the expansion mapping  $\mathcal{Q}(\varphi) : \mathcal{Q}(S) \rightarrow \mathcal{Q}(T)$  is a well-behaved prehomomorphism such that, moreover, the following diagram commutes.*

$$\begin{array}{ccc}
 \mathcal{Q}(S) & \xrightarrow{\mathcal{Q}(\varphi)} & \mathcal{Q}(T) \\
 \downarrow \sigma_S & & \downarrow \sigma_T \\
 S^0 & \xrightarrow{\varphi} & T^0
 \end{array}$$

In other words,  $\varphi \circ \sigma_S = \sigma_T \circ \mathcal{Q}(\varphi)$ .

*Proof.* Let  $S$  and  $T$  and  $\tau : S \rightarrow T$  as above. In order to simplify notation, let us write  $\varphi'$  in place of  $\mathcal{Q}(\varphi)$ .

Observe first that, by definition,  $\varphi'(0) = 0$  and we also have

$$\varphi'(1) = (T\varphi(S), \varphi(1), \varphi(S)T)$$

hence  $\varphi'(1) = 1$  since  $1 \in S$  and  $\varphi(1) = 1$  and thus  $1 \in \varphi(S)$ .

Monotonicity of  $\varphi'$  immediately follows from the characterization of the natural order provided by Lemma 28 and the fact that  $\varphi$  extended to  $\mathcal{P}(S)$  is monotonic.

Let then  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$  two non zero elements of  $\mathcal{Q}(S)$ . We have to show that  $\varphi'(u_1 u_2) \leq \varphi'(u_1) \varphi'(u_2)$ . By definition, we have:

$$\varphi'(u_1 u_2) = (T\varphi(L_1 \cap (L_2)x_1^{-1}), \varphi(x_1)\varphi(x_2), \varphi(x_2^{-1}(R_1) \cap R_2)T)$$

hence by distributivity of (the extension of)  $\varphi$  and product (over  $\mathcal{P}(S)$ ) w.r.t. the intersection

$$\begin{aligned} \varphi'(u_1 u_2) &= (T\varphi(L_1) \cap T\varphi((L_2)x_1^{-1}), \\ &\quad \varphi(x_1)\varphi(x_2), \varphi(x_2^{-1}(R_1))T \cap \varphi(R_2)T) \end{aligned}$$

By definition, we also have:

$$\begin{aligned} \varphi'(u_1)\varphi'(u_2) &= (T\varphi(L_1) \cap T\varphi(L_2)(\varphi(x_1))^{-1}, \\ &\quad \varphi(x_1)\varphi(x_2), (\varphi(x_2))^{-1}\varphi(R_1)T \cap \varphi(R_2)T) \end{aligned}$$

Now we conclude that  $\varphi'(u_1 u_2) \leq \varphi'(u_1)\varphi'(u_2)$  by applying Lemma 28, monotonicity of product and intersection in  $\mathcal{P}(S)$  and the fact that, for all  $X \subseteq S$  and all  $x \in S$  we have

$$\varphi(x^{-1}(X)) \subseteq (\varphi(x))^{-1}(\varphi(X))$$

and

$$\varphi((X)x^{-1}) \subseteq (\varphi(X))(\varphi(x))^{-1}$$

Indeed, let  $x \in S$  and  $X \subseteq S$  as above. Let  $y \in \varphi(x^{-1}(X))$ . This means that  $y = \varphi(z)$  for some  $z \in x^{-1}(X)$  hence some  $z$  such that  $xz \in X$ . But then, this also means, as  $\varphi$  extended to  $S$  is still a monoid morphism, that  $\varphi(x)\varphi(z) \in \varphi(X)$  hence  $y = \varphi(z) \in (\varphi(x))^{-1}(\varphi(X))$ . The other case is symmetric.

A similar argument as above for proving that  $\varphi'(1) = 1$  shows that for all maximal element  $u = (S, x, S)$  of  $\mathcal{Q}(S)$ , we have  $\varphi'(u) = (T, \varphi(x), T)$  hence  $\varphi'(u)$  is also maximal and, moreover, for two maximal elements  $u$  and  $v \in \mathcal{Q}(S)$ ,  $\varphi'(uv) = \varphi(u)\varphi(v)$ . In other words,  $\varphi'$  is well-behaved.

Diagram commutation immediately follows from the definitions of  $\sigma_S$ ,  $\sigma_T$  and  $\varphi' = \mathcal{Q}(\varphi)$ .  $\square$

**Corollary 37** *The extension that lift any trivial quasi-inverse monoid  $S^0$  to a non trivial quasi-inverse monoid  $\mathcal{Q}(S)$  turns out to be an expansion in the sense of Birget and Rhodes[13] with prehomomorphisms instead of morphisms.*

*Proof.* As any monoid morphism  $\varphi : S \rightarrow T$  can be lifted up to a trivial well-behaved prehomomorphism  $\varphi : S^0 \rightarrow T^0$  by taking  $\varphi(0) = 0$ , Lemma 36 and Lemma 35 just ensure that.  $\square$

**Remark.** This Lemma really says that, via this expansion, the standard notion of recognizability over monoids by means of morphisms can be

lifted up into a notion of quasi-recognizability over stably  $F^*$ -quasi-inverse monoids by means of well-behaved prehomomorphisms.

Our construction in [5], applied to languages of positive tiles, is an instance of that more general fact.

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