



## Quasi-inverse monoids

David Janin

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## Quasi-inverse monoids (DRAFT)

February 22, 2012

David Janin,  
LaBRI, IPB, Université de Bordeaux

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# Quasi-inverse monoids (DRAFT)

David Janin,  
Université de Bordeaux, LaBRI UMR 5800,  
351, cours de la libération,  
F-33405 Talence  
`janin@labri.fr`

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## Abstract

The notion of quasi-recognizability has been defined in [8] as a remedy to the collapse of standard algebraic recognizability on languages of one-dimensional overlapping tiles [7]. There quasi-recognizability makes an essential use of a peculiar class of ordered monoids. Our purpose here is to provide a presentation and a study of these monoids.

Called well-behaved ordered monoids in [8], they can also be seen as monoids  $S$  equipped with two unary mappings  $x \mapsto x_L$  and  $x \mapsto x_R$  such that, if  $S$  is an inverse monoid then  $x_L$  just equals  $x^{-1}x$  and  $x_R$  just equals  $xx^{-1}$ , and if  $S$  is not an inverse monoid then, as much as possible, both  $x_L$  and  $x_R$  still behave the same way.

These *quasi-inverse* monoids are thus presented and studied as such in this paper. It is shown in particular that quasi-inverse monoids are  $U$ -semiadequate monoids [9] and that stable quasi-inverse monoids are well-behaved ordered monoids [8].

The link between quasi-inverse monoids and prehomomorphisms is also studied a little further. We show in particular that the extension of arbitrary monoid  $S$  into a quasi-inverse monoid  $\mathcal{Q}(S)$ , used in [8], is actually an expansion in the sense of Birget and Rhodes [15] in the category of (ordered monoid) prehomomorphisms instead of (monoid) homomorphisms.

## 1 Introduction

The notion of quasi-recognizability has been defined in [8] as a remedy to the collapse of standard algebraic recognizability on languages of one-

dimensional overlapping tiles [7] or, equivalently, subsets of McAlister monoid [7, 12, 11].

Quasi-recognizability makes an essential use of prehomomorphisms and a peculiar class of ordered monoids. The purpose of the present paper is to study more in depth this class of monoids.

In order to do so, we first give an axiomatization of this class in terms of monoid equipped with two unary operators called left and right context operators. In this form they are called quasi-inverse monoids as they behave somehow like inverse monoids but without inverses themselves.

This bi-unary axiomatization permits to relate this class of monoids with other known classes. It finally turns out that quasi-inverse monoids are exactly  $U$ -semiadequate monoids [9]. As such, they also generalize inverse monoids or (two sided) restriction monoids [4, 1].

It is shown, as in [9], that quasi-inverse monoids can be equipped with a two sided variant of Nambooripad natural order [14]. In general this order is not stable under product. To remedy to this fact, we provide an additional axiom that turns out to capture stability.

The resulting notion of stable quasi-inverse monoids is then shown equivalent with the notion of well-behaved ordered monoid defined in [8].

The relationship between quasi-inver monoids and prehomomorphisms is then studied. Defined on ordered monoids, McAlister and Reilly prehomomorphisms [13] are monotonic mappings  $\varphi$  such that the morphism constraint  $\varphi(xy) = \varphi(x)\varphi(y)$  is relaxed in a submultiplicativity constraint  $\varphi(xy) \leq \varphi(x)\varphi(y)$ .

From a rough point of view, quasi-inverse monoids can be seen as monoids induced by images of inverse monoids [10] by prehomomorphisms.

More concretely, the extension of any monoid  $S$  into a non trivial quasi-inverse monoid  $\mathcal{Q}(S)$ , already presented in [8], is studied here a little further. In particular, we prove that this construction induces an expansion in the sense of Birget and Rhodes [15] in the category of ordered monoids and prehomomorphisms.

## Related works

As already mentioned in earlier works, we believe that quasi-inverse monoids together with prehomomorphisms can lead to many interesting developments in computer science. For instance, in theory, defining quasi-recognizability on top of quasi-inverse monoids, we already achieved a characterization of

MSO definable languages of tiles [8]. With more application oriented perspectives, this notion leads us to a rather new modeling metaphor for musical structures [6].

The purpose of this report is thus to give a rather self contained and complete overview of the underlying theory.

Of course this theory lays somewhere in the rich border of the even richer theory of inverse semigroup: a theory that has been developed for almost a century [10]. There are similarities between our development and the known theory. We already mentioned a few. It even occurs, as shown here, that quasi-inverse monoids turn out to be  $U$ -semiadequate monoids [9].

However, it must be noticed that in [9] and following studies, the emphasis is put on  $U$ -semiadequate semigroups that moreover satisfies the congruence property, i.e.  $U$ -semiadequate semigroups where both  $\tilde{\mathcal{L}}$  and  $\tilde{\mathcal{R}}$  are congruences.

In our study, we are rather concerned with stable quasi-inverse monoids i.e.  $U$ -semiadequate monoids where the induced (two-sided) natural order is stable under product. It follows that we do not need the congruence property. Our main construction  $\mathcal{Q}(S)$  leads to quasi-inverse monoids that do not satisfy that property. Thus it may still be the case that our work diverges from known studies.

The fact we are dealing with monoids instead of semigroups also make a somehow subtle difference. In particular, it is striking that in a stably  $U$ -semiadequate monoids (or equivalently well-behaved ordered monoids [8]),  $U$  just equals the set of subunits of  $S$ , i.e. those elements that are smaller than or equal to 1. In other words, a stably  $U$ -semiadequate monoid is completely an unambiguously determined by its induced order.

Moreover, although our construction of  $\mathcal{Q}(S)$  has many similarities with constructions *à la* McAlister, studied and developed further for instance in Cornock PhD [1], the product in  $\mathcal{Q}(S)$  just seems unrelated with the products defined there.

Despite its simplicity and robustness, despite the richness of the underlying field, it seems (so far) that our construction is still original.

Anyhow, as  $\mathcal{Q}(S)$  still looks like a sort of a double semi-direct product between  $S$  and the two lattices  $\mathcal{U}_s(S)$  and  $\mathcal{U}_p(S)$  of left and right ideals of  $S$  with intersection as product, a potential link with formerly defined constructions need to be further investigated.

## Notations

In the remainder of the text, given a monoid  $S$  with neutral written 1 and (possibly) absorbant element written 0, we write  $\leq_p$  for the prefix preorder and  $\leq_s$  for the suffix preorder, defined, for all  $x$  and  $y \in S$  by  $x \leq_p y$  when  $xz = y$  for some  $z \in S$  and  $x \leq_s y$  when  $zx = y$  for some  $z \in S$ .

Observe that, under both prefix and suffix preorder, 1 is the least element of  $S$  and 0 (if ever) is the greatest.

We also write  $x^{-1}(y) = \{z \in S : xz = y\}$  and  $(y)x^{-1} = \{z \in S : zx = y\}$ . This notation extend to sets as follows. For all  $x \in S$  and  $Y \subseteq S$  we write  $x^{-1}(Y) = \{z \in S : xz \in Y\}$  and  $(Y)x^{-1} = \{z \in S : zx \in Y\}$ .

## 2 Quasi-inverse monoids

In this section, we define quasi-inverse monoid that can be seen as almost inverse monoids where we only require, for all element  $x$ , the existence of a left context  $x_L$  behaving “like”  $x^{-1}x$  and a right context  $x_R$  behaving “like”  $xx^{-1}$ .

### 2.1 Axiomatic definition

We give here the axiomatic definition of quasi-inverse monoid and study some of its immediate properties.

**Definition 1** *A quasi-inverse monoid is a monoid  $S$  equipped with two mappings  $x \mapsto x_L$  and  $x \mapsto x_R$ , called the left and right context mappings, such for all  $x$  and  $y \in S$  :*

$$(Q0) \ (x_L y_L)_L = (x_L y_L)_R = x_L y_L \text{ and } (x_R y_R)_L = (x_R y_R)_R = x_R y_R,$$

$$(Q1) \ x_L x_L = x_L \text{ and } x_R x_R = x_R,$$

$$(Q2) \ x x_L = x \text{ and } x_R x = x,$$

$$(Q3) \ (xy)_L y_L = (xy)_L \text{ and } x_R (xy)_R = (xy)_R,$$

$$(Q4) \ x_L y_L = y_L x_L \text{ and } x_R y_R = y_R x_R,$$

**Lemma 2** *In a quasi-inverse monoid  $S$ , we have  $1_L = 1_R = 1$ .*

*Proof.* By axiom (Q2) we have  $1.1_L = 1$  hence  $1_L = 1$  since 1 is neutral. By symmetry we also have  $1_R = 1$ .  $\square$

**Lemma 3** *Let  $S$  be a quasi-inverse monoid, let  $C_L(S) = \{x_L \in S : x \in S\}$  and let  $C_R(S) = \{x_R \in S : x \in S\}$ . We have  $C_L(S) = C_R(S)$  from now on written  $C(S)$ .*

*Moreover,  $C(S)$  is a commutative submonoid of idempotents of  $S$  with, for all  $x \in C(S)$ ,  $x_L = x_R = x$ , i.e. left and right context operators are identities over context elements.*

*Proof.* Assume  $S$  is a quasi-inverse monoid. By Lemma 2 we have  $1_R = 1_L = 1$ . It follows that, by (Q0) taking  $y = 1$ , for all  $x \in C_R(S)$  (resp.  $x \in C_L(S)$ ) one has  $x_R = x_L = x$  hence  $x \in C_L(S)$  (resp.  $x \in C_R(S)$ ). Then closure of  $C(S)$  follows from (Q0) again. Idempotency is provided by (Q1). Commutativity is provided by (Q4).  $\square$

The importance of axiom (Q3) becomes clear in Lemma 12 below.

**Lemma 4** *Let  $S$  be a quasi-inverse monoid. If  $0 \in S$  then  $0_R = 0_L$  from now on written  $\perp$  and, for all  $x \in C(S)$ ,  $\perp x = x \perp = \perp$ .*

*Proof.* Assume  $0 \in S$ . By (Q3) for all  $y \in S$ , we have  $(0y)_L y_L = (0y)_R$  hence, for all  $y \in S$ ,  $0_L y_L = 0_L$ . By symmetry, for all  $y \in S$  we also have  $y_R 0_R = 0_R$ . Now by Lemma 3, taking  $y = 0_L$  or  $y = 0_R$  we deduce that  $0_R = 0_L$  with the announced property within context element.  $\square$

**Remark** We observe first that  $S = \{1, 0\}$  with  $0_L = 0_R = 0$  is a quasi-inverse monoid. We observe also that  $S = \{1, 0, \perp\}$  with  $0_L = 0_R = \perp$  is also a quasi-inverse monoid.

This suggests that taking  $\perp = 0$  or  $\perp \neq 0$  leads to two types of quasi-inverse monoids with zero. This remark is however not developed further.

**Lemma 5** *Let  $S$  be a quasi-inversive monoid  $S$  and let  $x$  and  $y \in S$ . If  $x \leq_s y$  (resp.  $x \leq_p y$ ) then  $yx_L = y$  (resp.  $x_R y = y$ ).*

*If both  $x \leq_s y$  and  $y \leq_s x$ , i.e.  $x$  and  $y$  are  $\mathcal{L}$ -equivalent, (resp. both  $x \leq_p y$  and  $y \leq_p x$ , i.e.  $x$  and  $y$  are  $\mathcal{R}$ -equivalent) then  $x_L = y_L$  (resp.  $x_R = y_R$ ).*

*Proof.* Let  $x$  and  $y$  as above. By symmetry, we only prove the suffix case. Assume  $x \leq_s y$  henceforth  $y = xz$ . By (Q2) on  $y$  we have  $yx_L = yy_L x_L$ , hence by (Q3)  $yx_L = yy_L$  hence by (Q2) again,  $yx_L = y$ .

If both  $x \leq_s y$  and  $y \leq_s x$  we have, by (Q3), both  $y_L x_L = y_L$  and  $x_L y_L = x_L$  hence  $x_L = y_L$  since, by (Q4), context elements commute.  $\square$



## 2.2 Trivial and non trivial Quasi-inverse monoids

One may ask if, when  $S$  is a quasi-inverse monoid, the left and right context mappings are uniquely determined. The answer is no in general as shown by the following definition.

**Definition 6** *Let  $S$  be an arbitrary monoid. Let  $S^0 = S + 0$  be the trivial extension of monoid  $S$  with a new zero element and the mappings  $x \mapsto x_L$  and  $x \mapsto x_R$  defined by  $0_L = 0_R = 0$  and, for all  $x \in S$ ,  $x_L = x_R = 1$ .*

**Lemma 7** *The trivial extension  $S^0$  of a monoid  $S$  is a quasi-inverse monoid.*

*Proof.* Straightforward. □

Before developing the notion of non trivial quasi-inverse monoid, we must say that, as proved in Lemma 22, as soon as  $C(S)$  is known, then both mappings  $x \mapsto x_L$  and  $x \mapsto x_R$  are uniquely determined.

The notion of trivial extension of a monoid and the fact that, in particular, inverse monoid are quasi-inverse monoid leads us to the following definition.

**Definition 8** *A quasi-inverse monoid  $S$  is said non trivial when, for every  $x \in S$ , if  $x_L = 1$  (resp. if  $x_R = 1$ ) then there is  $y \in S$  such that  $yx = 1$  (resp.  $xy = 1$ ), i.e.  $x$  has a left (resp. right) group inverse.*

**Remark** We observe that, in particular, in a non trivial monoid with zero, one must have  $0_L = 0_R = \perp < 1$ .

**Theorem 9** *An inverse monoid is a non trivial quasi-inverse monoid.*

*Proof.* Let  $S$  be an inverse monoid. Recall that this means  $S$  is a monoid such that for all  $x \in S$  there is a unique elements  $x^{-1} \in S$ , called the pseudo inverse of  $x$ , such that  $xx^{-1}x = x$  and  $x^{-1}xx^{-1} = x^{-1}$ .

In particular, observe that for all  $x \in S$  both  $x^{-1}x$  and  $xx^{-1}$  are idempotent. Denoting by  $E(S)$  the set of idempotent of  $S$ , it is well-known that  $E(S)$  is a commutative monoid. Even more, it can be shown [10] that idempotence commutation hypothesis is actually equivalent with the unicity of pseudo inverses.

Let us show that  $S$  is quasi-inverse. In order to do so, for every  $x \in S$ , let us take  $x_L = x^{-1}x$  and  $x_R = xx^{-1}$ . Since for all  $x$  and  $y \in S$  one has  $(xy)^{-1} = y^{-1}x^{-1}$  all axioms of quasi-inverse monoid are immediate.

For instance, for axiom (Q3), given  $x$  and  $y \in S$ , one has  $(xy)_L y_L = y^{-1} x^{-1} x y y^{-1} y$  hence by commutation  $(xy)_L y_L = y^{-1} y y^{-1} x^{-1} x y$  hence because  $y^{-1} y y^{-1} = y^{-1}$   $(xy)_L y_L = (xy)_L$ .

The fact it is non trivial then immediately follows from the definition since, for all  $x$ , if  $x_L = 1$  then  $x^{-1} x = 1$  and thus  $x^{-1}$  is a left group inverse of  $x$  and if  $x_R = 1$  then  $x x^{-1} = 1$  and thus  $x^{-1}$  is a right group inverse of  $x$ .  $\square$

**Remark** Observe that when  $S$  is an inverse monoid with left and right context operators defined as above, for all  $x$  and  $y \in S$ , if  $x_L y = y$  (resp.  $x_R y = y$ ) then  $x \leq_s y$  (resp.  $x \leq_p y$ ). In other words, an inverse monoid also satisfies the converse of the property stated in Lemma 5. For quasi-inverse monoids, even non trivial ones, this is no longer true as illustrated, in particular, by quasi-inverse extensions  $\mathcal{Q}(S)$  of arbitrary monoids  $S$  defined in Section 4.

**Remark** The monoid of positive tiles  $T_A$ , a non inverse submonoid of McAlister monoid (see [7]), is a non trivial quasi-inverse monoid.

### 2.3 Natural order

We define here, over quasi-inverse monoids, the analogous of the natural order in inverse monoid.

**Definition 10 (Natural order)** *In a quasi-inverse monoid  $S$ , let  $\leq$  be the natural relation defined, for every  $x$  and  $y \in S$  by  $x \leq y$  when  $x = x_R y x_L$ .*

**Lemma 11** *The relation  $\leq$  in a quasi-inverse monoid  $S$  is an order relation. When  $S$  is an inverse monoid, it coincides with Nambooripad natural order in  $S$ .*

*Proof.* Relation  $\leq$  is obviously a preorder, i.e. reflexive and transitive. Let us show it is anti-symmetric. Assume  $x \leq y$ . By definition,  $x = x_R y x_L$ . By left product with  $y_R$  and right product with  $y_L$  we have  $y_R x y_L = y_R x_R y x_L y_L$  hence by context element commutation  $y_R x y_L = x_R (y_R y y_L) x_L$  hence  $y_R x y_L = x_R y x_L$ . Assume now  $y \leq x$ , this means  $y = y_R x y_L$  hence  $x = y$ .

When  $S$  is inverse, we now that  $x = x_R y x_L$ ,  $x = x_R y$ ,  $x = y x_L$ ,  $x = e y$  for some  $e \in E(S)$  and  $x = y f$  for some  $f \in E(S)$  are all equivalent properties, the last two being Nambooripad [14] natural orders definitions.  $\square$

**Remark** In general, it is not true that, in a quasi-inverse monoid, if  $x \leq y$  then for all  $z \in S$ ,  $xz \leq yz$  and  $zx \leq zy$ , i.e. the natural order is not necessarily stable under product. The missing axiom (Q5) is provided in Section 3.1.

**Lemma 12** *Let  $S$  be a quasi-inverse monoid. The set  $C(S)$  of context elements ordered by natural order is a meet semi-lattice with 1 as maximum element,  $\perp = 0_R = 0_L$  as minimum element in the case  $0 \in S$ , and for all  $x$  and  $y \in C(S)$ ,  $x \wedge y = xy$ .*

*Proof.* Let  $x \in C(S)$ . By Lemma 3 one has  $x_L = x_R = x$  and, by (Q1),  $xx = x$  hence  $x = x_R 1 x_L$  hence  $x \leq 1$ .

Assume  $0 \in S$ . By Lemma 4, let  $\perp = 0_L = 0_R$ . For all  $x \in C(S)$  we have, again by Lemma 4,  $\perp = 0_R x 0_L$  hence  $\perp \leq x$ .

Let us prove  $C(S)$  is a meet semi-lattice with product as meet operator. In order to do so, let  $x$  and  $y \in C(S)$ . Observe first that, by commutation and idempotence of context elements we do have  $xy \leq x$  and  $xy \leq y$ . Conversely, let  $z \in C(S)$  such that (a)  $z \leq x$  and (b)  $z \leq y$ . By (a) we have  $zxy = zy$  hence by (b)  $zxy = z$  hence, by commutation of contexts and idempotence of  $z$ ,  $z = zxyz$  hence  $z \leq xy$ . Altogether, this means that  $x \wedge y$  is defined with  $x \wedge y = xy$ .  $\square$

**Remark** In particular, for all  $x$  and  $y \in C(S)$ ,  $x \leq y$  if and only if  $xy = x$ .

**Lemma 13** *When  $S$  is a quasi-inverse monoid the set  $C(S)$  of context elements of  $S$  equals the set  $U(S) = \{x \in S : x \leq 1\}$  of subunits of  $S$  ordered by the (quasi-inverse) natural order.*

*Proof.* Lemma 12 above already tells us that  $C(S) \subseteq U(S)$ . Let thus  $x \in S$  such that  $x \leq 1$ . By definition of the quasi-inverse natural order this means that  $x = x_R x_L$  hence, since  $C(S)$  is a submonoid of  $S$  (Lemma 3),  $x \in C(S)$ .  $\square$

**Lemma 14** *When  $S$  is a quasi-inverse monoid, for all  $x$  and  $y \in S$ , if  $x \leq y$  then  $x_R \leq (yx_L)_R \leq y_R$  and  $x_L \leq (x_R y)_L \leq y_L$ ,*

*Proof.* Let  $x$  and  $y \in S$  with  $S$  quasi-inverse.

Assume  $x \leq y$ . By symmetry, we only need to prove  $x_R \leq (yx_L)_R \leq y_R$ .

Since  $x \leq y$  this means that  $x = x_R y x_L$  hence  $y x_L \leq_s x$  hence, by (Q3),  $x_R (y x_L)_R = x_R$  that is to say  $x_R \leq (y x_L)_R$ . But we also have  $y_R (y x_L) = y x_L$  hence  $y_R \leq_p y x_L$  hence, by (Q3) again,  $(y x_L)_R y_R = (y x_L)_R$ .  $\square$

**Remark** One may ask about the converse. Assuming both  $x_L \leq y_L$  and  $x_R \leq y_R$  could we have  $x \leq y$  for any  $x$  and  $y \in S$ ? By Lemma 5 this would mean that whenever  $x$  and  $y \in S$  are  $\mathcal{H}$ -equivalent, i.e. both  $\mathcal{R}$  and  $\mathcal{L}$ -equivalent, then they are equal. This will not be true in general as already illustrated by inverse monoids.

## 2.4 Natural order revisited

We show here that the natural order definition can be further simplified as a two-sided variation of Nambooripad natural order [14] restricted to idempotents of  $C(S)$ .

**Lemma 15** *When  $S$  is a quasi-inverse monoid, for every  $e$  and  $f \in C(S)$  and  $x \in S$ ,  $exf \leq x$ .*

*Proof.* Let  $S$  be a quasi-inverse monoid. Let  $e$  and  $f \in C(S)$  and  $x \in S$ . Observe first that  $(exf)_R e = (exf)_R$ . In fact, we have  $e \leq_p exf$  hence, by property (Q3) of quasi-inverse monoid,  $(exf)_R (e)_R = (exf)_R$ . But, by Lemma 3 we have  $(e)_R = e$  hence the claim. Symmetrically, we have  $f(exf)_L = (exf)_L$ . Altogether,  $(exf)_R x (exf)_L = (exf)_R exf (exf)_L$  and thus  $exf \leq x$ .  $\square$

It follows that:

**Corollary 16** *In a quasi-inverse monoid  $S$ , for all  $x$  and  $y \in S$ ,  $x \leq y$  if and only if there exists two context elements  $e$  and  $f \in C(S)$  such that  $x = eyf$ .*

**Remark** We can also define two other orders in a quasi-inverse monoid by saying  $x \leq_L y$  (resp.  $x \leq_R y$ ) when there exists some  $e \in C(S)$  such that  $x = ye$  (resp.  $x = ey$ ).

It occurs however that these orders are both weaker than the natural order defined above, i.e.  $(\leq_L \cup \leq_R) \subseteq \leq$  and the inclusion can be strict as shown by  $\mathcal{Q}(S)$  defined below in Section 4.

As already observed in [9] for  $U$ -semiadequate semigroups, we have:

**Lemma 17** *In a quasi-inverse monoid  $\leq = (\leq_L \circ \leq_R) = (\leq_R \circ \leq_L)$  where  $\circ$  denotes relation composition.*

*Proof.* We obviously have  $\leq \subseteq (\leq_L \circ \leq_R)$  since whenever  $x = eyf$  then  $x = zf$  with  $z = ey$ . Conversely, both  $\leq_L \subseteq \leq$  and  $\leq_R \subseteq \leq$  as particular instance of  $\leq$  definition, hence, by transitivity  $(\leq_L \circ \leq_R) \subseteq \leq$ .  $\square$

## 2.5 $U$ -semiadequate monoids

We relate here quasi-inverse monoids with  $U$ -semiadequate semigroups as defined in [9].

**Definition 18** *Let  $S$  be a semigroup and let  $U \subseteq E(S)$  be a subset of idempotents of  $S$ . Extended Green relations  $\tilde{\mathcal{L}}$  and  $\tilde{\mathcal{R}}$  are defined from  $U$ , for all  $x$  and  $y \in S$ , by:*

$x\tilde{\mathcal{L}}y$  if for all  $e \in U$ ,  $xe = x \Leftrightarrow ye = y$ ,

$x\tilde{\mathcal{R}}y$  if for all  $e \in U$ ,  $ex = x \Leftrightarrow ey = y$ .

One can easily check that relation  $\tilde{\mathcal{L}}$  (resp.  $\tilde{\mathcal{R}}$ ) is a left (resp. right) congruence relation that generalizes Green's relation  $\mathcal{L}$  (resp.  $\mathcal{R}$ ).

Moreover, for all  $x \in S$  and  $y \in U$ , if  $y\tilde{\mathcal{L}}x$  (resp.  $y\tilde{\mathcal{R}}x$ ) then, since  $y$  is idempotent,  $yy = y$  hence, by definition,  $xy = x$  (resp.  $yx = x$ ).

**Definition 19** *A semigroup  $S$  is a  $U$ -semiabundant when for all  $x \in S$  there is at least one  $x^+ \in U$  such that  $x^+\tilde{\mathcal{R}}x$  and one  $x^* \in U$  such that  $x^*\tilde{\mathcal{L}}x$ .*

*A semigroup  $S$  is  $U$ -semiadequate when it is  $U$ -semiabundant and  $U$  is a commutative submonoid of  $S$ .*

**Lemma 20** *A quasi-inverse monoid  $S$  is a  $C(S)$ -semiadequate monoid.*

*Proof.* Let  $S$  be a quasi-inverse monoid. Let  $x \in S$ .

Let  $y \in C(S)$ . Since  $xx_L = x$ , if  $x_Ly = x_L$  then  $xy = xx_Ly = xx_L = x$ . Conversely, if  $xy = x$  then, by axiom (Q3),  $(xy)_Ly_L = (xy)_L$  with, by axiom (Q0),  $y_L = y$ , hence  $x_Ly = x_L$ .

In other words, we have  $x\tilde{\mathcal{L}}x_L$  and, by symmetrical arguments,  $x\tilde{\mathcal{R}}x_R$ .

Assume now there is some  $y \in C(S)$  such that  $y\tilde{\mathcal{L}}x$ . This means in particular, since, by axiom (Q2),  $xx_L = x$ , that  $yx_L = y$  hence  $y \leq x_L$ . But, as observed above, since  $y$  is idempotent, we have  $yy = y$  hence, by definition of  $\tilde{\mathcal{L}}$ ,  $xy = x$ .  $\square$

The following properties is proved in [9]:

**Lemma 21** *In a  $U$ -semiadequate semigroup each  $\tilde{\mathcal{L}}$ -class (resp.  $\tilde{\mathcal{R}}$ -class) has a unique element of  $U$ .*

*Proof.* Follows from the facts (see [9] Lemma1.3) that, in a  $U$ -semiadequate semigroup Green relation  $\mathcal{L}$  (resp.  $\mathcal{R}$ ) and extended Green  $\tilde{\mathcal{L}}$  (resp.  $\tilde{\mathcal{R}}$ ) are equal over  $U$  and the fact that, over commuting idempotents,  $\mathcal{L}$  and  $\mathcal{R}$  relations are just equality.  $\square$

**Lemma 22** *A  $U$ -semiadequate monoid  $S$  is a quasi-inverse monoid taking, for all  $x \in S$ ,  $x_L = x^+$ ,  $x_R = x^*$ .*

*Proof.* Let  $S$  be a  $U$ -semiadequate monoid. For all  $x \in S$ , we put  $x_L = x^+$  and  $x_R = x^*$ .

Observe first, since  $1 \in S$ , that  $1^* = 1^+ = 1$  henceforth  $1 \in U$ . Indeed, given  $1^* \in U$  we have  $1^*1 = 1$  hence  $1^* = 1$  since 1 is neutral. By symmetrical argument, we have  $1^+ = 1$ .

By hypothesis,  $U$  is a commutative subsemigroup (hence a submonoid) of idempotents with, by Lemma 21, for all  $x \in U$ ,  $x^* = x^+ = x$ , hence axioms (Q0), (Q1) and (Q4) of quasi-inverse monoids are satisfied.

As we have already observed that axiom (Q2) is satisfied it remains now to prove axiom (Q3). Let  $x$  and  $y \in S$ . Since  $yy^+ = y$  we also have  $(xy)y^+ = (xy)$  hence  $(xy)^+y^+ = (xy)^+$  since  $(xy)^+\tilde{\mathcal{L}}xy$ . A symmetrical argument shows that  $x^*(xy)^* = (xy)^*$ .  $\square$

In other words:

**Corollary 23** *Quasi-inverse monoids and  $U$ -semiadequate monoids are equivalent notions.*

### 3 Stable quasi-inverse monoids

We review here some additional properties of quasi-inverse monoid when the natural order is stable w.r.t. the monoid product.

#### 3.1 Stable quasi-inverse monoid

We provide here the missing axiom for natural order on quasi-inverse monoid to be stable by product.

**Definition 24** *A quasi-inverse monoid is called stable when for all  $x, y \in S$ , the following property is satisfied:*

(Q5) for all  $z \in C(S)$ ,  $(xzy)_R xy(xzy)_L = xzy$ .

**Lemma 25** *A quasi-inverse monoid is stable (as a quasi-inverse monoid) if and only if, ordered by the (quasi-inverse) natural order, it is a (stable) ordered monoid.*

*Proof.* Let  $S$  be a quasi-inverse monoid. Observe that axiom (Q5) essentially says that for all  $x$  and  $y \in S$ , for all  $z \in C(S)$ ,  $xzy \leq xy$ .

If  $S$  ordered by the natural order is a (stable) ordered monoid then axiom (Q5) is obviously satisfied.

Conversely assume (Q5) holds in  $S$ . Let  $x, x' \in S$  such that  $x \leq x'$  and let  $y \in S$ .

Since  $x = x_R x'_L$  we have  $xy = x_R x'_L y$ . By Lemma 15 this implies that  $xy \leq x'_L y$  hence by axiom (Q5),  $xy \leq x'y$ .

By symmetrical arguments, we can prove that  $yx \leq yx'$ .

□

### 3.2 Well-behaved ordered monoid

The following definition is adapted from [7].

**Definition 26** *A monoid  $S$  is a well-behaved ordered monoid when it is equipped with an order relation  $\leq$  such that:*

(W0) *for all  $x, y$  and  $z \in S$ , if  $x \leq y$  then  $xz \leq yz$  and  $zx \leq zy$ ,  
i.e. the order relation  $\leq$  is stable under product,*

*and given the set  $U(S) = \{x \in S : x \leq 1\}$  of subunits of  $S$ :*

(W1) *for all  $x \in U(S)$ ,  $xx = 1$ , i.e. subunit elements are idempotents,*

(W2) *for all  $x$  and  $y \in S$ , if  $x \leq y$  then there is  $e$  and  $f \in U(S)$  such that  $x = eyf$ , i.e. the order relation is a natural order,*

(W3) *for all  $x \in S$ , both sets  $L_x = \{e \in U(S) : xe = x\}$  and  $R_x = \{e \in U(S) : ex = x\}$  have least element,*

**Lemma 27** *Let  $S$  be a stable quasi-inverse monoid with natural order  $\leq$ . Monoid  $S$  ordered by  $\leq$  is a well-behaved ordered monoid.*

*Proof.* Axiom (W0) follows from axiom (Q5) and Lemma 25.

Axiom (W1). Assume there is some  $x \in S$  such that  $x \leq 1$ . This means, by definition of the natural order, that  $x = x_L x_R$  hence by Lemma 3,  $x_L = x$  (and  $x_R = x$ ) hence by axiom (Q1)  $x$  is idempotent.

Axiom (W2) follows from the definition of the natural order and the fact that  $C(S) = U(S)$ .

Axiom (W3). By symmetry, we only prove the left case. Let  $y \in U(S)$  such that  $xy = x$ . We have to show that  $x_L \leq y_R$  or, equivalently by Lemma 12, that  $x_L y_L = x_L$ . But, by (Q3) we have  $(xy)_L y_L = (xy)_L$  with  $(xy)_L = x_L$  since  $xy = x$  hence  $x_L y_L = x_L$ .  $\square$

**Lemma 28** *Let  $S$  be a well-ordered monoid  $S$  with order  $\leq$  and subunits  $U(S)$ . Defining, for all  $x \in S$ ,  $x_L = \bigwedge \{y \in U(S) : xy = x\}$  and  $x_R = \bigwedge \{y \in U(S) : yx = x\}$  turns  $S$  into a stable quasi-inverse monoid.*

*Proof.* Axiom (Q4) follows from stability (W0) and idempotence of subunits (W1).

Axioms (Q0) to (Q2) then immediately follows from axiom (W1) and (W3).

Axiom (Q3) follows from (W'') and the observation that for all  $x$  and  $y \in S$  and for all  $z \in U(S)$ , if  $zx = x$  (resp.  $yz = y$ ) then  $zxy = xy$  (resp.  $xyz = xy$ ).

Last, axioms (Q5) follows from Lemma 25.  $\square$

In other words, quasi-inverse monoid and well-ordered monoid are equivalent algebraic structures.

### 3.3 $F^*$ -property

An analogous of the  $F^*$ -property among  $E^*$ -unitary inverse monoid is worth being defined over quasi-inverse monoid.

**Definition 29** *A quasi-inverse monoid  $S$  is said to be  $F^*$  when any non zero element  $x \in S$  lies beneath a unique maximal element  $\hat{x} \in S$ . In particular, in such a monoid, for every  $x \in S$ , we have  $x = x_R \hat{x} x_L$ .*

**Lemma 30** *A  $F^*$ -quasi-inverse monoid ordered by natural order is a meet semi-lattice. Moreover, for all  $x$  and  $y \in S$ , either  $x \wedge y = 0$  or  $0 < x \wedge y = x_R y x_L = y_R x y_L$ .*

*Proof.* Let  $x$  and  $y \in S$ .



If there is no non zero  $z \in S$  such that  $z \leq x$  and  $z \leq y$ . In that case  $x \wedge y = 0$ .

Otherwise, let  $z$  such an element. By  $F^*$  property, we have  $\widehat{z} = \widehat{x} = \widehat{y}$  hence  $y = y_R \widehat{z} y_L$  and  $x = x_R \widehat{z} y_L$ .

Let then  $t = x_R y x_L$ . By commutation of idempotent, we also have  $t = y_R x y_L$ .

We claim that  $t \leq y$ . Indeed,  $x_R \leq_p t$  hence  $t_R x_R = t_R$ . Likewise,  $x_L \leq_s t$  hence  $t_L x_L = t_L$ . But then, because  $t = x_R y x_L$ , we also have  $t = t_R t t_L = t_R x_R y x_L t_L = t_R y t_L$  hence  $t \leq y$ .

A symmetrical argument show that  $t \leq x$ .

It remains to prove that  $z \leq t$ . Since  $z \leq x$  we have  $z = z_R x z_L$ , hence  $x_R z x_L = z$ . But since  $z \leq y$  we also have  $z = z_R y z_L$  hence  $z = x_R z_R y z_L x_R$  hence  $z = z_R t z_L$ .  $\square$

**Remark** The formulation of the previous Lemma, distinguishing the cases  $x \wedge y$  is zero or not, may seem a little strange. We must put the emphasis on the fact that in an  $F^*$ -quasi-inverse monoid  $S$ , even when both  $x_R y y_L$  and  $y_R x x_L$  are non zero for some  $x$  and  $y \in S$ , nothing ensure they are equal and thus nothing ensures they are equal. Henceforth it can still be the case that  $x \wedge y = 0$ . In Section 4 below, we will see that the quasi-inverse extension construction provides plenty of such examples.

### 3.4 Stable $F^*$ -property

We consider a stronger version of the  $F^*$ -property where we assume moreover that maximal elements of a  $F^*$ -quasi-inverse monoids form a submonoid.

**Definition 31** *An  $F^*$ -quasi-inverse monoid is said stably  $F^*$  when, for all non zero  $x$  and  $y \in S$ ,  $\widehat{x}\widehat{y} \neq 0$  and  $\widehat{\widehat{x}\widehat{y}} = \widehat{x}\widehat{y}$ , i.e. the set  $\widehat{S}$  of maximal elements of  $S$  is a submonoid of  $S$ .*

**Remark** Observe that the trivial extension  $S^0$  of an arbitrary monoid  $S$  is a stably  $F^*$ -quasi-inverse monoid. In Section 4 we will prove that this is true as well for the non trivial extension  $\mathcal{Q}(S)$  of  $S$ .

Applied to inverse monoid, the stably  $F^*$ -property induces a rather strong additional property.

**Lemma 32** *Let  $S$  be an inverse monoid ordered by natural order. If  $S$  is stably  $F^*$  then  $\widehat{S}$  is a group.*

*Proof.* Let  $x \in \widehat{S}$ . We have  $x^{-1} \in \widehat{S}$ . Otherwise, since  $x^{-1} \neq 0$ , by  $F^*$  property,  $x^{-1} < \widehat{x^{-1}}$  and thus, as the natural order is stable by inverse mapping,  $x < (\widehat{x^{-1}})^{-1}$  which contradicts the fact that  $x \in \widehat{S}$ .

We conclude then by stability assumption of  $\widehat{S}$ . Indeed, this means that both  $xx^{-1}$  and  $x^{-1}x \in \widehat{S}$  hence, since both  $x^{-1}x \leq 1$  and  $xx^{-1} \leq 1$ , by  $F^*$ -assumption,  $xx^{-1} = x^{-1}x = 1$ .  $\square$

**Remark** Though fairly simple, it seems that this fact was left unnoticed. This illustrates again the well-known fact that semigroup theories and related monoid theories sometimes slightly differs and the study presented here is concerned with monoids rather than semigroups.

## 4 A non trivial quasi-inverse extension

In this section, we define from arbitrary monoid  $S$ , sort of a monoid of positive tiles  $\mathcal{Q}(S)$  much in the same way the monoid  $T_A$  of positive tiles [7] over the alphabet  $A$  is been built upon  $A^*$  with a canonical injection from  $S$  to  $\mathcal{Q}(S)$ . It turns out that  $\mathcal{Q}(S)$  is a non trivial quasi-inverse monoid.

### 4.1 Prefix and suffix upper sets

Let  $S$  be a monoid. Let  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) defined to be the set of upward closed subsets of  $S$  preordered by  $\leq_p$  (resp.  $\leq_s$ ).

More precisely, as  $S$  is a monoid hence with  $1 \in S$ ,  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) is the set  $U \subseteq S$  such that  $US = U$  (resp.  $SU = U$ ).

For both  $x = p$  or  $x = s$ , elements of  $\mathcal{U}_x(S)$  are from now on called *x-upper set*. The set  $\mathcal{U}_x(S)$  is turned into a monoid by taking  $\cap$  as product. One can check that, indeed, the intersection of two  $x$ -upper sets is indeed a upper set. The neutral (or maximal) element is then  $S$  itself the absorbant is the empty set  $\emptyset$ .

**Remark** In semigroup theory non empty elements of  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) are sometimes called right ideals (resp. left ideals) of  $S$ . As ideals in order theory must satisfy some extra condition we prefer to stick to the notion of upper sets.

**Lemma 33** *Let  $S$  be some monoid and let  $x \in S$ . One have:*

(0)  $xS$  is a  $p$ -upper set (resp.  $Sx$  a  $s$ -upper set).

Moreover, for every  $p$ -upper set (resp.  $s$ -upper set)  $U \subseteq S$ :

- (1) if  $x \in U$  then  $x^{-1}(U) = S$  (resp.  $(U)x^{-1} = S$ ),
- (2)  $xU$  is a  $p$ -upper set (resp.  $Ux$  is a  $s$ -upper set),
- (3)  $x^{-1}(U)$  is a  $p$ -upper set (resp.  $(U)x^{-1}$  is a  $s$ -upper sets),
- (4)  $xx^{-1}(U) \subseteq U \subseteq x^{-1}(xU)$  (resp.  $(U)x^{-1}x \subseteq U \subseteq (Ux)x^{-1}$ ),

*Proof.* Straightforward. □

## 4.2 The extension

**Definition 34** Let  $S$  be a monoid. The quasi-inverse extension of  $S$ , written  $\mathcal{Q}(S)$  is defined to be

$$\mathcal{Q}(S) = (\mathcal{U}_s(S) - \emptyset) \times S \times (\mathcal{U}_p(S) - \emptyset) + 0$$

with product defined by

$$(L_1, x_1, R_1) \cdot (L_2, x_2, R_2) = (L_1 \cap (L_2)x_1^{-1}, x_1x_2, R_2 \cap x_2^{-1}(R_1))$$

when both  $L_1 \cap (L_2)x_1^{-1} \neq \emptyset$  and  $R_2 \cap x_2^{-1}(R_1) \neq \emptyset$ , and the product equals 0 otherwise. This product is extended to 0 as expected 0 being the absorbant elements.

Before proving in Theorem 37 that the monoid  $\mathcal{Q}(S)$  is a quasi-invers monoid, let us examine some properties of the idempotent elements of  $S$ .

**Lemma 35** A non zero triple  $(L, x, R) \in \mathcal{Q}(S)$  is idempotent if and only if  $x \in S$  is idempotent,  $L \subseteq (L)x^{-1}$  and  $R \subseteq x^{-1}(R)$ .

In particular any element of the form  $(L, 1, R) \in \mathcal{Q}(S)$  is idempotent.

*Proof.* Let  $(L, x, R)$  be an idempotent of  $\mathcal{Q}(S)$ . By definition of the product in  $\mathcal{Q}(S)$  this means that  $xx = x$  hence  $x$  is idempotent in  $S$ ,  $L \cap (L)x^{-1} = L$  hence  $L \subseteq (L)x^{-1}$  and  $R \cap x^{-1}(R) = R$  hence  $R \subseteq x^{-1}R$ .

Conversely, let  $(L, x, R)$  be a non zero element of  $\mathcal{Q}(S)$  with  $x$  idempotent,  $L \subseteq (L)x^{-1}$  and  $R \subseteq x^{-1}(R)$ . We have  $(L, x, R) \cdot (L, x, R) = (L \cap (L)x^{-1}, xx, R \cap x^{-1}(R))$ . By idempotence,  $xx = x$ . The fact that  $L = (L \cap (L)x^{-1})$  (resp.  $R = R \cap x^{-1}(R)$ ) immediately follows from the fact that  $L \subseteq (L)x^{-1}$  (resp.  $R \subseteq x^{-1}(R)$ ). □

**Lemma 36** The set  $C(\mathcal{Q}(S))$  of elements of the form  $(L, 1, R)$  in  $\mathcal{Q}(S)$  is a commutative monoid.

*Proof.* Let  $(L, 1, R)$  and  $(M, 1, N)$  be two elements in  $\mathcal{Q}(S)$  with the desired form.

Assume first the product  $(L, 1, R).(M, 1, N)$  is non zero. By definition of the product, this means that  $(L, 1, R).(M, 1, N) = (L \cap M, 1, R \cap N)$  hence, by symmetrical arguments,  $(L, 1, R).(M, 1, N) = (M, 1, N).(L, 1, R)$ .

In the case  $(L, x, R).(M, y, N) = 0$  similar arguments show that we also have  $(M, y, N).(L, x, R) = 0$  since either  $L \cap M = \emptyset$  or  $R \cap N = \emptyset$ .  $\square$

**Theorem 37** *When  $S$  be a monoid,  $\mathcal{Q}(S)$  with left and right context mappings defined, for every non zero element  $(L, x, R)$  by:*

$$(L, x, R)_L = (Lx, 1, R) \text{ and } (L, x, R)_R = (L, 1, xR)$$

*is a quasi-inverse monoid.*

*Moreover, the mapping  $i : S \rightarrow Qq(S)$  defined  $i(x) = (S, x, S)$  is a one-to-one morphism.*

*Proof.* Let  $S$  and  $\mathcal{Q}(S)$  defined as above.

Observe first that Lemma 33 ensures us that the left and right context mappings defined as above are indeed well-defined, i.e. for all  $q = (L, x, R) \in \mathcal{Q}(S)$ ,  $q_L = (Lx, 1, R) \in \mathcal{Q}(S)$  and  $q_R = (L, 1, xR) \in \mathcal{Q}(S)$ .

Axiom (Q0) and (Q4) follow from Lemma 36 above.

Axiom (Q1) follows from Lemma 35

Let us prove axiom (Q2). Let  $q = (L, x, R)$ . We need to prove that  $qq_L = q$ . By definition of the product  $qq_L = (L \cap (Lx)x^{-1}, x.1, x^{-1}(xR) \cap 1^{-1}(R))$  hence the result since, by Lemma 33 we have  $L \subseteq (Lx)x^{-1}$  and  $R \subseteq x^{-1}(xR)$ , and we obviously have  $x1 = x$ , and  $1^{-1}(R) = R$ .

Let us prove property (Q3). Let  $q = (L, x, R)$  and  $p = (M, y, N)$  be some non zero element of  $\mathcal{Q}(S)$ . Assume  $q \leq_s p$ . We need to prove that  $p_L q_L = p_L$ .

Since  $q \leq_s p$ , this means there is some  $r = (O, z, P)$  such that  $p = rq$ , equivalently (a)  $M = O \cap (L)z^{-1}$ , (b)  $y = zx$ , and (c)  $N = R \cap x^{-1}(P)$ .

Now, we have  $pq_L = (M \cap (Lx)y^{-1}, y, R \cap N)$ . We want to prove it equals  $(M, y, N)$ .

Observe first that, by (a) and (b),  $M \subseteq (Lx)y^{-1}$  hence  $M \cap (Lx)y^{-1} = M$ . In fact, let  $u \in M$ . By (a),  $u \in (L)z^{-1}$  hence  $uz \in L$  hence  $uzx \in Lx$ . But by (b)  $uzx = uy$  hence  $uy \in Lx$  hence  $u \in (Lx)y^{-1}$ .

By (c), we have  $N \subseteq R$  hence  $R \cap N = N$ .

The proof of the last statement is straightforward since, by Lemma 33,  $x^{-1}S = S = Sx^{-1}$  for arbitrary  $x \in S$  hence, over images by  $i$  of elements of  $S$ , the product in  $\mathcal{Q}(S)$  just mimics the product in  $S$ .  $\square$

The following theorem says that our construction above essentially extend to arbitrary monoid the construction of the monoid of positive tiles [7] from the free monoid  $A^*$ .

**Theorem 38** *There is a one to one morphism from  $T_A$  to  $\mathcal{Q}(A^*)$ .*

*Proof.* Observe that  $A^*$  is totally ordered by  $\leq_s$  and  $\leq_p$ . It follows that for  $x = p$  and  $x = s$ , the mapping  $\varphi_x : A^* + 0 \rightarrow \mathcal{U}_x(A^* + 0)$  defined, for every  $u \in A^* + 0$ , by  $\varphi_x(u) = \{v \in A^* + 0 : u \leq_x v\}$  is one-to-one.

Is is then an easy task to check that  $\varphi : T_A \rightarrow \mathcal{Q}(A^* + 0)$  defined by  $\varphi(0) = 0$  and, for every tile  $(u, v, w) \in A_T$ ,  $\varphi((u, v, w)) = (\varphi_s(u), v, \varphi_p(w))$  is a one-to-one morphism.  $\square$

### 4.3 Natural order in quasi-inverse extensions

Let us now review some additional properties of the natural order in  $\mathcal{Q}(S)$ .

**Lemma 39** *In  $\mathcal{Q}(S)$ , for every elements  $(L_1, x_1, R_1)$  and  $(L_2, x_2, R_2)$  we have  $(L_1, x_1, R_1) \leq (L_2, x_2, R_2)$  if and only if  $L_1 \subseteq L_2$ ,  $x_1 = x_2$  and  $R_1 \subseteq R_2$ .*

*Proof.* Let  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$ .

We have  $u_1 \leq u_2$  when, by definition,  $u_1 = (u_1)_R u_2 (u_1)_L$  or, stated explicitly:  $(L_1, x_1, R_1) = (L_1, 1, x_1 R_1) \cdot (L_2, x_2, R_2) \cdot (L_1 x_1, 1, R_1)$ .

It follows that  $u_1 \leq u_2$  if and only if (1)  $x_1 = x_2 = x$ , (2)  $L_1 = L_1 \cap L_2 \cap (L_1 x) x^{-1}$ , (3)  $R_1 = R_1 \cap R_2 \cap x^{-1} (x R_1)$ .

However, by Lemma 33, we have  $L_1 \subseteq (L_1 x) x^{-1}$  and  $R_2 \subseteq x^{-1} (x R_2)$  hence (2) can be rewritten  $L_1 = L_1 \cap L_2$  or equivalently  $L_1 \subseteq L_2$  and (3) can be rewritten  $R_1 = R_1 \cap R_2$  henceforth  $R_1 \subseteq R_2$ .  $\square$

**Theorem 40** *Quasi-inverse monoid  $\mathcal{Q}(S)$  is a stable, i.e. for every  $u, v$  and  $w \in \mathcal{Q}(S)$ , if  $u \leq v$  then  $uw \leq vw$  and  $wu \leq wv$ .*

*Proof.* Let  $u = (L_1, x_1, R_1)$ ,  $v = (L_2, x_2, R_2)$  and  $w = (M, y, N)$ . Assume  $u \leq v$ . By Lemma 39 above, we have  $x_1 = x_2$  from now on denoted by  $x$ ,  $L_1 \subseteq L_2$  and  $R_1 \subseteq R_2$ .

By definition of the product we have  $uw = (L_1 \cap (M)x^{-1}, xy, N \cap y^{-1}(R_1))$  and  $vw = (L_2 \cap (M_2)x^{-1}, xy, N \cap y^{-1}(R_2))$ . Hence  $uw \leq vw$  by applying Lemma 39 and stability of the inclusion order by intersection and residual.

Symmetrical arguments show that  $wu \leq wv$ .  $\square$

**Remark** By applying Lemma 25 we could have proven instead that the quasi-inverse extension  $\mathcal{Q}(S)$  of  $S$  satisfies axiom (Q5)

Lemma 39 has more consequences that are listed below.

**Lemma 41** *The extension  $\mathcal{Q}(S)$  of monoid  $S$  is a meet semi-lattice.*

*Proof.* By Lemma 39 we easily check that  $\mathcal{Q}(S)$  satisfies the  $F^*$ -property, i.e. any non zero element  $(L, x, R) \in \mathcal{Q}(S)$  lies beneath a unique maximal element  $(S, x, S)$ , hence this result follows from Lemma 30.  $\square$

**Lemma 42** *The extension  $\mathcal{Q}(S)$  is (upward) bounded complete and (upward) directed complete.*

*Proof.* Let  $X \subseteq \mathcal{Q}(S)$  a non empty either directed or bounded subset of  $\mathcal{Q}(S)$ . We have to show that  $X$  admits a greatest lower bound  $\bigvee X$ .

If  $X = \{0\}$  then  $\bigvee X = 0$  and we are done. Otherwise, in both directed and bounded case, by Lemma 39, there is some  $s \in S$  such that every  $u \in X$  is of the form  $(L_u, x, R_u)$ . It follows that  $\bigvee X = (\bigcup_{u \in X} L_u, s, \bigcup_{u \in X} R_u)$ .  $\square$

#### 4.4 Properties of maximal elements

Last, we study some properties of the maximal elements of  $\mathcal{Q}(S)$ . We proved in particular that  $\mathcal{Q}(S)$  is a non trivial quasi-inverse monoid in a strong sense.

**Lemma 43** *Let  $u$  and  $v \in \mathcal{Q}(S)$ . If both  $u$  and  $v$  are maximal then so is  $uv$ .*

*Moreover, if we only assume that  $v$  is maximal then the following properties are equivalent:*

$$(1) \ v_R = u_R v_R \text{ (resp. } v_L = v_L u_L),$$

$$(2) \ v = u_R v \text{ (resp. } v = v u_L),$$

$$(3) \ u \leq_p v \text{ (resp. } u \leq_s v).$$

*In particular, with  $v = 1$  this proves that  $\mathcal{Q}(S)$  is non trivial.*

*Proof.* Let  $u$  and  $v$  two maximal elements of  $S$ . By Theorem 37 and Lemma 39, maximal elements of  $\mathcal{Q}(S)$  are exactly the images of elements of  $S$  by canonical injection monoid. This proves that  $uv$  is also a maximal elements of  $\mathcal{Q}(S)$ .

Assume now  $u = (L, x, R)$  hence  $u_R = (L, , 1, xR)$  and  $v = (S, y, S)$ . Since  $\mathcal{Q}(S)$  is quasi-inverse (Theorem 37), (3) implies (1) and, by Lemma 5, (1) implies (2). Then it suffices to prove that (2) implies (3).

By definition of the product, we have  $u_R v = (L, y, y^{-1}(xR))$ . Assume  $u_R v = v$ . This means that (a)  $S = L$  and that  $S = y^{-1}(xR)$  with  $y^{-1}(xR) = \{z \in S : yz \in xR\}$ . Since  $1 \in S$ , this means that  $y \in xR$  and thus (b)  $y = xz$  for some  $z \in R$ . Taking then  $w = (S, z, S)$  we have, by definition of the product,

$$uw = (L \cap (S)x^{-1}, xz, S \cap z^{-1}(R))$$

Now by (a)  $L = S$  and since  $x \in S$ , by Lemma 33,  $(S)x^{-1} = S$ . By (b)  $xz = y$  and, since  $z \in R$ , again by Lemma 33,  $z^{-1}(R) = S$ . It follows that  $uw = v$  henceforth  $u \leq_p v$ .

The case of left context and suffix preorder is obtained by symmetrical arguments.  $\square$

Last, the following Lemma tells how extended Green relations behave on maximal elements of  $\mathcal{Q}(S)$ .

**Lemma 44** *In  $\mathcal{Q}(S)$ , restricted to maximal elements, relation  $\tilde{\mathcal{L}}$  equals  $\mathcal{L}$ -equivalence and relation  $\tilde{\mathcal{R}}$  equals  $\mathcal{R}$ -equivalence.*

*Proof.* Let  $u = (L_1, x_1, R_1)$ ,  $v = (L_2, x_2, R_3)$ . Assume  $u \tilde{\mathcal{R}} v$ . This means that for all  $e = (L, 1, R)$  one has  $eu = u$  if and only if  $ev = v$ .

On the left, this implies that for all  $L \in \mathcal{U}_s(S)$ , we have  $L_1 \subseteq L$  if and only if  $L_2 \subseteq L$ . As this must be true for  $L = L_1$  or  $L = L_2$  this implies that  $L_1 = L_2$ .

On the right, this implies that for all  $R \in \mathcal{U}_p(S)$ , we have  $R_1 \subseteq x_1^{-1}(R)$  if and only if  $R_2 \subseteq x_2^{-1}(R)$ .

Assume now  $R_1 = R_2 = S$ .

Taking  $R = x_2 S$  this implies  $1 \in x_1^{-1}(x_2 S)$  hence  $x_2 \leq_p x_1$ . Taking  $R = x_1 S$  this implies  $1 \in x_2^{-1}(x_1 S)$  hence  $x_1 \leq_p x_2$ . It follows that  $x_1$  and  $x_2$  are  $\mathcal{R}$ -equivalent and thus, so are  $u$  and  $v$ .  $\square$

## 5 Quasi-inverse monoids and prehomomorphisms

In this section we show that our extension construction  $\mathcal{Q}(S)$  induces an expansion in the sense of Birget and Rhodes [15] of the category of trivial

quasi-inverse monoid with (well-behaved) prehomomorphisms into the category of stable quasi-inverse monoids with well-behaved prehomomorphisms.

## 5.1 Prehomomorphisms

The following definition is adapted from McAlister and Reilly [13].

**Definition 45** *Let  $S$  and  $T$  be two ordered monoids with zeros. A mapping  $\varphi : S \rightarrow T$  is a prehomomorphism when  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , for all  $x$  and  $y \in S$ , if  $x \leq y$  then  $\varphi(x) \leq \varphi(y)$  and, for all  $x$  and  $y \in S$ ,  $\varphi(xy) \leq \varphi(x)\varphi(y)$*

*A prehomomorphism  $\varphi$  such that  $\varphi(xy) < \varphi(x)\varphi(y)$  if and only if  $xy = 0$  is called a trivial prehomomorphism.*

Observe that given a prehomomorphism  $\varphi : S \rightarrow T$ ,  $(\varphi(S))^*$  is a submonoid of  $T$  while, in general,  $\varphi(S)$  may not be closed under product.

**Lemma 46** *For every prehomomorphism  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow U$ , the mapping  $\varphi\psi : S \rightarrow U$  defined for all  $x \in S$  by  $\varphi\psi(x) = \psi(\varphi(x))$  is a prehomomorphism.*

In other words, ordered monoids and prehomomorphisms forms a category.

## 5.2 Well-behaved prehomomorphisms

**Definition 47** *Let  $S$  and  $T$  be two stably  $F^*$ -quasi-inverse monoid. A prehomomorphism  $\varphi : S \rightarrow T$  is a well-behaved prehomomorphism when the following condition are satisfied:*

(P1) *for all  $x$  and  $y \in \widehat{S}$ ,  $\varphi(xy) \in \widehat{T}$ ,*

(P2) *for all  $x, y$  and  $z \in \widehat{S}$ ,  $\varphi(x_L y z_R) = (\varphi(x))_L \varphi(y) (\varphi(z))_R$ ,*

*where  $\widehat{S}$  (resp.  $\widehat{T}$ ) denotes the set of maximal elements of  $S$  (resp.  $T$ ).*

In particular, as an immediate consequence of (P1) and (P2), for all non zero element  $x \in S$ , if  $\varphi(x) \neq 0$ , we have  $\widehat{\varphi(x)} = \varphi(\widehat{x})$  and, if  $x \in \widehat{S}$ ,  $\varphi(x_L) = (\varphi(x))_L$  and  $\varphi(x_R) = (\varphi(x))_R$ .

**Lemma 48** *Stably  $F^*$ -quasi-inverse monoids with well-behaved prehomomorphisms defined a category that contains the category of monoid and morphism as a (isomorphic) subcategory.*



*Proof.* Observe first that any monoid morphism  $\varphi : S \rightarrow T$  can be lifted to a trivial prehomomorphism  $\varphi : S^0 \rightarrow T^0$  by taking additionally  $\varphi(0) = 0$ . As both  $S^0$  and  $T^0$  are stably  $F^*$ -quasi-inverse monoids,  $\varphi$  extended in such a way is moreover a well-behaved prehomomorphism.

Let  $S$ ,  $T$  and  $U$  be three stably  $F^*$ -quasi-inverse monoids. Let then  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow U$  be two well-behaved prehomomorphisms. By Lemma 46,  $\psi \circ \varphi : S \rightarrow U$  is a prehomomorphism.

The fact it is well-behaved is immediate from the definitions and Lemma 48 above.  $\square$

### 5.3 Quasi-inverse extension and prehomomorphisms

We have already shown that for every monoid  $S$  there is a one-to-one morphism  $i : S \rightarrow \mathcal{Q}(S)$ , the canonical injection of  $S$  into  $\mathcal{Q}(S)$ . It can be extended to a trivial one-to-one prehomomorphism  $i : S^0 \rightarrow \mathcal{Q}(S)$  by taking  $i(0) = 0$ . Remember that  $S^0$  is the trivial extension of  $S$  with an *additional* new zero. What about a canonical (inverse) surjection from  $\mathcal{Q}(S)$  onto  $S^0$ ?

**Definition 49** *For all monoid  $S$ , let define the mapping  $\sigma_S : \mathcal{Q}(S) \rightarrow S^0$  that maps 0 to 0 and any non zero triple  $(L, x, R) \in \mathcal{Q}(S)$  to  $x$ .*

**Lemma 50** *For all monoid  $S$ , the mapping  $\sigma_S : \mathcal{Q}(S) \rightarrow S^0$  is a (trivial hence well-behaved) onto prehomomorphism.*

*Proof.* Mapping  $\sigma_S$  is obviously onto and monotonic. It remains to prove it is sub-multiplicative. Let  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$  two non zero elements of  $\mathcal{Q}(S)$ . If  $u_1 u_2 = 0$  we have  $\varphi(u_1 u_2) = 0 < \varphi(u_1) \varphi(u_2) = x_1 x_2$ . If  $u_1 u_2 \neq 0$  then  $\varphi(u_1 u_2) = x_1 x_2$  henceforth  $\varphi(u_1 u_2) = \varphi(u_1) \varphi(u_2)$ . In other words,  $\varphi$  is a trivial prehomomorphism.  $\square$

### 5.4 Expansion prehomomorphism

We provide here the last piece that show that our extension construction is actually an expansion in the sense of Birget and Rhodes in the category of ordered monoids and prehomomorphisms.

**Definition 51** *Let  $S$  and  $T$  be two monoids. Let  $\varphi : S \rightarrow T$  be a monoid morphism and let still denote by  $\varphi$  the lift of  $\varphi$  to  $\mathcal{P}(S)$  defined by as  $\varphi : \mathcal{P}(S) \rightarrow \mathcal{P}(T)$  by taking, for every  $X \subseteq S$ ,  $\varphi(X) = \{\varphi(x) : x \in X \cap S\}$ .*

*Let then  $\mathcal{Q}(\varphi)$  be the expansion mapping  $\mathcal{Q}(\varphi) : \mathcal{Q}(S) \rightarrow \mathcal{Q}(T)$  defined by  $\mathcal{Q}(\varphi)(0) = 0$  and for all non zero element  $u = (L, x, R) \in \mathcal{Q}(S)$ , by  $\mathcal{Q}(\varphi)(u) = (T\varphi(L), \varphi(x), \varphi(R)T)$ .*

**Lemma 52** *For all monoids  $S$  and  $T$ , for all (monoid) morphism  $\varphi : S \rightarrow T$ , the expansion mapping  $\mathcal{Q}(\varphi) : \mathcal{Q}(S) \rightarrow \mathcal{Q}(T)$  is a well-behaved prehomomorphism such that, moreover, the following diagram commutes.*

$$\begin{array}{ccc}
 \mathcal{Q}(S) & \xrightarrow{\mathcal{Q}(\varphi)} & \mathcal{Q}(T) \\
 \downarrow \sigma_S & & \downarrow \sigma_T \\
 S^0 & \xrightarrow{\varphi} & T^0
 \end{array}$$

In other words,  $\varphi \circ \sigma_S = \sigma_T \circ \mathcal{Q}(\varphi)$ .

*Proof.* Let  $S$  and  $T$  and  $\tau : S \rightarrow T$  as above. In order to simplify notation, let us write  $\varphi'$  in place of  $\mathcal{Q}(\varphi)$ .

Observe first that, by definition,  $\varphi'(0) = 0$  and we also have

$$\varphi'(1) = (T\varphi(S), \varphi(1), \varphi(S)T)$$

hence  $\varphi'(1) = 1$  since  $1 \in S$  and  $\varphi(1) = 1$  and thus  $1 \in \varphi(S)$ .

Monotonicity of  $\varphi'$  immediately follows from the characterization of the natural order provided by Lemma 39 and the fact that  $\varphi$  extended to  $\mathcal{P}(S)$  is monotonic.

Let then  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$  two non zero elements of  $\mathcal{Q}(S)$ . We have to show that  $\varphi'(u_1 u_2) \leq \varphi'(u_1) \varphi'(u_2)$ . By definition, we have:

$$\varphi'(u_1 u_2) = (T\varphi(L_1 \cap (L_2)x_1^{-1}), \varphi(x_1)\varphi(x_2), \varphi(x_2^{-1}(R_1) \cap R_2)T)$$

hence by distributivity of (the extension of)  $\varphi$  and product (over  $\mathcal{P}(S)$ ) w.r.t. the intersection

$$\begin{aligned}
 \varphi'(u_1 u_2) = & (T\varphi(L_1) \cap T\varphi((L_2)x_1^{-1}), \\
 & \varphi(x_1)\varphi(x_2), \varphi(x_2^{-1}(R_1))T \cap \varphi(R_2)T)
 \end{aligned}$$

By definition, we also have:

$$\begin{aligned}
 \varphi'(u_1) \varphi'(u_2) = & (T\varphi(L_1) \cap T\varphi(L_2)(\varphi(x_1))^{-1}, \\
 & \varphi(x_1)\varphi(x_2), (\varphi(x_2))^{-1}\varphi(R_1)T \cap \varphi(R_2)T)
 \end{aligned}$$

Now we conclude that  $\varphi'(u_1u_2) \leq \varphi'(u_1)\varphi'(u_2)$  by applying Lemma 39, monotonicity of product and intersection in  $\mathcal{P}(S)$  and the fact that, for all  $X \subseteq S$  and all  $x \in S$  we have

$$\varphi(x^{-1}(X)) \subseteq (\varphi(x))^{-1}(\varphi(X))$$

and

$$\varphi((X)x^{-1}) \subseteq (\varphi(X))(\varphi(x))^{-1}$$

Indeed, let  $x \in S$  and  $X \subseteq S$  as above. Let  $y \in \varphi(x^{-1}(X))$ . This means that  $y = \varphi(z)$  for some  $z \in x^{-1}(X)$  hence some  $z$  such that  $xz \in X$ . But then, this also means, as  $\varphi$  extended to  $S$  is still a monoid morphism, that  $\varphi(x)\varphi(z) \in \varphi(X)$  hence  $y = \varphi(z) \in (\varphi(x))^{-1}(\varphi(X))$ . The other case is symmetric.

A similar argument as above for proving that  $\varphi'(1) = 1$  shows that for all maximal element  $u = (S, x, S)$  of  $\mathcal{Q}(S)$ , we have  $\varphi'(u) = (T, \varphi(x), T)$  hence  $\varphi'(u)$  is also maximal and, moreover, for two maximal elements  $u$  and  $v \in \mathcal{Q}(S)$ ,  $\varphi'(uv) = \varphi(u)\varphi(v)$ . In other words,  $\varphi'$  is well-behaved.

Diagram commutation immediately follows from the definitions of  $\sigma_S$ ,  $\sigma_T$  and  $\varphi' = \mathcal{Q}(\varphi)$ .  $\square$

**Corollary 53** *The extension that lift any trivial quasi-inverse monoid  $S^0$  to a non trivial quasi-inverse monoid  $\mathcal{Q}(S)$  turns out to be an expansion in the sense of Birget and Rhodes[15] with prehomomorphisms instead of morphisms.*

*Proof.* As any monoid morphism  $\varphi : S \rightarrow T$  can be lifted up to a trivial well-behaved prehomomorphism  $\varphi : S^0 \rightarrow T^0$  by taking  $\varphi(0) = 0$ , Lemma 52 and Lemma 50 just ensure that.  $\square$

**Remark** This Lemma really says that, via this expansion, the standard notion of recognizability over monoids by means of morphisms can be lifted up into a notion of quasi-recognizability over stably  $F^*$ -quasi-inverse monoids by means of well-behaved morphisms.

Our construction in [8], applied to languages of positives tiles, is an instance of that more general fact.

*Our construction here is stlighthly simpler than in [8] since here we better understand the role played by zero.*

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## References

- [1] Claire Cornock. *Restriction Semigroups: Structure, Varieties and Presentations*. PhD thesis, Departement of Mathematics University of York, 2011.
- [2] Claire Cornock and Victoria Gould. Proper two-sided restriction semigroups and partial actions. *Journal of Pure and Applied Algebra*, 216:935–949, 2012.
- [3] J. Fountain, G M S Gomes, and Victoria Gould. A munn type representation for a class of e-semiadequate semigroups. *Journal of Algebra*, 218:693–714, 1999.
- [4] Victoria Gould. Restriction and Ehresmann semigroups. In *Proceedings of the International Conference on Algebra 2010*. World Scientific, 2010.
- [5] C. D. Hollings. From right PP monoids to restriction semigroups: a survey. *European Journal of Pure and Applied Mathematics*, 2(1):21–57, 2009.
- [6] David Janin. Modélisation compositionnelle des structures rythmiques : une exploration didactique. Technical Report RR-1455-11, LaBRI, Université de Bordeaux, 2011.
- [7] David Janin. On languages of one-dimensional overlapping tiles. Technical Report RR-1457-12, LaBRI, Université de Bordeaux, 2012.
- [8] David Janin. Quasi-recognizable vs MSO definable languages of one-dimensionnal overlapping tiles. Technical Report RR-1458-12, LaBRI, Université de Bordeaux, 2012.
- [9] Mark V. Lawson. Semigroups and ordered categories. i. the reduced case. *Journal of Algebra*, 141(2):422 – 462, 1991.

- [10] Mark V. Lawson. *Inverse Semigroups : The theory of partial symmetries*. World Scientific, 1998.
- [11] Mark V. Lawson. McAlister semigroups. *Journal of Algebra*, 202(1):276 – 294, 1998.
- [12] D.B. McAlister. Inverse semigroups which are separated over a sub-semigroups. *Trans. Amer. Math. Soc.*, 182:85–117, 1973.
- [13] D.B. McAlister and N. R. Reilly. E-unitary convers for inverse semigroups. *Pacific Journal of Mathematics*, 68:178–206, 1977.
- [14] K. S. S. Nambooripad. The natural partial order on a regular semigroup. *Proc. Edinburgh Math. Soc.*, 23:249–260, 1980.
- [15] J. Rhodes and J.-C. Birget. Almost finite expansions of arbitrary semigroups. *J. Pure and Appl. Algebra*, 32:239–287, 1984.