Recovering the Hamiltonian from spectral data

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RECOVERING THE HAMILTONIAN FROM SPECTRAL DATA

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Contents

Abstract. We show that the contributions to the Gutzwiller formula with observable associated to the iterates of a given elliptic nondegenerate periodic trajectory γ and to certain families of observables localized near γ determine the quantum Hamiltonian in a formal neighborhood of the trajectory γ, that is the full Taylor expansion of its total symbol near γ. We also treat the "bottom of a well" case both for general and Schrödinger operators.

1. Introduction and main results

It is well known that spectral properties of semiclassical Hamiltonians and dynamical properties of their principal symbols are linked. Even when there is no precise information "eigenvalue by eigenvalue" of the spectrum, the so-called Gutzwiller trace formula provide information on averages of the spectrum at scale of the Planck constant. More precisely, let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on a compact manifold X of dimension $n+1$, whose symbol $H(x, \xi)$ is proper (as a map from T^*X into $\mathbb{R}$). We will denote by $\sigma = \sigma(H(x, \hbar D_x))$ the spectrum of $H(x, \hbar D_x)$.

Let E be a regular value of H and γ a non-degenerate periodic trajectory of period T_γ lying on the energy surface $H = E$.

Consider the Gutzwiller trace (see [9])

$$Tr\left(\psi\left(\frac{H(x,\hbar D_x)-E}{\hbar}\right)\right) = \sum_\sigma \psi\left(\frac{E-E_i}{\hbar}\right) \tag{1.1}$$

where ψ is a C^∞ function whose Fourier transform is compactly supported with support in a small enough neighborhood of T_γ and is identically one in a still smaller neighborhood containing T_γ. As shown in [12], [13] (1.1) has an asymptotic expansion

$$e^{i\frac{S_\gamma}{\hbar}+\sigma_\gamma} \sum_{k=-n}^{\infty} a_k \hbar^k \tag{1.2}$$

In [6] was shown how to compute the terms of this expansion to all orders in terms of a microlocal Birkhoff canonical form for H in a formal neighborhood of γ, and that the constants $a_{k,r}, \kappa, r = 0, 1, ...$ determine the microlocal Birkhoff canonical form for H in a formal neighborhood of γ (and hence, a fortiori, determine the classical Birkhoff canonical form). When it is known "a priori" that $H(x, \hbar D_x)$ is a Schrödinger operator, it is known that the normal form near the bottom of a well determines part of the potential V [8]. But in the general case the Gutzwiller formula will determine only the normal form of the Hamiltonian, that is to say $H(x, \hbar D_x)$ only modulo unitary operators, and its principal symbol only modulo symplectomorphisms. Of course it cannot determine more, as the spectrum, and a fortiori the trace, is insensitive to unitary conjugation. The aim of this paper is to address the question of determining the *true* Hamiltonian from more precise spectral data, namely from the Gutzwiller trace formula with observables.

It is well know that, for any pseudodifferential operator $O(x, \hbar D_x)$ of symbol $\mathcal{O}(x, \xi)$, there is an equivalent result to (1.2) for the following quantity

$$Tr\left(O(x,\hbar D_x)\psi\left(\frac{H(x,\hbar D_x)-E}{\hbar}\right)\right)=\sum_{\sigma}\langle\varphi_j,O(x,\hbar D_x)\varphi_j\rangle\psi\left(\frac{E-E_i}{\hbar}\right), \tag{1.3}$$

(here φ_j is meant as the eigenvector of eigenvalue E_j) under the form of an asymptotic expansion of the form

$$e^{i\frac{S_\gamma}{\hbar}+\sigma_\gamma}\sum_{k=0}^{\infty}a_k^\gamma(\mathcal{O})\hbar^k \tag{1.4}$$

where a_k^γ are distribution supported on γ.

Through this article we will assume, without loss of generality, that the period of γ is equal to 1.

We will show in the present paper that the knowledge of the coefficients $a_k^\gamma(O)$ for a family (NOT all) of observables localized near γ is enough to determine the (full Taylor expansion of) the (total) symbol of $H(x, \hbar D_x)$ near γ, in other words $H(x, \hbar D_x)$ microlocally in a formal neighborhood of γ, when γ is non-degenerate elliptic in the following sense.

Definition 1.1. A periodic trajectory of the Hamiltonian flow generated by $H(x, \xi)$ is said to be non-degenerate elliptic if its linearized Poincaré map has eigenvalues $(e^{\pm i\theta_i})$, $i = 1, \dots, n$, $\theta_j \in \mathbb{R}$, and the rotation angles θ_i $(i = 1, \dots, n)$ and π are independent over $\mathbb{Q}$.

Definition 1.2 (Fermi coordinates)**.** We will denote by "Fermi coordinates" any system of local coordinates of $T^*\mathcal{M}$ near γ, $(x, \xi, t, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times S^1 \times \mathbb{R}$, such that $\gamma = \{x = \xi = \tau = 0\}$ and on which the principal symbol H_0 of $H(x, \hbar D_x)$ can be written as:

$$H_0(x,\xi,t,\tau)=H^0(x,\xi,t,\tau)+H_2 \tag{1.5}$$

where

$$H_2=O(|x|^3+|\xi^3|+|x\tau|+|\xi\tau|) \tag{1.6}$$

And

$$H^0(x,\xi,t,\tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2+\xi_i^2}{2} + \tau \tag{1.7}$$

The existence of such local coordinates, guaranteed by the Weinstein tubular neighborhood Theorem ([16]), was proved in [6] under the hypothesis of non-degeneracy mentioned earlier.

Our main result is the following.

Theorem 1.3. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by the principal symbol H_0 of $H(x,\hbar D_x)$ on the energy shell $H_0^{-1}(E)$, and let (x,ξ,t,τ) be* ***any*** *system of local coordinates near γ such that $\gamma = \{x=\xi=\tau=0\}$.*

For $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \{0,1\}$, let O_{mnds} be any pseudodifferential operator whose principal symbols $\mathcal{O}_{mnds}$ satisfies in a neighborhood of γ

$$\mathcal{O}_{mnds}(x,\xi,t,\tau) = e^{i2\pi dt}\tau^s \prod_{j}^{n} (x_j+i\xi_j)^{m_j}(x_j-i\xi_j)^{n_j}. \tag{1.8}$$

Then the knowledge of the coefficients $a_k(O_{mnds})$ in (1.3)-(1.4) for $k \le N$ and m,n,d,s satisfying

1. $|m|+|n| \le N$
2. $\forall j = 1\dots n,\ m_j = 0$ ***or*** $n_j = 0$
3. $s = 1$ *if* $m = n = 0$, *otherwise* $s = 0$
4. $d \in \mathbb{Z}$

determines the Taylor expansion near γ of the full symbol of $H(x,\hbar D_x)$ up to order N in any Fermi system of coordinates.

Corollary 1.4. *If one already determined some Fermi coordinates, then the knowledge of the $a_k(O_{mnds})$ with k,m,n,d,s as in Theorem 1.3 determines the Taylor expansion near γ of the full symbol of $H(x,\hbar D_x)$ up to order N in the coordinates (x,ξ,t,τ).*

Remark 1.5. It seems reasonable to think that spectral data with observable give enough information to recover the full Taylor expansion the Hamiltonian (without the quadratic part) without the knowledge of the Fermi coordinates, [10]. We will see below that this is true in the "bottom of a well" case.

Remark 1.6. Condition 2 implies that the number of observables (for each Fourier coefficient in t) needed for determining $H(x,\hbar D_x)$ up to order N is of order N^{n+1} and not N^{2n+2}, number of all polynomials of order N. The fact that not all observables are needed can be understood by the fact that we we know that the Hamiltonian we are looking for is conjugated to the normal form a unitary operator and not by any operator (see the discussion after Theorem 2.1). At the classical level this is a trace of the fact that we are looking for a symplectomorphism, and not any diffeomorphism (see section 5).

Remark 1.7. The asymptotic expansion of the trace (1.3) involves only the microlocalization of $H(x,\hbar D_x)$ in a formal neighborhood of γ. Therefore there is no hope to recover from spectral data more precise information that the Taylor expansion of its symbol near γ. The rest of the symbol concerns spectral data of order $\hbar^\infty$.

In the case where γ is reduced to one point, namely the "bottom of a well" case, the result is more complete.

Let us assume that the principal symbol H_0 of $H(x, \hbar D_x)$ has a global minimum at $z_0 \in T^*\mathcal{M}$, and let $d^2H_0(z_0)$ be the Hessian of H at z_0. Let us define matrix Ω defined by $d^2H_0(z_0)(\cdot,\cdot) =: \omega_{z_0}(\cdot, \Omega^{-1}\cdot)$ where $\omega_{z_0}(\cdot,\cdot)$ is the canonical symplectic form of $T^*\mathcal{M}$ at z_0. Ω's eigenvalues are purely imaginary, let us denote them by $\pm i\theta_j$ with $\theta_j > 0,\ j = 1 \ldots n$. Let us assume that $\theta_j, j = 1 \ldots n$ are rationally independent.

Theorem 1.8. *[Bottom of a well]*
*Let $(x, \xi) \in T^*\mathbb{R}^n$ be **any** system of local coordinates centered at z_0.*
For $(m, n) \in \mathbb{N}^{2n}$ and $1 \le i, j \le n,\ k = 1, 2, 3$, let O_{mn}, Q^k_{ij} be any pseudodifferential operator whose principal symbol $\mathcal{O}_{mn}$, $\mathcal{Q}^k_{ij}$ satisfies in a neighborhood of z_0

$$\mathcal{O}_{mn}(x, \xi) = \prod_{j=1}^{n} (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}, \tag{1.9}$$

$$\left\{ \begin{array}{lcl} \mathcal{Q}^1_{ij}(x, \xi) & = & x_i \xi_j \\ \mathcal{Q}^2_{ij}(x, \xi) & = & x_i x_j \\ \mathcal{Q}^3_{ij}(x, \xi) & = & \xi_i \xi_j . \end{array} \right. \tag{1.10}$$

Then, for all $\epsilon > 0$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[H_0(z_0), H_0(z_0) + \epsilon]$ and the diagonal matrix elements of O_{mn} and Q^k_{ij} between the corresponding eigenvectors, with

1. *$|m| + |n| \le N$*
2. *$\forall j = 1 \ldots n,\ m_j = 0$ **or** $n_j = 0$,*

determines the Taylor expansion up to order N of the full symbol of $H(x, \hbar D_x)$ at z_0 in the coordinates (x, ξ).

Let us notice that Remark 1.5 also holds in this case.

Remark 1.9. Although we will not prove it here, let us remark that Theorem 1.8 (and also Theorem 1.3) is also valid in the framework of quantization of Kälherian manifolds.

Remark 1.10. As it will be clear in the proof of Theorem 1.8, it is in fact enough, in order to get the same result, to know the matrix elements of the Q^k_{ij}s for eigenvectors corresponding to eigenvalues of $H(x, \hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max_{j=1\ldots n} \theta_j + \frac{n}{2} + \epsilon)\hbar]$ for some $\epsilon > 0$.

In the case where $H(x, \hbar D_x)$ is a Schrödinger operator, it is known, [8], that the (actually classical) normal form determines the Taylor expansion of the potential in the case where the latter is invariant, for each $i = 1 \ldots n$, by the symmetry $x_i \to -x_i$. Same result holds without the symmetry assumption in the case $n = 1$, with assumption $V'''(0) \neq 0$, as it has been shown in [3].

Our next result shows that the matrix elements of only a **finite number** of observables are necessary to recover the full Taylor expansion of the potential in the general case.

Theorem 1.11. *Let $H = -\hbar^2 \Delta + V$ be a Schrödinger operator and q_0 be a global non-degenerate minimum of V. Let us assume that the square-roots of the eigenvalues of $d^2V(q_0)$ are linearly independent over the rationals. Let $x \in \mathbb{R}^n$ be **any** local system* *of coordinates centered at q_0, and $(x, \xi) \in T^*\mathbb{R}^n$ the corresponding Darboux coordinates centered at $(q_0, 0)$.*

Then, for all $\epsilon > 0$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$ and the diagonal matrix elements between the corresponding eigenvectors of the $2^n - 1$ observables O_{m0}, $m = (m_1, \dots, m_n) \in \{0,1\}^n \setminus \{0\}$, Q^2_{ij}, $i, j = 1 \dots n$, defined in Theorem 1.8 determines the Taylor expansion of V at q_0 in the coordinates x.

Remark 1.12. Note that since we are dealing with observables localized near the bottoms of the wells, the hypothesis saying that z_0 in Theorem 1.8 and q_0 in Theorem 1.11 are *global* minima can be released and the corresponding results can be formulated in a straightforward way.

Let us notice that Remark 1.10 holds also for Theorem 1.11.

The proof of Theorem 1.3 relies on two results having their own interest per se: **Proposition 2.14** which shows that the coefficients of the trace formula determine the matrix elements $\langle \varphi_j, O(x, \hbar D_x)\varphi_j \rangle$ where φ_j are the eigenvectors of the normal form of the Hamiltonian, and **Proposition 2.15** which states that the knowledge of the matrix elements of the conjugation of a given known selfadjoint operator by a unitary one determines, in a certain sense, the latter.

As a byproduct of Proposition 2.15 we obtain also a purely classical result, somehow analog of it: the averages on Birkhoff angles associated to Birkhoff coordinates of the same classical observables than the ones in Theorem 1.3 determine the Taylor expansion of the (true) Hamiltonian. This is the content of **Theorem 5.1** below.

The paper is organized as follows. In Section 2 we show that, in the case where $X = \mathbb{R}^n \times S^1$, $\gamma = S^1$, the a_k determine the Taylor expansion of the Hamiltonian. In Section 3 we reduce the general situation to the case where $X = \mathbb{R}^n \times S^1$, $\gamma = S^1$. In Section 5 we show the classical equivalent of our quantum formulation. The "bottom of a well" case is treated in Section 4.

Through the whole paper $[\![l, m]\!]$, $l < m$, will stand for the set of integers $\{l, \dots, m\}$.

2. The $\mathbb{R}^n \times S^1$ case

The aim of this section is to prove following theorem in the flat case:

Theorem 2.1. *Let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on $L^2(\mathbb{R}^n \times S^1)$. Let $(x, \xi, t, \tau) \in \mathcal{U} \subset T^*(\mathbb{R}^n \times S^1)$ be the canonical symplectic coordinates near $\gamma = S^1$, non degenerate elliptic periodic orbit of the Hamiltonian flow generated by the principal symbol H_0 of $H(x, \hbar D_x)$ on the energy shell $H_0^{-1}(E)$.*

H_0 of $H(x, \hbar D_x)$ can be written in those coordinates as:

$$H_0(x, \xi, t, \tau) = H^0(x, \xi, t, \tau) + H_2 \tag{2.1}$$

where

$$H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|) \tag{2.2}$$

And H^0 *is equal to:*

$$H^0(x,\xi,t,\tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2+\xi_i^2}{2} + \tau \tag{2.3}$$

For $(m,d,s,n) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \{0,1\}$ *let us choose any pseudodifferential operators* O_{mnds} *whose principal symbols are*

$$\mathcal{O}_{mnds}(x,\xi,t,\tau) = e^{i2\pi dt}\tau^s \Pi_j (x_j + i\xi_j)^{m_j}(x_j - i\xi_j)^{n_j}. \tag{2.4}$$

Then the knowledge of the coefficients $a_k(O_{mnds}), k = 0 \ldots N$ *in ((1.3),(1.4) with*

(1) $|m| + |n| \leq N$
(2) $\forall j = 1 \ldots n,\ m_j = 0$ ***or*** $n_j = 0$
(3) $s = 1$ *if* $m = n = 0$*, otherwise* $s = 0$

determines the Taylor expansion near γ *of the full symbol (in the system of coordinates* (x,ξ,t,τ)*) of* $H(x,\hbar D_x)$ *up to order* N*.*

The proof of Theorem 2.1 will be essentially divided into three steps: first, we will prove in Proposition 2.2 the existence of the quantum Birkhoff normal form in a form convenient to our computations, especially concerning the discussion of orders. In Proposition 2.14, we will show that the trace formula with observable O determines the matrix elements of O in the eigenbasis of the normal form. Finally, in Proposition 2.15, we will show that these matrix elements determines $H(x,\hbar D_x)$ in a formal neighborhood of $x = \xi = \tau = 0$, which leads to Theorem 2.1.

For $i = 1 \ldots n$, let us consider on $L^2(\mathbb{R}^n \times S^1)$ the operators:

- $a_i = \frac{1}{\sqrt{2}}(x_i + \hbar\partial_{x_i})$
- $a_i^* = \frac{1}{\sqrt{2}}(x_i - \hbar\partial_{x_i})$
- $D_t = -i\hbar\partial_t$
- $P_i := \frac{1}{2}\left(-\hbar\partial_{x_i}^2 + x_i^2\right) = a_i^* a_i + \frac{\hbar}{2}$

Now for $\mu \in \mathbb{N}^n,\ \nu \in \mathbb{Z}$ we will denote by $|\mu,\nu\rangle$ a common eigenvector of the P_i's and D_t, namely the vectors such that:

$$P_i|\mu,\nu\rangle = (\mu_i + \frac{1}{2})\hbar|\mu,\nu\rangle \text{ and } D_t|\mu,\nu\rangle = 2\pi\hbar|\mu,\nu\rangle.$$

Those vectors can be explicitly constructed as follows:

$$|0,0\rangle(x,t) := \frac{1}{(\pi\hbar)^{\frac{n}{4}}} e^{\frac{-x^2}{2\hbar}} \tag{2.5}$$

and for any $\mu \in \mathbb{N}^n$

$$|\mu,\nu\rangle(x,t) := e^{i2\pi\nu t} \prod_{i=1}^{n} \frac{1}{\sqrt{\mu_i!}} (a_i^*)^{\mu_i} |0,0\rangle(x,t) \tag{2.6}$$

Let us recall the following:

$$
(2.7)\qquad \begin{cases} a_i|\mu,\nu\rangle = \sqrt{\mu_i\hbar}|\mu_1,\dots,\mu_{i-1},\mu_i-1,\mu_{i+1},\dots,\mu_n,\nu\rangle \\ a_i^*|\mu,\nu\rangle = \sqrt{(\mu_i+1)\hbar}|\mu_1,\dots,\mu_{i-1},\mu_i+1,\mu_{i+1},\dots,\mu_n,\nu\rangle \\ [a_i,a_j^*] = \delta_{ij}\hbar \\ [a_i,a_j] = 0 \end{cases}
$$

Also, we will write $|\mu| := \sum \mu_i$, and for $i = 1\dots n$, $z_i = \frac{x_i+i\xi_i}{\sqrt{2}}$, $p_i = \frac{x_i^2+\xi_i^2}{2}$. $\mathrm{Op}^W(a)$ will be the pseudo differential operator, whose Weyl total symbol is a.

Finally, let us denote by a, a^* or P the n-tuple of corresponding operators a_i, a_i^*, P_i, $i = 1\dots n$. We'll also use the usual convention that, if X is a n-tuple of complex numbers or operators, and j a n-tuple of nonnegative integers, X^j stands for $\prod_{i=1}^n X_i^{j_i}$.

Our construction of the normal form, inspired by [6], is the following:

Proposition 2.2. *Let $H(x,\hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on $L^2(\mathbb{R}^n\times\mathbb{S}^1)$, whose principal symbol is*

$$
(2.8)\qquad H_0(x,\xi,t,\tau) = H^0(p,\tau) + H_2
$$

where $H^0(p,\tau) = \sum_{i=1}^n \theta_i p_i + \tau$ and H_2 vanishes to the third order on $x=\xi=\tau=0$.

Then for any $N\geq 3$, there exists a self-adjoint semiclassical elliptic pseudodifferential operator $\widetilde{W}_{\leq N}$ and a smooth function $h(p_1,\dots,p_n,\tau,\hbar)$ such that microlocally in a neighborhood of $x=\xi=\tau=0$:

$$
(2.9)\qquad \begin{aligned} &\forall M>0,\ \exists C_N>0, \forall(\mu,\nu,\hbar)\in\mathbb{N}^n\times\mathbb{Z}\times[0,1[, |\mu\hbar|+|\nu\hbar|<M,\\ &\left|\left|\left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}He^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1,\dots,P_n,D_t,\hbar)\right)|\mu,\nu\rangle\right|\right| \leq C_N(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}}\end{aligned}
$$

The operators can be computed recursively in the form:

$$
(2.10)\qquad \widetilde{W}_{\leq N} = W_{\leq N} + (|D_t|^2 + \sum_{i=1}^n P_i)^{N+1}
$$

where

$$
(2.11)\qquad \begin{cases} W_{\leq N} = \sum_{3\leq q\leq N} W_q \\ W_q := \sum\limits_{2p+|j|+|k|+2m=q} \alpha_{p,j,k,m}(t)\hbar^p\, Op^W(z^j\bar{z}^k)D_t^m \end{cases}
$$

where for any index (p,j,k,m), $\alpha_{p,j,k,m}\in C^\infty(\mathbb{S}^1,\mathbb{C})$ and for any $q\geq 3$, W_q is symmetric.

Remark 2.3. We are only interested in recovering the Hamiltonian in a formal neighborhood of γ: every asymptotic expansion is meant microlocally and we'll be rewriting equations such as (2.9) simply as:

$$
\left|\left|\left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}He^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1,\dots,P_n,D_t,\hbar)\right)|\mu,\nu\rangle\right|\right| = O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}}\right)
$$

Also, by abuse of notation, we'll identify any operator with its version microlocalized near γ.

Remark 2.4. One passes from $W_{\leq N}$ to $\widetilde{W}_{\leq N}$ in order to gain ellipticity and self-adjointness, like it has been done in Lemma 4.5 of [6].

The proof of Proposition 2.2 will need several preliminaries:

Definition 2.5. We will say that a pseudodifferential operator A on $L^2(\mathbb{R}^n \times S^1)$ is "polynomial of order $r \in \mathbb{N}$" (PO(r)) if there exists $\beta_{p,j,k,m} \in C^\infty(\mathbb{S}^1, \mathbb{C})$ such that:

$$A = \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m \tag{2.12}$$

Let us remark that those operators have the following interesting properties:

Proposition 2.6. *Let A be a pseudodifferential operator on $L^2(\mathbb{R}^n \times S^1)$ Then, there exists a family of operators A_r, $r \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, A_r is PO(r) and*

$$\forall N \in \mathbb{N}, \left\| \left(A - \sum_{r=0}^{N} A_r \right) |\mu, \nu\rangle \right\| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right) \tag{2.13}$$

Definition 2.7. Let us introduce for any operator A the notations $\lfloor A \rfloor_r$ et $\lfloor A \rfloor_{\leq N}$ which represents respectively the terms of order r and of order smaller or equal to N of his preceding expansion (2.13).
If A and B are two operators, we'll write that: $A \sim B$ if for any $r \in \mathbb{N}$, $\lfloor A \rfloor_r = \lfloor B \rfloor_r$.
Also, if $(A_n)_{n\in\mathbb{N}}$ is a family of operators, we'll write that:

$$A \sim \sum_{n=0}^{+\infty} A_n \tag{2.14}$$

if for any $N \in \mathbb{N}$, $\lfloor A_n \rfloor_{\leq N}$ is zero for n sufficiently large, and the finite sum:

$$\sum_{n=0}^{+\infty} \lfloor A_n \rfloor_{\leq N} = \lfloor A \rfloor_{\leq N} \tag{2.15}$$

Proof. Let $a(z, \bar{z}, t, \tau)$ be the total symbol of A, which has the following Taylor expansion around γ:

$$\forall N \in \mathbb{N},\ a(z, \bar{z}, t, \tau) = \sum_{r=0}^{N} \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p z^j \bar{z}^k \tau^m + \sum_{p=0}^{\frac{N+1}{2}} O\left(\hbar^p (|z|^2 + |\tau|)^{\frac{N+1}{2}-p} \right)$$

Now, for any $r \in \mathbb{N}$, let us notice that the pseudodifferential operator A_r with symbol $\sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p z^j \bar{z}^k \tau^m$ is PO(r), and therefore:

$$\begin{aligned} \forall N \in \mathbb{N}, \quad \left\| \left(A - \sum_{r=0}^{N} A_r \right) |\mu, \nu\rangle \right\| &= \sum_{p=0}^{\frac{N+1}{2}} \hbar^p O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}-p} \right) \\ &= O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right) \end{aligned} \tag{2.16}$$

which concludes the proof. □

Let us remark the following corollary:

Corollary 2.8. *If the expansion* (2.13) *of an operator A contains no PO(r), $r = 0 \dots N$, then:*

$$\|A|\mu,\nu\rangle\| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}}\right)$$

It will also be convenient to our calculations to notice that:

Lemma 2.9. *Let F and G be PO(r) and PO(r') respectively then $\frac{[F,G]}{i\hbar}$ is PO($r + r' - 2$).*

Proof. Our proof will be a direct consequence of the two following lemmas, whose proof will be given at the end of the proof of Lemma 2.9

Lemma 2.10. *Any monomial operator of order r, that is of the form $\alpha(t)\hbar^p b_1 \dots b_l D_t^m$, where:*

- *for $j = 1 \dots l$, $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$*
- $2p + l + 2m = r$

is PO(r).

Lemma 2.11. *If F and G are monomials of order r and r' respectively, then $\frac{[F,G]}{i\hbar}$ is PO($r + r' - 2$)*

Indeed, any PO(r) is a finite sum of monomials of the same order, hence if F and G are PO(r) and PO(r') respectively, then $\frac{[F,G]}{i\hbar}$ is a finite sum of quantities of type $\frac{[\widetilde{F},\widetilde{G}]}{i\hbar}$ where $\widetilde{F}$ and $\widetilde{G}$ are monomials of order r and r' respectively. Any of those quantities are PO($r + r' - 2$) by Lemma 2.11, and a finite sum of PO($r + r' - 2$) is PO($r + r' - 2$). Therefore, Lemma 2.9 is proved. Let us now prove the two lemmas:

Proof of Lemma 2.10. Since for any $i, j = 1 \dots n$, $i \neq j$, a_i and a_i^* commute with both a_j and a_j^*, it is sufficient in order to prove Lemma 2.10 the following assertion (Ass$_l$) for any positive integer l: " any ordered product $b_1 \dots b_l$, where for any $j = 1 \dots l$, $b_j \in \{a_1, a_1^*\}$ can be written as a finite sum of the quantities $\hbar^p Op^W(z_1^j \bar{z}_1^k)$ with $2p + j + k = l$ and $j - k = l - 2\sharp\{m \in [\![1, l]\!], b_m = a_1^*\}$" More precisely, let us proceed by induction, and introduce for any ordered product $b_1 \dots b_l$, $k(b_1 \dots b_l) = \sharp\{m \in [\![1, l]\!], b_m = a_1^*\}$

- If $l = 1$, there is nothing to prove since $a_1 = Op^W(z_1)$ and $a_1^* = Op^W(\bar{z}_1)$.
- If $l = 2$,
$$\begin{cases} a_1^2 = Op^W(z_1^2) \\ a_1^{*2} = Op^W(\bar{z}_1^2) \\ a_1 a_1^* = P_1 + \frac{\hbar}{2} = Op^W(z_1\bar{z}_1) + \frac{\hbar}{2} \\ a_1^* a_1 = Op^W(z_1\bar{z}_1) - \frac{\hbar}{2} \end{cases}$$
and therefore, the assertion is proved for $l = 2$.
- Now, let l be a positive integer, let us assume (Ass$_k$) up to order $k = l$, and let $B = b_1 \dots b_{l+1}$ be an ordered product, where for any $j = 1 \dots l+1$, $b_j \in \{a_1, a_1^*\}$. If for any $j = 1 \dots l$, $b_j = b_{j+1}$, then $B = Op^W(z_1^{l+1})$ or $B = Op^W(\bar{z}_1^{l+1})$.
Otherwise, the proof of the symmetric case being identical, let us can assume that $b_1 = a_1$, and set $j_0 = \max\{j \in [\![1, l+1]\!], b_j = a_1\}$. Let us remark that: $1 \leq j_0 \leq l$ and $[a_1^{j_0}, a_1^*] = j_0\hbar a_1^{j_0-1}$, so that:

$$b_1 \dots b_{l+1} = a_1^{j_0} a_1^* b_{j_0+2} \dots b_{l+1} = a_1^* a_1^{j_0} b_{j_0+2} \dots b_{l+1} + \hbar j_0 a_1^{j_0-1} b_{j_0+2} \dots b_{l+1} \tag{2.17}$$

Hence if one sets $k := k(b_1 \dots b_{l+1})$

$$\begin{aligned}\binom{l+1}{k} b_1 \dots b_{l+1} =& \binom{l}{k} a_1^{j_0} a_1^* b_{j_0+2} \dots b_{l+1} + \binom{l}{k-1} a_1^* a_1^{j_0} b_{j_0+2} \dots b_{l+1} \\ &+ \hbar \binom{l}{k_b - 1} j_0 a_1^{j_0-1} b_{j_0+2} \dots b_{l+1}\end{aligned}$$

Now, because we assumed (Ass_{l-1}):

$$(l-1) - 2k(a_1^{j_0-1} b_{j_0+2} \dots b_{l+1}) = (l+1) - 2k(b_1 \dots b_{l+1})$$

we only need to observe that the $\binom{l+1}{k}$ ordered monomials in the sum $Op^W(z^{l+1-k}\bar{z}^k)$ can be divided in two parts: the $\binom{l}{k}$ ones whose first term is a_1, whose sum is $\binom{l}{k} a_1 Op^W(z^{l-k}\bar{z}^k)$ and the $\binom{l}{k-1}$ who forms $\binom{l}{k-1} a_1^* Op^W(z^{l+1-k}\bar{z}^{k-1})$, and since:

$$\binom{l+1}{k} Op^W(z^{l+1-k}\bar{z}^k) = \binom{l}{k} a_1 Op^W(z^{l-k}\bar{z}^k) + \binom{l}{k-1} a_1^* Op^W(z^{l+1-k}\bar{z}^{k-1})$$

the assumption of (Ass_l) will be enough to conclude our proof by induction.

□

Proof of Lemma 2.11. It is now sufficient in order to prove Lemma 2.11 that if F and G are of the form:

$$F = \alpha(t) b_1 \dots b_l D_t^m \text{ and } G = \beta(t) b'_1 \dots b'_{l'} D_t^{m'}$$

where:

- α and β are smooth
- $l + 2m = r$, $l' + 2m' = r'$
- For $j = 1 \dots l$, for $j' = 1 \dots l'$, $b_j, b'_{j'} \in \{a_1, a_1^*\}$

then $\frac{[F,G]}{i\hbar}$ is a finite sum of monomials of order $r + r' - 2$ since, by Lemma 2.10, each of them is $\mathrm{PO}(r + r' - 2)$. □

With those assumptions on F and G, we get:

$$\begin{aligned}\frac{[F,G]}{i\hbar} =& \frac{[\alpha(t) b_1 \dots b_l D_t^m, \beta(t) b'_1 \dots b'_{l'} D_t^{m'}]}{i\hbar} \\ =& \alpha(t)\beta(t) \frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar} D_t^{m+m'} + \alpha(t) b_1 \dots b_l \frac{[D_t^m, \beta(t)]}{i\hbar} b'_1 \dots b'_{l'} D_t^{m'} \\ &- \beta(t) b'_1 \dots b'_{l'} \frac{[D_t^{m'}, \alpha(t)]}{i\hbar} b_1 \dots b_l D_t^m\end{aligned} \tag{2.18}$$

Therefore it is sufficient to prove that $\frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar}$, $\frac{[D_t^m, \beta(t)]}{i\hbar}$ and $\frac{[D_t^{m'}, \alpha(t)]}{i\hbar}$ are respectively: $\mathrm{PO}(l + l' - 2)$, $\mathrm{PO}(2m - 2)$ and $\mathrm{PO}(2m' - 2)$ (with the convention that a $\mathrm{PO}(j)$ with $j < 0$ is 0).
For the two last, it is quite obvious, since:

$$\frac{[D_t^m, \beta(t)]}{i\hbar} = \sum_{k=0}^{m-1} \binom{m}{k} (i\hbar)^{m-k-1} \beta^{(m-k)}(t) D_t^k \tag{2.19}$$

Now, for $j = 1 \dots l'$, let us set $\epsilon_j = 1$ if $b'_j = a_1^*$, otherwise $\epsilon_j = -1$. Since $[a_1, a_1^*] = \hbar$, we get:

$$b_1 \dots b_l b'_1 \dots b'_{l'} = b'_1 b_1 \dots b_l b'_2 \dots b'_{l'} + \frac{\epsilon_1 + 1}{2}\hbar \sum_{\substack{k=1 \\ b_k = a_1}}^{l} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_2 \dots b'_{l'}$$
$$+ \frac{\epsilon_1 - 1}{2}\hbar \sum_{\substack{j=1 \\ b_k = a_1^*}}^{l} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_2 \dots b'_{l'}$$

Hence by induction on $j = 1 \dots l'$:

$$\begin{aligned} \frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar} = &- i \sum_{j=1}^{l'} \frac{\epsilon_j + 1}{2} \sum_{\substack{k=1 \\ b_k = a_1}}^{l} b'_1 \dots b'_{j-1} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_{j+1} \dots b'_{l'} \\ &- i \sum_{j=1}^{l'} \frac{\epsilon_j - 1}{2} \sum_{\substack{k=1 \\ b_k = a_1^*}}^{l} b'_1 \dots b'_{j-1} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_{j+1} \dots b'_{l'} \end{aligned} \tag{2.20}$$

The right-hand side of (2.20) is a finite sum of monomials of order $l + l' - 2$, hence $\mathrm{PO}(l + l' - 2)$ by Lemma 2.10, hence Lemma 2.11 is proved.

□

Lemma 2.12. *Let G be PO(r).*
Then there exists F an operator PO(r), and $G_1 = G_1(P_1, \dots, P_n, D_t, \hbar)$ such that:

$$\frac{[H^0(P, D_t), F]}{i\hbar} = G + G_1 \tag{2.21}$$

where if G is symmetric, F is also symmetric, if r is odd, $G_1 = 0$, and if r is even G_1 is an homogeneous polynomial function of total order $\frac{r}{2}$.

Remark 2.13. If $F = \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t)\hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m$, one can choose:

$$\int_{\mathbb{S}^1} \alpha_{p,j,j,m}(t) dt = 0 \tag{2.22}$$

Indeed, any $Op^W(z^j \bar{z}^j) D_t^m$ commutes with $H^0(P, D_t, \hbar)$

Proof of Lemma 2.12. Let us first assume that G is a monomial of order r: $G = \beta(t) b_1 \dots b_l D_t^m$ where:

- α is smooth
- $l + 2m = r$
- For $j = 1 \dots l$, $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$

and let us look for F under the form: $F=\alpha(t)b_1\dots b_lD_t^m$ We have:

$$\begin{aligned}\frac{[H^0,F]}{i\hbar} &= \frac{[H^0,\alpha(t)b_1\dots b_lD_t^m]}{i\hbar}\\ &= \alpha(t)\sum_{i=1}^n\theta_i\frac{[P_i,b_1\dots b_l]}{i\hbar}D_t^m+\frac{[D_t,\alpha(t)]}{i\hbar}b_1\dots b_lD_t^m\\ &= \alpha(t)\sum_{i=1}^n\theta_i\frac{[P_i,b_1\dots b_l]}{i\hbar}D_t^m+\alpha'(t)b_1\dots b_lD_t^m\end{aligned}\tag{2.23}$$

If for $i=1\dots n$, $k_i=\sharp\{m\in[\![1,l]\!],b_m=a_i^*\}$ and $j_i=\sharp\{m\in[\![1,l]\!],b_m=a_i\}$, we deduce from (2.20) that:

$$\frac{[P_i,b_1\dots b_l]}{i\hbar}=\sqrt{-1}(j_i-k_i)b_1\dots b_l\tag{2.24}$$

Hence:

$$\frac{[H^0,F]}{\sqrt{-1}\hbar}=\sqrt{-1}\sum_{i=1}^n\theta_i(j_i-k_i)\alpha(t)b_1\dots b_lD_t^m+\alpha'(t)b_1\dots b_lD_t^m\tag{2.25}$$

The problem: $\frac{[H^0,F]}{\sqrt{-1}\hbar}=G$ admits a solution if there exists α such that:

$$\sqrt{-1}\sum_{i=1}^n\theta_i(j_i-k_i)\alpha(t)+\alpha'(t)=\beta(t)\tag{2.26}$$

If $c_p(\alpha)$ and $c_p(\beta)$ are the Fourier coefficients of α and β, it is sufficient for the $c_p(\alpha)$ to be solution of:

$$\sqrt{-1}\left(\sum_{i=1}^n\theta_i(j_i-k_i)+2\pi p\right)c_p(\alpha)=c_p(\beta)\tag{2.27}$$

and

$$c_p(\alpha)\underset{p\to+\infty}{=}O\left(\frac{1}{|p|^\infty}\right)\tag{2.28}$$

If the n-tuples j and k are different, the non-degeneracy condition on the θ_i's together with the fact that $c_p(\beta)\underset{p\to+\infty}{=}O\left(\frac{1}{|p|^\infty}\right)$ (because β is smooth), gives the existence of $c_p(\alpha)$ satisfying (2.27) and (2.28).
If r is odd, j and k can't be equal, hence Lemma 2.12 is proved in this case (r odd and G monomial)
If r is even, and $j=k$, there exists a family $(c_p(\alpha))_{p\in\mathbb{Z}^*}$ satisfying (2.27) and (2.28). Hence, if α is the smooth function with Fourier coefficients $c_p(\alpha)$ for $p\neq 0$ and $c_0(\alpha)=0$, we get:

$$\frac{[H^0,F]}{\sqrt{-1}\hbar}=G+c_0(\beta)b_1\dots b_lD_t^m\tag{2.29}$$

And from the proof of Lemma 2.10, we know that $c_0(\beta)b_1\dots b_lD_t^m$ that is a linear combination of $G_1(P,D_t,\hbar):=c_0(\beta)\sum_{2p+2|k|=l}a_{p,k}\hbar^pP^kD_t^m$, and Lemma 2.12 is proved in the case where r is even and G is monomial.
The general case is easily deduced from the case where G is monomial, since G is a finite

sum of monomials of the same order.
Also, the form of F allows us to conclude immediately that F is symmetric if G is so. $\square$

Now we have everything we need for the proof by induction of Proposition 2.2.

Proof of Proposition 2.2. Microlocally near $x = \xi = \tau = 0$, $H(x, \hbar D_x)$ satisfies

$$H := H(x, \hbar D_x) \sim H^0(P_1, \dots, P_n, \hbar D_t) + \sum_{q \geq 3} H_q \tag{2.30}$$

where:

$$H_q := \lfloor H(x, \hbar D_x) \rfloor_q \tag{2.31}$$

Let us look for $\widetilde{W}_{\leq 3}$ under the form predicted in Proposition 2.2, that is:

$$\widetilde{W}_{\leq 3} = W_3 + (|D_t|^2 + \sum_{i=1}^{n} P_i)^4 \tag{2.32}$$

where W_3 is PO(3).

$$\begin{aligned}
e^{\frac{i\widetilde{W}_{\leq 3}}{\hbar}} H(x, \hbar D_x) e^{\frac{-i\widetilde{W}_{\leq 3}}{\hbar}} &\sim H(x, \hbar D_x) + \frac{i}{\hbar}[\widetilde{W}_{\leq 3}, H] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq 3}, \dots, \widetilde{W}_{\leq 3}}^{l \text{ times}}, H] \\
&\sim H^0 + H_3 + \frac{i}{\hbar}[W_3, H^0] \\
&+ \frac{i}{\hbar}[W_3, H - H^0] + \frac{i}{\hbar}[\widetilde{W}_{\leq 3} - W_3, H(x, \hbar D_x)] \\
&+ \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq 3}, \dots, \widetilde{W}_{\leq 3}}^{l \text{ times}}, H(x, \hbar D_x)] + \sum_{q \geq 4} H_q
\end{aligned}$$

Since H_3 is polynomial of order 3, let us choose W_3, as in Lemma 2.12, such that:

$$H_3 + \frac{i}{\hbar}[W_3, H^0] = H^1(P_1, \dots, P_n, D_t, \hbar) \equiv 0 \tag{2.33}$$

Since W_3 is PO(3) and the expansion of $H - H^0$ in PO(r) contains no PO(r) of order less or equal to 2, the expansion of $\widetilde{W}_{\leq 3} - W_3$ no term order less or equal to 3, and the one of $H(x, \hbar D_x)$ no term of order less or equal to 1, we know from Lemma 2.11 that the expansion of:

$$\frac{i}{\hbar}[W_3, H - H^0] + \frac{i}{\hbar}[\widetilde{W}_{\leq 3} - W_3, H] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq 3}, \dots, \widetilde{W}_{\leq 3}}^{l \text{ times}}, H] + \sum_{q \geq 4} H_q \tag{2.34}$$

contains no term of order less or equal to 3.
Therefore, Corollary 2.8 gives us:

$$\left|\left| \left(e^{\frac{i\widetilde{W}_{\leq 3}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq 3}}{\hbar}} - H^0(P, D_t, \hbar) \right) |\mu, \nu\rangle \right|\right| = O\left(|\mu\hbar| + |\nu\hbar| \right)^2 \tag{2.35}$$

We can construct by induction $(W_q)_{q \geq 3}$ and $(H^q)_{q \geq 1}$, such that:

- for $q \geq 3$, W_q is PO(q) and for H^{q-2} is zero if q is odd, an homogeneous polynomial function of total order $\frac{q}{2}$ if q is even.
-

$$H_3 + \frac{i}{\hbar}[W_3, H^0] = H^1(P, D_t, \hbar) \tag{2.36}$$

- and for any $q \geq 4$:

$$\frac{i}{\hbar}[W_q, H^0]+H_q+\left\lfloor \frac{i}{\hbar}[W_{\leq q-1}, H - H^0] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{W_{\leq q-1}, \dots, W_{\leq q-1}}^{l \text{ times}}, H] \right\rfloor_q = H^{q-2}(P, D_t, \hbar)$$

Let us now set: $\widetilde{W}_{\leq N} := \sum_{q=3}^{N} W_q + (|D_t|^2 + \sum_{i=1}^{n} P_i)^{\frac{N+1}{2}}$. Also, as for any $q \geq 0$, H^{2q} is an homogeneous polynomial function of total order $q+1$, we can choose by Borel's lemma a smooth function h such that for any $N \geq 1$, in a neighborhood of $p = \tau = 0$.

$$\left| h(p, \tau, \hbar) - \sum_{q=0}^{N-1} H^{2q}(p, \tau, \hbar) \right| = O\left((|p| + |\tau| + |\hbar|)^{N+1} \right) \tag{2.37}$$

Now, let us write, for any $N \geq 4$

$$\begin{aligned} e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} &\sim H + \frac{i}{\hbar}[\widetilde{W}_{\leq N}, H] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H] \\ &\sim H + \frac{i}{\hbar}[W_{\leq N}, H^0] + \frac{i}{\hbar}[W_{\leq N}, H - H^0] + \sum_{l\geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H] \\ &+ \frac{i}{\hbar}[\widetilde{W}_{\leq N} - W_{\leq N}, H] \end{aligned}$$

Let us also observe that Lemma 2.9 gives us for $q \leq N$:

$$\begin{cases} \left\lfloor \frac{i}{\hbar}[W_{\leq N}, H - H^0] \right\rfloor_q = \lfloor \frac{i}{\hbar}[W_{\leq q-1}, H - H^0] \rfloor_q \\ \left\lfloor \sum_{l\geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H] \right\rfloor_q = \left\lfloor \sum_{l\geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{W_{\leq q-1}, \dots, W_{\leq q-1}}^{l \text{ times}}, H] \right\rfloor_q \end{cases} \tag{2.38}$$

Therefore for any $q \leq N$:

$$\left\lfloor e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right\rfloor_q = H^{q-2}(P, D_t, \hbar) = \lfloor h(P, D_t, \hbar) \rfloor_q \tag{2.39}$$

And Corollary 2.8 gives us:

$$\left\| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O\left(|\mu\hbar| + |\nu\hbar| \right)^{\frac{N+1}{2}} \Big) \tag{2.40}$$

which concludes the proof. □

The next result is the first inverse result needed for the proof of our main result.

Proposition 2.14. *Let O be a pseudodifferential operator, whose principal symbol vanishes on γ.*

(1) *There exists a smooth function f vanishing at $(0,0,0)$ such that for any $N \geq 3$:*

$$\langle \mu,\nu| e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} |\mu,\nu\rangle = f\left((\mu+\frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar\right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N}{2}}\right) \tag{2.41}$$

Moreover let, for any integer l, ϕ_l be a Schwartz function whose Fourier transform is compactly supported in $(l-1, l+1)$ and let $(a_j^l(O))_{l\geq 0}$ provided by the trace formula:

$$Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right) \sim \sum_{j=0}^{+\infty} a_j^l(O)\hbar^j \tag{2.42}$$

(2) *The Taylor expansion of f up to order N is entirely determined by the family $(a_j^l(O))$, $0 \leq j \leq N$, $l \in \mathbb{N}$.*

Proof. Let us first prove point 1.
Let us consider a monomial $G = \alpha(t) b_1 \dots b_l D_t^m$ where:

- α is smooth
- $l + 2m = r$
- For $j = 1 \dots l$, $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$

Let us set for $i = 1 \dots n$, $k_i = \sharp\{m \in [\![1,l]\!], b_m = a_i^*\}$
and $j_i = \sharp\{m \in [\![1,l]\!], b_m = a_i\}$.

If $j \neq k$ or $\alpha \notin \mathbb{C}$, then: $\langle \mu,\nu|G|\mu,\nu\rangle = 0$ for any $(\mu,\nu) \in \mathbb{N}^n \times \mathbb{Z}$.

If now $j = k$ and $\alpha \in \mathbb{C}$, then there exists complex numbers α_l ($0 \leq l_i \leq j_i$ for $i = 1 \dots n$), such that:

$$G = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} P_1^{j_1 - l_1} \dots P_n^{j_n - l_n} D_t^m \tag{2.43}$$

and: $\alpha_0 = \alpha$.
Therefore for any $(\mu,\nu) \in \mathbb{N}^n \times \mathbb{Z}$:

$$\langle \mu,\nu|G|\mu,\nu\rangle = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} \left(\left(\mu + \frac{1}{2}\right)\hbar\right)^{j-l} (2\pi\nu\hbar)^m \tag{2.44}$$

Hence, if G is PO(r), then for any $(\mu,\nu) \in \mathbb{N}^n \times \mathbb{Z}$:

- $\langle \mu,\nu|G|\mu,\nu\rangle = 0$ if r is odd.
- If r is even, there exists an homogeneous polynomial function g of order $\frac{r}{2}$ such that:

$$\langle \mu,\nu|G|\mu,\nu\rangle = g\left((\mu+\frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar\right) \tag{2.45}$$

From Proposition 2.6, Corollary 2.8 and Borel's lemma, we get that that for any operator A, there exists a function g such that for any $(\mu,\nu) \in \mathbb{N}^n \times \mathbb{Z}$:

$$\langle \mu,\nu|A|\mu,\nu\rangle = g\left((\mu+\frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar\right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}}\right) \tag{2.46}$$

Hence, the only point remaining to prove, is that function f in point 1 does not depend on N. It is therefore sufficient to prove that for any $q \leq N-1$,

$$\left\lfloor e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right\rfloor_q = \left\lfloor e^{\frac{i\widetilde{W}_{\leq q+1}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq q+1}}{\hbar}} \right\rfloor_q \tag{2.47}$$

But (2.47) is a direct consequence of Lemma 2.9. Indeed,

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim O + \sum_{l\geq 1} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, O] \tag{2.48}$$

and since the principal symbol of O vanishes on γ, Lemma 2.9 gives us for any $l \geq 1$ and any $q \leq N-1$:

$$\left\lfloor \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, O] \right\rfloor_q = \left\lfloor \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq q+1}, \ldots, \widetilde{W}_{\leq q+1}}^{l \text{ times}}, O] \right\rfloor_q \tag{2.49}$$

Let us now move on to the proof of point 2

Since $\hat{\phi}_l$ is supported near a single period of the flow, we know from the general theory of Fourier integral operators that one can microlocalize the trace formula with observables near γ:

$$Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right) = Tr\left(O\int_{\mathbb{R}} \hat{\phi}_l(t)\rho(P_1+\cdots+P_n+|\zeta|)e^{it\frac{H-E}{\hbar}}dt\right) + O(\hbar^\infty) \tag{2.50}$$

where $\rho \in C_0^\infty(\mathbb{R})$ is compactly supported and $\rho = 1$ in a neighborhood of $p = \tau = 0$. Therefore we can conjugate (2.50) by the microlocally unitary operator $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}$:

$$Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right) =$$
$$= Tr\left((e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \int_{\mathbb{R}} \hat{\phi}_l(t)\rho(P_1+\cdots+P_n+|\zeta|)e^{it\frac{e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}He^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}-E}{\hbar}}dt\right) + O(\hbar^\infty)$$

Thanks to Proposition 2.2, we can lighten the r.h.s. for any $(\mu,\nu) \in \mathbb{N}^n \times \mathbb{Z}$

$$\begin{aligned}&\int_{\mathbb{R}} \hat{\phi}_l(t)\rho(P_1+\cdots+P_n+|\zeta|)e^{it\frac{e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}He^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}-E}{\hbar}}dt|\mu,\nu\rangle \\ &= \left(\int_{\mathbb{R}} \hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar\right)e^{it\frac{h((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar)-E+O(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}})}{\hbar}}dt\right)|\mu,\nu\rangle\end{aligned} \tag{2.51}$$

As $\hat{\phi}_l$ is smooth and compactly supported, together with the non-degeneracy condition on the θ_i's, we can assure that if we choose a sufficiently small support for ρ, we have for any $\eta > 0$:

$$\left(\int_{\mathbb{R}} \hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar\right)e^{it\frac{h((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar)-E+O(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}})}{\hbar}}dt\right)|\mu,\nu\rangle$$
$$=\left(\int_{\mathbb{R}} \hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar^\eta\right)e^{it\frac{h((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar)-E+O(|\mu\hbar|+|\nu\hbar|)^{\frac{N+1}{2}})}{\hbar}}dt\right)|\mu,\nu\rangle+O(\hbar^\infty)$$

Hence, choosing $\eta<\frac{1}{2}$:

$$Tr\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right)+O(\hbar^\infty)$$
$$=\sum_{\mu,\nu}\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}Oe^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}|\mu,\nu\rangle\times\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|\nu|)\hbar^\eta\right)e^{it(2\pi\nu+\theta.(\mu+\frac{1}{2}))}\dots$$
$$\dots\exp\left(\frac{it}{\hbar}\sum_{1\leq q\leq N-2}H^q\left((\mu+\frac{1}{2})\hbar,\nu\hbar,\hbar\right)+O\left((|\mu|+|\nu|)^{\frac{N+1}{2}}\hbar^{\frac{N-1}{2}}\right)\right)dt$$
$$=\sum_{\mu,\nu}\int_{\mathbb{R}}\hat{\phi}_l(t)\rho\left((|\mu|+\frac{n}{2}+|2\pi\nu|)\hbar^\eta\right)e^{it(2\pi\nu+\theta.(\mu+\frac{1}{2}))}$$
$$\left(1+\sum_{i\geq 1}^{\frac{N-1}{2}}\hbar^iQ_i(\mu+\frac{1}{2},\nu,t)\right)\times\sum_{p\geq 1}^{\frac{N+1}{2}}\sum_{|k|+m\leq p}b_{k,m,p-|k|-m}(\mu+\frac{1}{2})^k(2\pi\nu)^m\hbar^pdt+O(\hbar^{\frac{N+1}{2}})$$

where for any $i\leq\frac{N-1}{2}$, Q_i is a determined polynomial function, of degree in $\left(\mu+\frac{1}{2},\nu\right)$ less or equal to $i+1$, which depends on the H^q's and the Taylor expansion of exp, and the $b_{k,m,s}$ ($(k,m,s)\in\mathbb{N}^{n+2}\backslash\{0\}$) come from the Taylor expansion at $(0,0,0)$ of the function f defined in the first point of Proposition 2.14, *i.e.* for any $N\geq 1$:

$$f(x,y,z)=\sum_{1\leq|k|+m+s\leq N}b_{k,m,s}x^ky^mz^s+O\left(|x|+|y|+|z|\right)^{N+1}\tag{2.52}$$

Now, let us set:

$$\forall t\in\mathbb{R}^*,\forall\alpha\in(\mathbb{R}\backslash\frac{2\pi}{t}\mathbb{Z})^n,g(t,\alpha):=\frac{e^{i\frac{t}{2}(\alpha_1+\dots+\alpha_n)}}{\prod_i(1-e^{it\alpha_i})}\tag{2.53}$$

By the non-degeneracy condition on the θ_i's, g is well defined on the compact support of $\hat{\phi}_l$ around a single period, which is precisely l. It also implies that $\theta_i.\mu$ is bounded below by $C|\mu|$ (where $C>0$) as $|\mu|$ goes to ∞.

Therefore we get from the Poisson formula and the Riemann-Lebesgue lemma that the following quantity $X_p(l)$ can be computed recursively on $p\leq\frac{N+1}{2}$ from the $a_j^l(O)$, $j=0,\dots,p$:

$$\begin{aligned}X_p(l)&=\sum_{|k|+m\leq p}b_{k,m,p-|k|-m}\left[\left(-i\frac{\partial}{\partial t}\right)^m\left(\hat{\phi}_l(t)\left(\frac{-i}{t}\right)^k\frac{\partial^kg}{\partial\alpha^k}(t,\alpha)\right)\right](l,\theta)\\&=\sum_{|k|+m\leq p}b_{k,m,p-|k|-m}\left[\left(-i\frac{\partial}{\partial t}\right)^m\left(-i\frac{\partial}{t\partial\alpha}\right)^kg\right](l,\theta)\end{aligned}\tag{2.54}$$

since $\hat{\phi}_l$ is identically 1 around l.
Now, let us set, for any $i \in [\![1,n]\!]$, any $t \in \mathbb{R}$ and any $\alpha \in (\mathbb{R}\backslash \frac{2\pi}{t}\mathbb{Z})^n$, $x_i(t,\alpha) = e^{i\frac{t\alpha_i}{2}}$. and also define holomorphic function h on $\mathbb{C}\backslash\{-1,1\}$ by $h(z) = \frac{z}{1-z^2}$ for $z \in \mathbb{C}\backslash\{-1,1\}$. We have for any $k \in \mathbb{N}^n$:

$$\left(-i\frac{\partial}{t\partial\alpha}\right)^k g = \prod_{i=1}^n \left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) \tag{2.55}$$

For any $i \in [\![1,n]\!]$, an easy induction on $k_i \in \mathbb{N}$ leads to the following, since for any $z \in \mathbb{C}\backslash\{-1,1\}$, $h(z) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{1+z}\right)$, and $-i\frac{\partial x_i}{t\partial\alpha_i} = \frac{1}{2}x_i$:

$$\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) = \frac{k_i!}{2^{k_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+1}} + \frac{x_i}{(1+x_i)^{k_i+1}}\right) \tag{2.56}$$

Now, since $-i\frac{\partial x_i}{\partial t} = \frac{\alpha_i}{2}x_i$, an induction on $s_i \in \mathbb{N}$ shows that:

$$\left(-i\frac{\partial}{\partial t}\right)^{s_i}\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) = \frac{(k_i+s_i)!\alpha_i^{s_i}}{2^{k_i+s_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}}\right) \tag{2.57}$$

Let us now introduce for any n-tuple s such that $|s| = m$, the multinomial coefficient:

$$\binom{m}{s} = \frac{m!}{s_1!\dots s_n!}$$

We have:

$$\left(-i\frac{\partial}{\partial t}\right)^m\left(-i\frac{\partial}{t\partial\alpha}\right)^k g = \sum_{|s|=m}\binom{m}{s}\prod_{i=1}^n\left(-i\frac{\partial}{\partial t}\right)^{s_i}\left(-i\frac{\partial}{t\partial\alpha_i}\right)^{k_i}(h\circ x_i) \tag{2.58}$$

Let us use Kronecker theorem, whose hypothesis is precisely the non-degeneracy condition on the θ_i's: for any n-tuple $(x_1,\dots,x_n) \in \mathbb{S}_1^n$, one can find a sequence of integers $(l_p)_{p\in\mathbb{Z}}$, such that:

$$\forall j \in [\![1,n]\!],\ x_j(l_p,\theta) \underset{p\to+\infty}{\longrightarrow} x_j$$

Therefore, if one sets, for any $(x_1,\dots,x_n) \in (\mathbb{S}_1\backslash\{-1,1\})^n$ and $(k,m) \in \mathbb{N}^{n+1}$:

$$u^{(k,m)} = \sum_{|s|=m}\binom{m}{s}\prod_{i=1}^n\frac{(k_i+s_i)!\theta_i^{s_i}}{2^{k_i+s_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}}\right)$$

Then (2.54), (2.57) and (2.58) together with Kronecker theorem allows us to conclude that the following quantity is determined by the $a_j^l(O)$, $j = 0,\dots,p$:

$$X_p = \sum_{|k|+m\le p} b_{k,m,p-|k|-m}u^{(k,m)} \tag{2.59}$$

Hence, the only thing remaining to prove is that, if one chooses the x_i's tending to 1 in a way convenient to us, the $|u^{(k,m)}|$'s will tend to ∞ to different orders.
Let us be more precise:

Let the x_i's tend to 1 in a way such that:

$$\forall i \in [\![1,n-1]\!], |1-x_i| \ll |1-x_{i+1}|^p \tag{2.60}$$

If $\simeq$ means that two functions are equivalent, as the x_i's tend to 1 as in (2.60), up a multiplicative constant, we have for any $(k,m) \in \mathbb{N}^{n+1}$:

$$(1-x_1)^m u^{(k,m)} \simeq \prod_{i=1}^{n} \frac{1}{(1-x_i)^{k_i+1}} \tag{2.61}$$

Hence, if one sets $\widetilde{m} = (m, 0, \dots, 0)$:

$$u^{(k,m)} \ll u^{(k',m')} \text{ si } k + \widetilde{m} < k' + \widetilde{m'} \tag{2.62}$$

where $<$ is the lexicographical order on $\mathbb{N}^n$. Therefore, for any $p \in \mathbb{N}$ and for any $(k,m) \in \mathbb{N}^{n+1}$ such that $|k_0| + m_0 \leq p$, the following quantity can be recursively determined from X_p:

$$X_{k_0,m_0} = \sum_{k' + \widetilde{m'} = k + \widetilde{m}} b_{k,m,p-|k|-m} u^{(k,m)} \tag{2.63}$$

Reversing for example the roles of $i = 1$ and $i = 2$ in (2.60), and observing that $k_2 + m \neq k'_2 + m'$ if $k + \widetilde{m} = k' + \widetilde{m'}$ and $(k,m) \neq (k',m')$, one determines $b_{k,m,p-|k|-m}$ from (2.63) recursively on m. Finally, each $b_{k,m,s}$ with $|k| + m + s \leq N$ is determined by the $a_j^l(O)$, with $j = 0 \dots N$ and $l \in \mathbb{N}$ and the point 2 is proved, which ends the proof of Proposition 2.14.

□

Our next result shows how the knowledge of the matrix elements of the conjugation of a given known selfadjoint operator by a unitary one determines the latter (in the framework of asymptotic expansions).
For any $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$, and any $(x,\xi,t,\tau) \in T^*(\mathbb{R}^n \times S^1)$, let us define:

$$\mathcal{O}_{mnds}(x,\xi,t,\tau) = e^{i2\pi dt} \tau^s \prod_{j=1}^{n} (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}. \tag{2.64}$$

and let O_{mnds} be a pseudodifferential operator whose Weyl principal symbol is $\mathcal{O}_{mnds}$.

By Proposition 2.14, there exists a smooth function f_{mnds} vanishing at $(0,0,0)$ such that for any $N \geq 3$:

(2.65)
$$\langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle = f_{mnds}\left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N}{2}} \right)$$

Theorem 2.1 will now be a direct consequence of Proposition 2.14 and following proposition:

Proposition 2.15. *Let $N \geq 3$. The Taylor expansion of f_{mnds} up to order $N-1$ for any $(m,n,d,s) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$ satisfying conditions*

(1) $|m| + |n| \leq N$
(2) $\forall j = 1 \dots n,\ m_j = 0$ ***or*** $n_j = 0$
(3) $s = 1$ *if* $m = n = 0$, *otherwise* $s = 0$

determines completely $W_{\leq N}$

Remark 2.16. Let us remark, like it will be seen in the proof of Proposition 2.15, that the only relevant information is the asymptotic expansion of $\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}|\mu,\nu\rangle$ as $\hbar$ tends to 0 and μ,ν go to ∞ slower than any negative power of $\hbar$.

Proof of Proposition 2.15. Let $N\geq 3$ and $(m,n,d,s)\in(\mathbb{N}^n)^2\times\mathbb{Z}\times\{0,1\}$ satisfy conditions (1), (2) and (3).
Then, we have:

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}\sim O_{mnds}+\frac{i}{\hbar}[\widetilde{W}_{\leq N},O_{mnds}]+\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l\text{ times}},O_{mnds}]$$

Therefore:

$$\begin{aligned}&\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}|\mu,\nu\rangle-\langle\mu,\nu|O_{mnds}|\mu,\nu\rangle\\ &=\frac{i}{\hbar}\langle\mu,\nu|[\widetilde{W}_{\leq N},O_{mnds}]|\mu,\nu\rangle+\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle\mu,\nu|[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l\text{ times}},O_{mnds}]|\mu,\nu\rangle\\ &+O\left((|\mu\hbar|+|\nu\hbar|)^\infty\right)\end{aligned}\tag{2.66}$$

Now, since $\widetilde{W}_{\leq N}$ is a sum of polynomial operators of order greater that 3, we get from Proposition 2.9 that for any $l\geq 2$

$$\frac{i^l}{\hbar^l}\langle\mu,\nu|[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l-1\text{ times}},\cdot]\tag{2.67}$$

maps a PO(r) into a sum of polynomial operators of order strictly larger than r. Therefore, if A is a PO(r), we have:

$$\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle\mu,\nu|[\overbrace{\widetilde{W}_{\leq N},\dots,\widetilde{W}_{\leq N}}^{l-1\text{ times}},A]|\mu,\nu\rangle=O\left((|\mu\hbar|+|\nu\hbar|)^{\frac12}\right)\langle\mu,\nu|A|\mu,\nu\rangle\tag{2.68}$$

Finally, let us recall that:

$$\begin{aligned}W_N&=\sum_{2p+|j|+|k|+2q=N}\alpha_{p,j,k,q}(t)\hbar^p Op^W(z^j\bar{z}^k)D_t^q\\ &:=\sum_{2p+|j|+|k|+2q=N}\sum_{r\in\mathbb{Z}}\alpha_{p,j,k,q,r}\hbar^p e^{-i2\pi rt}Op^W(z^j\bar{z}^k)D_t^q\end{aligned}\tag{2.69}$$

Let us also state the following lemma, whose proof will be given after the end of the present proof.

Lemma 2.17.

$$\langle\mu,\nu|[e^{-i2\pi dt}Op^W(z^j\bar{z}^k)D_t^q,O_{mnds}]|\mu,\nu\rangle=\hbar g_{jkqrmnds}\left(\left(\mu+\frac12\right)\hbar,\nu\hbar\right)+O(\hbar^2)\tag{2.70}$$

where, if $j+m=k+n$ *and* $r=d$*:*
(2.71)

$$g_{jkqrmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right)=(2\pi\nu\hbar)^{q+s}(\mu\hbar)^{\max(j,k)}\left(\sum_{\substack{i=1\\|j_i|+|k_i|>0}}^{n}\frac{j_in_i-k_im_i}{\mu_i\hbar}+\frac{d(q+s)}{\nu\hbar}\right)$$

and if $j+m\neq k+n$ *or* $r\neq d$*,* $g_{jkqrmnds}\equiv 0$

Let us now proceed by induction on $N\geq 3$, and first assume $N=3$.

Equation (2.65) gives us that the Taylor expansion up to order 2 of function f_{mnds} determines modulo $O\left((|\mu\hbar|+|\nu\hbar|)^3\right)$:

$$\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq 6}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq 6}}{\hbar}}|\mu,\nu\rangle-\langle\mu,\nu|O_{mnds}|\mu,\nu\rangle \tag{2.72}$$

Thanks to (2.68), (2.72) is equal, modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)$, to:
(2.73)

$$\sum_{\substack{2p+|j|\\+|k|+2q\\=3}}\sum_{r\in\mathbb{Z}}\alpha_{p,j,k,q,r}\hbar^p\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)\langle\mu,\nu|\frac{i}{\hbar}[e^{-i2\pi rt}Op^W(z^j\bar{z}^k)D_t^q,O_{mnds}]|\mu,\nu\rangle$$

and with the lemma's notations modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)+O(\hbar)$ to:

$$\sum_{\substack{|j|+|k|+2q=3\\j+m=k+n}}i\alpha_{0,j,k,q,d}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.74}$$

Let us assume we already proved (assertion ($\star$)) that quantity (2.74) determines coefficients $\alpha_{0,j,k,q,d}$ ($|j|+|k|+2q=3$, $j+m=k+n$).
We'll have determined every function $\alpha_{0,j,k,q}$ ($|j|+|k|+2q=3$). Indeed, for any $(j,k,q)\in\mathbb{N}^{2n+1}$ such that $|j|+|k|+2q=3$, and for any $i\in[\![1,n]\!]$, let us choose:

$$n_i=\max(j_i-k_i,0)\text{ and }m_i=\max(k_i-j_i,0) \tag{2.75}$$

$d\in\mathbb{Z}^*$ and $s=1$ if $m=n=0$, $d\in\mathbb{Z}$ and $s=0$ otherwise.
We have for any $i\in[\![1,n]\!]$, $m_i=0$ or $n_i=0$, and

$$|m|+|n|=\sum_{i=1}^{n}|j_i-k_i|\leq|j|+|k|\leq 3$$

Therefore, (m,n,d,s) verifies the three assumptions (1), (2), and (3): (2.74) will hence determine $\alpha_{0,j,k,q,d}$ and letting d describe $\mathbb{Z}$ if $j\neq k$, $\mathbb{Z}^*$ if $j=k$, we will have determined functions $\alpha_{0,j,k,q}$ (thanks to remark 2.13 for the case $j=k$)

Let us prove assertion ($\star$) in the two cases: $m\neq n$ and $m=n$.
Let us also define the set Γ of (j,k,q) such that: $|j|+|k|+2q=3$ and $j+m=k+n$.
Let us first assume that $m\neq n$, and choose $\mu_1(\hbar),\ldots\mu_n(\hbar),\nu(\hbar)$ such that, as $\hbar$ tends to 0:

$$1\ll\mu_1,\ \mu_n^{2N}\ll\nu,\text{ and }\forall i\in[\![1,n-1]\!],\ \mu_i^{2N}\ll\mu_{i+1} \tag{2.76}$$

Let us also define $i_0 := \min\{i \in [\![1,n]\!], m_i \neq n_i\}$. We have, for $(j,k,q) \in \Gamma$:

$$g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right) \underset{\hbar\to 0}{\sim} \frac{j_{i_0}n_{i_0} - k_{i_0}m_{i_0}}{\mu_{i_0}\hbar}(2\pi\nu\hbar)^q \prod_{i=1}^{n}(\mu_i\hbar)^{\max(j_i,k_i)} \tag{2.77}$$

and $j_{i_0}n_{i_0} - k_{i_0}m_{i_0}$ never vanishes.
Also, (2.76) in addittition to (2.77) gives us that:

$$g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right) \ll g_{j'k'q'dmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right) \tag{2.78}$$

if $(j,k,q) < (j',k',q')$, where $<$ is a strict total order on Γ defined by the lexicographical order of $(\max(j_1,k_1),\dots,\max(j_n,k_n),q)$. It is indeed asymmetric since for $i = 1\dots n$, the sign of $m_i - n_i$ determines whether $\max(j_i,k_i)$ is equal to j_i or k_i.

Therefore, making additional assumption on function $\mu_1(\hbar)$ that: $\hbar = O(\mu_1(\hbar)^3\hbar^3)$, we get that qunatity (2.74) is determined modulo $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)$ and assertion $(\star)$ easily follows by induction on $(\Gamma, <)$ in the case $m \neq n$.

If now $m = n$, we may assume that $d \neq 0$ like seen before. Also, $s = 1$, thus for any q, $(q+s)d \neq 0$.
Hence,

$$g_{jjqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right) = (2\pi\nu\hbar)^q(q+1)d\prod_{i=1}^{n}(\mu_i\hbar)^{j_i} \tag{2.79}$$

and assertion $(\star)$ is proved just as before.

Finally, all functions $\alpha_{0,j,k,q}$ are determined for (j,k,q) satisfying $|j|+|k|+2q = 3$. Let (m,n,d,s) satisfy conditions (1), (2), and (3) with $N = 1$.
Therefore, we obtain from (2.73), that the Taylor expansion of f_{mnds} up to order 2 also determines, modulo $O((|\mu\hbar| + |\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}})$:

(2.80)
$$\sum_{\substack{|j|+|k|\\+2q=1}}\sum_{r\in\mathbb{Z}}\alpha_{1,j,k,q,r}\hbar\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)\langle\mu,\nu|\frac{i}{\hbar}[e^{-i2\pi rt}Op^W(z^j\bar{z}^k)D_t^q, O_{mnds}]|\mu,\nu\rangle$$

Just as before, with assumptions (2.76) and $|\mu\hbar| + |\nu\hbar| \ll \hbar^{\frac{2}{3}}$, we can determine every $\alpha_{1,j,k,q,d}$ with $|j|+|k|+2q = 1$ and $j+m = k+n$ (there is actually just one corresponding to $q = 0$, and $(j,k) = (n,m)$), and finally, every function $\alpha_{1,j,k,q}$ with $|j|+|k|+2q = 1$)

This prove the statement for $N = 3$.

Let us now assume that we already every $\alpha_{p,j,k,q}$ up to order $2p+|j|+|k|+2q = N \geq 3$. Let (m,n,d,s) conditions (1) (with $N+1$), (2), and (3).
The Taylor expansion up to order N of function f_{mnds} determines modulo $O\left((|\mu\hbar| + |\nu\hbar|)^{N+1}\right)$:

$$\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq 2N+2}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq 2N+2}}{\hbar}}|\mu,\nu\rangle - \langle\mu,\nu|O_{mnds}|\mu,\nu\rangle \tag{2.81}$$

which is equal, thanks to (2.68) and Lemma 2.17 modulo $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}\right) + O(\hbar)$, to:

$$\sum_{\substack{|j|+|k|+2q\leq N+1\\ j+m=k+n}} i\alpha_{0,j,k,q,d}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right)$$

and by induction hypothesis, the following quantity is determined modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}\right)+O(\hbar)$:

$$\sum_{\substack{|j|+|k|+2q=N+1\\ j+m=k+n}} i\alpha_{0,j,k,q,d}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.82}$$

Now, making assumptions (2.76) and $\hbar = O\left((|\mu\hbar|+|\nu\hbar|)^N\right)$, we determine every $\alpha_{0,j,k,q,d}$ with $|j|+|k|+2q=N+1$ and $j+m=k+n$, and like before, letting (m,n,d,s) run over all possible values (under conditions (1), (2), and (3)), we determine every function $\alpha_{0,j,k,q}$.

Functions $\alpha_{p,j,k,q}$ $(2p+|j|+|k|+2q=N+1)$ will now be determined by induction on p. Let $0\leq p_0\leq \frac{N-1}{2}$ and let us assume we determined functions $\alpha_{p,j,k,q}$ $(0\leq p\leq p_0$ and $|j|+|k|+2q=N+1-2p)$.
Let (m,n,d,s) satisfy conditions (1) (with $N+1-2(p_0+1)$), (2), and (3). Thus, the Taylor expansion of f_{minds} up to order N determines modulo $O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}\right)+O(\hbar^{p_0+2})$

$$\sum_{\substack{2p_0+2+|j|+|k|+2q=N+1\\ j+m=k+n}} i\alpha_{p,j,k,q,d}\hbar^{p_0+1}\left(1+O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{1}{2}}\right)\right)g_{jkqdmnds}\left(\left(\mu+\frac{1}{2}\right)\hbar,\nu\hbar\right) \tag{2.83}$$

And with assumptions (2.76) and $|\mu\hbar|+|\nu\hbar|\ll\hbar^{\frac{2(p_0+1)}{2p_0+3}}$, heredity can be proved just as before, which concludes the proof.

□

Proof of Lemma 2.17. The principal symbol of $\frac{1}{i\hbar}[e^{-i2\pi dt}\mathrm{Op}^W z^j\bar{z}^k D_t^q, O_{mnds}]$ is:

$$\sigma_{jkdq}(z,\bar{z},t,\tau)=\left\{e^{-i2\pi dt}z^j\bar{z}^k\tau^q,\mathcal{O}_{mnds}\right\}=\left\{e^{-i2\pi dt}z^j\bar{z}^k\tau^q,e^{i2\pi dt}z^m\bar{z}^n\tau^s\right\} \tag{2.84}$$

where $e^{-i2\pi dt}z^j\bar{z}^k\tau^q$ is meant for the function $(z,\bar{z},t,\tau)\mapsto e^{-i2\pi dt}z^j\bar{z}^k\tau^q$.

Hence

$$\begin{aligned}
\sigma_{jkdq}(z,\bar z,t,\tau) =& -i\sum_{i=1}^{n}\frac{\partial}{\partial z_i}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial \bar z_i}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\
&+i\sum_{i=1}^{n}\frac{\partial}{\partial \bar z_i}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial z_i}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\
&+\frac{\partial}{\partial t}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial \tau}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\
&-\frac{\partial}{\partial \tau}(e^{-i2\pi dt}z^j\bar z^k\tau^q)\frac{\partial}{\partial t}(e^{i2\pi dt}z^m\bar z^n\tau^s)\\
&=-iz\bar z^{|\max(j,k)|}\tau^{q+s}\left(\sum_{i=1}^{n}\frac{j_in_i-k_im_i}{z_i\bar z_i}+2\pi\frac{d(s+q)}{\tau}\right)
\end{aligned} \tag{2.85}$$

Which means that:

(2.86)

$$\begin{aligned}
\frac{1}{\hbar}[e^{-i2\pi dt}\mathrm{Op}^W z^j\bar z^k D_t^q, O_{mnds}] =&D_t^{q+s}\sum_{\substack{i=1\\ |j_i|+|k_i|>0}}^{n}(j_in_i-k_im_i)P_i^{\max(j_i,k_i)-1}\prod_{\substack{i=1\\ i'\neq i}}^{n}P_{i'}^{\max(j_{i'},k_{i'})}\\
&+2\pi(q+s)D_t^{q+s-1}P^{\max(j,k)}+O(\hbar)
\end{aligned}$$

Hence,

(2.87)

$$\begin{aligned}
\frac{1}{\hbar}\langle\mu,\nu|[e^{-i2\pi dt}\mathrm{Op}^W z^j\bar z^k D_t^q, O_{mnds}]|\mu,\nu\rangle =&(2\pi\nu\hbar)^{q+s}(\mu\hbar)^{\max(j,k)}\sum_{\substack{i=1\\ |j_i|+|k_i|>0}}^{n}\frac{j_in_i-k_im_i}{\mu_i\hbar}\\
&+2\pi(q+s)(2\pi\nu\hbar)^{q+s-1}(\mu\hbar)^{\max(j,k)}+O(\hbar)
\end{aligned}$$

□

3. Reduction to the $\mathbb{R}^n\times S^1$ case

The aim of this section is to prove that Theorem 1.3 is a consequence of his analog in the flat case: Theorem 2.1.

Let $H(x,\hbar D_x)$ be as in Theorem 1.3: a self-adjoint semiclassical elliptic pseudodifferential operator, on a compact manifold X of dimension $n+1$, whose symbol, $H(x,\xi)$, is proper (as a map from T^*X into $\mathbb{R}$). Let E be a regular value of H and γ a non-degenerate periodic trajectory of period T_γ lying on the energy surface $H=E$.

As in [6], thanks to [16], there exists a symplectomorphism ϕ from a neighborhood of S^1 in $T^*(\mathbb{R}^n\times S^1)$ in a neighborhood of γ in $T^*(X)$ such that in the standard symplectic coordinates of $T^*(S^1\times\mathbb{R}^n)$

$$H_0\circ\phi(x,\xi,t,\tau)=H^0+H_2 \text{ and } \phi\circ\gamma(t)=(0,0,t,0) \tag{3.1}$$

where H^0 is defined as in (2.3):

$$H^0(x,\xi,t,\tau)=E+\sum_{i=1}^{n}\theta_i\frac{x_i^2+\xi_i^2}{2}+\tau$$

and H_2 satisfies condition (2.2):

$$H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|)$$

Moreover, one can assume that:

$$\phi(t,\tau,z,\bar{z}) = (t,\tau,(z,\bar{z})A(t)) \tag{3.2}$$

where $z_i = x_i + i\xi_i$ for $i = 1 \dots n$ and $A(t)$ is a complex symplectic matrix of size $2n$.

Expressing our original symplectic coordinates in some Fermi coordinates (t,τ,x,ξ) on $T^*(X)$ (in which the principal Hamiltonian can be written $H^0 + H_2$ as before ((2.2) and (2.3))) determines the matrix A satisfying the conditions above. Hence, identifying those Fermi coordinates with the canonical symplectic coordinates of $T^*(\mathbb{R}^n \times S^1)$, one can assume that $X = \mathbb{R}^n \times S^1$ and it is sufficient to prove that Theorem 2.1 holds for operators O_{mnds} whose principal $\mathcal{O}_{mnds}$ can be written as in (2.4) in some symplectic coordinates:

$$\mathcal{O}_{mnds}(x,\xi,t,\tau) = e^{i2\pi dt}\tau^s \Pi_j (x_j + i\xi_j)^{m_j}(x_j - i\xi_j)^{n_j} \tag{3.3}$$

where, given $N \geq 3$, we assume again

(1) $|m| + |n| \leq N$
(2) $\forall j = 1 \dots n,\ m_j = 0$ **or** $n_j = 0$
(3) $s = 1$ if $m = n = 0$, otherwise $s = 0$.

Let us therefore consider any symplectic coordinates on $T^*(\mathbb{R}^n \times S^1)$, operators O_{mnds} satisfying conditions above for a given $N \geq 3$ and a microlocally unitary Fourier integral operator $A_\phi :\ C_0^\infty(\mathbb{R}^n \times S^1) \to C^\infty(\mathbb{R}^n \times S^1)$ implementing symplectomorphism ϕ.

Let us finally assume that the coefficients intervening in the trace formula associated to our Hamiltonian $H(x,\hbar D_x)$ and the O_{mnds} are known up to order N, or equivalently, the coefficients of the trace formula associated to $A_\phi^{-1}H(x,\hbar D_x)A_\phi$ and $\widehat{O}_{mnds} = A_\phi^{-1}O_{mnds}A_\phi$.
According to Proposition 2.14, one can determine the asymptotic expansion up to order N of following matrix elements:

$$\langle \mu,\nu| e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} \widehat{O}_{mnds} e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} |\mu,\nu\rangle \tag{3.4}$$

where $\widetilde{W}_{\leq N}$ is defined in Proposition 2.2: $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} A_\phi^{-1}H(x,\hbar D_x)A_\phi e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}$ and the quantum Birkhoff normal form have the same expansion in PO (2.13) up to order N. And thanks to Proposition 2.15, it is enough to determine the asymptotic expansion up to order N of following matrix elements:

$$\langle \mu,\nu| e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} \widetilde{O}_{mnds} e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} |\mu,\nu\rangle \tag{3.5}$$

for operators $\widetilde{O}_{mnds}$ whose principal symbol in the standard symplectic coordinates are precisely functions $\mathcal{O}_{mnds}$ (with conditions (1), (2) and (3) on the index) in order to determine $\widetilde{W}_{\leq N}$, and hence conclude the proof.

Now it is enough to remark that the principal symbol of $\widehat{O}_{mnds}$ in the standard symplectic coordinates is $\mathcal{O}_{mnds} \circ \phi$. Therefore, the linearized form we have chosen for ϕ allows to conclude that $\mathcal{O}_{mnds}$ can be expressed as a infinite (due to Fourier coefficients of matrix A) sum of functions $\mathcal{O}_{m'n'd's'} \circ \phi$ times polynomials in the $z_i\bar{z}_i$, $i = 1 \dots n$, where

$|m'|+|n'| \leq |m|+|n|$. Since the P_i's each commute with $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}$, we determined

$$\langle \mu,\nu| e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} \widetilde{O}_{mnds} e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} |\mu,\nu\rangle \tag{3.6}$$

for operators $\widetilde{O}_{mnds}$ satisfying the conditions above, which leads to the conclusion of the proof.

4. The "bottom of a well" case

In this section we prove Theorems 1.8 and 1.11.

The proof of Theorem 1.8 is essentially a consequence of lemmas on linear and bilinear algebra, which are stated and proven in the appendix: Lemmas A.1, A.2, and A.4.

Proof of Theorem 1.8. Let $(x,\xi) \in \mathcal{U} \subset T^*(\mathbb{R}^n)$ be a system of Darboux coordinates centered at z_0 $d^2H_0(z_0)$ is a positive bilinear form on $T_{z_0}(T^*\mathcal{M})$, therefore, by lemma A.1, one can find a symplectomorphism ϕ, defined on a neighborhood of z_0 such that $\phi(z_0)=z_0$ and linear once read in the charts, such that : $\phi^*H_0(x,\xi) = H_0(z_0) + \sum_{i=1}^n \theta_i \frac{x_i^2+\xi_i^2}{2} + O((x,\xi)^3)$.

Let us prove that the diagonal matrix elements of the family of pseudodifferential operators $(Q_{ij}^k)_{1\leq i,j\leq n, 1\leq k\leq 3}$ in the system of eigenvectors corresponding to eigenvalues of $H(x,\hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max\limits_{1\leq j\leq n} \theta_j + \frac{n}{2}+\epsilon).\hbar]$ (for some $\epsilon>0$) allow the explicit construction of such a symplectomorphism ϕ. The remaining part of the Theorem is then very similar the proof of Theorem 1.3.

Let S be the matrix of $d\phi_{z_0}$ in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n})$. Then, we have for $(i,j) \in [\![1,n]\!]^2$ and $s \in \{1,2,3\}$.

$$\begin{aligned}
\phi^*\mathcal{Q}_{i,j}^s(x,\xi) &= \left(\sum_{k=1}^n S_{i^s,2k-1}x_k + S_{i^s,2k}\xi_k\right)\left(\sum_{k=1}^n S_{j^s,2k-1}x_k + S_{j^s,2k}\xi_k\right)\\
&= \sum_{k,k'=1}^n S_{i^s,2k-1}S_{j^s,2k'-1}x_kx_{k'} + \sum_{k,k'=1}^n S_{i^s,2k}S_{j^s,2k'-1}\xi_kx_{k'}\\
&+ \sum_{k,k'=1}^n S_{i^s,2k-1}S_{j^s,2k'}x_k\xi_{k'} + \sum_{k,k'=1}^n S_{i^s,2k}S_{j^s,2k}\xi_k\xi_{k'}\\
&= \sum_{k=1}^n \left[S_{i^s,2k-1}S_{j^s,2k-1} + S_{i^s,2k}S_{j^s,2k}\right] z_k\bar{z}_k + \dots
\end{aligned} \tag{4.1}$$

where for $(i,j)\in[\![1,n]\!]^2$, $i^s = \begin{cases} 2i-1 & \text{if } s\in\{1,2\} \\ 2i & \text{if } s=3\end{cases}$ and $j^s = \begin{cases} 2i & \text{if } s\in\{1,3\} \\ 2i-1 & \text{if } s=2\end{cases}$

and $\dots$ stands for a linear combination of the terms $z_kz_{k'}$ $((k,k')\in[\![1,n]\!])$ and $z_k\bar{z}_{k'}$ $((k,k')\in[\![1,n]\!],\ k\neq k')$.

Therefore, if M_ϕ is the metaplectic representation of $d\phi_{z_0}$, then for $\mu\in\mathbb{N}^n$:

$$\langle\mu|M_\phi Q_{i,j}^s M_\phi^{-1}|\mu\rangle = \sum_{k=1}^n \left[S_{i^s,2k-1}S_{j^s,2k-1} + S_{i^s,2k}S_{j^s,2k}\right]\left(\mu_k+\frac{1}{2}\right)\hbar + O(\hbar|\mu\hbar|) \tag{4.2}$$

Therefore, we only need eigenvectors corresponding to $\mu_k\in\{0,1\}$ for $k\in[\![1,n]\!]$ to determine, for $(i,j)\in[\![1,2n]\!]^2$ and $k\in[\![1,n]\!]$ the values of : $S_{i,2k-1}S_{j,2k-1}+S_{i,2k}S_{j,2k}$. By

Lemma A.2, it is enough to determine a possible choice $d\phi_{z_0}$, hence for ϕ defined on a neighborhood of z_0 with $\phi(z_0)$ and the condition that ϕ is linear once read in the charts. □

Let us now move on to the proof of Theorem 1.11.

Proof of Theorem 1.11. Let $(x,\xi) \in \mathcal{U} \subset T^*(\mathbb{R}^n)$ be a system of Darboux coordinates centered at $z_0 = (q_0, 0) \in T^*\mathcal{M}$ $d^2V(q_0)$ is a positive bilinear form on $T_{q_0}\mathcal{M}$. Hence, by Lemma A.4, there exists a fonction u defined on a neighborhood of q_0, such that $u(q_0) = q_0$, linear one read in the charts, $du(q_0)$ is an orthogonal matrix U written in the basis $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, and:

$$V \circ u(x) = \frac{1}{2}\sum_{i=1}^{n} \theta_i^2 x_i^2 + O(x^3) \tag{4.3}$$

where the θ_i^2's are the eigenvalues of $d^2V(q_0)$.

Let us define a symplectomorphism ϕ by : $\phi(z_0) = z_0$, ϕ is linear once read in the charts and $d\phi_{z_0} = (du, (du^*)^{-1})$, *i.e.* $\phi(x,\xi) = (Ux, U\xi)$. If ϕ_0 is the symplectomorphism that sends (x,ξ) to $(\frac{x_1}{\sqrt{\theta_1}}, \dots, \frac{x_n}{\sqrt{\theta_n}}, \sqrt{\theta_1}\xi_1, \dots, \sqrt{\theta_i}\xi_n)$, then:

$$(\phi \circ \phi_0)^* H(x,\xi) = V(q_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + O(x^3) \tag{4.4}$$

Just as in proof of Theorem 1.8, the diagonal matrix elements of the family of pseudodifferential operators $(Q_{ij}^2)_{1 \le i,j \le n}$ in the system of eigenvectors corresponding to eigenvalues of $H(x,\hbar D_x)$ in $[H_0(z_0), H_0(z_0) + (\max_{1 \le j \le n} \theta_j + \frac{n}{2} + \epsilon).\hbar]$ (for some $\epsilon > 0$) determine the family $(U_{ik}U_{jk})_{1 \le i,j,k \le n}$. Therefore, one can determine, for any $k \in [\![1,n]\!]$, an index $i_k \in [\![1,n]\!]$, such that $U_{i_k k} \neq 0$. The choice of the sign of $U_{i_k k}$ determines the sign of every other coefficient of the k-column. And this n choices do not change property (4.4) by Lemma A.4. Therefore one can construct explicitely from those matrix elements a suitable matrix U, hence a suitable symplectomorphism ϕ.

Hence, in order to prove Theorem 1.11, one can now assume that:

$$H(x,\xi) = V(q_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + R(x) \tag{4.5}$$

where $R(x) = O(x^3)$ and prove that the knowledge of the spectrum of $H(x,\hbar D_x)$ in $[0,\epsilon]$, $\epsilon > 0$ and the diagonal matrix elements of the finite $(2^n - 1)$ number of observables whose principal symbols in this system of local coordinates are: $\prod_{j=1}^{n} (x_j + i\xi_j)^{m_j}$, where for any $j = 1 \dots n$, $m_j \in \{0,1\}$ determine the Taylor expansion near q_0 of R.

Let H_0 be the function defined by $H_0(x,\xi) = \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2}$.

Let us state the following lemma, classical analog of Lemma 2.12, and which uses the hypothesis of rational independance of the θ_i's.

Lemma 4.1. *Let $G \in \mathcal{C}^\infty(T^*(\mathbb{R}^n), \mathbb{R})$ be an homogeneous polynomial of degree $k \geq 3$. There exists a unique couple of functions $G_1 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $F \in \mathcal{C}^\infty(T^*(\mathbb{R}^n), \mathbb{R})$ satisfying :*

$$\forall (x,\xi) \in T^*(\mathbb{R}^n),\ \{H_0, F\}(x,\xi) = G(x,\xi) - G_1(p) \tag{4.6}$$

*and F is polynomial with no diagonal term when written as a function of $(z, \bar{z})$ (*i.e.* of the form $z^l\bar{z}^l$)*

Moreover :

1. *F is an homogeneous polynomial of degree k and is entierely determined by the extradiagonal termes of G,* i.e. *of the form $z^l\bar{z}^m$ $(l \neq m)$*
2. *G_1 is an homogeneous polynomial of degree $\frac{k}{2}$ if k is even, zero if else. Moreover, $G_1(z\bar{z})$ is equal to the sum of the diagonal terms of G.*

Just like in the proof of proposition 2.2, one shows recursively the existence of a family of real numbers $\alpha_{lm})_{l,m\in\mathbb{N}}$ such that for any $l \geq 0$, $\alpha_{ll} = 0$, such that if the functions $(F_N)_{N\geq 3}$ are defined for $N \geq 3$ by :

$$F_N(z,\bar{z}) = \sum_{|l|+|m|=N} \alpha_{lm} z^l \bar{z}^m \tag{4.7}$$

there exists homogeneous polynomials of degree i denoted by $H^i \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying, for $N \geq 3$:

$$(\exp \chi_{F_{\leq N}})^* H(x,\xi) = \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} H^i(p) + O((x,\xi)^{N+1}) \tag{4.8}$$

where $p = p(x,\xi) = (\frac{x_i^2+\xi_i^2}{2})_{1\leq i\leq n}$, and $F_{\leq N} = \sum_{k=1}^N F_k$, and $\chi_{F_{\leq N}}$ is the vector field :

$$\chi_{F_{\leq N}} = \sum_{i=1}^n \frac{\partial F_{\leq N}}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial F_{\leq N}}{\partial x_i}\frac{\partial}{\partial \xi_i} \tag{4.9}$$

If $H_1 \sim \sum_{i=1}^{+\infty} H^i$, then $(x,\xi) \mapsto H_1(p)$ is the classical Birkhoff normal form of H (defined modulo a flat function), which is equal to the principal symbol to the quantum one.

Let us also define for $k \in \mathbb{N}^n$, $|k| \geq 3$: $a_k = \frac{1}{k!}\frac{\partial^{|k|} R}{\partial x^k}(0)$

Let $k \in \mathbb{N}^n$. Let us observe that :

$$x^k = \left(\frac{z+\bar{z}}{\sqrt{2}}\right)^k = \frac{1}{\sqrt{2}^{|k|}} \sum_{\substack{(l,m)\in\mathbb{N}^n \\ l+m=k}} \prod_{j=1}^n \binom{k_j}{m_j} z^l \bar{z}^m \tag{4.10}$$

Let us define $\mathcal{K} = \{k \in \mathbb{N}^n, |k| \geq 3\} \setminus 2\mathbb{N}^n$, and let $k \in \mathbb{N}^n$, $|k| \geq 3$. By lemma 4.1, there exists an homogeneous polynomial of degree $|k|$ with no diagonal terms, such that :

$$\{H_0, I_k\}(x,\xi) = \begin{cases} x^k & \text{if } k \in \mathcal{K} \\ x^k - \frac{1}{\sqrt{2}^{|k|}} \prod_{j=1}^n \binom{k_j}{k_j/2} |z|^k & \text{if } k \in 2\mathbb{N}^n \end{cases} \tag{4.11}$$

Functions $(F_N)_{N\geq 3}$ and $(H^i)_{i\geq 1}$ are constructed recursively as follows : let $N \geq 2$ and let us assume we already constructed $F_3, \dots, F_N$ $(F_2 = 0)$, and $H_1, \dots, H^{\lfloor \frac{N}{2} \rfloor}$ $(H_1(p) = \sum_{i=1}^n \theta_i p_i)$, and let us set :

$$G_{N+1}(x,\xi) = (\exp \chi_{F_{\leq N}})^* H(x,\xi) - \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} H^i(p) + O(\|(x,\xi)\|^{N+1}) \tag{4.12}$$

$$\{H_0, F_{N+1}\}(x,\xi) = \begin{cases} G_{N+1}(x,\xi) & \text{if } N \text{ is even} \\ G_{N+1}(x,\xi) - H^{\frac{N+1}{2}}(p) & \text{if } N \text{ is odd} \end{cases} \tag{4.13}$$

Let us remark that, in our case, $G_{N+1}(x,\xi) - \sum_{|k|=N+1} a_k x^k$ is a sum of terms that depends only on $F_{\leq N}$, $(H^i)_{1\leq i \leq \lfloor \frac{N}{2} \rfloor}$ and $(a_k)_{|k|\leq N}$. Therefore, F_{N+1} and $H^{\frac{N+1}{2}}$ if N is odd depend only on $(a_k)_{|k|\leq N+1}$.

More precisely,

$$\begin{cases} F_{N+1} & = \sum_{|k|=N+1} a_k I_k + \dots \\ H^{\frac{N+1}{2}}(p) & = \sum_{|l|=\frac{N+1}{2}} \frac{a_{2l}}{2^{|l|}} \prod_{j=1}^n \binom{2l_j}{l_j} p^l + \dots (N \text{ odd}) \end{cases} \tag{4.14}$$

where $\dots$ stands for terms that depend only on $(a_k)_{|k|\leq N}$.

Now, let us denote by Λ the set $\{0,1\}^n \setminus \{0\}$. Let us also set, for $k \in \mathbb{N}^n$, $(l_k, m_k) \in \mathbb{N}^{2n}$ as follows : for any $i \in [\![1,n]\!]$, $(l_k)_i = \lfloor \frac{k_i}{2} \rfloor$, $(m_k)_i = k_i - \lfloor \frac{k_i}{2} \rfloor$. $k \mapsto (l_k, m_k)$ is a biunique correspondance between $\mathcal{K}$ and the set $A = \{(l,m) \in \mathbb{N}^{2n} \mid m - l \in \Lambda, |l| + |m| \geq 3\}$.

Moreover, for any $k \in \mathcal{K}$, the coefficient of I_k is $\frac{1}{\theta.(m_k - 1_k)}$ (well defined by the rational independance of the θ_i's. Therefore, the family $(a_k)_{|k|=N+1}$ can be determined recursively from the Taylor expansion from the classical Birkhoff normal form and family $(\alpha_{l_k m_k})_{|k|=N+1}$. The Taylor expansion the Taylor expansion from the classical Birkhoff normal form is determined by the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$, $\epsilon > 0$ as it is already known. Now, let $N \geq 2$, and $s \in \Lambda$.

$$\begin{aligned} (\exp \chi_{F_{\leq N+1}})^* \left(\frac{x + i\xi}{\sqrt{2}}\right)^s &= (x+i\xi)^s + \{F_{\leq N+1}, z^s\} + O((x,\xi)^{N+|s|}) \\ &= - \sum_{\substack{|k|=N+1 \\ m_k - l_k = s}} \alpha_{l_k m_k} |z|^{2m_k} \sum_{i=1}^n \frac{(m_k)_i}{|z_i|^2} + \dots + O((x,\xi)^{N+|s|}) \end{aligned} \tag{4.15}$$

where $\dots$ stands for terms depending only on $(a_k)_{|k|\leq N}$ (hence already determined by induction hypothesis) or extradiagonal terms. Therefore, the diagonal matrix elements of an observable whose principal symbol is z^s, $s \in \Lambda$ will be equal modulo $O(\hbar)$ and for any $N \geq 3$ to :

$$\sum_{\substack{|k|=N+1 \\ m_k - l_k = s}} \alpha_{l_k m_k} (\mu\hbar)^{m_k} \sum_{i=1}^n \frac{(m_k)_i}{\mu_i \hbar} + O(\hbar) + O(|\mu\hbar|^{\frac{N+|s|}{2}}) \tag{4.16}$$

which, just in like the proof of Theorem 2.1 determines $(\alpha_{l_k m_k})_{|k|=N+1}$. Therefore the Taylor expansion of R, hence of V near q_0 is completely determined, which concludes the proof. $\square$

5. Classical analogs

In this section we want to prove a classical analog to Proposition 2.15. It is well known that matrix elements of quantum observables between eigenvectors of integrable Hamiltonians are given at the classical limit by Fourier coefficients in action-angle variables of the classical Hamiltonian. More precisely in the case of diagonal matrix elements the result states that, with the notation of section 2, for any bounded pseudodifferential operator O on $L^2(\mathbb{R}^n \times S^1)$,

$$\langle \mu, \nu | O | \mu, \nu \rangle \sim \int_{\mathbb{T}^n \times S^1} \mathcal{O}'(\mu\hbar, \nu\hbar; \varphi, s) d\varphi ds, \tag{5.1}$$

where $\mathcal{O}'(p, \tau : \varphi, s)$ is the principal symbol of O expressed in the action angles variables (p_i, φ_i) such that $x_l + i\xi_l = \sqrt{p_l} e^{i\varphi_l}$. Therefore it is natural to ask if angle-averages of observables expressed in Birkhoff coordinates determine the original Hamiltonian. Our result is the following.

Theorem 5.1. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by a proper Hamiltonian function H. Let (x, ξ, t, τ) be **any** system of local coordinates near γ such that $\gamma = \{x = \xi = \tau = 0\}$.*

For $(m, n, d, s) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \{0, 1\}$ let $\mathcal{O}_{mnds}$ be functions satisfying in a neighborhood of γ:

$$\mathcal{O}_{mnds}(x, \xi; s, \tau) := e^{i2\pi dt} \tau^s \Pi_j (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}. \tag{5.2}$$

Let Φ be the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near γ and $(p, \varphi; \tau_0, s)$ the corresponding Birkhoff coordinates such that $\gamma = \{p = \tau = 0\}$. Let us define near $p = \tau = 0$ the following "average" quantities

$$\overline{\mathcal{O}}_{mnds}(p, \tau_0) := \int_{\mathbb{T}^n \times S^1} \mathcal{O}_{mnds} \circ \Phi(p, \tau_0; \varphi, s) d\varphi ds. \tag{5.3}$$

Then the knowledge of the Taylor expansion at $p = \tau = 0$ up to order N of $\overline{\mathcal{O}}_{mnds}$ for

1. $|m| + |n| \leq N$
2. $\forall j = 1 \dots n,\ m_j = 0$ ***or*** $n_j = 0$
3. $s = 1$ *if* $m = n = 0$, *otherwise* $s = 0$
4. $d \in \mathbb{Z}$

determines the Taylor expansion of Φ near γ up to order N. Therefore the knowledge of these expansions together with the normal form up to order N determine the Taylor expansion of the "true" Hamiltonian H up to the same order.

Remark 5.2. A Corollary of Theorem 5.1 in the line of Corollary 1.4 (??) can be obtained in a straightforward way. We omit it here.

Proof. We saw in the preceding sections that the diagonal matrix elements of the quantum observables O_{mnds} determine the full semiclassical expansion of the Taylor expansion of the total symbol of the Hamiltonian. What's left to be done is, roughly speaking, to check that the classical limit of the matrix elements determine the one of the symbol. We will need the following lemma (see [14] for a proof)

Lemma 5.3. *Let O be an pseudodifferential operator on $L^2(\mathbb{R}^n\times S^1)$ whose Weyl symbol, expressed in polar and cylindrical coordinates is the function $\mathcal{O}(p,\tau;\varphi,s)$. Then*

$$\langle\mu,\nu|O|\mu,\nu\rangle=\int_{\mathbb{T}^n\times S^1}\mathcal{O}(\mu\hbar,\nu\hbar;\varphi,s)d\varphi ds+O(\hbar). \tag{5.4}$$

Let O_{mnds} be the pseudodifferential operator on $L^2(\mathbb{R}^n\times S^1)$ whose Weyl symbol is the function $\mathcal{O}_{mnds}$. In order to prove Theorem 5.1, it is enough to see that one can recover from the Taylor expansion of the averages O^0_{mnds} up to order N the principal symbol of $\widetilde{W}_{\leq N}$ up to order N. We will proceed by induction on N just as in the proof of Proposition 2.15.

Let us first remark that the principal symbols of $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}$, and $\mathcal{O}_{mnds}\circ\Phi$ have the same Taylor expansion up to order N.

Hence, using Lemma 5.3 we get:

$$\begin{aligned}
&\mathcal{O}^0_{mnds}(\mu\hbar,\nu\hbar)=\langle\mu,\nu|e^{\frac{i\widetilde{W}_{\leq N+1}}{\hbar}}O_{mnds}e^{\frac{-i\widetilde{W}_{\leq N+1}}{\hbar}}|\mu,\nu\rangle+O\left(((|\mu|+|\nu|)\hbar)^{N/2+1}\right)+O(\hbar)\\
&=\frac{i}{\hbar}\langle\mu,\nu|[\widetilde{W}_{\leq N+1},O_{mnds}]|\mu,\nu\rangle+\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle\mu,\nu|[\overbrace{\widetilde{W}_{\leq N+1},\dots,\widetilde{W}_{\leq N+1}}^{l\text{ times}},O_{mnds}]|\mu,\nu\rangle\\
&+O\left(((|\mu|+|\nu|)\hbar)^{N/2+1}\right)+O(\hbar)\\
&=\sum_{\substack{|j|+|k|+2q=N+1\\ j+m=k+n}}\alpha_{0,j,k,q,d}\,i\hbar^{-1}\langle\mu,\nu|[e^{-i2\pi dt}a^j(a^*)^kD_t^q,O_{mnds}]|\mu,\nu\rangle\\
&+\frac{i}{\hbar}\langle\mu,\nu|[W_{\leq N},O_{mnds}]|\mu,\nu\rangle+\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle\mu,\nu|[\overbrace{W_{\leq N},\dots,W_{\leq N}}^{l\text{ times}},O_{mnds}]|\mu,\nu\rangle\\
&+O\left(((|\mu|+|\nu|)\hbar)^{N/2+1}\right)+O(\hbar)
\end{aligned}$$

We now remark that the Taylor expansion of the principal symbol $\sigma_N(z,\bar{z},t\tau)$ of $\widetilde{W}_{\leq N}$ up to order N is exactly $\sigma_N(z,\bar{z},t,\tau)=\sum_{|j|+|k|+2q\leq N}\alpha_{0,j,k,q}(t)z^j\bar{z}^k\tau^q$ up to $(|z|^2+|\tau|)^{\frac{N+1}{2}}$.

We therefore have the

Lemma 5.4.

$$\frac{i}{\hbar}\langle\mu,\nu|[W_{\leq N},O_{mnds}]|\mu,\nu\rangle+\sum_{l\geq 2}\frac{i^l}{\hbar^l l!}\langle\mu,\nu|[\overbrace{W_{\leq N},\dots,W_{\leq N}}^{l\ \mathit{times}},O_{mnds}]|\mu,\nu\rangle \tag{5.5}$$

depends only on $\alpha_{0,j,k,q}(t)$, $|j|+|k|+2q\leq N$, up to $O\left(((|\mu|+|\nu|)\hbar)^{N/2+1}\right)+O(\hbar)$.

For $N=2$ we have that $W_{\leq N}=0$. Therefore

$$\begin{aligned}
\mathcal{O}^0_{mnds}(\mu\hbar,\nu\hbar)=&\sum_{\substack{|j|+|k|+2q=3\\ j+m=k+n}}\alpha_{0,j,k,q,d}\,i\hbar^{-1}\langle\mu,\nu|[e^{-i2\pi dt}a^j(a^*)^kD_t^q,O_{mnds}]|\mu,\nu\rangle\\
&+O\left(((|\mu|+|\nu|)\hbar)^2\right)+O(\hbar).
\end{aligned}$$

So the same argument that the one in the proof of Proposition 2.15, in particular using Lemma 2.17, allows to conclude the case $N = 2$.

Moreover Lemma 5.4 shows clearly that we can conclude by induction again just like in Proposition 2.15.

□

The last result of this paper will be the classical analog of Theorems 1.8 1.11.

Let us remind, [8], that in the case where $H(x,\xi) = \xi^2 + V(x)$ the classical normal form determines the Taylor expansion of the potential when the latter is invariant, for each $i = 1 \dots n$ by the symmetry $x_i \to -x_i$, In the general case the Taylor expansion of the averages, in the sense of (5.3), of a finite number of classical observables are necessary to recover the full potential.

Let us assume $H \in \mathcal{C}^\infty(T^*\mathcal{M}, \mathbb{R})$ has a global minimum at $z_0 \in T^*\mathcal{M}$, and let $d^2H_0(z_0)$ be the Hessian of H at z_0. Let us define matrix Ω defined by $d^2H_0(z_0)(\cdot,\cdot) =: \omega_{z_0}(\cdot, \Omega^{-1}\cdot)$ where $\omega_{z_0}(\cdot,\cdot)$ is the canonical symplectic form of $T^*\mathcal{M}$ at z_0. Ω's eigenvalues are purely imaginary, let us denote them by $\pm i\theta_j$ with $\theta_j > 0$, $j = 1 \dots n$. Let us assume that $\theta_j, j = 1 \dots n$ are rationally independent.

Theorem 5.5. *Let $(x,\xi) \in T^*\mathbb{R}^n$ be **any** system of local coordinates centered at z_0.*

For $(m,n) \in \mathbb{N}^{2n}$, $1 \le i,j \le n$ and $k \in \{1,2,3\}$, let $\mathcal{O}_{mn}$, $\mathcal{Q}^k_{ij}$ be functions satisfying in a neighborhood of z_0:

$$\mathcal{O}_{mn}(x,\xi) = \prod_{j=1}^{n} (x_j + i\xi_j)^{m_j}(x_j - i\xi_j)^{n_j}, \tag{5.6}$$

$$\left\{ \begin{array}{rcl} \mathcal{Q}^1_{ij}(x,\xi) &=& x_i\xi_j \\ \mathcal{Q}^2_{ij}(x,\xi) &=& x_ix_j \\ \mathcal{Q}^3_{ij}(x,\xi) &=& \xi_i\xi_j. \end{array} \right. \tag{5.7}$$

Let us denote by Φ the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near z_0 and (p,φ) the corresponding Birkhoff coordinates such that $\{z_0\} = \{p = 0\}$. Then the knowledge of the Taylor expansion at $p = 0$ up to order $N \ge 3$ of the (finite number) "average" quantities

$$\int_{\mathbb{T}^n} \mathcal{O}_{mn} \circ \Phi(p,\varphi) d\varphi,$$

with

(1) $|m| + |n| \le N$
(2) $\forall j \in [\![1,n]\!]$, $m_j = 0$ ***or*** $n_j = 0$

and the Taylor expansion up to order 2 of the quantities

$$\int_{\mathbb{T}^n} \mathcal{Q}^k_{ij} \circ \Phi(p,\varphi) d\varphi, \quad (i,j) \in [\![1,n]\!]^2, \ k \in \{1,2,3\},$$

together with the Bikhoff normal form itself, determines the Taylor expansion up to order N of H at z_0 in the system of coordinates (x,ξ).

Let us now enunciate the classical of Theorem 1.11 in the case of a Schrödinger operator with potential $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$:

Theorem 5.6. *Let q_0 be a global non-degenerate minimum of V on $\mathcal{M}$. Let us assume that the square roots of the eigenvalues of $d^2V(q_0)$ are linearly independent over the rationnals.*

Let $x \in\subset \mathbb{R}^n$ be ***any*** *system of local coordinates centered at q_0, and (x,ξ) the corresponding Darboux coordinates centered at $(q_0,0)$.*

Let us denote by Φ the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near $(q_0,0)$ and (p,φ) the corresponding Birkhoff coordinates.

With the notations of Theorem 5.5, the knowledge of the Taylor expansion at $p=0$ up to order $N \geq 3$ of the (finite number) "average" quantities

$$\int_{\mathbb{T}^n} \mathcal{O}_{m0} \circ \Phi(p,\varphi) d\varphi, \quad m = (m_1,\dots,m_n) \in \{0,1\}^n \setminus \{0\}$$

and the Taylor expansion up to order 2 of the quantities

$$\int_{\mathbb{T}^n} \mathcal{Q}^2_{ij} \circ \Phi(p,\varphi) d\varphi, \quad (i,j) \in [\![1,n]\!]^2,$$

together with the Bikhoff normal form itself, determines the Taylor expansion up to order N of V at q_0 in the system of coordinates x.

In the line of the proof of Theorem 5.1 the proofs of Theorem 5.6 and 5.5 are easy adaptations of the proofs of Theorem 1.11 and 1.8. We omit them here.

Appendix A. Lemmas on linear and bilinear algebra

Lemma A.1. *Let q be a positive quadratic form on $\mathbb{R}^{2n}$. Then there exists a canonical endomorphism ϕ on $\mathbb{R}^{2n}$, and a n-tuple of positive real numbers $(\lambda_1,\dots,\lambda_n)$, defined as the imaginary part of the eigenvalues of positive imaginary part of the endomorphism defined by:*

$$\langle \cdot ; a(\cdot)\rangle_q = \omega(\cdot,\cdot) \tag{A.1}$$

where $\langle \cdot;\cdot\rangle_q$ be the scalar product associated to q and ω the canonical form on $\mathbb{R}^{2n}$, and such that:

$$\forall (x,\xi) \in \mathbb{R}^{2n},\ q(\phi(x,\xi)) = \sum_{i=1}^{n} \lambda_i (x_i^2 + \xi_i^2) \tag{A.2}$$

Moreover, if the real numbers $\lambda_1,\dots,\lambda_n$ are pairwise different, and ϕ' is an endomorphism of $\mathbb{R}^{2n}$. Then ϕ' is canonical and satisfies (A.2) *if and only there exists an orthogonal isomophism u on $\mathbb{R}^{2n}$ whose restriction to the plane spanned by $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i})$ (for any $i \in [\![1,n]\!]$) is a rotation, such that $\phi' = \phi \circ u$.*

Proof of Lemma A.1. a is antisymmetric respective to q, and therefore there exists a q-orthonormal basis of $\mathbb{R}^{2n}$ $(u_1,\dots,u_n,v_1,\dots,v_n)$ and a n-tuple of positive real numbers $(\lambda_1,\dots,\lambda_n)$ such that, for $j \in [\![1,n]\!]$:

$$\lambda_j a(u_j) = -v_j \text{ and } \lambda_j a(v_j) = u_j \tag{A.3}$$

Now let us set, for $j \in [\![1,n]\!]$:

$$\tilde{u}_j = \sqrt{\lambda_j} u_j \text{ and } \tilde{v}_j = \sqrt{\lambda_j} v_j \tag{A.4}$$

Then, $(\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n)$ is a q-orthogonal basis of $\mathbb{R}^{2n}$ satisfying, for $j \in [\![1,n]\!]$, $q(\tilde{u}_j) = \lambda_j$ and $q(\tilde{v}_j) = \lambda_j$, and the preceeding properties together with (A.2) implies that it is also a symplectic basis, which concludes the proof of the first part of Lemma A.1.

To prove the second part of Lemma A.1, let us consider another symplectic and orthogonal basis $(u'_1, \dots, u'_n, v'_1, \dots, v'_n)$ where, for $j \in [\![1,n]\!]$, the q-norm of u'_j and v'_j is λ_j. Then, by (A.2), for any $j \in [\![1,n]\!]$, $a(u'_j)$ is orthogonal to any vector of the basis but v'_j and $\langle v'_j, a(u'_j)\rangle_q = w(v'_j, u'_j) = -1$, therefore $\lambda_j a(u'_j) = -v'_j$, and by the same argument, $\lambda_j a(v'_j) = u'_j$.

Therefore, the plane spanned by (u_j, v_j) and the plane by (u'_j, v'_j) both are the kernel of $a^2 + \lambda_j^2$ (2-dimensional since we made the additional assumption the λ_i's are pairwise different). Therefore, if ϕ and ϕ' are the endomorphisms which send the canonical basis of $\mathbb{R}^{2n}$ to basis $(\tilde{u}_1, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_n)$ and basis $(u'_1, \dots, u'_n, v'_1, \dots, v'_n)$ respectively, then one can considerer the restriction to any plane spanned by $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i})$ (for any $i \in [\![1,n]\!]$) is an orthogonal symplectomorphism from the plane to itself, that is a rotation.

□

Let σ be the permutation of $[\![1,2n]\!]$ defined by:

$$\forall i \in [\![1,2n]\!],\ \sigma(i) = \begin{cases} 2i-1 & \text{si } i \le n \\ 2(i-n) & \text{si } i \ge n+1 \end{cases} \tag{A.5}$$

and M_σ be the associated permutation matrix (*i.e.* for any $(i,j) \in [\![1,2n]\!]^2$, $(M_\sigma)_{ij} = \delta_{\sigma(i),j}$.

Now, let us set, for any matrix $S \in \mathcal{M}_{2n}(\mathbb{R})$, $S_\sigma = M_\sigma^{-1} S M_\sigma$.

Let us also, for $(i,k) \in [\![1,2n]\!] \times [\![1,n]\!]$, denote by $L_{S,i,k}$ the vector of $\mathbb{R}^2$ defined by $L_{S,i,k} = \begin{pmatrix} (S_\sigma)_{i,2k-1} \\ (S_\sigma)_{i,2k} \end{pmatrix} \in \mathbb{R}^2$. Then, for $(i,k) \in [\![1,n]\!]^2$, $S_{i,k}$ will be the matrix of size 2 whose first line is ${}^tL_{S,2i-1,k}$ and second line ${}^tL_{S,2i,k}$.

Lemma A.2. *Let A be a positive matrix of size $2n$. Let $\mathcal{S}$ be the (non-empty by lemma A.1) set of symplectic matrices satisfying*

$$ {}^tSAS = \left(\begin{array}{c|c} D_\lambda & 0 \\ \hline 0 & D_\lambda \end{array}\right) \tag{A.6}$$

where D_λ is the diagonal matrix with $(\lambda_1, \dots, \lambda_n)$ as n-tuple of positive diagonal elements, which we assume pairwise different. Then:

(1) *Family $(\langle L_{S,i,k}; L_{S,j,k}\rangle)_{i,j)\in[\![1,2n]\!]^2, k\in[\![1,n]\!]}$ is independant of matrix $S \in \mathcal{S}$.*
(2) *Once the preceeding invariants of $\mathcal{S}$ given, one can construct explicitely a particular matrix of $\mathcal{S}$ (hence all of them by Lemma A.1).*

Proof of Lemma A.2. Let us first prove the first point. Let $(S,T) \in \mathcal{S}^2$. By Lemma A.1, there exists n matrices belonging to $SO_2(\mathbb{R})$ and denoted by $O_1, \dots, O_n$, such that:

$$T_\sigma = S_\sigma \begin{pmatrix} O_1 & & \\ & \ddots & \\ & & O_n \end{pmatrix} = \begin{pmatrix} S_{1,1}O_1 & \cdots & S_{1,n}O_n \\ \vdots & & \vdots \\ S_{n,1}O_1 & \cdots & S_{n,n}O_n \end{pmatrix} \tag{A.7}$$

and (A.7) is equivalent to :

$$\forall (i,k) \in [\![1,2n]\!] \times [\![1,n]\!], L_{T,i,k} = {}^tO_k L_{S,i,k} \tag{A.8}$$

Hence $(\langle L_{S,i,k}; L_{S,j,k}\rangle)_{i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$ does not depend of matrix $S \in \mathcal{S}$ and the first point of Lemma A.2 is proven.

Now, let $S \in \mathcal{S}$, and let $(a_{ijk})_{(i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$ be the family defined by:

$$\forall (i,j) \in [\![1,2n]\!]^2,\ \forall k \in [\![1,n]\!], a_{ijk} = \langle L_{S,i,k}; L_{S,j,k}\rangle \tag{A.9}$$

Let us assume that this family is given. Two vectors u and v of $\mathbb{R}^2$ are independant iif: $\langle u;v\rangle^2 < \langle u;u\rangle\langle v;v\rangle$. Since matrix S is invertible, on can choose, for any $k \in [\![1,n]\!]$, a couple of indices $(i_k, j_k) \in [\![1,2n]\!]^2$ such that:

$$a^2_{i_k j_k k} < a_{i_k i_k k} a_{j_k j_k k} \tag{A.10}$$

Let $k \in [\![1,n]\!]$ Let us choose a vector $v_{i_k k}$, whose norm is $\sqrt{a_{i_k i_k k}} > 0$. The following system of equations with unknown $v \in \mathbb{R}^2$:

$$\begin{cases} \langle v_{i_k k}; v\rangle & = a_{i_k j_k k} \\ \langle v; v\rangle & = a_{j_k j_k k} \end{cases} \tag{A.11}$$

admits exactly two solutions (by (A.10)), denoted by $v^+_{j_k k}$ et $v^-_{j_k k}$ obtained from one another by orthogonal symmetry R_k of axis the line spanned by $v_{i_k k}$.

Let us set $v^-_{i_k k} = v^+_{i_k k} = v_{i_k k}$. Since families $(v^+_{i_k k}, v^+_{j_k k})$ et $(v^-_{i_k k}, v^-_{j_k k})$ are basis of $\mathbb{R}^2$, for any $i \in [\![1,2n]\!] \setminus \{i_k, j_k\}$, each one of the two systems:

$$\begin{cases} \langle v_{i_k k}; v\rangle & = a_{i_k ik} \\ \langle v^+_{j_k k}; v\rangle & = a_{j_k ik} \end{cases} \quad \text{et} \quad \begin{cases} \langle v_{i_k k}; v\rangle & = a_{i_k ik} \\ \langle v^-_{j_k k}; v\rangle & = a_{j_k ik} \end{cases} \tag{A.12}$$

admits exactly one solution denoted respectively by v^+_{ik} and v^-_{ik}, and satisfying relation $v^-_{ik} = R_k v^+_{ik}$.

We are now able to construct 2^n matrices $(T_A)_{A\in\mathcal{P}([\![1,n]\!])}$ defined, for $A \in \mathcal{P}([\![1,n]\!])$, by:

$$\forall (i,k) \in [\![1,2n]\!] \times [\![1,n]\!], L_{T_A,i,k} = \begin{cases} v^+_{ik} & \text{if } k \in A \\ v^-_{ik} & \text{if else} \end{cases} \tag{A.13}$$

In order to prove the second point of Lemma A.2, it is sufficient to prove following assertions:

(1) There exists at least one set $A \in \mathcal{P}([\![1,n]\!])$, such that: $T_A \in \mathcal{S}$.
(2) There is at most one set $A \in \mathcal{P}([\![1,n]\!])$, such that T_A is symplectic (and A is determined by family $(a_{ijk})_{(i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$)

Indeed, one those two assertions proven, there will be exactly one set $A \in \mathcal{P}([\![1,n]\!])$ such that T_A is symplectic, and it will be an element of $\mathcal{S}$, constructed from the values of family $(a_{ijk})_{(i,j)\in[\![1,2n]\!]^2,k\in[\![1,n]\!]}$ only.

Let us prove the first assertion. Let, for any $k \in [\![1,n]\!]$, O_k be the unique element of $SO_2(\mathbb{R})$ tel que: $L_{S,i_k,k} = O_k v_{i_k k}$ (where S is a particular matrix of $\mathcal{S}$).

The system (A.11) is equivalent to:

$$\begin{cases} \langle L_{S,i_k,k}; O_k v\rangle & = a_{i_k j_k k} \\ \langle O_k v; O_k v\rangle & = a_{j_k j_k k} \end{cases} \tag{A.14}$$

which admits exactly two solutions: $v^+_{j_k k}$ et $v^-_{j_k k}$. Hence, for any $k \in [\![1,n]\!]$:

$$L_{S,j_k,k} = O_k v^+_{j_k k} \ \text{ or } L_{S,j_k,k} = O_k v^-_{j_k k} \tag{A.15}$$

Let us define the set A by:

$$A = \{k \in \mathbb{N} \mid L_{S,j_k,k} = O_k v^+_{j_k k}\} \tag{A.16}$$

Since each system (A.12) admit a unique solution, we obtain:

$$\begin{aligned}\forall (i,k) \in [\![1,2n]\!] \times [\![1,n]\!], L_{S,i,k} &= \begin{cases} O_k v^+_{ik} & \text{if } \ k \in A \\ O_k v^-_{ik} & \text{if else} \end{cases} \\ &= O_k L_{T_A,i,k}\end{aligned} \tag{A.17}$$

that is:

$$T_{A,\sigma} = S_\sigma \begin{pmatrix} O_1 & & \\ & \ddots & \\ & & O_n \end{pmatrix} \tag{A.18}$$

and $T_A \in \mathcal{S}$ by Lemma A.1.

In order to prove the second assertion, let us use following lemma:

Lemma A.3. *For any symplectic matrix B of size $2n$, we have:*

$$\forall k \in [\![1,n]\!], \ \sum_{i=1}^{n} \det(B_{i,k}) = 1 \tag{A.19}$$

If A_1 and A_2 are two parts of $[\![1,n]\!]$, we get from (A.13) and relation $v^-_{ik} = R_k v^+_{ik}$ that:

$$\forall (i,k) \in [\![1,2n]\!] \times [\![1,n]\!], L_{T_{A_2},i,k} = \begin{cases} R_k L_{T_{A_1},i,k} & \text{if } \ k \in A_1 \Delta A_2 \\ L_{T_{A_1},i,k} & \text{if else} \end{cases} \tag{A.20}$$

where $A_1 \Delta A_2$ is the symmetric difference of A_1 and A_2: $A_1 \Delta A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$. Hence:

$$\forall k \in [\![1,n]\!], \ \sum_{i=1}^{n} \det\left((T_{A_2})_{i,k}\right) = \epsilon_k \sum_{i=1}^{n} \det\left((T_{A_1})_{i,k}\right) \tag{A.21}$$

where, for $k \in [\![1,n]\!]$, $\epsilon_k = -1$ if $k \in A_1 \Delta A_2$, $\epsilon_k = 1$ if else. Since $A_1 \Delta A_2 = \emptyset$ if and only if $A_1 = A_2$, there exists at most one part A of $[\![1,n]\!]$ such T_A is symplectic. The second assertion, hence the second point of Lemma A.3, is proven. □

Proof of Lemma A.3. Since B is a symplectic matrix, matrix B_σ satisfies:

$${}^t B_\sigma J_\sigma B_\sigma = J_\sigma \tag{A.22}$$

It is sufficient, for $k \in [\![1,n]\!]$, to read equality (A.22) at line $2k$ and $2k-1$ to obtain:

$$\sum_{i=1}^{n} \det(B_{i,k}) = 1 \tag{A.23}$$

□

Lemma A.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive matrix whose eigenvalues are pairwise different. Then there exists exactly 2^n orthogonal matrices conjugating A to the diagonal matrix of its ordered eigenvalues, and they are obtained from one another by eventually changing the sign of their columns.*

Proof of Lemma A.4. Let D be the diagonal matrix of the ordered eigenvalues of A. As A is positive. There exists an orthogonal matrix Q_1 such that:

$$Q_1^{-1} A Q_1 = {}^t Q_1 A Q_1 = D \tag{A.24}$$

Let $Q_2 \in GL_n(\mathbb{R})$. Then Q_2 is orthogonal and satisfies : $Q_2^{-1} A Q_2 = D$ if and only if $Q_2^{-1} Q_1$ is an orthogonal matrix which commutes to D, that is, because the diagonal elements of D are pairwise different, if and only if $Q_2^{-1} Q_1$ is an orthogonal diagonal matrix. Finally, Q_2 is orthogonal and satisfies : $Q_2^{-1} A Q_2 = D$ if and only if $Q_2^{-1} Q_1$ is diagonal and its elements belong to $\{-1, 1\}$, that is if Q_2 is obtained from Q_1 by eventually changing the sign of its columns. □

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