Recovering the Hamiltonian from spectral data

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RECOVERING THE HAMILTONIAN FROM SPECTRAL DATA

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Contents

Abstract. We show that the contributions to the Gutzwiller formula with observable associated to the iterates of a given elliptic nondegenerate periodic trajectory γ and to certain families of observables localized near γ determine the quantum Hamiltonian in a formal neighborhood of the trajectory γ, that is the full Taylor expansion of its total symbol near γ. We also treat the "bottom of a well" case both for general and Schrödinger operators.

1. Introduction and main results

It is well known that spectral properties of semiclassical Hamiltonians and dynamical properties of their principal symbols are linked. Even when there is no precise information "eigenvalue by eigenvalue" of the spectrum, the so-called Gutzwiller trace formula provide information on averages of the spectrum at scale of the Planck constant. More precisely, let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on a compact manifold X of dimension $n+1$, whose symbol $H(x,\xi)$ is proper (as a map from T^*X into $\mathbb{R}$). We will denote by $\sigma = \sigma(H(x,\hbar D_x))$ the spectrum of $H(x,\hbar D_x)$.

Let E be a regular value of H and γ a non-degenerate periodic trajectory of period T_γ lying on the energy surface $H = E$.

Consider the Gutzwiller trace (see [9])

$$Tr\left(\psi\left(\frac{H(x,\hbar D_x)-E}{\hbar}\right)\right) = \sum_{\sigma} \psi\left(\frac{E-E_i}{\hbar}\right) \tag{1.1}$$

where ψ is a C^∞ function whose Fourier transform is compactly supported with support in a small enough neighborhood of T_γ and is identically one in a still smaller neighborhood containing T_γ. As shown in [12], [13] (1.1) has an asymptotic expansion

$$e^{i\frac{S_\gamma}{\hbar}+\sigma_\gamma}\sum_{k=-n}^{\infty}a_k\hbar^k \tag{1.2}$$

In [6] was shown how to compute the terms of this expansion to <u>all</u> orders in terms of a microlocal Birkhoff canonical form for H in a formal neighborhood of γ, and that the constants $a_{k,r}, k, r = 0, 1, \dots$ determine the microlocal Birkhoff canonical form for H in a formal neighborhood of γ (and hence, a fortiori, determine the classical Birkhoff canonical form). When it is known "a priori" that $H(x, \hbar D_x)$ is a Schrödinger operator, it is known that the normal form near the bottom of a well determines part of the potential V [8]. But in the general case the Gutzwiller formula will determine only the normal form of the Hamiltonian, that is to say $H(x, \hbar D_x)$ only modulo unitary operators, and its principal symbol only modulo symplectomorphisms. Of course it cannot determine more, as the spectrum, and a fortiori the trace, is insensitive to unitary conjugation. The aim of this paper is to address the question of determining the *true* Hamiltonian from more precise spectral data, namely from the Gutzwiller trace formula with observables.

It is well know that, for any pseudodifferential operator $O(x, \hbar D_x)$ of symbol $\mathcal{O}(x, \xi)$, there is a result equivalent to (1.2) for the following quantity

$$Tr\left(O(x,\hbar D_x)\psi\left(\frac{H(x,\hbar D_x)-E}{\hbar}\right)\right)=\sum_{\sigma}\langle\varphi_j, O(x,\hbar D_x)\varphi_j\rangle\psi\left(\frac{E-E_i}{\hbar}\right), \tag{1.3}$$

(here φ_j is meant as the eigenvector of eigenvalue E_j) under the form of an asymptotic expansion of the form

$$e^{i\frac{S_\gamma}{\hbar}+\sigma_\gamma}\sum_{k=0}^{\infty}a_k^\gamma(\mathcal{O})\hbar^k \tag{1.4}$$

where a_k^γ are distribution supported on γ.

Through this article we will assume, without loss of generality, that the period of γ is equal to 1.

We will show in the present paper that the knowledge of the coefficients $a_k^\gamma(O)$ for a family of observables (and NOT all) localized near γ is enough to determine the (full Taylor expansion of) the (total) symbol of $H(x, \hbar D_x)$ near γ, in other words $H(x, \hbar D_x)$ microlocally in a formal neighborhood of γ, when γ is non-degenerate elliptic in the following sense.

Definition 1.1. A periodic trajectory of the Hamiltonian flow generated by $H(x, \xi)$ is said to be non-degenerate elliptic if its linearized Poincaré map has eigenvalues $(e^{\pm i\theta_i})_{1\leq i\leq n}$, $\theta_j \in \mathbb{R}$, and the rotation angles θ_i $(1 \leq i \leq n)$ and π are independent over $\mathbb{Q}$.

Definition 1.2 (Fermi coordinates)**.** We will denote by "Fermi coordinates" any system of local coordinates of $T^*\mathcal{M}$ near γ, $(x, t, \xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^1 \times \mathbb{R}$, such that $\gamma = \{x = \xi = \tau = 0\}$ and on which the principal symbol H_0 of $H(x, \hbar D_x)$ can be written as:

$$H_0(x, t, \xi, \tau) = H^0(x, t, \xi, \tau) + H_2 \tag{1.5}$$

where

$$H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|) \tag{1.6}$$

And

$$H^0(x,t,\xi,\tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau \tag{1.7}$$

The existence of such local coordinates, guaranteed by the Weinstein tubular neighborhood Theorem ([16]), was proved in [5, 6, 17] under the hypothesis of non-degeneracy mentioned earlier.

Our main results are the following:

Theorem 1.3. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by the principal symbol H_0 of $H(x,\hbar D_x)$ on the energy shell $H_0^{-1}(E)$, and let (x,t,ξ,τ) be any system of local coordinates near γ, such that $\gamma = \{x = \xi = \tau = 0\}$.*

For $1 \leq i,j \leq n$, $p \in \mathbb{Z}$, and $k \in \{1,2,3\}$, let Q_{ijp}^k and Q_p be pseudodifferential operators whose respective principal symbol $\mathcal{Q}_{ijp}^k$ and $\mathcal{Q}_p$ satisfy in a neighborhood of z_0:

$$\left\{ \begin{array}{rcl} \mathcal{Q}_{ij}^1(x,t,\xi,\tau) & = & e^{-2i\pi t} x_i \xi_j \\ \mathcal{Q}_{ij}^2(x,t,\xi,\tau) & = & e^{-2i\pi t} x_i x_j \\ \mathcal{Q}_{ij}^3(x,t,\xi,\tau) & = & e^{-2i\pi t} \xi_i \xi_j \\ \mathcal{Q}_p(x,t,\xi,\tau) & = & e^{-2i\pi t} \tau. \end{array} \right. \tag{1.8}$$

Then for any $\epsilon > 0$, the knowledge of the spectrum of $H(x,\hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max_{j=1\dots n} \theta_j + \frac{n}{2} + \epsilon)\hbar]$ and the diagonal matrix elements of Q_{ijp}^k, Q_p, $1 \leq i,j \leq n, k = 1,2,3, p \in \mathbb{Z}$ between the corresponding eigenvectors of $H(x,\hbar D_x)$ allows the explicit construction of a particular local symplectic change of variable which sends the above system to a system of Fermi coordinates near γ.

Theorem 1.4. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by the principal symbol H_0 of $H(x,\hbar D_x)$ on the energy shell $H_0^{-1}(E)$, and let (x,t,ξ,τ) be a system of Fermi coordinates near γ.*

For $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \{0,1\}$, let O_{mnds} be any pseudodifferential operator whose principal symbols $\mathcal{O}_{mnds}$ satisfies in a neighborhood of γ

$$\mathcal{O}_{mnds}(x,t,\xi,\tau) = e^{i2\pi dt} \tau^s \prod_j^n (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}. \tag{1.9}$$

Then the knowledge of the coefficients $a_k(O_{mnds})$ in (1.3)-(1.4) for $k \leq N$ and m,n,d,s satisfying

1. *$|m| + |n| \leq N$*
2. *$\forall j = 1 \dots n,\ m_j = 0$ **or** $n_j = 0$*
3. *$s = 1$ if $m = n = 0$, otherwise $s = 0$*

determines the Taylor expansion near γ of the full symbol of $H(x,\hbar D_x)$ up to order N in this system of Fermi coordinates.

Remark 1.5. Condition 2 implies that the number of observables (for each Fourier coefficient in t) needed for determining $H(x,\hbar D_x)$ up to order N is of order N^{n+1} and not N^{2n+2}, number of all polynomials of order N. The fact that not all observables are needed can be understood by the fact that we know that the Hamiltonian we are looking for is

conjugated to the normal form by a unitary operator and not by any operator (see the discussion after Theorem 2.1). At the classical level this is a trace of the fact that we are looking for a symplectomorphism, and not any diffeomorphism (see section 4).

Remark 1.6. The asymptotic expansion of the trace (1.3) involves only the microlocalization of $H(x, \hbar D_x)$ in a formal neighborhood of γ. Therefore there is no hope to recover from spectral data more precise information that the Taylor expansion of its symbol near γ. The rest of the symbol concerns spectral data of order $\hbar^\infty$.

Let us now consider the case where γ is reduced to one point, namely the "bottom of a well" case.

Let us assume that the principal symbol H_0 of $H(x, \hbar D_x)$ has a global minimum at $z_0 \in T^*\mathcal{M}$, and let $d^2H_0(z_0)$ be the Hessian of H at z_0. Let us define matrix Ω defined by $d^2H_0(z_0)(\cdot,\cdot) =: \omega_{z_0}(\cdot, \Omega^{-1}\cdot)$ where $\omega_{z_0}(\cdot,\cdot)$ is the canonical symplectic form of $T^*\mathcal{M}$ at z_0. Ω's eigenvalues are purely imaginary, let us denote them by $\pm i\theta_j$ with $\theta_j > 0,\ j = 1 \dots n$. Let us assume that $\theta_j, j = 1 \dots n$ are rationally independent.

Definition 1.7. By extension of definition 1.2, we will also denote by Fermi coordinates any system of Darboux coordinates $(x, \xi) \in T^*\mathbb{R}^n$ centered at z_0 and such that:

$$H_0(x,\xi) = H_0(z_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + O((x,\xi)^3). \tag{1.10}$$

The existence of such local coordinates will be proved in section 3, and Theorem 1.8 below proves that one can explicitely construct Fermi coordinates out of any system of Darboux coordinates.

Theorem 1.8. *Let $(x, \xi) \in T^*\mathbb{R}^n$ be **any** system of Darboux coordinates centered at z_0.*

For $1 \leq i, j \leq n$, $k \in \{1, 2, 3\}$, let Q_{ij}^k be any pseudodifferential operator whose principal symbol $\mathcal{Q}_{ij}^k$ satisfy in a neighborhood of z_0:

$$\left\{ \begin{array}{lcl} \mathcal{Q}_{ij}^1(x,\xi) & = & x_i\xi_j \\ \mathcal{Q}_{ij}^2(x,\xi) & = & x_ix_j \\ \mathcal{Q}_{ij}^3(x,\xi) & = & \xi_i\xi_j. \end{array} \right. \tag{1.11}$$

Then for any $\epsilon > 0$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max_{j=1\dots n} \theta_j + \frac{n}{2} + \epsilon)\hbar]$ and the diagonal matrix elements of $Q_{ij}^k, 1 \leq i, j \leq n, k = 1, 2, 3$ between the corresponding eigenvectors allows the explicit construction of a particular local change of variable ϕ, linear and symplectic in the above Darboux coordinates, which sends this system to a system of Fermi coordinates.

Theorem 1.9. *Let $(x, \xi) \in T^*\mathbb{R}^n$ be a system of Fermi coordinates centered at z_0.*

For $(m, n) \in \mathbb{N}^{2n}$, let O_{mn} be any pseudodifferential operator whose principal symbol $\mathcal{O}_{mn}$ satisfy in a neighborhood of z_0:

$$\mathcal{O}_{mn}(x,\xi) = \prod_{j=1}^{n} (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}, \tag{1.12}$$

Then, for all $\epsilon > 0$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[H_0(z_0), H_0(z_0) + \epsilon]$ and the diagonal matrix elements of O_{mn} between the corresponding eigenvectors, with

(1) $|m| + |n| \leq N$

(2) $\forall j = 1 \dots n,\ m_j = 0$ ***or*** $n_j = 0$,

determines the Taylor expansion up to order N of the full symbol of $H(x, \hbar D_x)$ at z_0 in the coordinates (x, ξ).

Remark 1.10. Although we will not prove it here, let us remark that Theorem 1.9 (and also Theorem 1.4) is also valid in the framework of quantization of Kälherian manifolds.

In the case where $H(x, \hbar D_x)$ is a Schrödinger operator, it is known, [8], that the (actually classical) normal form determines the Taylor expansion of the potential in the case where the latter is invariant, for each $i = 1 \dots n$, by the symmetry $x_i \to -x_i$. Same result holds without the symmetry assumption in the case $n = 1$, with assumption $V'''(0) \neq 0$, as it has been shown in [3].

Let now $H = -\hbar^2 \Delta + V$ be a Schrödinger operator and q_0 be a global non-degenerate minimum of V. Let us assume that the square-roots $(\theta_i)_{1 \leq i \leq n}$ of the eigenvalues of $d^2V(q_0)$ are linearly independent over the rationals. In that precise case, we will denote by Fermi coordinates any system of Darboux coordinates $(x, \xi) \in T^*\mathbb{R}^n$, in which the (principal or total, both notions are equivalent here) symbol H of our Schrödinger operator can be written as:

$$H(x, \xi) = V(q_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + R(x) \tag{1.13}$$

where $R(x) = O(x^3)$. The existence of such local coordinates will also be proved in section 3, and Theorem 1.11 below proves that one can explicitely construct Fermi coordinates out of any system of local coordinates centered at q_0.

Theorem 1.12 shows that the matrix elements of only a **finite number** of observables are necessary to recover the full Taylor expansion of the potential in the general case.

Theorem 1.11. *Let $x \in \mathbb{R}^n$ be **any** local system of coordinates centered at q_0, and $(x, \xi) \in T^*\mathbb{R}^n$ the corresponding Darboux coordinates centered at $(q_0, 0)$.*

Then, for any $\epsilon > 0$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + (\max_{1 \leq j \leq n} \theta_j + \frac{n}{2} + \epsilon)\hbar]$ and the diagonal matrix elements between the corresponding eigenvectors of Q_{ij}^2, $1 \leq i, j \leq n$, defined in Theorem 1.8, allows the explicit construction of a particular local change of variable ϕ, linear and symplectic in the above Darboux coordinates, which sends this system to a system of Fermi coordinates.

Theorem 1.12. *Let $(x, \xi) \in T^*\mathbb{R}^n$ be a system of Fermi coordinates centered at $(q_0, 0)$.*

Then, for all $\epsilon > 0$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$ and the diagonal matrix elements between the corresponding eigenvectors of the $2^n - 1$ observables O_{m0}, $m = (m_1, \dots, m_n) \in \{0, 1\}^n \setminus \{0\}$, defined in Theorem 1.9, determines the full Taylor expansion of V at q_0 in the coordinates x.

Remark 1.13. Note that since we are dealing with observables localized near the bottoms of the wells, the hypothesis that z_0 in Theorems 1.8-1.9 and q_0 in Theorems 1.11-1.12 are *global* minima can be released and the corresponding results can be formulated in a straightforward way.

The proof of Theorem 1.4 relies on two results having their own interest per se: **Proposition 2.13** which shows that the coefficients of the trace formula determine the matrix elements $\langle \varphi_j, O(x, \hbar D_x)\varphi_j \rangle$ where φ_j are the eigenvectors of the normal form of the Hamiltonian, and **Proposition 2.14** which states that the knowledge of the matrix elements of the conjugation of a given known selfadjoint operator by a unitary one determines, in a certain sense, the latter.

As a byproduct of Proposition 2.14 we obtain also a purely classical result, somehow analog of it: the averages on Birkhoff angles associated to Birkhoff coordinates of the same classical observables than the ones in Theorem 1.4 determine the Taylor expansion of the (true) Hamiltonian. This is the content of **Theorem 4.1** below.

The paper is organized as follows. In the first part of section 2, we give a detailed proof of Theorem 1.4, while the last subsection contains the proof of Theorems 1.9 and 1.12. In section 3, we give an explicit construction of some Fermi coordinates out of any system of local coordinates in both the periodic and "Bottom of the well" case: this is the content of Theorems 1.3, 1.8 and 1.11. In Section 4 we show the classical equivalent of our quantum formulation.

Through the whole paper, $[\![l, m]\!]$, $l < m$, will stand for the set of integers $\{l, \dots, m\}$.

2. Recovering the Hamiltonian in some given Fermi coordinates

Let us start this section by observing that, since we are only interested in a microlocal recovery of our Hamiltonian, it is enough, in order to prove Theorem 1.4, to prove following Theorem 2.1. Its proof will first need a construction of the quantum Birkoff normal form, which we give in subsection 2.1. The rest of the proof is then a consequence of Proposition 2.13 (subsection 2.2) and Proposition 2.14 (subsection 2.3). Subsection 2.4 contains the proof of the analogs of Theorem 1.4 as γ is reduced to a single point, both in the general and "Schrödinger" case: Theorems 1.9 and 1.12.

Theorem 2.1. *Let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$. Let $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$ be the canonical symplectic coordinates near $\gamma = \mathbb{S}^1$, non degenerate elliptic periodic orbit of the Hamiltonian flow generated by the principal symbol H_0 of $H(x, \hbar D_x)$ on the energy shell $H_0^{-1}(E)$.*

Let us assume that H_0 can be written in these coordinates as:

$$H_0(x, t, \xi, \tau) = H^0(x, t, \xi, \tau) + H_2 \tag{2.1}$$

where

$$H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|) \tag{2.2}$$

And H^0 is equal to:

$$H^0(x, t, \xi, \tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau \tag{2.3}$$

For $(m,d,s,n) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \{0,1\}$ let us choose any pseudodifferential operators O_{mnds} whose principal symbols are

$$\mathcal{O}_{mnds}(x,t,\xi,\tau) = e^{i2\pi dt}\tau^s \Pi_j (x_j + i\xi_j)^{m_j}(x_j - i\xi_j)^{n_j}. \tag{2.4}$$

Then the knowledge of the coefficients $a_k(O_{mnds}), k = 0 \dots N$ in ((1.3),(1.4) with

1. *$|m| + |n| \leq N$*
2. *$\forall j \in [\![1,n]\!],\ m_j = 0$ **or** $n_j = 0$*
3. *$s = 1$ if $m = n = 0$, otherwise $s = 0$*

determines the Taylor expansion near γ of the full symbol (in the system of coordinates (x,t,ξ,τ)) of $H(x,\hbar D_x)$ up to order N.

The proof of Theorem 2.1 will be essentially divided into three steps: first, we will prove in Proposition 2.2 the existence of the quantum Birkhoff normal form in a form convenient to our computations, especially concerning the discussion of orders. In Proposition 2.13, we will show that the trace formula with any observable O determines the matrix elements of O in the eigenbasis of the normal form. Finally, in Proposition 2.14, we will show that these matrix elements determines $H(x,\hbar D_x)$ in a formal neighborhood of $x = \xi = \tau = 0$, which will lead to Theorem 2.1.

For $i \in [\![1,n]\!]$, let us consider on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ the operators:

- $a_i = \frac{1}{\sqrt{2}}(x_i + \hbar\partial_{x_i})$
- $a_i^* = \frac{1}{\sqrt{2}}(x_i - \hbar\partial_{x_i})$
- $D_t = -i\hbar\partial_t$
- $P_i := \frac{1}{2}\left(-\hbar\partial_{x_i}^2 + x_i^2\right) = a_i^* a_i + \frac{\hbar}{2}$

For $\mu \in \mathbb{N}^n,\ \nu \in \mathbb{Z}$ we will denote by $|\mu,\nu\rangle$ a common eigenvector of the P_i's and D_t, namely the vectors such that:

$$P_i|\mu,\nu\rangle = (\mu_i + \frac{1}{2})\hbar|\mu,\nu\rangle \text{ and } D_t|\mu,\nu\rangle = 2\pi\hbar|\mu,\nu\rangle.$$

Those vectors can be explicitly constructed as follows:

$$|0,0\rangle(x,t) := \frac{1}{(\pi\hbar)^{\frac{n}{4}}} e^{\frac{-x^2}{2\hbar}} \tag{2.5}$$

and for any $\mu \in \mathbb{N}^n$

$$|\mu,\nu\rangle(x,t) := e^{i2\pi\nu t} \prod_{i=1}^{n} \frac{1}{\sqrt{\mu_i!}} (a_i^*)^{\mu_i} |0,0\rangle(x,t) \tag{2.6}$$

Let us recall the following:

$$\begin{cases} a_i|\mu,\nu\rangle = \sqrt{\mu_i\hbar}|\mu_1,\dots,\mu_{i-1},\mu_i - 1,\mu_{i+1},\dots,\mu_n,\nu\rangle \\ a_i^*|\mu,\nu\rangle = \sqrt{(\mu_i+1)\hbar}|\mu_1,\dots,\mu_{i-1},\mu_i + 1,\mu_{i+1},\dots,\mu_n,\nu\rangle \\ [a_i, a_j^*] = \delta_{ij}\hbar \\ [a_i, a_j] = 0. \end{cases} \tag{2.7}$$

We will write $|\mu| := \sum_{i=1}^{1=n} \mu_i$, $z_i = \frac{x_i+i\xi_i}{\sqrt{2}}$, $p_i = \frac{x_i^2+\xi_i^2}{2}$ and denote by $\mathrm{Op}^W(f)$ the pseudo-differential operator whose total Weyl symbol is f.

Finally, let us denote by a, a^* or P the n-tuple of corresponding operators a_i, a_i^*, P_i, $i \in [\![1, n]\!]$. We'll also use the usual convention that, if X is a n-tuple of complex numbers or operators, and j a n-tuple of nonnegative integers, X^j stands for $\prod_{i=1}^n X_i^{j_i}$.

2.1. **Construction of the Quantum Birkhoff normal form.** Our construction of the normal form, inspired by [6], is the following:

Proposition 2.2. *Let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$, whose principal symbol is*

$$H_0(x, t, \xi, \tau) = H^0(p, \tau) + H_2 \tag{2.8}$$

where $H^0(p, \tau) = \sum_{i=1}^n \theta_i p_i + \tau$ and H_2 vanishes to the third order on $x = \xi = \tau = 0$.

Then for any $N \geq 3$, there exists a self-adjoint semiclassical elliptic pseudodifferential operator $\widetilde{W_{\leq N}}$ and a smooth function $h(p_1, \dots, p_n, \tau, \hbar)$ satisfying microlocally in a neighborhood of $x = \xi = \tau = 0$ the following statement:

$$\begin{aligned} &\forall M > 0,\ \exists C_N > 0, \forall (\mu, \nu, \hbar) \in \mathbb{N}^n \times \mathbb{Z} \times [0, 1[, |\mu\hbar| + |\nu\hbar| < M, \\ &\left\| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1, \dots, P_n, D_t, \hbar) \right) |\mu, \nu\rangle \right\| \leq C_N (|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \end{aligned} \tag{2.9}$$

The operators can be computed recursively in the form:

$$\widetilde{W_{\leq N}} = W_{\leq N} + (D_t^2 + \sum_{i=1}^n P_i)^{N+1} \tag{2.10}$$

where

$$\begin{cases} W_{\leq N} = \sum_{3 \leq q \leq N} W_q \\ W_q := \sum\limits_{2p+|j|+|k|+2m=q} \alpha_{p,j,k,m}(t) \hbar^p \, Op^W(z^j \bar{z}^k) D_t^m \end{cases} \tag{2.11}$$

with $\alpha_{p,j,k,m})$smooth and W_q is symmetric.

Remark 2.3 (**important convention**). We are only interested in recovering the Hamiltonian in a formal neighborhood of γ: every asymptotic expansion is meant microlocally and we'll be rewriting equations such as (2.9) simply as:

$$\left\| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1, \dots, P_n, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right)$$

By abuse of notation, we'll identify the same way any operator with its version microlocalized near γ.

Remark 2.4. We introduce $\widetilde{W_{\leq N}}$ in order to gain ellipticity and self-adjointness like it has been done in Lemma 4.5 of [6].

The proof of Proposition 2.2 will need several preliminaries:

Definition 2.5. We will say that a pseudodifferential operator A on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ is "polynomial of order $r \in \mathbb{N}$" (PO(r)) if there exists $\beta_{p,j,k,m} \in C^\infty(\mathbb{S}^1, \mathbb{C})$ such that:

$$A = \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m \tag{2.12}$$

These operators have the following properties.

Proposition 2.6. *Let A be a pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ Then, there exists a family of operators A_r, $r \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, A_r is PO(r) and*

$$\forall N \in \mathbb{N}, \left\| \left(A - \sum_{r=0}^{N} A_r \right) |\mu, \nu\rangle \right\| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right) \tag{2.13}$$

Proof. We start by defining a notion of suitable asymptotic equivalence.

Definition 2.7. Let us introduce for any operator A the notations $\lfloor A \rfloor_r$ et $\lfloor A \rfloor_{\leq N}$ which represent respectively the terms of order r and of order smaller or equal to N in the expansion (2.13).
If A and B are two operators, we'll write that: $A \sim B$ if for any $r \in \mathbb{N}$, $\lfloor A \rfloor_r = \lfloor B \rfloor_r$.
Also, if $(A_n)_{n\in\mathbb{N}}$ is a family of operators, we'll write that:

$$A \sim \sum_{n=0}^{+\infty} A_n \tag{2.14}$$

if for any $N \in \mathbb{N}$, $\lfloor A_n \rfloor_{\leq N}$ is zero for n sufficiently large, and the finite sum:

$$\sum_{n=0}^{+\infty} \lfloor A_n \rfloor_{\leq N} = \lfloor A \rfloor_{\leq N}. \tag{2.15}$$

Let $a(z, t, \bar{z}, \tau)$ be the total symbol of A, the following Taylor expansion of which we split in two terms:

$$\forall N \in \mathbb{N},\ a(z,t,\bar{z},\tau) = \sum_{r=0}^{N} \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p z^j \bar{z}^k \tau^m + \sum_{p=0}^{\frac{N+1}{2}} O\left(\hbar^p (|z|^2 + |\tau|)^{\frac{N+1}{2}-p} \right)$$

Now, for any $r \in \mathbb{N}$, let us notice that the pseudodifferential operator A_r of symbol $\sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p z^j \bar{z}^k \tau^m$ is PO(r), and therefore:

$$\begin{aligned} \forall N \in \mathbb{N},\ \left\| \left(A - \sum_{r=0}^{N} A_r \right) |\mu, \nu\rangle \right\| &= \sum_{p=0}^{\frac{N+1}{2}} \hbar^p O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}-p} \right) \\ &= O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right). \end{aligned} \tag{2.16}$$

This concludes the proof. □

The following lemma will be crucial for our computations.

Lemma 2.8. *Let F and G be PO(r) and PO(r') respectively then $\frac{[F,G]}{i\hbar}$ is PO($r + r' - 2$).*

Proof. The proof of Lemma 2.8 will be a direct consequence of the two following lemmas, whose proof will be given at the end of this proof.

Lemma 2.9. *Any monomial operator of order r, that is of the form $\alpha(t)\hbar^p b_1 \dots b_l D_t^m$, where:*

- *for $j \in [\![1,l]\!]$, $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$*
- *$2p + l + 2m = r$*

is PO(r).

Lemma 2.10. *If F and G are monomials of order r and r' respectively, then $\frac{[F,G]}{i\hbar}$ is PO($r + r' - 2$)*

Indeed, any PO(r) is a finite sum of monomials of the same order, hence if F and G are PO(r) and PO(r') respectively, then $\frac{[F,G]}{i\hbar}$ is a finite sum of quantities of type $\frac{[\widetilde{F},\widetilde{G}]}{i\hbar}$ where $\widetilde{F}$ and $\widetilde{G}$ are monomials of order r and r' respectively. Any of those quantities are PO($r+r'-2$) by Lemmas 2.9 and 2.10, and a finite sum of PO($r+r'-2$) is PO($r+r'-2$). Therefore, Lemma 2.8 is proved. □

Let us now prove Lemmas 2.9 and 2.10:

Proof of Lemma 2.9. Since for any $i, j \in [\![1,n]\!]$, $i \neq j$, a_i and a_i^* commute with both a_j and a_j^*, it is sufficient, in order to prove Lemma 2.9, to prove the following assertion (Ass$_l$) for any positive integer l: " any ordered product $b_1 \dots b_l$, where for any $j \in [\![1,l]\!]$, $b_j \in \{a_1, a_1^*\}$ can be written as a finite sum of the quantities $\hbar^p Op^W(z_1^j \bar{z}_1^k)$ with $2p + j + k = l$ and $j - k = l - 2\sharp\{m \in [\![1,l]\!], b_m = a_1^*\}$" More precisely, let us proceed by induction, and introduce for any ordered product $b_1 \dots b_l$, the integer $k(b_1 \dots b_l) = \sharp\{m \in [\![1,l]\!], b_m = a_1^*\}$.

- If $l = 1$, there is nothing to prove since $a_1 = Op^W(z_1)$ and $a_1^* = Op^W(\bar{z}_1)$.
- If $l = 2$,
$$\begin{cases} a_1^2 = Op^W(z_1^2) \\ a_1^{*2} = Op^W(\bar{z}_1^2) \\ a_1 a_1^* = P_1 + \frac{\hbar}{2} = Op^W(z_1\bar{z}_1) + \frac{\hbar}{2} \\ a_1^* a_1 = Op^W(z_1\bar{z}_1) - \frac{\hbar}{2} \end{cases}$$
and therefore, the assertion is proved for $l = 2$.
- Now, let l be a positive integer, let us assume (Ass$_k$) up to order $k = l$, and let $B = b_1 \dots b_{l+1}$ be an ordered product, where for any $j \in [\![1, l+1]\!]$, $b_j \in \{a_1, a_1^*\}$. If for any $j \in [\![1,l]\!]$, $b_j = b_{j+1}$, then $B = Op^W(z_1^{l+1})$ or $B = Op^W(\bar{z}_1^{l+1})$. Otherwise, the proof of the symmetric case being identical, let us can assume that $b_1 = a_1$, and set $j_0 = \max\{j \in [\![1, l+1]\!], b_j = a_1\}$. Let us remark that: $1 \leq j_0 \leq l$ and $[a_1^{j_0}, a_1^*] = j_0 \hbar a_1^{j_0 - 1}$, so that:

$$b_1 \dots b_{l+1} = a_1^{j_0} a_1^* b_{j_0+2} \dots b_{l+1} = a_1^* a_1^{j_0} b_{j_0+2} \dots b_{l+1} + \hbar j_0 a_1^{j_0-1} b_{j_0+2} \dots b_{l+1} \tag{2.17}$$

Hence if one sets $k := k(b_1 \dots b_{l+1})$

$$\begin{aligned} \binom{l+1}{k} b_1 \dots b_{l+1} = &\binom{l}{k} a_1^{j_0} a_1^* b_{j_0+2} \dots b_{l+1} + \binom{l}{k-1} a_1^* a_1^{j_0} b_{j_0+2} \dots b_{l+1} \\ &+ \hbar \binom{l}{k_b - 1} j_0 a_1^{j_0-1} b_{j_0+2} \dots b_{l+1} \end{aligned}$$

Now, because we assumed (Ass_{l-1}):

$$(l-1)-2k(a_1^{j_0-1}b_{j_0+2}\dots b_{l+1})=(l+1)-2k(b_1\dots b_{l+1})$$

we only need to observe that the $\binom{l+1}{k}$ ordered monomials in the sum $_{Op^W(z^{l+1-k}\bar{z}^k)}$ can be divided in two parts: the $\binom{l}{k}$ ones whose first term is a_1, whose sum is $\binom{l}{k}a_1Op^W(z^{l-k}\bar{z}^k)$ and the $\binom{l}{k-1}$ who forms $\binom{l}{k-1}a_1^*Op^W(z^{l+1-k}\bar{z}^{k-1})$, and since:

$$\binom{l+1}{k}Op^W(z^{l+1-k}\bar{z}^k)=\binom{l}{k}a_1Op^W(z^{l-k}\bar{z}^k)+\binom{l}{k-1}a_1^*Op^W(z^{l+1-k}\bar{z}^{k-1})$$

the assumption of (Ass_l) will be enough to conclude our proof by induction.

□

Proof of Lemma 2.10. It is now sufficient in order to prove Lemma 2.10 to remark that if F and G are of the form:

$$F=\alpha(t)b_1\dots b_lD_t^m \text{ and } G=\beta(t)b'_1\dots b'_{l'}D_t^{m'}$$

where:

- α and β are smooth
- $l+2m=r$, $l'+2m'=r'$
- For $j\in[\![1,l]\!]$, for $j'\in 1,l'[\![$, $b_j,b'_{j'}\in\{a_1,a_1^*\}$

then $\frac{[F,G]}{i\hbar}$ is a finite sum of monomials of order $r+r'-2$ since, by Lemma 2.9, each of them is $\mathrm{PO}(r+r'-2)$. With those assumptions on F and G, we get:

$$\begin{aligned}\frac{[F,G]}{i\hbar}=&\frac{[\alpha(t)b_1\dots b_lD_t^m,\beta(t)b'_1\dots b'_{l'}D_t^{m'}]}{i\hbar}\\ =&\alpha(t)\beta(t)\frac{[b_1\dots b_l,b'_1\dots b'_{l'}]}{i\hbar}D_t^{m+m'}+\alpha(t)b_1\dots b_l\frac{[D_t^m,\beta(t)]}{i\hbar}b'_1\dots b'_{l'}D_t^{m'}\\ &-\beta(t)b'_1\dots b'_{l'}\frac{[D_t^{m'},\alpha(t)]}{i\hbar}b_1\dots b_lD_t^m\end{aligned}\tag{2.18}$$

Therefore it is sufficient to prove that $\frac{[b_1\dots b_l,b'_1\dots b'_{l'}]}{i\hbar}$, $\frac{[D_t^m,\beta(t)]}{i\hbar}$ and $\frac{[D_t^{m'},\alpha(t)]}{i\hbar}$ are respectively: $\mathrm{PO}(l+l'-2)$, $\mathrm{PO}(2m-2)$ and $\mathrm{PO}(2m'-2)$ (with the convention that a $\mathrm{PO}(j)$ with $j<0$ is 0).
For the two last, it is quite obvious, since:

$$\frac{[D_t^m,\beta(t)]}{i\hbar}=\sum_{k=0}^{m-1}\binom{m}{k}(i\hbar)^{m-k-1}\beta^{(m-k)}(t)D_t^k\tag{2.19}$$

Now, for $j\in[\![1,l']\!]$, let us set $\epsilon_j=1$ if $b'_j=a_1^*$, otherwise $\epsilon_j=-1$. Since $[a_1,a_1^*]=\hbar$, we get:

$$\begin{aligned}b_1\dots b_lb'_1\dots b'_{l'}=b'_1b_1\dots b_lb'_2\dots b'_{l'}&+\frac{\epsilon_1+1}{2}\hbar\sum_{\substack{k=1\\b_k=a_1}}^{l}b_1\dots b_{k-1}b_{k+1}\dots b_lb'_2\dots b'_{l'}\\ &+\frac{\epsilon_1-1}{2}\hbar\sum_{\substack{j=1\\b_k=a_1^*}}^{l}b_1\dots b_{k-1}b_{k+1}\dots b_lb'_2\dots b'_{l'}\end{aligned}$$

Hence by induction on $j \in [\![1, l']\!]$:

$$\begin{aligned}\frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar} =& -i \sum_{j=1}^{l'} \frac{\epsilon_j + 1}{2} \sum_{\substack{k=1 \\ b_k = a_1}}^{l} b'_1 \dots b'_{j-1} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_{j+1} \dots b'_{l'} \\ & -i \sum_{j=1}^{l'} \frac{\epsilon_j - 1}{2} \sum_{\substack{k=1 \\ b_k = a_1^*}}^{l} b'_1 \dots b'_{j-1} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_{j+1} \dots b'_{l'}\end{aligned} \tag{2.20}$$

The right-hand side of (2.20) is a finite sum of monomials of order $l + l' - 2$, hence $\mathrm{PO}(l + l' - 2)$ by Lemma 2.9, and Lemma 2.10 is proved.

□

Lemma 2.11. *Let G be PO(r). There exists F PO(r) and $G_1 = G_1(P_1, \dots, P_n, D_t, \hbar)$ such that:*

$$\frac{[H^0(P, D_t), F]}{i\hbar} = G + G_1 \tag{2.21}$$

Moreover, F is symmetric if G is symmetric, $G_1 = 0$ if r is odd, and G_1 is an homogeneous polynomial function of total order $\frac{r}{2}$ if r is even.

Remark 2.12. If $F = \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t) \hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m$, one can choose:

$$\int_{\mathbb{S}^1} \alpha_{p,j,j,m}(t) dt = 0 \tag{2.22}$$

Indeed, any $Op^W(z^j \bar{z}^j) D_t^m$ commutes with $H^0(P, D_t, \hbar)$

Proof of Lemma 2.11. Let us first assume that G is a monomial of order r: $G = \beta(t) b_1 \dots b_l D_t^m$ where:

- α is smooth
- $l + 2m = r$
- For $j \in [\![1, l]\!]$, $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$

and let us look for F under the form: $F = \alpha(t) b_1 \dots b_l D_t^m$. We have:

$$\begin{aligned}\frac{[H^0, F]}{i\hbar} =& \frac{[H^0, \alpha(t) b_1 \dots b_l D_t^m]}{i\hbar} \\ =& \alpha(t) \sum_{i=1}^{n} \theta_i \frac{[P_i, b_1 \dots b_l]}{i\hbar} D_t^m + \frac{[D_t, \alpha(t)]}{i\hbar} b_1 \dots b_l D_t^m \\ =& \alpha(t) \sum_{i=1}^{n} \theta_i \frac{[P_i, b_1 \dots b_l]}{i\hbar} D_t^m + \alpha'(t) b_1 \dots b_l D_t^m\end{aligned} \tag{2.23}$$

If for $i \in [\![1, n]\!]$, $k_i = \sharp\{m \in [\![1, l]\!], b_m = a_i^*\}$ and $j_i = \sharp\{m \in [\![1, l]\!], b_m = a_i\}$, we deduce from (2.20) that:

$$\frac{[P_i, b_1 \dots b_l]}{i\hbar} = \sqrt{-1}(j_i - k_i) b_1 \dots b_l \tag{2.24}$$

Hence:

$$\frac{[H^0, F]}{\sqrt{-1}\hbar} = \sqrt{-1} \sum_{i=1}^{n} \theta_i (j_i - k_i) \alpha(t) b_1 \dots b_l D_t^m + \alpha'(t) b_1 \dots b_l D_t^m \tag{2.25}$$

The problem: $\frac{[H^0,F]}{\sqrt{-1}\hbar} = G$ admits a solution if there exists α such that:

$$\sqrt{-1}\sum_{i=1}^{n}\theta_i(j_i-k_i)\alpha(t)+\alpha'(t)=\beta(t) \tag{2.26}$$

If $(c_p(\alpha))_{p\in\mathbb{Z}}$ and $(c_p(\beta))_{p\in\mathbb{Z}}$ are the Fourier coefficients of α and β, it is sufficient that, for $p\in\mathbb{Z}$, $c_p(\alpha)$ is solution of:

$$\sqrt{-1}\left(\sum_{i=1}^{n}\theta_i(j_i-k_i)+2\pi p\right)c_p(\alpha)=c_p(\beta) \tag{2.27}$$

and

$$c_p(\alpha) \underset{p\to+\infty}{=} O\left(\frac{1}{|p|^\infty}\right) \tag{2.28}$$

If the n-tuples j and k are different, the non-degeneracy condition on the θ_i's together with the fact that $c_p(\beta) \underset{p\to+\infty}{=} O\left(\frac{1}{|p|^\infty}\right)$ (because β is smooth), gives the existence of $c_p(\alpha)$ satisfying (2.27) and (2.28).
If r is odd, j and k can't be equal, hence Lemma 2.11 is proved in this case (r odd and G monomial)
If r is even, and $j=k$, there exists a family $(c_p(\alpha))_{p\in\mathbb{Z}^*}$ satisfying (2.27) and (2.28). Hence, if α is the smooth function with Fourier coefficients $c_p(\alpha)$ for $p\neq 0$ and $c_0(\alpha)=0$, we get:

$$\frac{[H^0,F]}{\sqrt{-1}\hbar} = G + c_0(\beta)b_1\dots b_l D_t^m \tag{2.29}$$

And from the proof of Lemma 2.9, we know that $c_0(\beta)b_1\dots b_l D_t^m$ can be reordered as the sum: $G_1(P,D_t,\hbar) := c_0(\beta)\sum_{2p+2|k|=l} a_{p,k}\hbar^p P^k D_t^m$. Therefore, Lemma 2.11 is proved in the case where r is even and G is monomial.
The general case is easily deduced from the case where G is monomial, since G is a finite sum of monomials of the same order.
Also, the form of F allows us to conclude immediately that F is symmetric if G is so. □

Now we have everything we need for the proof by induction of Proposition 2.2.

Proof of Proposition 2.2. Microlocally near $x=\xi=\tau=0$, $H(x,\hbar D_x)$ satisfies

$$H := H(x,\hbar D_x) \sim H^0(P_1,\dots,P_n,\hbar D_t) + \sum_{q\geq 3} H_q \tag{2.30}$$

where:

$$H_q := \lfloor H(x,\hbar D_x)\rfloor_q \tag{2.31}$$

Let us look for $\widetilde{W}_{\leq 3}$ under the form predicted in Proposition 2.2, that is:

$$\widetilde{W}_{\leq 3} = W_3 + (|D_t|^2 + \sum_{i=1}^{n} P_i)^4 \tag{2.32}$$

where W_3 is PO(3).

$$e^{\frac{i\widetilde{W}_{\leq 3}}{\hbar}} H(x,\hbar D_x) e^{\frac{-i\widetilde{W}_{\leq 3}}{\hbar}} \sim H(x,\hbar D_x) + \frac{i}{\hbar}[\widetilde{W}_{\leq 3}, H] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq 3},\dots,\widetilde{W}_{\leq 3}}^{l \text{ times}}, H]$$

$$\sim H^0 + H_3 + \frac{i}{\hbar}[W_3, H^0]$$
$$+ \frac{i}{\hbar}[W_3, H - H^0] + \frac{i}{\hbar}[\widetilde{W}_{\leq 3} - W_3, H(x,\hbar D_x)]$$
$$+ \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq 3},\dots,\widetilde{W}_{\leq 3}}^{l \text{ times}}, H(x,\hbar D_x)] + \sum_{q\geq 4} H_q$$

Since H_3 is polynomial of order 3, let us choose W_3, as in Lemma 2.11, such that:

$$H_3 + \frac{i}{\hbar}[W_3, H^0] = H^1(P_1,\dots,P_n,D_t,\hbar) \equiv 0 \tag{2.33}$$

Since W_3 is PO(3) and the expansion of $H - H^0$ in PO(r) contains no PO(r) of order less or equal to 2, the expansion of $\widetilde{W}_{\leq 3} - W_3$ no term order less or equal to 3, and the one of $H(x,\hbar D_x)$ no term of order less or equal to 1, we know from Lemma 2.10 that the expansion of:

$$\frac{i}{\hbar}[W_3, H - H^0] + \frac{i}{\hbar}[\widetilde{W}_{\leq 3} - W_3, H] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq 3},\dots,\widetilde{W}_{\leq 3}}^{l \text{ times}}, H] + \sum_{q\geq 4} H_q \tag{2.34}$$

contains no term of order less or equal to 3.
Therefore, Proposition 2.6 gives us:

$$\left|\left| \left(e^{\frac{i\widetilde{W}_{\leq 3}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq 3}}{\hbar}} - H^0(P,D_t,\hbar) \right) |\mu,\nu\rangle \right|\right| = O\left((|\mu\hbar| + |\nu\hbar|)^2 \right) \tag{2.35}$$

We can construct by induction $(W_q)_{q\geq 3}$ and $(H^q)_{q\geq 1}$, such that:

- for $q \geq 3$, W_q is PO(q) and for H^{q-2} is zero if q is odd, an homogeneous polynomial function of total order $\frac{q}{2}$ if q is even.
-

$$H_3 + \frac{i}{\hbar}[W_3, H^0] = H^1(P, D_t, \hbar) \tag{2.36}$$

- and for any $q \geq 4$:

$$\frac{i}{\hbar}[W_q, H^0] + H_q + \left[\frac{i}{\hbar}[W_{\leq q-1}, H - H^0] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{W_{\leq q-1},\dots,W_{\leq q-1}}^{l \text{ times}}, H] \right]_q = H^{q-2}(P, D_t, \hbar)$$

Let us now set: $\widetilde{W}_{\leq N} := \sum_{q=3}^{N} W_q + (|D_t|^2 + \sum_{i=1}^{n} P_i)^{\frac{N+1}{2}}$. Also, as for any $q \geq 0$, H^{2q} is an homogeneous polynomial function of total order $q+1$, we can choose by Borel's lemma a smooth function h such that for any $N \geq 1$, in a neighborhood of $p = \tau = 0$.

$$\left| h(p,\tau,\hbar) - \sum_{q=0}^{N-1} H^{2q}(p,\tau,\hbar) \right| = O\left((|p| + |\tau| + |\hbar|)^{N+1} \right) \tag{2.37}$$

Now, let us write, for any $N \geq 4$

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim H + \frac{i}{\hbar}[\widetilde{W}_{\leq N}, H] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H]$$

$$\sim H + \frac{i}{\hbar}[W_{\leq N}, H^0] + \frac{i}{\hbar}[W_{\leq N}, H - H^0] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H]$$

$$+ \frac{i}{\hbar}[\widetilde{W}_{\leq N} - W_{\leq N}, H]$$

Let us also observe that Lemma 2.8 gives us for $q \leq N$:

$$\begin{cases} \lfloor \frac{i}{\hbar}[W_{\leq N}, H - H^0] \rfloor_q = \lfloor \frac{i}{\hbar}[W_{\leq q-1}, H - H^0] \rfloor_q \\ \left\lfloor \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H] \right\rfloor_q = \left\lfloor \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{W_{\leq q-1}, \dots, W_{\leq q-1}}^{l \text{ times}}, H] \right\rfloor_q \end{cases} \tag{2.38}$$

Therefore for any $q \leq N$:

$$\left\lfloor e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right\rfloor_q = H^{q-2}(P, D_t, \hbar) = \lfloor h(P, D_t, \hbar) \rfloor_q \tag{2.39}$$

And Proposition 2.6 gives us:

$$\left|\left| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P, D_t, \hbar) \right) |\mu, \nu\rangle \right|\right| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right) \tag{2.40}$$

which concludes the proof. □

2.2. **Recovering the matrix elements from the Trace formula.** The next result is the first inverse result needed for the proof of our main result.

Proposition 2.13. *Let O be a pseudodifferential operator, whose principal symbol vanishes on γ.*

(1) *There exists a smooth function f vanishing at $(0,0,0)$ such that for any $N \geq 3$:*

$$\langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} |\mu, \nu\rangle = f\left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N}{2}} \right) \tag{2.41}$$

Moreover let, for any integer l, ϕ_l be a Schwartz function whose Fourier transform is compactly supported in $(l-1, l+1)$ and let $(a_j^l(O))_{l\geq 0}$ provided by the trace formula:

$$Tr\left(O\phi_l\left(\frac{H - E}{\hbar} \right) \right) \sim \sum_{j=0}^{+\infty} a_j^l(O)\hbar^j \tag{2.42}$$

(2) *The Taylor expansion of f up to order N is entirely determined by the family $(a_j^l(O))$, $0 \leq j \leq N$, $l \in \mathbb{N}$.*

Proof. Let us first prove point 1.
Let us consider a monomial $G = \alpha(t) b_1 \dots b_l D_t^m$ where:

- α is smooth
- $l + 2m = r$
- For $j \in [\![1, l]\!]$, $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$

Let us set for $i \in [\![1, n]\!]$, $k_i = \sharp\{m \in [\![1, l]\!], b_m = a_i^*\}$
and $j_i = \sharp\{m \in [\![1, l]\!], b_m = a_i\}$.

If $j \neq k$ or $\alpha \notin \mathbb{C}$, then: $\langle \mu, \nu | G | \mu, \nu \rangle = 0$ for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$.

If now $j = k$ and $\alpha \in \mathbb{C}$, then there exists complex numbers α_l ($0 \leq l_i \leq j_i$ for $i \in [\![1, n]\!]$), such that:

$$G = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} P_1^{j_1 - l_1} \dots P_n^{j_n - l_n} D_t^m \tag{2.43}$$

and: $\alpha_0 = \alpha$.
Therefore for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

$$\langle \mu, \nu | G | \mu, \nu \rangle = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} \left(\left(\mu + \frac{1}{2} \right) \hbar \right)^{j-l} (2\pi\nu\hbar)^m \tag{2.44}$$

Hence, if G is $\mathrm{PO}(r)$, then for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

- $\langle \mu, \nu | G | \mu, \nu \rangle = 0$ if r is odd.
- If r is even, there exists an homogeneous polynomial function g of order $\frac{r}{2}$ such that:

$$\langle \mu, \nu | G | \mu, \nu \rangle = g \left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) \tag{2.45}$$

From Proposition 2.6 and Borel's lemma, we get that that for any operator A, there exists a function g such that for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

$$\langle \mu, \nu | A | \mu, \nu \rangle = g \left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right) \tag{2.46}$$

Hence, the only point remaining to prove, is that function f in point 1 does not depend on N. It is therefore sufficient to prove that for any $q \leq N - 1$,

$$\left\lfloor e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right\rfloor_q = \left\lfloor e^{\frac{i\widetilde{W}_{\leq q+1}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq q+1}}{\hbar}} \right\rfloor_q \tag{2.47}$$

But (2.47) is a direct consequence of Lemma 2.8. Indeed,

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim O + \sum_{l \geq 1} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, O] \tag{2.48}$$

and since the principal symbol of O vanishes on γ, Lemma 2.8 gives us for any $l \geq 1$ and any $q \leq N-1$:

$$\left[\frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l \text{ times}},O]\right]_q = \left[\frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq q+1},\dots,\widetilde{W}_{\leq q+1}}^{l \text{ times}},O]\right]_q \tag{2.49}$$

Let us now move on to the proof of point 2.

Since $\hat{\phi}_l$ is supported near a single period of the flow, we know from the general theory of Fourier integral operators that one can microlocalize the trace formula with observables near γ:

$$Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right) = Tr\left(O\int_{\mathbb{R}}\hat{\phi}_l(t)\rho(P_1+\cdots+P_n+|\zeta|)e^{it\frac{H-E}{\hbar}}dt\right)+O(\hbar^\infty) \tag{2.50}$$

where $\rho \in C_0^\infty(\mathbb{R})$ is compactly supported and $\rho = 1$ in a neighborhood of $p = \tau = 0$. Therefore we can conjugate (2.50) by the microlocally unitary operator $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}$:

$$\begin{aligned} &Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right) = \\ &= Tr\left((e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}Oe^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}\int_{\mathbb{R}}\hat{\phi}_l(t)\rho(P_1+\cdots+P_n+|\zeta|)e^{it\frac{e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}He^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}-E}{\hbar}}dt\right)+O(\hbar^\infty)\end{aligned}$$

Thanks to Proposition 2.2, we can lighten the r.h.s. for any $(\mu,\nu)\in\mathbb{N}^n\times\mathbb{Z}$

$$\begin{aligned} &\int_{\mathbb{R}}\hat{\phi}_l(t)\rho(P_1+\cdots+P_n+|\zeta|)e^{it\frac{e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}He^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}-E}{\hbar}}dt|\mu,\nu\rangle \\ &= \left(\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar\right)e^{it\frac{h((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar)-E+O(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}})}{\hbar}}dt\right)|\mu,\nu\rangle\end{aligned} \tag{2.51}$$

As $\hat{\phi}_l$ is smooth and compactly supported, together with the non-degeneracy condition on the θ_i's, we can assure that if we choose a sufficiently small support for ρ, we have for any $\eta > 0$:

$$\begin{aligned} &\left(\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar\right)e^{it\frac{h((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar)-E+O(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}})}{\hbar}}dt\right)|\mu,\nu\rangle \\ &= \left(\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar^\eta\right)e^{it\frac{h((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar)-E+O(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}})}{\hbar}}dt\right)|\mu,\nu\rangle+O(\hbar^\infty)\end{aligned}$$

Hence, choosing $\eta < \frac{1}{2}$:

$$\begin{aligned}
&Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right)+O(\hbar^\infty)\\
=&\sum_{\mu,\nu}\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}Oe^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}|\mu,\nu\rangle\times\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|\nu|)\hbar^\eta\right)e^{it(2\pi\nu+\theta.(\mu+\frac{1}{2}))}\dots\\
&\dots\exp\left(\frac{it}{\hbar}\sum_{1\leq q\leq N-2}H^q\left((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar\right)+O\left((|\mu|+|\nu|)^{\frac{N+1}{2}}\hbar^{\frac{N-1}{2}}\right)\right)dt\\
&=\sum_{\mu,\nu}\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar^\eta\right)e^{it(2\pi\nu+\theta.(\mu+\frac{1}{2}))}\\
&\left(1+\sum_{i\geq1}^{\frac{N-1}{2}}\hbar^iQ_i(\mu+\frac{1}{2},\nu,t)\right)\times\sum_{p\geq1}^{\frac{N+1}{2}}\sum_{|k|+m\leq p}b_{k,m,p-|k|-m}(\mu+\frac{1}{2})^k(2\pi\nu)^m\hbar^pdt+O(\hbar^{\frac{N+1}{2}})
\end{aligned}$$

where for any $i\leq\frac{N-1}{2}$, Q_i is a determined polynomial function, of degree in $\left(\mu+\frac{1}{2},\nu\right)$ less or equal to $i+1$, which depends on the H^q's and the Taylor expansion of exp, and the $b_{k,m,s}$ ($(k,m,s)\in\mathbb{N}^{n+2}\backslash\{0\}$) come from the Taylor expansion at $(0,0,0)$ of the function f defined in the first point of Proposition 2.13, *i.e.* for any $N\geq1$:

$$f(x,y,z)=\sum_{1\leq|k|+m+s\leq N}b_{k,m,s}x^ky^mz^s+O\left(|x|+|y|+|z|\right)^{N+1}\right) \tag{2.52}$$

Now, let us set:

$$\forall t\in\mathbb{R}^*,\forall\alpha\in(\mathbb{R}\backslash\frac{2\pi}{t}\mathbb{Z})^n,g(t,\alpha):=\frac{e^{i\frac{t}{2}(\alpha_1+\dots+\alpha_n)}}{\prod_i(1-e^{it\alpha_i})} \tag{2.53}$$

By the non-degeneracy condition on the θ_i's, g is well defined on the compact support of $\hat{\phi}_l$ around a single period, which is precisely l. It also implies that $\theta_i.\mu$ is bounded below by $C|\mu|$ (where $C>0$) as $|\mu|$ goes to ∞.
Therefore we get from the Poisson formula and the Riemann-Lebesgue lemma that the following quantity $X_p(l)$ can be computed recursively on $p\leq\frac{N+1}{2}$ from the $a_j^l(O)$, $j=0,\dots,p$:

$$\begin{aligned}
X_p(l)&=\sum_{|k|+m\leq p}b_{k,m,p-|k|-m}\left[\left(-i\frac{\partial}{\partial t}\right)^m\left(\hat{\phi}_l(t)\left(\frac{-i}{t}\right)^k\frac{\partial^kg}{\partial\alpha^k}(t,\alpha)\right)\right](l,\theta)\\
&=\sum_{|k|+m\leq p}b_{k,m,p-|k|-m}\left[\left(-i\frac{\partial}{\partial t}\right)^m\left(-i\frac{\partial}{t\partial\alpha}\right)^kg\right](l,\theta)
\end{aligned} \tag{2.54}$$

since $\hat{\phi}_l$ is identically 1 around l.
Now, let us set, for any $i\in[\![1,n]\!]$, any $t\in\mathbb{R}$ and any $\alpha\in(\mathbb{R}\backslash\frac{2\pi}{t}\mathbb{Z})^n$, $x_i(t,\alpha)=e^{i\frac{t\alpha_i}{2}}$. and also define holomorphic function h on $\mathbb{C}\backslash\{-1,1\}$ by $h(z)=\frac{z}{1-z^2}$ for $z\in\mathbb{C}\backslash\{-1,1\}$. We

have for any $k \in \mathbb{N}^n$:

$$\left(-i\frac{\partial}{t\partial\alpha}\right)^k g = \prod_{i=1}^{n}\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) \tag{2.55}$$

For any $i \in [\![1,n]\!]$, an easy induction on $k_i \in \mathbb{N}$ leads to the following, since for any $z \in \mathbb{C}\backslash\{-1,1\}$, $h(z) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{1+z}\right)$, and $-i\frac{\partial x_i}{t\partial\alpha_i} = \frac{1}{2}x_i$:

$$\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) = \frac{k_i!}{2^{k_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+1}} + \frac{x_i}{(1+x_i)^{k_i+1}}\right) \tag{2.56}$$

Now, since $-i\frac{\partial x_i}{\partial t} = \frac{\alpha_i}{2}x_i$, an induction on $s_i \in \mathbb{N}$ shows that:

$$\left(-i\frac{\partial}{\partial t}\right)^{s_i}\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) = \frac{(k_i+s_i)!\alpha_i^{s_i}}{2^{k_i+s_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}}\right) \tag{2.57}$$

Let us now introduce for any n-tuple s such that $|s| = m$, the multinomial coefficient:

$$\binom{m}{s} = \frac{m!}{s_1!\dots s_n!}$$

We have:

$$\left(-i\frac{\partial}{\partial t}\right)^m\left(-i\frac{\partial}{t\partial\alpha}\right)^k g = \sum_{|s|=m}\binom{m}{s}\prod_{i=1}^{n}\left(-i\frac{\partial}{\partial t}\right)^{s_i}\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) \tag{2.58}$$

Let us use Kronecker theorem, whose hypothesis is precisely the non-degeneracy condition on the θ_i's: for any n-tuple $(x_1,\dots,x_n) \in \mathbb{S}_1^n$, one can find a sequence of integers $(l_p)_{p\in\mathbb{Z}}$, such that:

$$\forall j \in [\![1,n]\!],\ x_j(l_p,\theta) \underset{p\to+\infty}{\longrightarrow} x_j$$

Therefore, if one sets, for any $(x_1,\dots,x_n) \in (\mathbb{S}_1\backslash\{-1,1\})^n$ and $(k,m) \in \mathbb{N}^{n+1}$:

$$u^{(k,m)} = \sum_{|s|=m}\binom{m}{s}\prod_{i=1}^{n}\frac{(k_i+s_i)!\theta_i^{s_i}}{2^{k_i+s_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}}\right)$$

Then (2.54), (2.57) and (2.58) together with Kronecker theorem allows us to conclude that the following quantity is determined by the $a_j^l(O)$, $j = 0,\dots,p$:

$$X_p = \sum_{|k|+m\le p} b_{k,m,p-|k|-m}u^{(k,m)} \tag{2.59}$$

Hence, the only thing remaining to prove is that, if one chooses the x_i's tending to 1 in a way convenient to us, the $|u^{(k,m)}|$'s will tend to ∞ to different orders.
Let us be more precise:

Let the x_i's tend to 1 in a way such that:

$$\forall i \in [\![1,n-1]\!], |1-x_i| \ll |1-x_{i+1}|^p \tag{2.60}$$

If $\simeq$ means that two functions are equivalent, as the x_i's tend to 1 as in (2.60), up a multiplicative constant, we have for any $(k,m) \in \mathbb{N}^{n+1}$:

$$(1-x_1)^m u^{(k,m)} \simeq \prod_{i=1}^{n} \frac{1}{(1-x_i)^{k_i+1}} \tag{2.61}$$

Hence, if one sets $\widetilde{m} = (m, 0, \dots, 0)$:

$$u^{(k,m)} \ll u^{(k',m')} \text{ si } k + \widetilde{m} < k' + \widetilde{m'} \tag{2.62}$$

where $<$ is the lexicographical order on $\mathbb{N}^n$. Therefore, for any $p \in \mathbb{N}$ and for any $(k,m) \in \mathbb{N}^{n+1}$ such that $|k_0| + m_0 \leq p$, the following quantity can be recursively determined from X_p:

$$X_{k_0,m_0} = \sum_{k'+\widetilde{m'}=k+\widetilde{m}} b_{k,m,p-|k|-m} u^{(k,m)} \tag{2.63}$$

Reversing for example the roles of $i=1$ and $i=2$ in (2.60), and observing that $k_2 + m \neq k_2' + m'$ if $k + \widetilde{m} = k' + \widetilde{m'}$ and $(k,m) \neq (k',m')$, one determines $b_{k,m,p-|k|-m}$ from (2.63) recursively on m. Finally, each $b_{k,m,s}$ with $|k| + m + s \leq N$ is determined by the $a_j^l(O)$, with $j = 0 \dots N$ and $l \in \mathbb{N}$ and the point 2 is proved, which ends the proof of Proposition 2.13.

□

2.3. **Recovering the Hamiltonian from matrix elements.** Our next result shows how the knowledge of the matrix elements of the conjugation of a given known selfadjoint operator by a unitary one determines the latter (in the framework of asymptotic expansions).
For any $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$, and any $(x,t,\xi,\tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$, let us define:

$$\mathcal{O}_{mnds}(x,t,\xi,\tau) = e^{i2\pi dt} \tau^s \prod_{j=1}^{n} (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}. \tag{2.64}$$

and let O_{mnds} be a pseudodifferential operator whose Weyl principal symbol is $\mathcal{O}_{mnds}$.

By Proposition 2.13, there exists a smooth function f_{mnds} vanishing at $(0,0,0)$ such that for any $N \geq 3$:

$$\langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle = f_{mnds}\left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N}{2}} \right) \tag{2.65}$$

Theorem 2.1 will now be a direct consequence of Proposition 2.13 and following proposition:

Proposition 2.14. *Let $N \geq 3$. The Taylor expansion of f_{mnds} up to order $N-1$ for any $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$ satisfying conditions*

1. $|m| + |n| \leq N$
2. $\forall j \in [\![1,n]\!]$, $m_j = 0$ ***or*** $n_j = 0$
3. $s = 1$ *if* $m = n = 0$, *otherwise* $s = 0$

determines completely $W_{\leq N}$

Remark 2.15. Let us remark, like it will be seen in the proof of Proposition 2.14, that the only relevant information is the asymptotic expansion of $\langle \mu,\nu|e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}|\mu,\nu\rangle$ as $\hbar$ tends to 0 and μ,ν go to ∞ slower than any negative power of $\hbar$.

Proof of Proposition 2.14. Let $N \geq 3$ and $(m,n,d,s) \in (\mathbb{N}^n)^2 \times \mathbb{Z} \times \{0,1\}$ satisfy conditions (1), (2) and (3).
Then, we have:

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim O_{mnds} + \frac{i}{\hbar}[\widetilde{W}_{\leq N}, O_{mnds}] + \sum_{l\geq 2}\frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l \text{ times}}, O_{mnds}]$$

Therefore:

$$\begin{aligned}
&\langle \mu,\nu|e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}|\mu,\nu\rangle - \langle \mu,\nu|O_{mnds}|\mu,\nu\rangle \\
&= \frac{i}{\hbar}\langle \mu,\nu|[\widetilde{W}_{\leq N}, O_{mnds}]|\mu,\nu\rangle + \sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle \mu,\nu|[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l \text{ times}}, O_{mnds}]|\mu,\nu\rangle \\
&+ O\left((|\mu\hbar| + |\nu\hbar|)^{\infty}\right)
\end{aligned} \tag{2.66}$$

Now, since $\widetilde{W}_{\leq N}$ is a sum of polynomial operators of order greater that 3, we get from Proposition 2.8 that for any $l \geq 2$

$$\frac{i^l}{\hbar^l}\langle \mu,\nu|[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l-1 \text{ times}}, \cdot] \tag{2.67}$$

maps a $\mathrm{PO}(r)$ into a sum of polynomial operators of order strictly larger than r. Therefore, if A is a $\mathrm{PO}(r)$, we have:

$$\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle \mu,\nu|[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l-1 \text{ times}}, A]|\mu,\nu\rangle = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}\right)\langle \mu,\nu|A|\mu,\nu\rangle \tag{2.68}$$

Finally, let us recall that:

$$\begin{aligned}
W_N &= \sum_{2p+|j|+|k|+2q=N} \alpha_{p,j,k,q}(t)\hbar^p Op^W(z^j\bar{z}^k)D_t^q \\
&:= \sum_{2p+|j|+|k|+2q=N}\sum_{r\in\mathbb{Z}} \alpha_{p,j,k,q,r}\hbar^p e^{-i2\pi rt}Op^W(z^j\bar{z}^k)D_t^q
\end{aligned} \tag{2.69}$$

Let us also state the following lemma, whose proof will be given after the end of the present proof.

Lemma 2.16.

$$\langle \mu,\nu|[e^{-i2\pi dt}Op^W(z^j\bar{z}^k)D_t^q, O_{mnds}]|\mu,\nu\rangle = \hbar g_{jkqrmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right) + O(\hbar^2) \tag{2.70}$$

where, if $j+m=k+n$ and $r=d$:
(2.71)

$$g_{jkqrmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right)=(2\pi\nu\hbar)^{q+s}(\mu\hbar)^{\max(j,k)}\left(\sum_{\substack{i=1\\|j_i|+|k_i|>0}}^{n}\frac{j_in_i-k_im_i}{\mu_i\hbar}+\frac{d(q+s)}{\nu\hbar}\right)$$

and if $j+m\neq k+n$ or $r\neq d$, $g_{jkqrmnds}\equiv 0$

Let us now proceed by induction on $N\geq 3$, and first assume $N=3$.

Equation (2.65) gives us that the Taylor expansion up to order 2 of function f_{mnds} determines modulo $O\left((|\mu\hbar|+|\nu\hbar|)^3\right)$:

$$\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq 6}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq 6}}{\hbar}}|\mu,\nu\rangle-\langle\mu,\nu|O_{mnds}|\mu,\nu\rangle \tag{2.72}$$

Thanks to (2.68), (2.72) is equal, modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)$, to:
(2.73)

$$\sum_{\substack{2p+|j|\\+|k|+2q\\=3}}\sum_{r\in\mathbb{Z}}\alpha_{p,j,k,q,r}\hbar^p\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)\langle\mu,\nu|\frac{i}{\hbar}[e^{-i2\pi rt}Op^W(z^j\bar{z}^k)D_t^q,O_{mnds}]|\mu,\nu\rangle$$

and with the lemma's notations modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)+O(\hbar)$ to:

$$\sum_{\substack{|j|+|k|+2q=3\\j+m=k+n}}i\alpha_{0,j,k,q,d}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.74}$$

Let us assume we already proved (assertion ($\star$)) that quantity (2.74) determines coefficients $\alpha_{0,j,k,q,d}$ ($|j|+|k|+2q=3$, $j+m=k+n$).
We'll have determined every function $\alpha_{0,j,k,q}$ ($|j|+|k|+2q=3$). Indeed, for any $(j,k,q)\in\mathbb{N}^{2n+1}$ such that $|j|+|k|+2q=3$, and for any $i\in[\![1,n]\!]$, let us choose:

$$n_i=\max(j_i-k_i,0)\text{ and }m_i=\max(k_i-j_i,0) \tag{2.75}$$

$d\in\mathbb{Z}^*$ and $s=1$ if $m=n=0$, $d\in\mathbb{Z}$ and $s=0$ otherwise.
We have for any $i\in[\![1,n]\!]$, $m_i=0$ or $n_i=0$, and

$$|m|+|n|=\sum_{i=1}^{n}|j_i-k_i|\leq|j|+|k|\leq 3$$

Therefore, (m,n,d,s) verifies the three assumptions (1), (2), and (3): (2.74) will hence determine $\alpha_{0,j,k,q,d}$ and letting d describe $\mathbb{Z}$ if $j\neq k$, $\mathbb{Z}^*$ if $j=k$, we will have determined functions $\alpha_{0,j,k,q}$ (thanks to remark 2.12 for the case $j=k$)

Let us prove assertion ($\star$) in the two cases: $m\neq n$ and $m=n$.
Let us also define the set Γ of (j,k,q) such that: $|j|+|k|+2q=3$ and $j+m=k+n$.
Let us first assume that $m\neq n$, and choose $\mu_1(\hbar),\dots\mu_n(\hbar),\nu(\hbar)$ such that, as $\hbar$ tends to 0:

$$1\ll\mu_1,\ \mu_n^{2N}\ll\nu,\text{ and }\forall i\in[\![1,n-1]\!\},\ \mu_i^{2N}\ll\mu_{i+1} \tag{2.76}$$

Let us also define $i_0 := \min\{i \in [\![1,n]\!], m_i \neq n_i\}$. We have, for $(j,k,q) \in \Gamma$:

$$g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \underset{\hbar\to 0}{\sim} \frac{j_{i_0}n_{i_0}-k_{i_0}m_{i_0}}{\mu_{i_0}\hbar}(2\pi\nu\hbar)^q\prod_{i=1}^{n}(\mu_i\hbar)^{\max(j_i,k_i)} \tag{2.77}$$

and $j_{i_0}n_{i_0}-k_{i_0}m_{i_0}$ never vanishes.
Also, (2.76) in addtition to (2.77) gives us that:

$$g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \ll g_{j'k'q'dmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.78}$$

if $(j,k,q) < (j',k',q')$, where $<$ is a strict total order on Γ defined by the lexicographical order of $(\max(j_1,k_1),\dots,\max(j_n,k_n),q)$. It is indeed asymmetric since for $i \in [\![1,n]\!]$, the sign of $m_i - n_i$ determines whether $\max(j_i,k_i)$ is equal to j_i or k_i.

Therefore, making additional assumption on function $\mu_1(\hbar)$ that: $\hbar = O(\mu_1(\hbar)^3\hbar^3)$, we get that qunatity (2.74) is determined modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)$ and assertion $(\star)$ easily follows by induction on $(\Gamma,<)$ in the case $m \neq n$.

If now $m = n$, we may assume that $d \neq 0$ like seen before. Also, $s = 1$, thus for any q, $(q+s)d \neq 0$.
Hence,

$$g_{jjqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) = (2\pi\nu\hbar)^q(q+1)d\prod_{i=1}^{n}(\mu_i\hbar)^{j_i} \tag{2.79}$$

and assertion $(\star)$ is proved just as before.

Finally, all functions $\alpha_{0,j,k,q}$ are determined for (j,k,q) satisfying $|j|+|k|+2q = 3$. Let (m,n,d,s) satisfy conditions (1), (2), and (3) with $N = 1$.
Therefore, we obtain from (2.73), that the Taylor expansion of f_{mnds} up to order 2 also determines, modulo $O((|\mu\hbar|+|\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}})$:

$$\sum_{\substack{|j|+|k|\\+2q=1}}\sum_{r\in\mathbb{Z}}\alpha_{1,j,k,q,r}\hbar\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)\langle\mu,\nu|\frac{i}{\hbar}[e^{-i2\pi rt}Op^W(z^j\bar{z}^k)D_t^q,O_{mnds}]|\mu,\nu\rangle \tag{2.80}$$

Just as before, with assumptions (2.76) and $|\mu\hbar|+|\nu\hbar| \ll \hbar^{\frac{2}{3}}$, we can determine every $\alpha_{1,j,k,q,d}$ with $|j|+|k|+2q = 1$ and $j+m = k+n$ (there is actually just one corresponding to $q = 0$, and $(j,k) = (n,m)$), and finally, every function $\alpha_{1,j,k,q}$ with $|j|+|k|+2q = 1$).
This prove the statement for $N = 3$.

Now, let $N \geq 3$, and let us assume that we already determined the family $(\alpha_{p,j,k,q})_{2p+|j|+|k|+2q=N}$.
Let (m,n,d,s) conditions (1) (with $N+1$), (2), and (3).
The Taylor expansion up to order N of function f_{mnds} determines modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{N+1}\right)$:

$$\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq 2N+2}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq 2N+2}}{\hbar}}|\mu,\nu\rangle - \langle\mu,\nu|O_{mnds}|\mu,\nu\rangle \tag{2.81}$$

which is equal, thanks to (2.68) and Lemma 2.16 and modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}\right)+O(\hbar)$, to:

$$\sum_{\substack{|j|+|k|+2q\leq N+1\\ j+m=k+n}} i\alpha_{0,j,k,q,d}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right) g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right)$$

and by induction hypothesis, the following quantity is determined modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}\right)+O(\hbar)$:

$$\sum_{\substack{|j|+|k|+2q=N+1\\ j+m=k+n}} i\alpha_{0,j,k,q,d}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right) g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.82}$$

Now, making assumptions (2.76) and $\hbar = O\left((|\mu\hbar|+|\nu\hbar|)^N\right)$, we determine every $\alpha_{0,j,k,q,d}$ with $|j|+|k|+2q=N+1$ and $j+m=k+n$, and like before, letting (m,n,d,s) run over all possible values (under conditions (1), (2), and (3)), we determine every function $\alpha_{0,j,k,q}$.

Functions $\alpha_{p,j,k,q}$ $(2p+|j|+|k|+2q=N+1)$ will now be determined by induction on p. Let $0\leq p_0\leq\frac{N-1}{2}$ and let us assume we determined functions $\alpha_{p,j,k,q}$ $(0\leq p\leq p_0$ and $|j|+|k|+2q=N+1-2p)$.
Let (m,n,d,s) satisfy conditions (1) (with $N+1-2(p_0+1)$), (2), and (3). Thus, the Taylor expansion of f_{minds} up to order N determines modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}\right)+O(\hbar^{p_0+2})$

$$\sum_{\substack{2p_0+2+|j|+|k|+2q=N+1\\ j+m=k+n}} i\alpha_{p,j,k,q,d}\hbar^{p_0+1}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right) g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.83}$$

And with assumptions (2.76) and $|\mu\hbar|+|\nu\hbar|\ll\hbar^{\frac{2(p_0+1)}{2p_0+3}}$, heredity can be proved just as before, which concludes the proof.

□

Proof of Lemma 2.16. The principal symbol of $\frac{1}{i\hbar}[e^{-i2\pi dt}\mathrm{Op}^W z^j\bar{z}^k D_t^q, O_{mnds}]$ is:

$$\sigma_{jkdq}(z,t,\bar{z},\tau)=\left\{e^{-i2\pi dt}z^j\bar{z}^k\tau^q,\mathcal{O}_{mnds}\right\}=\left\{e^{-i2\pi dt}z^j\bar{z}^k\tau^q,e^{i2\pi dt}z^m\bar{z}^n\tau^s\right\} \tag{2.84}$$

where $e^{-i2\pi dt}z^j\bar{z}^k\tau^q$ is meant for the function $(z,t,\bar{z},\tau)\mapsto e^{-i2\pi dt}z^j\bar{z}^k\tau^q$.

Hence

$$\begin{aligned}\sigma_{jkdq}(z,t,\bar z,\tau) =& -i\sum_{i=1}^n \frac{\partial}{\partial z_i}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial \bar z_i}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\ &+i\sum_{i=1}^n \frac{\partial}{\partial \bar z_i}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial z_i}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\ &+\frac{\partial}{\partial t}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial \tau}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\ &-\frac{\partial}{\partial \tau}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial t}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\ =&-iz\bar z^{|\max(j,k)|}\tau^{q+s}\left(\sum_{i=1}^n\frac{j_in_i-k_im_i}{z_i\bar z_i}+2\pi\frac{d(s+q)}{\tau}\right)\end{aligned} \tag{2.85}$$

which means that:

$$\begin{aligned}\frac{1}{\hbar}[e^{-i2\pi dt}\mathrm{Op}^W z^j\bar z^k D_t^q, O_{mnds}] =&D_t^{q+s}\sum_{\substack{i=1\\ |j_i|+|k_i|>0}}^n (j_in_i-k_im_i)P_i^{\max(j_i,k_i)-1}\prod_{\substack{i=1\\ i'\neq i}}^n P_{i'}^{\max(j_{i'},k_{i'})}\\ &+2\pi(q+s)D_t^{q+s-1}P^{\max(j,k)}+O(\hbar)\end{aligned} \tag{2.86}$$

and finally:

$$\begin{aligned}\frac{1}{\hbar}\langle\mu,\nu|[e^{-i2\pi dt}\mathrm{Op}^W z^j\bar z^k D_t^q, O_{mnds}]|\mu,\nu\rangle =&(2\pi\nu\hbar)^{q+s}(\mu\hbar)^{\max(j,k)}\sum_{\substack{i=1\\ |j_i|+|k_i|>0}}^n\frac{j_in_i-k_im_i}{\mu_i\hbar}\\ &+2\pi(q+s)(2\pi\nu\hbar)^{q+s-1}(\mu\hbar)^{\max(j,k)}+O(\hbar)\end{aligned} \tag{2.87}$$

□

2.4. **"Bottom of the well".** In this subsection, we treat the "Bottom of well" analogs of Theorem 1.4, namely Theorems 1.9 and 1.12. The proof of Theorem 1.9 is a line by line analog of Theorem 1.4: we omit it here. However, Theorem 1.12, that needs less assumptions in the particular case of a Schrödinger operator, deserves a proper proof, which we give below.

Proof of Theorem 1.12. In a system of Fermi coordinates, the (principal and total) symbol of our Schrödinger operator can be written as:

$$H(x,\xi)=V(q_0)+\sum_{i=1}^n\theta_i\frac{x_i^2+\xi_i^2}{2}+R(x) \tag{2.88}$$

where $R(x)=O(x^3)$.

Let ϵ be a positive real number, and let us prove that the knowledge of the spectrum of $H(x,\hbar D_x)$ in $[V(q_0),V(q_0)+\epsilon]$ and the diagonal matrix elements of the finite (2^n-1) number of observables whose principal symbols in this system of local coordinates are:

$\prod_{j=1}^{n}(x_j+i\xi_j)^{m_j}$, where for any $j\in[\![1,n]\!]$, $m_j\in\{0,1\}$ determine the Taylor expansion near q_0 of R.

Let H^0 be the function defined by $H^0(x,\xi)=\sum_{i=1}^{n}\theta_i\frac{x_i^2+\xi_i^2}{2}$.

Let us state the following lemma, which is a classical analog of Lemma 2.11 and uses the hypothesis of rational independance of the θ_i's (we therefore omit its proof).

Lemma 2.17. *Let $G\in\mathcal{C}^\infty(T^*(\mathbb{R}^n),\mathbb{R})$ be an homogeneous polynomial of degree $k\geq 3$. There exists a unique couple of functions $G_1\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ and $F\in\mathcal{C}^\infty(T^*(\mathbb{R}^n),\mathbb{R})$ satisfying:*

$$\forall(x,\xi)\in T^*(\mathbb{R}^n),\ \{H^0,F\}(x,\xi)=G(x,\xi)-G_1(p) \tag{2.89}$$

*and F is polynomial with no diagonal term when written as a function of $(z,\bar{z})$ (*i.e. *of the form $z^l\bar{z}^l$)*

Moreover:

1. *F is an homogeneous polynomial of degree k and is entierely determined by the extradiagonal termes of G,* i.e. *of the form $z^l\bar{z}^m$ $(l\neq m)$*
2. *G_1 is an homogeneous polynomial of degree $\frac{k}{2}$ if k is even, zero if else. Moreover, $G_1(z\bar{z})$ is equal to the sum of the diagonal terms of G.*

Just like in the proof of proposition 2.2, one shows recursively using Lemma 2.17 the existence of a family of real numbers $(\alpha_{lm})_{l,m\in\mathbb{N}}$ such that for any $l\geq 0$, $\alpha_{ll}=0$, such that if the functions $(F_N)_{N\geq 3}$ are defined for $N\geq 3$ by:

$$F_N(z,\bar{z})=\sum_{|l|+|m|=N}\alpha_{lm}z^l\bar{z}^m \tag{2.90}$$

there exists homogeneous polynomials of degree i denoted by $H^i\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ satisfying, for $N\geq 3$:

$$(\exp\chi_{F_{\leq N}})^*H(x,\xi)=\sum_{i=1}^{\lfloor\frac{N}{2}\rfloor}H^i(p)+O((x,\xi)^{N+1}) \tag{2.91}$$

where $p=p(x,\xi)=(\frac{x_i^2+\xi_i^2}{2})_{1\leq i\leq n}$, and $F_{\leq N}=\sum_{k=1}^{N}F_k$, and $\chi_{F_{\leq N}}$ is the vector field:

$$\chi_{F_{\leq N}}=\sum_{i=1}^{n}\frac{\partial F_{\leq N}}{\partial\xi_i}\frac{\partial}{\partial x_i}-\frac{\partial F_{\leq N}}{\partial x_i}\frac{\partial}{\partial\xi_i} \tag{2.92}$$

If $H_1\sim\sum_{i=1}^{+\infty}H^i$, then $(x,\xi)\mapsto H_1(p)$ is the classical Birkhoff normal form of H (defined modulo a flat function), which is equal to the principal symbol to the quantum Birkhoff normal form.

Let us also define for $k\in\mathbb{N}^n$, $|k|\geq 3$: $a_k=\frac{1}{k!}\frac{\partial^{|k|}R}{\partial x^k}(0)$

Let $k\in\mathbb{N}^n$. Let us observe that:

$$x^k=\left(\frac{z+\bar{z}}{\sqrt{2}}\right)^k=\frac{1}{\sqrt{2}^{|k|}}\sum_{\substack{(l,m)\in\mathbb{N}^n\\ l+m=k}}\prod_{j=1}^{n}\binom{k_j}{m_j}z^l\bar{z}^m \tag{2.93}$$

Let us define $\mathcal{K} = \{k \in \mathbb{N}^n, |k| \geq 3\} \setminus 2\mathbb{N}^n$, and let $k \in \mathbb{N}^n$, $|k| \geq 3$. By lemma 2.17, there exists an homogeneous polynomial of degree $|k|$ with no diagnonal terms, such that:

$$\{H_0, I_k\}(x,\xi) = \begin{cases} x^k & \text{if } k \in \mathcal{K} \\ x^k - \frac{1}{\sqrt{2}^{|k|}} \prod_{j=1}^n \binom{k_j}{k_j/2} |z|^k & \text{if } k \in 2\mathbb{N}^n \end{cases} \tag{2.94}$$

Functions $(F_N)_{N\geq 3}$ and $(H^i)_{i\geq 1}$ are constructed recursively as follows: let $N \geq 2$ and let us assume we already constructed $F_3, \dots, F_N$ ($F_2 = 0$), and $H_1, \dots, H^{\lfloor \frac{N}{2} \rfloor}$ ($H_1(p) = \sum_{i=1}^n \theta_i p_i$), and let us set:

$$G_{N+1}(x,\xi) = (\exp \chi_{F_{\leq N}})^* H(x,\xi) - \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} H^i(p) + O(\|(x,\xi)\|^{N+1}) \tag{2.95}$$

$$\{H_0, F_{N+1}\}(x,\xi) = \begin{cases} G_{N+1}(x,\xi) & \text{if } N \text{ is even} \\ G_{N+1}(x,\xi) - H^{\frac{N+1}{2}}(p) & \text{if } N \text{ is odd} \end{cases} \tag{2.96}$$

Let us remark that, in our case, $G_{N+1}(x,\xi) - \sum_{|k|=N+1} a_k x^k$ is a sum of terms that depends only on $F_{\leq N}$, $(H^i)_{1\leq i\leq \lfloor \frac{N}{2} \rfloor}$ and $(a_k)_{|k|\leq N}$. Therefore, F_{N+1} and $H^{\frac{N+1}{2}}$ if N is odd depend only on $(a_k)_{|k|\leq N+1}$.

More precisely,

$$\begin{cases} F_{N+1} & = \sum_{|k|=N+1} a_k I_k + \dots \\ H^{\frac{N+1}{2}}(p) & = \sum_{|l|=\frac{N+1}{2}} \frac{a_{2l}}{2^{|l|}} \prod_{j=1}^n \binom{2l_j}{l_j} p^l + \dots (N \text{ odd}) \end{cases} \tag{2.97}$$

where $\dots$ stands for terms that depend only on $(a_k)_{|k|\leq N}$.

Now, let us denote by Λ the set $\{0,1\}^n \setminus \{0\}$. Let us also set, for $k \in \mathbb{N}^n$, $(l_k, m_k) \in \mathbb{N}^{2n}$ as follows: for any $i \in [\![1,n]\!]$, $(l_k)_i = \lfloor \frac{k_i}{2} \rfloor$, $(m_k)_i = k_i - \lfloor \frac{k_i}{2} \rfloor$. $k \mapsto (l_k, m_k)$ is a biunique correspondance between $\mathcal{K}$ and the set $A = \{(l,m) \in \mathbb{N}^{2n} \mid m - l \in \Lambda, |l| + |m| \geq 3\}$.

Moreover, for any $k \in \mathcal{K}$, the coefficient of I_k is $\frac{1}{\theta.(m_k - 1_k)}$ (well defined by the rational independance of the θ_i's). Therefore, the family $(a_k)_{|k|=N+1}$ can be determined recursively from the Taylor expansion of the classical Birkhoff normal form and family $(\alpha_{l_k m_k})_{|k|=N+1}$. The Taylor expansion of the classical Birkhoff normal form is determined by the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$, $\epsilon > 0$ as it is already known. Now, let $N \geq 2$, and $s \in \Lambda$.

$$\begin{aligned} (\exp \chi_{F_{\leq N+1}})^* \left(\frac{x + i\xi}{\sqrt{2}} \right)^s &= (x + i\xi)^s + \{F_{\leq N+1}, z^s\} + O((x,\xi)^{N+|s|}) \\ &= - \sum_{\substack{|k|=N+1 \\ m_k - l_k = s}} \alpha_{l_k m_k} |z|^{2m_k} \sum_{i=1}^n \frac{(m_k)_i}{|z_i|^2} + \dots + O((x,\xi)^{N+|s|}) \end{aligned} \tag{2.98}$$

where $\dots$ stands for terms depending only on $(a_k)_{|k|\leq N}$ (hence already determined by induction hypothesis) or extradiagonal terms. Therefore, the diagonal matrix elements of an observable whose principal symbol is z^s, $s \in \Lambda$ will be equal modulo $O(\hbar)$ and for any $N \geq 3$ to:

$$\sum_{\substack{|k|=N+1\\ m_k-l_k=s}} \alpha_{l_k m_k}(\mu\hbar)^{m_k} \sum_{i=1}^{n} \frac{(m_k)_i}{\mu_i \hbar} + O(\hbar) + O(|\mu\hbar|^{\frac{N+|s|}{2}}) \tag{2.99}$$

which, just in like the proof of Theorem 2.1 determines $(\alpha_{l_k m_k})_{|k|=N+1}$. Therefore the Taylor expansion of R, hence of V near q_0 is completely determined, which concludes the proof.

□

3. Explicit construction of Fermi coordinates

In this section we prove Theorems 1.3, 1.8, and 1.11, whose proofs are essentially a consequence of lemmas on linear and bilinear algebra, which are stated and proven in the appendix: Lemmas A.1, A.2, and A.4.

3.1. **General "Bottom of the well" case.**

The aim of this subsection is to prove Theorem 1.8, that is the existence and the explicit constructivity of some Fermi coordinates in the case where our trajectory is reduced to a single point.

Proof of Theorem 1.8. Let $(x,\xi) \in T^*(\mathbb{R}^n)$ be a system of Darboux coordinates centered at z_0. $d^2H_0(z_0)$ is a positive bilinear form on $T_{z_0}(T^*\mathcal{M})$, therefore, by lemma A.1, there exists a local change of variable ϕ, linear and symplectic in the Darboux coordinates, such that:

$$H_0 \circ \phi(x,\xi) = H_0(z_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2+\xi_i^2}{2} + O((x,\xi)^3). \tag{3.1}$$

Let us prove that the diagonal matrix elements of the family of pseudodifferential operators $(Q_{ij}^k)_{1\le i,j\le n, 1\le k\le 3}$ in the system of eigenvectors corresponding to eigenvalues of $H(x,\hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max_{1\le j\le n}\theta_j + \frac{n}{2} + \epsilon).\hbar]$ (for some $\epsilon > 0$) allow the explicit construction of such a symplectomorphism ϕ (which is not unique).

Let S be the matrix of $d\phi_{z_0}$ in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n})$. We have for $(i,j) \in [\![1,n]\!]^2$ and $s \in \{1,2,3\}$:

$$\begin{aligned}
\mathcal{Q}_{i,j}^s \circ \phi(x,\xi) &= \left(\sum_{k=1}^{n} S_{i^s,2k-1}x_k + S_{i^s,2k}\xi_k\right)\left(\sum_{k=1}^{n} S_{j^s,2k-1}x_k + S_{j^s,2k}\xi_k\right)\\
&= \sum_{k,k'=1}^{n} S_{i^s,2k-1}S_{j^s,2k'-1}x_k x_{k'} + \sum_{k,k'=1}^{n} S_{i^s,2k}S_{j^s,2k'-1}\xi_k x_{k'}\\
&+ \sum_{k,k'=1}^{n} S_{i^s,2k-1}S_{j^s,2k'}x_k \xi_{k'} + \sum_{k,k'=1}^{n} S_{i^s,2k}S_{j^s,2k}\xi_k\xi_{k'}\\
&= \sum_{k=1}^{n} \left[S_{i^s,2k-1}S_{j^s,2k-1} + S_{i^s,2k}S_{j^s,2k}\right] z_k\bar{z}_k + R,
\end{aligned} \tag{3.2}$$

where, for $(i,j) \in [\![1,n]\!]^2$, $i^s = \begin{cases} 2i-1 & \text{if } s \in \{1,2\} \\ 2i & \text{if } s = 3 \end{cases}$ and $j^s = \begin{cases} 2j & \text{if } s \in \{1,3\} \\ 2j-1 & \text{if } s = 2 \end{cases}$,
and R is a linear combination of terms of the form $z_k z_{k'}$ ($(k,k') \in [\![1,n]\!]$) and $z_k \bar{z}_{k'}$ ($(k,k') \in [\![1,n]\!]$, $k \neq k'$).

Therefore, if M_ϕ is the metaplectic representation of $d\phi_{z_0}$, then for $\mu \in \mathbb{N}^n$:

$$\langle \mu | M_\phi Q^s_{i,j} M_\phi^{-1} | \mu \rangle = \sum_{k=1}^{n} [S_{i^s,2k-1} S_{j^s,2k-1} + S_{i^s,2k} S_{j^s,2k}] \left(\mu_k + \frac{1}{2} \right) \hbar + O(\hbar|\mu\hbar|). \tag{3.3}$$

hypothèse à rajouter: sous-symbole non constant ?

Therefore, we only need eigenvectors corresponding to $|\mu| = 1$ to determine the values of $S_{i,2k-1}S_{j,2k-1} + S_{i,2k}S_{j,2k}$ for $(i,j) \in [\![1,2n]\!]^2$ and $k \in [\![1,n]\!]$. As it was already claimed by Lemma A.2, the preceeding quantities are independent of the choice of a symplectic matrix S satisfying (3.1). Since, as we already said, such a matrix S is not unique, it is not possible to determine S out of the preceeding matrix elements. However, by Lemma A.2, the family $(S_{i,2k-1}S_{j,2k-1} + S_{i,2k}S_{j,2k})_{(i,j)\in[\![1,2n]\!]^2, k\in[\![1,n]\!]}$ (determined by the preceeding matrix elements) allows us to construct explicitely a suitable matrix S, hence a suitable symplectomorphism ϕ.

□

3.2. **The "Schrödinger case".** In this subsection, we prove Theorem 1.11, that is the existence and the explicit constructivity of some Fermi coordinates in the case where our trajectory is reduced to a single point, with less assumptions than Theorem 1.8 but in the particular case where our Hamiltonian is a Schrödinger operator.

Proof of Theorem 1.11. Let $x \in \mathbb{R}^n$ be **any** system of local coordinates centered at $q_0 \in \mathcal{M}$, and $(x,\xi) \in T^*(\mathbb{R}^n)$ the corresponding Darboux coordinates centered at $(q_0,0) \in T^*\mathcal{M}$. $d^2V(q_0)$ being a positive bilinear form on $T_{q_0}\mathcal{M}$, there exists, by Lemma A.4, a local change of variable u, linear and orthogonal in the Darboux coordinates, such that:

$$V \circ u(x) = \frac{1}{2} \sum_{i=1}^{n} \theta_i^2 x_i^2 + O(x^3) \tag{3.4}$$

where the θ_i^2's are the eigenvalues of $d^2V(q_0)$.

Let us denote by U the matrix of du_{q_0} written in the basis $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, and define a symplectomorphism ϕ locally by its expression in the Darboux coordinates: $\phi(x,\xi) = (Ux, U\xi)$.

If ϕ_0 is the symplectomorphism that sends (x,ξ) to $(\frac{x_1}{\sqrt{\theta_1}}, \dots, \frac{x_n}{\sqrt{\theta_n}}, \sqrt{\theta_1}\xi_1, \dots, \sqrt{\theta_i}\xi_n)$, and H is the (principal and total) symbol of the considered Schrödinger operator then:

$$H \circ \phi \circ \phi_0(x,\xi) = V(q_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + O(x^3) \tag{3.5}$$

Just as in proof of Theorem 1.8, the diagonal matrix elements of the family of pseudodifferential operators $(Q^2_{ij})_{1\leq i,j\leq n}$ in the system of eigenvectors corresponding to eigenvalues of $H(x,\hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max_{1\leq j\leq n} \theta_j + \frac{n}{2} + \epsilon).\hbar]$ (for some $\epsilon > 0$) determine the family $(U_{ik}U_{jk})_{1\leq i,j,k\leq n}$. An orthogonal matrix U such that (3.5) is verified is not unique, therefore it is not possible to determine the matrix U from the preceeding diagonal matrix elements. However, by Lemma A.4, the family $(U_{ik}U_{jk})_{1\leq i,j,k\leq n}$ does not depend

on the suitable matrix U (*i.e.* orthogonal and satisfying (3.5)), and as we just saw it is determined by the preceeding matrix elements. Therefore, one can determine the absolute values of the coefficients of any suitable matrix U, and also, for any $k \in [\![1, n]\!]$, an index $i_k \in [\![1, n]\!]$, such that $U_{i_k k} \neq 0$. The choice of the sign of $U_{i_k k}$ then determines the sign of every other coefficient of the k-th column. Therefore, one can determine the 2^n suitable matrices, corresponding to n choices of signs, as claimed by Lemma A.4. Choosing one of them determines (explicitely) a suitable symplectomorphism ϕ.

□

3.3. **Periodic case.** Let us finally finish this section by proving Theorem 1.3, that is the explicit construct of some particular Fermi coordinates, obtained from our original system of local coordinates by a symplectic change of variable.

Proof of Theorem 1.3. Let $X, H(x, \hbar D_x), E, \gamma$ be as in Theorem 1.3.

We first recall [5, 6, 16, 17] that there exists a symplectomorphism ϕ from a neighborhood of $\mathbb{S}^1$ in $T^*(\mathbb{R}^n \times \mathbb{S}^1)$ in a neighborhood of γ in $T^*(X)$ such that in the standard symplectic coordinates of $T^*(\mathbb{R}^n \times \mathbb{S}^1)$,

$$H_0 \circ \phi(x, t, \xi, \tau) = H^0 + H_2 \quad \text{and} \quad \gamma(t) = \phi(0, 0, t, 0). \tag{3.6}$$

Here H^0 is defined as in (2.3) by

$$H^0(x, t, \xi, \tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau$$

and H_2 satisfies condition (2.2):

$$H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|).$$

Such a symplectomorphism is not unique. Expressing ϕ in a system a local coordinates (x', ξ', t', τ') near γ such that $\gamma = \{x' = \xi' = \tau' = 0\}$, one can assume that:

$$\phi(x, t, \xi, \tau) = \phi_S(x, t, \xi, \tau) = (S(t)(x, \xi), t, \tau + q_S(t, x, \xi)) \tag{3.7}$$

where for any $t \in \mathbb{S}^1$, $S(t)$ is a linear symplectic change of variable (identified with its matrix in our system of coordinates), $q_S(t, \cdot, \cdot)$ is quadratic and satisfies: $q_u(t, 0, 0) = 0$ and

$$dq_u = \left(\sum_{i=1}^{n} \dot{L}_{i+n}(t).(x, \xi) L_i(t) - \dot{L}_i(t).(x, \xi) L_{i+n}(t) \right).(dx, d\xi) \tag{3.8}$$

where for $i \in [\![1, 2n]\!]$ and $t \in \mathbb{S}^1$, $L_i(t)$ is the i-th line of the matrix $S(t)$, $\dot{}$ the derivation with respect to t, and for two line vectors of size $2n$, $u.v$ is their canonical scalar product.

Now, just as in the proof of Theorem 1.8, for $(i, j) \in [\![1, n]\!]^2$, $p \in \mathbb{Z}$ and $s \in \{1, 2, 3\}$, the matrix elements corresponding to operators of principal symbol $(x, t, \xi, \tau) \mapsto e^{-2i\pi pt} \mathcal{Q}^s_{i,j}(x, \xi)$ and to eigenvectors indexed by $(\mu, 0)$ with $|\mu| = 1$, determine the p-th Fourier coefficient of:

$$t \mapsto S^\sigma_{i^s, 2k-1}(t) S^\sigma_{j^s, 2k-1}(t) + S^\sigma_{i^s, 2k}(t) S^\sigma_{j^s, 2k}(t) \tag{3.9}$$

where, for $(i, j) \in [\![1, n]\!]^2$, $i^s = \begin{cases} 2i - 1 & \text{if } s \in \{1, 2\} \\ 2i & \text{if } s = 3 \end{cases}$, $j^s = \begin{cases} 2j & \text{if } s \in \{1, 3\} \\ 2j - 1 & \text{if } s = 2 \end{cases}$, σ is the permutation defined by (A.5), and S^σ is defined by conjugation by the permutation

matrix associated to σ just as in (A.6).Therefore, those matrix elements determine the functions:

$$A_{i,j,k} := S^{\sigma}_{i,2k-1}S^{\sigma}_{j,2k-1} + S^{\sigma}_{i,2k}S^{\sigma}_{j,2k} \tag{3.10}$$

for $(i,j) \in [\![1,2n]\!]^2$ and $k \in [\![1,n]\!]$.

An easy adaptation of the proof of Lemma A.2 shows that, once the set of functions $(A_{i,j,k})_{(i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$ is given, one can construct explicitely a particular smooth function $\mathbb{S}^1 \ni t \mapsto S_0(t)$ with values in the set of symplectic matrices, such that equality (3.10) holds. We also get that any matrix S such that equality (3.10) holds is related to S_0 by the equality $S^{\sigma} = S_0^{\sigma}U$ where $t \mapsto U(t)$ is a smooth function that takes his values in the set of block diagonal matrices whose diagonal block matrices are 2 by 2 rotations.

Now let us consider this particular S_0 and let U be any smooth function that takes his values in the set of block diagonal matrices whose diagonal block matrices are 2 by 2 rotations. Let us finally define S by the relation $S^{\sigma} = S_0^{\sigma}U$.

Now let $p \in \mathbb{Z}$, and let us consider Q_p an operator with principal symbol $\mathcal{Q}_p(x,t,\xi,\tau) = e^{-2i\pi pt}\tau$. Since for any $t \in \mathbb{S}^1$, $q_S(t,\cdot,\cdot)$ is quadratic:

$$\begin{aligned}\mathcal{Q}_p \circ \phi_S(x,t,\xi,\tau) &= e^{-2i\pi pt}\tau + e^{-2i\pi pt}q_S(t,x,\xi)\\ &= e^{-2i\pi pt}\tau + e^{-2i\pi pt}\sum_{k=1}^{n}\left(\frac{\partial^2 q_S}{\partial x_k^2} + \frac{\partial^2 q_S}{\partial \xi_k^2}\right) z_k\bar{z}_k + R\end{aligned} \tag{3.11}$$

where R is a linear combination of terms of the form $e^{-2i\pi pt}z_kz_{k'}$ $((k,k') \in [\![1,n]\!])$ and $e^{-2i\pi pt}z_k\bar{z}_{k'}$ $((k,k') \in [\![1,n]\!],\ k \neq k')$.

As in the proof of Theorem 1.9, for $p \in \mathbb{Z}$, the diagonal matrix element corresponding to operator Q_p and eigenvectors indexed by $(\mu,0)$ with $|\mu| = 1$ determines the p-th Fourier coefficient of the functions $\left(t \mapsto \frac{\partial^2 q_S}{\partial x_k^2} + \frac{\partial^2 q_S}{\partial \xi_k^2}\right)_{k\in[\![1,n]\!]}$. Hence, the family of matrix elements associated to $(Q_p)_{p\in\mathbb{Z}}$ determines thes function $\left(t \mapsto \frac{\partial^2 q_S}{\partial x_k^2} + \frac{\partial^2 q_S}{\partial \xi_k^2}\right)_{k\in[\![1,n]\!]}$.

Now, we get from equation (3.8) that, for $k \in [\![1,n]\!]$ and $t \in \mathbb{S}^1$:

$$\begin{aligned}\frac{\partial^2 q_S}{\partial x_k^2}(t) + \frac{\partial^2 q_S}{\partial \xi_k^2}(t) &= \sum_{i=1}^{n}\dot{S}_{i+n,k}(t)S_{i,k}(t) + \dot{S}_{i+n,k+n}(t)S_{i,k+n}(t)\\ &\quad - \sum_{i=1}^{n}\dot{S}_{i,k}(t)S_{i+n,k}(t) + \dot{S}_{i,k+n}(t)S_{i+n,k+n}(t)\end{aligned} \tag{3.12}$$

Now, for $k \in [\![1,n]\!]$ and $t \in \mathbb{S}^1$, let us denote by $U_k(t) = \begin{pmatrix}\cos\theta_k(t) & -\sin\theta_k(t)\\ \sin\theta_k(t) & \cos\theta_k(t)\end{pmatrix}$ the k-th diagonal block of $U(t)$. Then, for $j \in [\![1,2n]\!]$, $k \in [\![1,n]\!]$ and $t \in \mathbb{S}^1$:

$$\begin{pmatrix}S_{j,k}(t)\\ S_{j,k+n}(t)\end{pmatrix} = {}^tU_k(t)\begin{pmatrix}S_{0,j,k}(t)\\ S_{0,j,k+n}(t)\end{pmatrix} \tag{3.13}$$

Therefore:

$$\begin{pmatrix}\dot{S}_{j,k}(t)\\ \dot{S}_{j,k+n}(t)\end{pmatrix} = {}^tU_k(t)\begin{pmatrix}\dot{S}_{0,j,k}(t)\\ \dot{S}_{0,j,k+n}(t)\end{pmatrix} + {}^t\dot{U}_k(t)\begin{pmatrix}S_{0,j,k}(t)\\ S_{0,j,k+n}(t)\end{pmatrix} \tag{3.14}$$

Let us now observe that for $k \in [\![1, n]\!]$, and any $t \in \mathbb{S}^1$:

$$\dot{U}_k(t)\ {}^tU_k(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.15}$$

Therefore, because for any $k \in [\![1, n]\!]$, and $t \in \mathbb{S}^1$, $U_k(t)$ is an orthogonal matrix and $S_0(t)$ is a symplectic matrix, we get from equations (3.14) and (3.15):

$$\begin{aligned} \frac{\partial^2 q_S}{\partial x_k^2}(t) + \frac{\partial^2 q_S}{\partial \xi_k^2}(t) = \frac{\partial^2 q_{S_0}}{\partial x_k^2}(t) + \frac{\partial^2 q_{S_0}}{\partial \xi_k^2}(t) \\ + 2\dot{\theta}_k(t) \end{aligned} \tag{3.16}$$

Since the function $t \mapsto \frac{\partial^2 q_S}{\partial x_k^2}(t) + \frac{\partial^2 q_S}{\partial \xi_k^2}(t)$ has been determined above by matrix elements, and the function $t \mapsto frac\partial^2 q_{S_0} \partial x_k^2(t) + \frac{\partial^2 q_{S_0}}{\partial \xi_k^2}(t)$ is entirely determined by the explicitely contructed function $t \mapsto S_0$, equation (3.16) then determines the function $\dot{\theta}_k$. Therefore, the function $t \mapsto U(t)$, hence the function $t \mapsto S^\sigma(t)$, is determined up to right multiplication by a constant block diagonal matrix U_0 whose diagonal block matrices are 2 by 2 rotations. It is now sufficient to observe, that if two functions $t \mapsto S_1(t)$ and $t \mapsto S_2(t)$ are related by the equation:

$$S_2^\sigma = S_1^\sigma U_0 \tag{3.17}$$

where U_0 is a constant matrix, then

$$\phi_{S_2} = \phi_{S_1} \circ \phi_{U_0^{\sigma -1}} \tag{3.18}$$

and, if U_0 is a constant block diagonal matrix whose diagonal block matrices are 2 by 2 rotations:

$$H^0 \circ \phi_{U_0^{\sigma -1}} = H^0 \tag{3.19}$$

Finally, the choice of U_0 in the determination of $t \mapsto S(t)$ does not change the validity of equation (3.6) for $\phi = \phi_S$, and Theorem 1.3 is proved.

□

4. Classical analogs

In this section we want to prove a classical analog to Proposition 2.14. It is well known that matrix elements of quantum observables between eigenvectors of integrable Hamiltonians are given at the classical limit by Fourier coefficients in action-angle variables of the classical Hamiltonian. More precisely in the case of diagonal matrix elements the result states that, with the notation of section 2, for any bounded pseudodifferential operator O on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$,

$$\langle \mu, \nu | O | \mu, \nu \rangle \sim \int_{\mathbb{T}^n \times \mathbb{S}^1} \mathcal{O}'(\mu\hbar, \nu\hbar; \varphi, s) d\varphi ds, \tag{4.1}$$

where $\mathcal{O}'(p, \tau : \varphi, s)$ is the principal symbol of O expressed in the action angles variables (p_i, φ_i) such that $x_l + i\xi_l = \sqrt{p_l} e^{i\varphi_l}$. Therefore it is natural to ask if angle-averages of observables expressed in Birkhoff coordinates determine the original Hamiltonian. Our result is the following.

Theorem 4.1. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by a proper smooth Hamiltonian function H. Let (x,t,ξ,τ) be **any** system of local coordinates near γ such that $\gamma = \{x = \xi = \tau = 0\}$.*

For $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \{0,1\}$ let $\mathcal{O}_{mnds}$ be functions satisfying in a neighborhood of γ:

$$\mathcal{O}_{mnds}(x,\xi;t,\tau) := e^{i2\pi dt}\tau^s \prod_{j=1}^{n}(x_j + i\xi_j)^{m_j}(x_j - i\xi_j)^{n_j}. \tag{4.2}$$

Let Φ be the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near γ and $(p,\varphi;\tau,s)$ the corresponding Birkhoff coordinates such that $\gamma = \{p = \tau = 0\}$. Let us define near $p = \tau = 0$ the following "average" quantities

$$\overline{\mathcal{O}}_{mnds}(p,\tau) := \int_{\mathbb{T}^n \times \mathbb{S}^1} \mathcal{O}_{mnds} \circ \Phi(p,\tau;\varphi,s) d\varphi ds. \tag{4.3}$$

Then the knowledge of the Taylor expansion at $p = \tau = 0$ up to order N of $\overline{\mathcal{O}}_{mnds}$ for

1. *$|m| + |n| \leq N$*
2. *$\forall j \in [\![1,n]\!]$, $m_j = 0$ **or** $n_j = 0$*
3. *$s = 1$ if $m = n = 0$, otherwise $s = 0$*

determines the Taylor expansion of Φ near γ up to order N in any system of Fermi coordinates. Therefore the knowledge of these expansions together with the normal form up to order N determine the Taylor expansion of the "true" Hamiltonian H up to the same order again in any system of Fermi coordinates.

Remark 4.2. A Corollary of Theorem 4.1 in the line of Corollary **??** can be obtained in a straightforward way. We omit it here.

Proof. We saw in the preceding sections that the diagonal matrix elements of the quantum observables O_{mnds} determine the full semiclassical expansion of the Taylor expansion of the total symbol of the Hamiltonian. What's left to be done is, roughly speaking, to check that the classical limit of the matrix elements determine the one of the symbol. We will need the following lemma (see [14] for a proof)

Lemma 4.3. *Let O be an pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ whose Weyl symbol, expressed in polar and cylindrical coordinates is the function $\mathcal{O}(p,\tau;\varphi,s)$. Then*

$$\langle \mu,\nu|O|\mu,\nu\rangle = \int_{\mathbb{T}^n \times \mathbb{S}^1} \mathcal{O}(\mu\hbar,\nu\hbar;\varphi,s) d\varphi ds + O(\hbar). \tag{4.4}$$

Let O_{mnds} be the pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ whose Weyl symbol is the function $\mathcal{O}_{mnds}$. In order to prove Theorem 4.1, it is enough to see that one can recover from the Taylor expansion of the averages O^0_{mnds} up to order N the principal symbol $\sigma_N(z,t,\bar{z},\tau)$ of $\widetilde{W}_{\leq N}$ up to order N. We will proceed by induction on N just as in the proof of Proposition 2.14.

Let us first remark, by Egorov's Theorem and the link between the construction of the quantum Birkhoff normal forms and the classical normal form that the principal symbols of $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}$, and $\mathcal{O}_{mnds} \circ \Phi$ have the same Taylor expansion up to order $N - 1 + |m| + |n| + 2s$. More precisely, the Taylor expansion of the principal symbol $\sigma_N(z,t,\bar{z},\tau)$

of $\widetilde{W}_{\leq N}$ up to order N is exactly:

$$\sigma_N(z,t,\bar{z},\tau) = \sum_{|j|+|k|+2q\leq N} \alpha_{0,j,k,q}(t) z^j \bar{z}^k \tau^q + O\left(|z|^2+|\tau|\right)^{\frac{N+1}{2}} \tag{4.5}$$

and by Lemma 4.3, we get:

$$\begin{aligned}
\mathcal{O}^0_{mnds}(\mu\hbar,\nu\hbar) &= \langle \mu,\nu| e^{\frac{i\widetilde{W}_{\leq N+1}}{\hbar}} O_{mnds} e^{\frac{-i\widetilde{W}_{\leq N+1}}{\hbar}} |\mu,\nu\rangle + O\left((|\mu|+|\nu|)\hbar\right)^{N/2+1}\Big) + O(\hbar) \\
&= \frac{i}{\hbar}\langle \mu,\nu|[\widetilde{W}_{\leq N+1}, O_{mnds}]|\mu,\nu\rangle + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} \langle \mu,\nu|[\overbrace{\widetilde{W}_{\leq N+1},\dots,\widetilde{W}_{\leq N+1}}^{l \text{ times}}, O_{mnds}]|\mu,\nu\rangle \\
&+ O\left((|\mu|+|\nu|)\hbar\right)^{\frac{N+|m|+|n|+2s}{2}}\Big) + O(\hbar) \\
&= \sum_{\substack{|j|+|k|+2q=N+1 \\ j+m=k+n}} \alpha_{0,j,k,q,d} i\hbar^{-1} \langle \mu,\nu|[e^{-i2\pi dt}\mathrm{Op}^W(z^j\bar{z}^k)D_t^q, O_{mnds}]|\mu,\nu\rangle \\
&+ \frac{i}{\hbar}\langle \mu,\nu|[W_{\leq N}, O_{mnds}]|\mu,\nu\rangle + \sum_{l=2}^{N} \frac{i^l}{\hbar^l l!} \langle \mu,\nu|[\overbrace{W_{\leq N},\dots,W_{\leq N}}^{l \text{ times}}, O_{mnds}]|\mu,\nu\rangle \\
&+ O\left((|\mu|+|\nu|)\hbar\right)^{\frac{N+|m|+|n|+2s}{2}}\Big) + O(\hbar)
\end{aligned}$$

Now because the principal symbol of:

$$\frac{i}{\hbar}[W_{\leq N}, O_{mnds}] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{W_{\leq N},\dots,W_{\leq N}}^{l \text{ times}}, O_{mnds}] \tag{4.6}$$

is precisely:

$$\sum_{l=1}^{N} \frac{1}{l!} \{\overbrace{\sigma_N,\dots,\sigma_N}^{l \text{ times}}, \mathcal{O}_{mnds}\}, \tag{4.7}$$

it depends, by equation (4.5), only on $\alpha_{0,j,k,q}(t)$, $|j|+|k|+2q \leq N$, up to $O\left((z\bar{z}+|\tau|)^{\frac{N+|m|+|n|+2s}{2}}\right)$.

Therefore, one can conclude by induction again just like in the proof of Proposition 2.14.

□

The last result of this paper will be the classical analog of Theorems 1.9 and 1.12.

Let us remind, [8], that in the case where $H(x,\xi) = \xi^2 + V(x)$ the classical normal form determines the Taylor expansion of the potential when the latter is invariant, for each $i \in [\![1,n]\!]$ by the symmetry $x_i \to -x_i$, In the general case the Taylor expansion of the averages, in the sense of (4.3), of a finite number of classical observables are necessary to recover the full potential.

Let us assume $H \in \mathcal{C}^\infty(T^*\mathcal{M},\mathbb{R})$ has a global minimum at $z_0 \in T^*\mathcal{M}$, and let $d^2H_0(z_0)$ be the Hessian of H at z_0. Let us define matrix Ω defined by $d^2H_0(z_0)(\cdot,\cdot) =: \omega_{z_0}(\cdot,\Omega^{-1}\cdot)$ where $\omega_{z_0}(\cdot,\cdot)$ is the canonical symplectic form of $T^*\mathcal{M}$ at z_0. Ω's eigenvalues are purely imaginary, let us denote them by $\pm i\theta_j$ with $\theta_j > 0$, $j \in [\![1,n]\!]$. Let us assume that $\theta_j, j \in [\![1,n]\!]$ are rationally independent.

Theorem 4.4. *Let $(x,\xi) \in T^*\mathbb{R}^n$ be **any** system of local coordinates centered at z_0.*

For $(m,n) \in \mathbb{N}^{2n}$, $1 \leq i,j \leq n$ and $k \in \{1,2,3\}$, let $\mathcal{O}_{mn}$, $\mathcal{Q}^k_{ij}$ be functions satisfying in a neighborhood of z_0:

$$\mathcal{O}_{mn}(x,\xi) = \prod_{j=1}^{n} (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}, \tag{4.8}$$

$$\left\{ \begin{array}{lcl} \mathcal{Q}^1_{ij}(x,\xi) & = & x_i\xi_j \\ \mathcal{Q}^2_{ij}(x,\xi) & = & x_ix_j \\ \mathcal{Q}^3_{ij}(x,\xi) & = & \xi_i\xi_j. \end{array} \right. \tag{4.9}$$

Let us denote by Φ the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near z_0 and (p,φ) the corresponding Birkhoff coordinates such that $\{z_0\} = \{p = 0\}$. Then the knowledge of the Taylor expansion at $p = 0$ up to order $N \geq 3$ of the (finite number) "average" quantities

$$\int_{\mathbb{T}^n} \mathcal{O}_{mn} \circ \Phi(p,\varphi) d\varphi,$$

with

(1) $|m| + |n| \leq N$
(2) $\forall j \in [\![1,n]\!],\ m_j = 0$ ***or*** $n_j = 0$

and the Taylor expansion up to order 2 of the quantities

$$\int_{\mathbb{T}^n} \mathcal{Q}^k_{ij} \circ \Phi(p,\varphi) d\varphi, \quad (i,j) \in [\![1,n]\!]^2,\ k \in \{1,2,3\},$$

together with the Bikhoff normal form itself, determines the Taylor expansion up to order N of H at z_0 in the system of coordinates (x,ξ).

Let us now enunciate the classical analog of Theorem 1.12 in the case of a Schrödinger operator with potential $V \in \mathcal{C}^\infty(\mathcal{M},\mathbb{R})$:

Theorem 4.5. *Let q_0 be a global non-degenerate minimum of V on $\mathcal{M}$. Let us assume that the square roots of the eigenvalues of $d^2V(q_0)$ are linearly independent over the rationnals.*

*Let $x \in \mathbb{R}^n$ be **any** system of local coordinates centered at q_0, and (x,ξ) the corresponding Darboux coordinates centered at $(q_0,0)$.*

Let us denote by Φ the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near $(q_0,0)$ and (p,φ) the corresponding Birkhoff coordinates.

With the notations of Theorem 4.4, the knowledge of the Taylor expansion at $p = 0$ up to order $N \geq 3$ of the (finite number) "average" quantities

$$\int_{\mathbb{T}^n} \mathcal{O}_{m0} \circ \Phi(p,\varphi) d\varphi, \quad m = (m_1,\ldots,m_n) \in \{0,1\}^n \setminus \{0\}$$

and the Taylor expansion up to order 2 of the quantities

$$\int_{\mathbb{T}^n} \mathcal{Q}^2_{ij} \circ \Phi(p,\varphi) d\varphi, \quad (i,j) \in [\![1,n]\!]^2,$$

together with the Bikhoff normal form itself, determines the Taylor expansion up to order N of V at q_0 in the system of coordinates x.

In the line of the proof of Theorem 4.1 the proofs of Theorem 4.5 and 4.4 are easy adaptations of the proofs of Theorem 1.12 and 1.9. We omit them here.

Appendix A. Lemmas on linear and bilinear algebra

Lemma A.1. *Let q be a positive quadratic form on $\mathbb{R}^{2n}$. Then there exists a canonical endomorphism ϕ on $\mathbb{R}^{2n}$, and a n-tuple of positive real numbers $(\lambda_1, \dots, \lambda_n)$, defined as the imaginary part of the eigenvalues of positive imaginary part of the endomorphism defined by:*

$$\langle \cdot ; a(\cdot) \rangle_q = \omega(\cdot, \cdot) \tag{A.1}$$

where $\langle \cdot ; \cdot \rangle_q$ be the scalar product associated to q and ω the canonical form on $\mathbb{R}^{2n}$, and such that:

$$\forall (x, \xi) \in \mathbb{R}^{2n},\ q(\phi(x, \xi)) = \sum_{i=1}^{n} \lambda_i (x_i^2 + \xi_i^2) \tag{A.2}$$

Moreover, if the real numbers $\lambda_1, \dots, \lambda_n$ are pairwise different, and ϕ' is an endomorphism of $\mathbb{R}^{2n}$. Then ϕ' is canonical and satisfies (A.2) *if and only there exists an orthogonal isomophism u on $\mathbb{R}^{2n}$ whose restriction to the plane spanned by $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i})$ (for any $i \in [\![1, n]\!]$) is a rotation, such that $\phi' = \phi \circ u$.*

Proof of Lemma A.1. a is antisymmetric respective to q, and therefore there exists a q-orthonormal basis of $\mathbb{R}^{2n}$ $(u_1, \dots, u_n, v_1, \dots, v_n)$ and a n-tuple of positive real numbers $(\lambda_1, \dots, \lambda_n)$ such that, for $j \in [\![1, n]\!]$:

$$\lambda_j a(u_j) = -v_j \text{ and } \lambda_j a(v_j) = u_j \tag{A.3}$$

Now let us set, for $j \in [\![1, n]\!]$:

$$\tilde{u}_j = \sqrt{\lambda_j} u_j \text{ and } \tilde{v}_j = \sqrt{\lambda_j} v_j \tag{A.4}$$

Then, $(\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n)$ is a q-orthogonal basis of $\mathbb{R}^{2n}$ satisfying, for $j \in [\![1, n]\!]$, $q(\tilde{u}_j) = \lambda_j$ and $q(\tilde{v}_j) = \lambda_j$, and the preceeding properties together with (A.2) implies that it is also a symplectic basis, which concludes the proof of the first part of Lemma A.1.

To prove the second part of Lemma A.1, let us consider another symplectic and orthogonal basis $(u'_1, \dots, u'_n, v'_1, \dots, v'_n)$ where, for $j \in [\![1, n]\!]$, the q-norm of u'_j and v'_j is λ_j. Then, by (A.2), for any $j \in [\![1, n]\!]$, $a(u'_j)$ is orthogonal to any vector of the basis but v'_j and $\langle v'_j, a(u'_j) \rangle_q = w(v'_j, u'_j) = -1$, therefore $\lambda_j a(u'_j) = -v'_j$, and by the same argument, $\lambda_j a(v'_j) = u'_j$.

Therefore, the plane spanned by (u_j, v_j) and the plane by (u'_j, v'_j) both are the kernel of $a^2 + \lambda_j^2$ (2-dimensional since we made the additional assumption the λ_i's are pairwise different). Therefore, if ϕ and ϕ' are the endomorphisms which send the canonical basis of $\mathbb{R}^{2n}$ to basis $(\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n)$ and basis $(u'_1, \dots, u'_n, v'_1, \dots, v'_n)$ respectively, then one can considerer the restriction to any plane spanned by $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i})$ (for any $i \in [\![1, n]\!]$) is an orthogonal symplectomorphism from the plane to itself, that is a rotation.

□

Let σ be the permutation of $[\![1, 2n]\!]$ defined by:

$$\forall i \in [\![1, 2n]\!],\ \sigma(i) = \begin{cases} 2i - 1 & \text{si } i \leq n \\ 2(i - n) & \text{si } i \geq n + 1 \end{cases} \tag{A.5}$$

and M_σ be the associated permutation matrix (*i.e.* for any $(i, j) \in [\![1, 2n]\!]^2$, $(M_\sigma)_{ij} = \delta_{\sigma(i), j}$.

Now, let us set, for any matrix $S \in \mathcal{M}_{2n}(\mathbb{R})$:

$$S_\sigma = M_\sigma^{-1} S M_\sigma. \tag{A.6}$$

Let us also, for $(i,k) \in [\![1,2n]\!] \times [\![1,n]\!]$, denote by $L_{S,i,k}$ the vector of $\mathbb{R}^2$ defined by $L_{S,i,k} = \begin{pmatrix} (S_\sigma)_{i,2k-1} \\ (S_\sigma)_{i,2k} \end{pmatrix} \in \mathbb{R}^2$. Then, for $(i,k) \in [\![1,n]\!]^2$, $\mathfrak{s}_{i,k}$ will be the matrix of size 2 whose first line is ${}^t L_{S,2i-1,k}$ and second line ${}^t L_{S,2i,k}$.

Lemma A.2. *Let A be a positive matrix of size $2n$. Let $\mathcal{S}$ be the (non-empty by lemma A.1) set of symplectic matrices satisfying*

$${}^t S A S = \left(\begin{array}{c|c} D_\lambda & 0 \\ \hline 0 & D_\lambda \end{array}\right) \tag{A.7}$$

where D_λ is the diagonal matrix with $(\lambda_1, \ldots, \lambda_n)$ as n-tuple of positive diagonal elements, which we assume pairwise different. Then:

1. *Family $(\langle L_{S,i,k}; L_{S,j,k}\rangle)_{i,j)\in[\![1,2n]\!]^2, k\in[\![1,n]\!]}$ is independent of matrix $S \in \mathcal{S}$.*
2. *Once the preceeding invariants of $\mathcal{S}$ given, one can construct explicitely a particular matrix of $\mathcal{S}$ (hence all of them by Lemma A.1).*

Proof of Lemma A.2. Let us first prove the first point. Let $(S,T) \in \mathcal{S}^2$. By Lemma A.1, there exists n matrices belonging to $SO_2(\mathbb{R})$ and denoted by $O_1, \ldots, O_n$, such that:

$$T_\sigma = S_\sigma \begin{pmatrix} O_1 & & \\ & \ddots & \\ & & O_n \end{pmatrix} = \begin{pmatrix} \mathfrak{s}_{1,1} O_1 & \cdots & \mathfrak{s}_{1,n} O_n \\ \vdots & & \vdots \\ \mathfrak{s}_{n,1} O_1 & \cdots & \mathfrak{s}_{n,n} O_n \end{pmatrix} \tag{A.8}$$

and (A.8) is equivalent to:

$$\forall (i,k) \in [\![1,2n]\!] \times [\![1,n]\!], L_{T,i,k} = {}^t O_k L_{S,i,k} \tag{A.9}$$

Hence $(\langle L_{S,i,k}; L_{S,j,k}\rangle)_{i,j)\in[\![1,2n]\!]^2, k\in[\![1,n]\!]}$ does not depend of matrix $S \in \mathcal{S}$ and the first point of Lemma A.2 is proven.

Now, let $S \in \mathcal{S}$, and let $(a_{ijk})_{(i,j)\in[\![1,2n]\!]^2, k\in[\![1,n]\!]}$ be the family defined by:

$$\forall (i,j) \in [\![1,2n]\!]^2, \ \forall k \in [\![1,n]\!], a_{ijk} = \langle L_{S,i,k}; L_{S,j,k}\rangle \tag{A.10}$$

Let us assume that this family is given. Two vectors u and v of $\mathbb{R}^2$ are independent iif: $\langle u;v\rangle^2 < \langle u;u\rangle\langle v;v\rangle$. Since matrix S is invertible, on can choose, for any $k \in [\![1,n]\!]$, a couple of indices $(i_k, j_k) \in [\![1,2n]\!]^2$ such that:

$$a_{i_k j_k k}^2 < a_{i_k i_k k} a_{j_k j_k k} \tag{A.11}$$

Let $k \in [\![1,n]\!]$ Let us choose a vector $v_{i_k k}$, whose norm is $\sqrt{a_{i_k i_k k}} > 0$. The following system of equations with unknown $v \in \mathbb{R}^2$:

$$\left\{ \begin{array}{ll} \langle v_{i_k k}; v\rangle & = a_{i_k j_k k} \\ \langle v; v\rangle & = a_{j_k j_k k} \end{array} \right. \tag{A.12}$$

admits exactly two solutions (by (A.11)), denoted by $v_{j_k k}^+$ et $v_{j_k k}^-$ obtained from one another by orthogonal symmetry R_k of axis the line spanned by $v_{i_k k}$.

Let us set $v^-_{i_k k} = v^+_{i_k k} = v_{i_k k}$. Since families $(v^+_{i_k k}, v^+_{j_k k})$ et $(v^-_{i_k k}, v^-_{j_k k})$ are basis of $\mathbb{R}^2$, for any $i \in [\![1, 2n]\!] \setminus \{i_k, j_k\}$, each one of the two systems:

$$\text{(A.13)} \qquad \left\{ \begin{array}{ll} \langle v_{i_k k}; v\rangle & = a_{i_k i k} \\ \langle v^+_{j_k k}; v\rangle & = a_{j_k i k} \end{array} \right. \text{ et } \left\{ \begin{array}{ll} \langle v_{i_k k}; v\rangle & = a_{i_k i k} \\ \langle v^-_{j_k k}; v\rangle & = a_{j_k i k} \end{array} \right.$$

admits exactly one solution denoted respectively by v^+_{ik} and v^-_{ik}, and satisfying relation $v^-_{ik} = R_k v^+_{ik}$.

We are now able to construct 2^n matrices $(T_A)_{A\in\mathcal{P}([\![1,n]\!])}$ defined, for $A \in \mathcal{P}([\![1, n]\!])$, by:

$$\text{(A.14)} \qquad \forall (i,k) \in [\![1, 2n]\!] \times [\![1, n]\!], L_{T_A,i,k} = \left\{ \begin{array}{ll} v^+_{ik} & \text{if } \ k \in A \\ v^-_{ik} & \text{if else} \end{array} \right.$$

In order to prove the second point of Lemma A.2, it is sufficient to prove following assertions:

(1) There exists at least one set $A \in \mathcal{P}([\![1, n]\!])$, such that: $T_A \in \mathcal{S}$.
(2) There is at most one set $A \in \mathcal{P}([\![1, n]\!])$, such that T_A is symplectic (and A is determined by family $(a_{ijk})_{(i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$)

Indeed, one those two assertions proven, there will be exactly one set $A \in \mathcal{P}([\![1, n]\!])$ such that T_A is symplectic, and it will be an element of $\mathcal{S}$, constructed from the values of family $(a_{ijk})_{(i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$ only.

Let us prove the first assertion. Let, for any $k \in [\![1, n]\!]$, O_k be the unique element of $SO_2(\mathbb{R})$ tel que: $L_{S,i_k,k} = O_k v_{i_k k}$ (where S is a particular matrix of $\mathcal{S}$).

The system (A.12) is equivalent to:

$$\text{(A.15)} \qquad \left\{ \begin{array}{ll} \langle L_{S,i_k,k}; O_k v\rangle & = a_{i_k j_k k} \\ \langle O_k v; O_k v\rangle & = a_{j_k j_k k} \end{array} \right.$$

which admits exactly two solutions: $v^+_{j_k k}$ et $v^-_{j_k k}$. Hence, for any $k \in [\![1, n]\!]$:

$$\text{(A.16)} \qquad L_{S,j_k,k} = O_k v^+_{j_k k} \text{ or } L_{S,j_k,k} = O_k v^-_{j_k k}$$

Let us define the set A by:

$$\text{(A.17)} \qquad A = \{k \in \mathbb{N} \mid L_{S,j_k,k} = O_k v^+_{j_k k}\}$$

Since each system (A.13) admit a unique solution, we obtain:

$$\text{(A.18)} \qquad \begin{array}{rl} \forall (i,k) \in [\![1, 2n]\!] \times [\![1, n]\!], L_{S,i,k} = & \left\{ \begin{array}{ll} O_k v^+_{ik} & \text{if } \ k \in A \\ O_k v^-_{ik} & \text{if else} \end{array} \right. \\ & = O_k L_{T_A,i,k} \end{array}$$

that is:

$$\text{(A.19)} \qquad T_{A,\sigma} = S_\sigma \begin{pmatrix} O_1 & & \\ & \ddots & \\ & & O_n \end{pmatrix}$$

and $T_A \in \mathcal{S}$ by Lemma A.1.

In order to prove the second assertion, let us use following lemma:

Lemma A.3. *For any symplectic matrix B of size $2n$, we have:*

$$\forall k \in [\![1,n]\!],\ \sum_{i=1}^{n} \det(\mathfrak{b}_{i,k}) = 1 \tag{A.20}$$

If A_1 and A_2 are two parts of $[\![1,n]\!]$, we get from (A.14) and relation $v_{ik}^- = R_k v_{ik}^+$ that:

$$\forall (i,k) \in [\![1,2n]\!] \times [\![1,n]\!], L_{T_{A_2},i,k} = \begin{cases} R_k L_{T_{A_1},i,k} & \text{if } \ k \in A_1 \Delta A_2 \\ L_{T_{A_1},i,k} & \text{if else} \end{cases} \tag{A.21}$$

where $A_1 \Delta A_2$ is the symmetric difference of A_1 and A_2: $A_1 \Delta A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$. Hence:

$$\forall k \in [\![1,n]\!],\ \sum_{i=1}^{n} \det\left((\mathfrak{t}_{A_2})_{i,k}\right) = \epsilon_k \sum_{i=1}^{n} \det\left((\mathfrak{t}_{A_1})_{i,k}\right) \tag{A.22}$$

where, for $k \in [\![1,n]\!]$, $\epsilon_k = -1$ if $k \in A_1 \Delta A_2$, $\epsilon_k = 1$ if else. Since $A_1 \Delta A_2 = \emptyset$ if and only if $A_1 = A_2$, there exists at most one part A of $[\![1,n]\!]$ such T_A is symplectic. The second assertion, hence the second point of Lemma A.3, is proven. □

Proof of Lemma A.3. Since B is a symplectic matrix, matrix B_σ satisfies:

$${}^t B_\sigma J_\sigma B_\sigma = J_\sigma \tag{A.23}$$

It is sufficient, for $k \in [\![1,n]\!]$, to read equality (A.23) at line $2k$ and column $2k-1$ to obtain:

$$\sum_{i=1}^{n} \det(\mathfrak{b}_{i,k}) = 1 \tag{A.24}$$

□

Lemma A.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive matrix whose eigenvalues are pairwise different. Then there exists exactly 2^n orthogonal matrices conjugating A to the diagonal matrix of its ordered eigenvalues, and they are obtained from one another by eventually changing the sign of their columns.*

Proof of Lemma A.4. Let D be the diagonal matrix of the ordered eigenvalues of A. As A is positive. There exists an orthogonal matrix Q_1 such that:

$$Q_1^{-1} A Q_1 = {}^t Q_1 A Q_1 = D \tag{A.25}$$

Let $Q_2 \in GL_n(\mathbb{R})$. Then Q_2 is orthogonal and satisfies: $Q_2^{-1} A Q_2 = D$ if and only if $Q_2^{-1} Q_1$ is an orthogonal matrix which commutes to D, that is, because the diagonal elements of D are pairwise different, if and only if $Q_2^{-1} Q_1$ is an orthogonal diagonal matrix. Finally, Q_2 is orthogonal and satisfies: $Q_2^{-1} A Q_2 = D$ if and only if $Q_2^{-1} Q_1$ is diagonal and its elements belong to $\{-1, 1\}$, that is if Q_2 is obtained from Q_1 by eventually changing the sign of its columns. □

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