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# RECOVERING THE HAMILTONIAN FROM SPECTRAL DATA

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ABSTRACT. We show that the contributions to the Gutzwiller formula with observable associated to the iterates of a given elliptic nondegenerate periodic trajectory  $\gamma$  and to certain families of observables localized near  $\gamma$  determine the quantum Hamiltonian in a formal neighborhood of the trajectory  $\gamma$ , that is the full Taylor expansion of its total symbol near  $\gamma$ .

## 1. INTRODUCTION AND MAIN RESULTS

It is well known that spectral properties of semiclassical Hamiltonians and dynamical properties of their principal symbols are linked. Even when there is no precise information “eigenvalue by eigenvalue” of the spectrum, the so-called Gutzwiller trace formula provide information on averages of the spectrum at scale of the Planck constant. More precisely, let  $H(x, \hbar D_x)$  be a self-adjoint semiclassical elliptic pseudodifferential operator, on a compact manifold  $X$  of dimension  $n + 1$ , whose symbol,  $H(x, \xi)$ , is proper (as a map from  $T^*X$  into  $\mathbb{R}$ ). Let  $E$  be a regular value of  $H$  and  $\gamma$  a non-degenerate periodic trajectory of period  $T_\gamma$  lying on the energy surface  $H = E$ .

Consider the Gutzwiller trace (see [7])

$$(1.1) \quad \sum \psi \left( \frac{E - E_i}{\hbar} \right)$$

where  $\psi$  is a  $C^\infty$  function whose Fourier transform is compactly supported with support in a small neighborhood of  $T_\gamma$  and is identically one in a still smaller neighborhood. As shown in [10], [11] (1.1) has an asymptotic expansion

$$(1.2) \quad e^{i\frac{S_\gamma}{\hbar} + \sigma_\gamma} \sum_{k=0}^{\infty} a_k \hbar^k$$

In [5] was shown how to compute the terms of this expansion to all orders in terms of a microlocal Birkhoff canonical form for  $H$  in a formal neighborhood of  $\gamma$ , and that the constants  $a_{k,r}, \kappa, r = 0, 1, \dots$  determine the microlocal Birkhoff canonical form for  $H$  in a formal neighborhood of  $\gamma$  (and hence, a fortiori, determine the classical Birkhoff canonical form). When it is known “a priori” that  $H(x, \hbar D_x)$  is a Schrödinger operator,

it is known that the normal form determines the potential  $V$  [6]. But in the general case the Gutzwiller formula will determine only the normal form of the Hamiltonian, that is to say  $H(x, \hbar D_x)$  only modulo unitary operators, and its principal symbol only modulo symplectomorphisms. Of course it cannot determine more, as the spectrum, and a fortiori the trace, is insensitive to unitary conjugation. The aim of this paper is to address the question of determining the *true* Hamiltonian from more precise spectral data, namely from the Gutzwiller trace formula with observables.

It is well known that, for any pseudodifferential operator  $O(x, \hbar D_x)$  of symbol  $\mathcal{O}(x, \xi)$ , there is an equivalent result to (1.2) for the following quantity

$$(1.3) \quad \text{Tr} \left( O(x, \hbar D_x) \frac{H(x, \hbar D_x) - E}{\hbar} \right) = \sum \langle \varphi_j, O(x, \hbar D_x) \varphi_j \rangle \psi \left( \frac{E - E_j}{\hbar} \right),$$

(here  $\varphi_j$  is meant as the eigenvector of eigenvalue  $E_j$ ) under the form of an asymptotic expansion of the form

$$(1.4) \quad e^{i \frac{S_\gamma}{\hbar} + \sigma_\gamma} \sum_{k=0}^{\infty} a_k^\gamma(\mathcal{O}) \hbar^k$$

where  $a_k^\gamma$  are distribution supported on  $\gamma$ .

Through this article we will assume, without loss of generality, that the period of  $\gamma$  is equal to 1.

We will show in the present paper that the knowledge of the coefficients  $a_k^\gamma(O)$  for a family (NOT all) of observables localized near  $\gamma$  is enough to determine the (full Taylor expansion of) the (total) symbol of  $H(x, \hbar D_x)$  near  $\gamma$ , or in other words  $H(x, \hbar D_x)$  microlocally in a formal neighborhood of  $\gamma$ , when  $\gamma$  is non-degenerate elliptic, which means that linearized Poincaré map has eigenvalues  $(e^{\pm i\theta_i})$ ,  $i = 1, \dots, n$ , where the rotation angles  $\theta_i$  ( $i = 1, \dots, n$ ) and  $\pi$  are independent over the rationals. The vector field corresponding to a basis of eigenvectors of the linearized Poincaré map will form a family of local symplectic coordinates which are tangent to this vector field. Let us define these coordinates more precisely, out of which follows one of the main result of this article.

**Definition 1.1** (Fermi coordinates). We will denote by Fermi coordinates any system of local coordinates  $(x, \xi, t, \tau)$  near  $\gamma$  in which the principal symbol  $H_0$  of  $H(x, \hbar D_x)$  can be written as:

$$(1.5) \quad H_0(x, \xi, t, \tau) = H^0(x, \xi, t, \tau) + H_2$$

where

$$(1.6) \quad H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|)$$

And

$$(1.7) \quad H^0(x, \xi, t, \tau) = E + \sum_{i=1}^n \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau$$

The existence of such local coordinates, guaranteed by the Weinstein tubular neighborhood theorem ([14]), was proved in [5] under the hypothesis of non degeneracy mentioned earlier.

**Theorem 1.2.** *Let  $(x, \xi, t, \tau) \in T^*(\mathbb{R}^n \times S^1)$  be **any** system of local coordinates near  $\gamma$ , non degenerate elliptic periodic orbit of the Hamiltonian flow generated by the principal symbol  $H_0$  of  $H(x, \hbar D_x)$  on the energy shell  $H_0^{-1}(E)$ .*

*For  $(m, d, s, n) \in \mathbb{N}^n \times \mathbb{Z} \times \{0, 1\}$  let us choose any pseudodifferential operators  $O_{mnds}$  whose principal symbols are*

$$(1.8) \quad \mathcal{O}_{mnds}(x, \xi, t, \tau) = e^{i2\pi dt} \tau^s \prod_j (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}.$$

*Then the knowledge of the coefficients  $a_k(O_{mnds}), k = 0 \dots N$  in ((1.3),(1.4) with*

- (1)  $|m| + |n| \leq N$
- (2)  $\forall j = 1 \dots n, m_j = 0$  **or**  $n_j = 0$
- (3)  $s = 1$  if  $m = n = 0$ , otherwise  $s = 0$

*determines the Taylor expansion near  $\gamma$  of the full symbol of  $H(x, \hbar D_x)$  up to order  $N$  in any Fermi system of coordinates.*

**Corollary 1.3.** *If one already determined some Fermi coordinates, then we can recover from the knowledge of the  $a_k(O_{mnds})$  (with order less or equal to  $N$ ) the Taylor expansion near  $\gamma$  of the full symbol of  $H(x, \hbar D_x)$  up to order  $N$  in the given system of coordinates.*

*Remark 1.4.* It seems reasonable to think that spectral data with observable give enough information to recover of the full Taylor expansion the Hamiltonian (without the quadratic part) without the knowledge of the Fermi coordinates, [8].

*Remark 1.5.* The condition 2 implies that the number of observables (for each Fourier coefficient in  $t$ ) needed for determining  $H(x, \hbar D_x)$  up to order  $N$  is of order  $N^{n+1}$  and not  $N^{2n+2}$ , number of all polynomials of order  $N$ . The fact that not all observables are needed can be understood by the fact that we we knows that the Hamiltonian we are looking for is conjugated to the normal form a unitary operator and not by any operator (see the discussion after theorem 2.1). At the classical level this is a trace of the fact that we are looking for at a symplectomorphism, and not any diffeomorphism (see section 4).

*Remark 1.6.* The asymptotic expansion of the trace (1.3) involves only the microlocalization of  $H(x, \hbar D_x)$  in a formal neighborhood of  $\gamma$ . Therefore there is no hope to recover from spectral data more precise information that the Taylor expansion of its symbol near  $\gamma$ . The rest of the symbol concerns spectral data of order  $\hbar^\infty$

The proof of theorem 1.2 will rely on two other results, expressed in the flat case but easily extendable to the general setting: proposition 2.14 which shows that the coefficients of the trace formula determine the matrix elements  $\langle \varphi_j, O(x, \hbar D_x) \varphi_j \rangle$  where  $\varphi_j$  are the eigenvectors of the normal form of the Hamiltonian, and proposition 2.15 which states that the knowledge of the matrix elements of the conjugation of a given known selfadjoint operator by a unitary one determines, in a certain sense, the latter.

As a byproduct of our main theorem we obtain also a purely classical result, somehow analog of it: the averages on Birkhoff angles associated to Birkhoff coordinates of the same classical observables than the ones in Theorem 1.2 determine the Taylor expansion of the (true) Hamiltonian, Theorem 4.1 below.

The paper is organized as follows. In Section 3 we reduce the problem to the case where  $X = \mathbb{R}^n \times S^1, \gamma = S^1$ . In Section 2 we show that, in the latter case, the  $a_k$  determine the

Taylor expansion of the Hamiltonian and in Section 4 we show the classical equivalent of our quantum formulation.

## 2. PROOF OF THEOREM 2.1

The aim of this section is to prove following theorem in the flat case:

**Theorem 2.1.** *Let  $H(x, \hbar D_x)$  be a self-adjoint semiclassical elliptic pseudodifferential operator on  $L^2(\mathbb{R}^n \times S^1)$ . Let  $(x, \xi, t, \tau) \in T^*(\mathbb{R}^n \times S^1)$  be the canonical symplectic coordinates near  $\gamma = S^1$ , non degenerate elliptic periodic orbit of the Hamiltonian flow generated by the principal symbol  $H_0$  of  $H(x, \hbar D_x)$  on the energy shell  $H_0^{-1}(E)$ .*

*$H_0$  of  $H(x, \hbar D_x)$  can be written in those coordinates as:*

$$(2.1) \quad H_0(x, \xi, t, \tau) = H^0(x, \xi, t, \tau) + H_2$$

where

$$(2.2) \quad H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|)$$

And  $H^0$  is equal to:

$$(2.3) \quad H^0(x, \xi, t, \tau) = E + \sum_{i=1}^n \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau$$

For  $(m, d, s, n) \in \mathbb{N}^n \times \mathbb{Z} \times \{0, 1\}$  let us choose any pseudodifferential operators  $\mathcal{O}_{mnds}$  whose principal symbols are

$$(2.4) \quad \mathcal{O}_{mnds}(x, \xi, t, \tau) = e^{i2\pi dt} \tau^s \prod_j (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}.$$

Then the knowledge of the coefficients  $a_k(\mathcal{O}_{mnds})$ ,  $k = 0 \dots N$  in ((1.3),(1.4) with

- (1)  $|m| + |n| \leq N$
- (2)  $\forall j = 1 \dots n$ ,  $m_j = 0$  **or**  $n_j = 0$
- (3)  $s = 1$  if  $m = n = 0$ , otherwise  $s = 0$

determines the Taylor expansion near  $\gamma$  of the full symbol (in the system of coordinates  $(x, \xi, t, \tau)$ ) of  $H(x, \hbar D_x)$  up to order  $N$ .

The proof of theorem 2.1 will be essentially divided into three steps: first, we will prove in Proposition 2.2 the existence of the quantum Birkhoff normal form in a form convenient to our computations, especially concerning the discussion of orders. In proposition 2.14, we will show that the trace formula with observable  $O$  determines the matrix elements of  $O$  in the eigenbasis of the normal form. Finally, in proposition 2.15, we will show that these matrix elements determines  $H(x, \hbar D_x)$  in a formal neighborhood of  $x = \xi = \tau = 0$ , which leads to theorem 2.1.

For  $i = 1 \dots n$ , let us consider on  $L^2(\mathbb{R}^n \times S^1)$  the operators:

- $a_i = \frac{1}{\sqrt{2}}(x_i + \hbar \partial_{x_i})$
- $a_i^* = \frac{1}{\sqrt{2}}(x_i - \hbar \partial_{x_i})$
- $D_t = -i\hbar \partial_t$
- $P_i := \frac{1}{2}(-\hbar \partial_{x_i}^2 + x_i^2) = a_i^* a_i + \frac{\hbar}{2}$

Now for  $\mu \in \mathbb{N}^n$ ,  $\nu \in \mathbb{Z}$  we will denote by  $|\mu, \nu\rangle$  a common eigenvector of the  $P_i$ 's and  $D_t$ , namely the vectors such that:

$$P_i|\mu, \nu\rangle = (\mu_i + \frac{1}{2})\hbar|\mu, \nu\rangle \text{ and } D_t|\mu, \nu\rangle = 2\pi\hbar|\mu, \nu\rangle.$$

Those vectors can be explicitly constructed as follows:

$$(2.5) \quad |0, 0\rangle(x, t) := \frac{1}{(\pi\hbar)^{\frac{n}{4}}} e^{-\frac{x^2}{2\hbar}}$$

and for any  $\mu \in \mathbb{N}^n$

$$(2.6) \quad |\mu, \nu\rangle(x, t) := e^{i2\pi\nu t} \prod_{i=1}^n \frac{1}{\sqrt{\mu_i!}} (a_i^*)^{\mu_i} |0, 0\rangle(x, t)$$

Let us recall the following:

$$(2.7) \quad \begin{cases} a_i|\mu, \nu\rangle = \sqrt{\mu_i\hbar}|\mu_1, \dots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \dots, \mu_n, \nu\rangle \\ a_i^*|\mu, \nu\rangle = \sqrt{(\mu_i + 1)\hbar}|\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n, \nu\rangle \\ [a_i, a_j^*] = \delta_{ij}\hbar \\ [a_i, a_j] = 0 \end{cases}$$

Also, we will write  $|\mu| := \sum \mu_i$ , and for  $i = 1 \dots n$ ,  $z_i = \frac{x_i + i\xi_i}{\sqrt{2}}$ ,  $p_i = \frac{x_i^2 + \xi_i^2}{2}$ .  $\text{Op}^W(a)$  will be the pseudo differential operator, whose Weyl total symbol is  $a$ .

Finally, let us denote by  $a$ ,  $a^*$  or  $P$  the  $n$ -tuple of corresponding operators  $a_i$ ,  $a_i^*$ ,  $P_i$ ,  $i = 1 \dots n$ . We'll also use the usual convention that, if  $X$  is a  $n$ -tuple of complex numbers or operators, and  $j$  a  $n$ -tuple of nonnegative integers,  $X^j$  stands for  $\prod_{i=1}^n X_i^{j_i}$ .

Our construction of the normal form, inspired by [5], is the following:

**Proposition 2.2.** *Let  $H(x, \hbar D_x)$  be a self-adjoint semiclassical elliptic pseudodifferential operator on  $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ , whose principal symbol is*

$$(2.8) \quad H_0(x, \xi, t, \tau) = H^0(p, \tau) + H_2$$

where  $H^0(p, \tau) = \sum_{i=1}^n \theta_i p_i + \tau$  and  $H_2$  vanishes to the third order on  $x = \xi = \tau = 0$ .

Then for any  $N \geq 3$ , there exists a self-adjoint semiclassical elliptic pseudodifferential operator  $\widetilde{W}_{\leq N}$  and a smooth function  $h(p_1, \dots, p_n, \tau, \hbar)$  such that microlocally in a neighborhood of  $x = \xi = \tau = 0$ :

$$(2.9) \quad \begin{aligned} & \forall M > 0, \exists C_N > 0, \forall (\mu, \nu, \hbar) \in \mathbb{N}^n \times \mathbb{Z} \times [0, 1[, |\mu\hbar| + |\nu\hbar| < M, \\ & \left\| \left( e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{-\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1, \dots, P_n, D_t, \hbar) \right) |\mu, \nu\rangle \right\| \leq C_N (|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \end{aligned}$$

The operators can be computed recursively in the form:

$$(2.10) \quad \widetilde{W}_{\leq N} = W_{\leq N} + (|D_t|^2 + \sum_{i=1}^n P_i)^{N+1}$$

where

$$(2.11) \quad \begin{cases} W_{\leq N} = \sum_{3 \leq q \leq N} W_q \\ W_q := \sum_{2p+|j|+|k|+2m=q} \alpha_{p,j,k,m}(t) \hbar^p \text{Op}^W(z^j \bar{z}^k) D_t^m \end{cases}$$

where for any index  $(p, j, k, m)$ ,  $\alpha_{p,j,k,m} \in C^\infty(\mathbb{S}^1, \mathbb{C})$  and for any  $q \geq 3$ ,  $W_q$  is symmetric.

*Remark 2.3.* We are only interested in recovering the Hamiltonian in a formal neighborhood of  $\gamma$ : every asymptotic expansion is meant microlocally and we'll be rewriting equations such as (2.9) simply as:

$$\left\| \left( e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1, \dots, P_n, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N+1}{2}}$$

Also, by abuse of notation, we'll identify any operator with its version microlocalized near  $\gamma$ .

*Remark 2.4.* One passes from  $W_{\leq N}$  to  $\widetilde{W}_{\leq N}$  in order to gain ellipticity and self-adjointness, like it has been done in lemma 4.5 of [5].

The proof of proposition 2.2 will need several preliminaries:

**Definition 2.5.** We will say that a pseudodifferential operator  $A$  on  $L^2(\mathbb{R}^n \times S^1)$  is "polynomial of order  $r \in \mathbb{N}$ " ( $\text{PO}(r)$ ) if there exists  $\beta_{p,j,k,m} \in C^\infty(\mathbb{S}^1, \mathbb{C})$  such that:

$$(2.12) \quad A = \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t) \hbar^p \text{Op}^W(z^j \bar{z}^k) D_t^m$$

Let us remark that those operators have the following interesting properties:

**Proposition 2.6.** *Let  $A$  be a pseudodifferential operator on  $L^2(\mathbb{R}^n \times S^1)$ . Then, there exists a family of operators  $A_r$ ,  $r \in \mathbb{N}$  such that for any  $i \in \mathbb{N}$ ,  $A_r$  is  $\text{PO}(r)$  and*

$$(2.13) \quad \forall N \in \mathbb{N}, \left\| \left( A - \sum_{r=0}^N A_r \right) |\mu, \nu\rangle \right\| = O\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N+1}{2}}$$

**Definition 2.7.** Let us introduce for any operator  $A$  the notations  $[A]_r$  et  $[A]_{\leq N}$  which represents respectively the terms of order  $r$  and of order smaller or equal to  $N$  of his preceding expansion (2.13).

If  $A$  and  $B$  are two operators, we'll write that:  $A \sim B$  if for any  $r \in \mathbb{N}$ ,  $[A]_r = [B]_r$ .

Also, if  $(A_n)_{n \in \mathbb{N}}$  is a family of operators, we'll write that:

$$(2.14) \quad A \sim \sum_{n=0}^{+\infty} A_n$$

if for any  $N \in \mathbb{N}$ ,  $[A_n]_{\leq N}$  is zero for  $n$  sufficiently large, and the finite sum:

$$(2.15) \quad \sum_{n=0}^{+\infty} [A_n]_{\leq N} = [A]_{\leq N}$$

*Proof.* Let  $a(z, \bar{z}, t, \tau)$  be the total symbol of  $A$ , which has the following Taylor expansion around  $\gamma$ :

$$\forall N \in \mathbb{N}, a(z, \bar{z}, t, \tau) = \sum_{r=0}^N \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t) \hbar^p z^j \bar{z}^k \tau^m + \sum_{p=0}^{\frac{N+1}{2}} O\left(\hbar^p (|z|^2 + |\tau|)^{\frac{N+1}{2}-p}\right)$$

Now, for any  $r \in \mathbb{N}$ , let us notice that the pseudodifferential operator  $A_r$  with symbol  $\sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t) \hbar^p z^j \bar{z}^k \tau^m$  is  $PO(r)$ , and therefore:

$$(2.16) \quad \forall N \in \mathbb{N}, \left\| \left( A - \sum_{r=0}^N A_r \right) |\mu, \nu\rangle \right\| = \sum_{p=0}^{\frac{N+1}{2}} \hbar^p O\left(\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N+1}{2}-p}\right) \\ = O\left(\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N+1}{2}}\right)$$

which concludes the proof.  $\square$

Let us remark the following corollary:

**Corollary 2.8.** *If the expansion (2.13) of an operator  $A$  contains no  $PO(r)$ ,  $r = 0 \dots N$ , then:*

$$\|A|\mu, \nu\rangle\| = O\left(\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N+1}{2}}\right)$$

It will also be convenient to our calculations to notice that:

**Lemma 2.9.** *Let  $F$  and  $G$  be  $PO(r)$  and  $PO(r')$  respectively then  $\frac{[F,G]}{i\hbar}$  is  $PO(r+r'-2)$ .*

*Proof.* Our proof will be a direct consequence of the two following lemmas, whose proof will be given at the end of the proof of lemma 2.9

**Lemma 2.10.** *Any monomial operator of order  $r$ , that is of the form  $\alpha(t)\hbar^p b_1 \dots b_l D_t^m$ , where:*

- for  $j = 1 \dots l$ ,  $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$
- $2p + l + 2m = r$

is  $PO(r)$ .

**Lemma 2.11.** *If  $F$  and  $G$  are monomials of order  $r$  and  $r'$  respectively, then  $\frac{[F,G]}{i\hbar}$  is  $PO(r+r'-2)$*

Indeed, any  $PO(r)$  is a finite sum of monomials of the same order, hence if  $F$  and  $G$  are  $PO(r)$  and  $PO(r')$  respectively, then  $\frac{[F,G]}{i\hbar}$  is a finite sum of quantities of type  $\frac{[\tilde{F}, \tilde{G}]}{i\hbar}$  where  $\tilde{F}$  and  $\tilde{G}$  are monomials of order  $r$  and  $r'$  respectively. Any of those quantities are  $PO(r+r'-2)$  by lemma 2.11, and a finite sum of  $PO(r+r'-2)$  is  $PO(r+r'-2)$ . Therefore, lemma 2.9 is proved. Let us now prove the two lemmas:

*Proof of lemma 2.10.* Since for any  $i, j = 1 \dots n$ ,  $i \neq j$ ,  $a_i$  and  $a_i^*$  commute with both  $a_j$  and  $a_j^*$ , it is sufficient in order to prove lemma 2.10 the following assertion (Ass $_l$ ) for any positive integer  $l$ : "any ordered product  $b_1 \dots b_l$ , where for any  $j = 1 \dots l$ ,  $b_j \in \{a_1, a_1^*\}$  can be written as a finite sum of the quantities  $\hbar^p Op^W(z_1^j \bar{z}_1^k)$  with  $2p + j + k = l$  and  $j - k = l - 2\#\{m \in \{1, \dots, l\}, b_m = a_1^*\}$ " More precisely, let us proceed by induction, and introduce for any ordered product  $b_1 \dots b_l$ ,  $k(b_1 \dots b_l) = \#\{m \in \{1, \dots, l\}, b_m = a_1^*\}$

- If  $l = 1$ , there is nothing to prove since  $a_1 = Op^W(z_1)$  and  $a_1^* = Op^W(\bar{z}_1)$ .
- If  $l = 2$ ,

$$\begin{cases} a_1^2 = Op^W(z_1^2) \\ a_1^{*2} = Op^W(\bar{z}_1^2) \\ a_1 a_1^* = P_1 + \frac{\hbar}{2} = Op^W(z_1 \bar{z}_1) + \frac{\hbar}{2} \\ a_1^* a_1 = Op^W(z_1 \bar{z}_1) - \frac{\hbar}{2} \end{cases}$$

and therefore, the assertion is proved for  $l = 2$ .

- Now, let  $l$  be a positive integer, let us assume (Ass $_k$ ) up to order  $k = l$ , and let  $B = b_1 \dots b_{l+1}$  be an ordered product, where for any  $j = 1 \dots l+1$ ,  $b_j \in \{a_1, a_1^*\}$ . If for any  $j = 1 \dots l$ ,  $b_j = b_{j+1}$ , then  $B = Op^W(z_1^{l+1})$  or  $B = Op^W(\bar{z}_1^{l+1})$ . Otherwise, the proof of the symmetric case being identical, let us can assume that  $b_1 = a_1$ , and set  $j_0 = \max\{j \in \{1, \dots, l+1\}, b_j = a_1\}$ . Let us remark that:  $1 \leq j_0 \leq l$  and  $[a_1^{j_0}, a_1^*] = j_0 \hbar a_1^{j_0-1}$ , so that:

$$(2.17) \quad b_1 \dots b_{l+1} = a_1^{j_0} a_1^* b_{j_0+2} \dots b_{l+1} = a_1^* a_1^{j_0} b_{j_0+2} \dots b_{l+1} + \hbar j_0 a_1^{j_0-1} b_{j_0+2} \dots b_{l+1}$$

Hence if one sets  $k := k(b_1 \dots b_{l+1})$

$$\begin{aligned} \binom{l+1}{k} b_1 \dots b_{l+1} &= \binom{l}{k} a_1^{j_0} a_1^* b_{j_0+2} \dots b_{l+1} + \binom{l}{k-1} a_1^* a_1^{j_0} b_{j_0+2} \dots b_{l+1} \\ &\quad + \hbar \binom{l}{k_b-1} j_0 a_1^{j_0-1} b_{j_0+2} \dots b_{l+1} \end{aligned}$$

Now, because we assumed (Ass $_{l-1}$ ):

$$(l-1) - 2k(a_1^{j_0-1} b_{j_0+2} \dots b_{l+1}) = (l+1) - 2k(b_1 \dots b_{l+1})$$

we only need to observe that the  $\binom{l+1}{k}$  ordered monomials in the sum  $Op^W(z^{l+1-k} \bar{z}^k)$  can be divided in two parts: the  $\binom{l}{k}$  ones whose first term is  $a_1$ , whose sum is  $\binom{l}{k} a_1 Op^W(z^{l-k} \bar{z}^k)$  and the  $\binom{l}{k-1}$  who forms  $\binom{l}{k-1} a_1^* Op^W(z^{l+1-k} \bar{z}^{k-1})$ , and since:

$$\binom{l+1}{k} Op^W(z^{l+1-k} \bar{z}^k) = \binom{l}{k} a_1 Op^W(z^{l-k} \bar{z}^k) + \binom{l}{k-1} a_1^* Op^W(z^{l+1-k} \bar{z}^{k-1})$$

the assumption of (Ass $_l$ ) will be enough to conclude our proof by induction.  $\square$

*Proof of lemma 2.11.* It is now sufficient in order to prove lemma 2.11 that if  $F$  and  $G$  are of the form:

$$F = \alpha(t) b_1 \dots b_l D_t^m \text{ and } G = \beta(t) b'_1 \dots b'_{l'} D_t^{m'}$$

where:

- $\alpha$  and  $\beta$  are smooth
- $l + 2m = r$ ,  $l' + 2m' = r'$
- For  $j = 1 \dots l$ , for  $j' = 1 \dots l'$ ,  $b_j, b'_{j'} \in \{a_1, a_1^*\}$

then  $\frac{[F, G]}{i\hbar}$  is a finite sum of monomials of order  $r + r' - 2$  since, by lemma 2.10, each of them is PO( $r + r' - 2$ ).  $\square$

With those assumptions on  $F$  and  $G$ , we get:

$$(2.18) \quad \begin{aligned} \frac{[F, G]}{i\hbar} &= \frac{[\alpha(t)b_1 \dots b_l D_t^m, \beta(t)b'_1 \dots b'_{l'} D_t^{m'}]}{i\hbar} \\ &= \alpha(t)\beta(t) \frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar} D_t^{m+m'} + \alpha(t)b_1 \dots b_l \frac{[D_t^m, \beta(t)]}{i\hbar} b'_1 \dots b'_{l'} D_t^{m'} \\ &\quad - \beta(t)b'_1 \dots b'_{l'} \frac{[D_t^{m'}, \alpha(t)]}{i\hbar} b_1 \dots b_l D_t^m \end{aligned}$$

Therefore it is sufficient to prove that  $\frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar}$ ,  $\frac{[D_t^m, \beta(t)]}{i\hbar}$  and  $\frac{[D_t^{m'}, \alpha(t)]}{i\hbar}$  are respectively:  $\text{PO}(l+l'-2)$ ,  $\text{PO}(2m-2)$  and  $\text{PO}(2m'-2)$  (with the convention that a  $\text{PO}(j)$  with  $j < 0$  is 0).

For the two last, it is quite obvious, since:

$$(2.19) \quad \frac{[D_t^m, \beta(t)]}{i\hbar} = \sum_{k=0}^{m-1} \binom{m}{k} (i\hbar)^{m-k-1} \beta^{(m-k)}(t) D_t^k$$

Now, for  $j = 1 \dots l'$ , let us set  $\epsilon_j = 1$  if  $b'_j = a_1^*$ , otherwise  $\epsilon_j = -1$ . Since  $[a_1, a_1^*] = \hbar$ , we get:

$$\begin{aligned} b_1 \dots b_l b'_1 \dots b'_{l'} &= b'_1 b_1 \dots b_l b'_2 \dots b'_{l'} + \frac{\epsilon_1 + 1}{2} \hbar \sum_{\substack{k=1 \\ b_k = a_1}}^l b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_2 \dots b'_{l'} \\ &\quad + \frac{\epsilon_1 - 1}{2} \hbar \sum_{\substack{j=1 \\ b_k = a_1^*}}^l b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_2 \dots b'_{l'} \end{aligned}$$

Hence by induction on  $j = 1 \dots l'$ :

$$(2.20) \quad \begin{aligned} \frac{[b_1 \dots b_l, b'_1 \dots b'_{l'}]}{i\hbar} &= -i \sum_{j=1}^{l'} \frac{\epsilon_j + 1}{2} \sum_{\substack{k=1 \\ b_k = a_1}}^l b'_1 \dots b'_{j-1} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_j b'_{j+1} \dots b'_{l'} \\ &\quad - i \sum_{j=1}^{l'} \frac{\epsilon_j - 1}{2} \sum_{\substack{k=1 \\ b_k = a_1^*}}^l b'_1 \dots b'_{j-1} b_1 \dots b_{k-1} b_{k+1} \dots b_l b'_j b'_{j+1} \dots b'_{l'} \end{aligned}$$

The right-hand side of (2.20) is a finite sum of monomials of order  $l + l' - 2$ , hence  $\text{PO}(l + l' - 2)$  by lemma 2.10, hence lemma 2.11 is proved.  $\square$

**Lemma 2.12.** *Let  $G$  be  $\text{PO}(r)$ .*

*Then there exists  $F$  an operator  $\text{PO}(r)$ , and  $G_1 = G_1(P_1, \dots, P_n, D_t, \hbar)$  such that:*

$$(2.21) \quad \frac{[H^0(P, D_t), F]}{i\hbar} = G + G_1$$

*where if  $G$  is symmetric,  $F$  is also symmetric, if  $r$  is odd,  $G_1 = 0$ , and if  $r$  is even  $G_1$  is an homogenous polynomial function of total order  $\frac{r}{2}$ .*

*Remark 2.13.* If  $F = \sum_{2p+|j|+|k|+2m=r} \alpha_{p,j,k,m}(t) \hbar^p \text{Op}^W(z^j \bar{z}^k) D_t^m$ , one can choose:

$$(2.22) \quad \int_{\mathbb{S}^1} \alpha_{p,j,j,m}(t) dt = 0$$

Indeed, any  $\text{Op}^W(z^j \bar{z}^j) D_t^m$  commutes with  $H^0(P, D_t, \hbar)$

*Proof of lemma 2.12.* Let us first assume that  $G$  is a monomial of order  $r$ :  $G = \beta(t) b_1 \dots b_l D_t^m$  where:

- $\alpha$  is smooth
- $l + 2m = r$
- For  $j = 1 \dots l$ ,  $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$

and let us look for  $F$  under the form:  $F = \alpha(t) b_1 \dots b_l D_t^m$ . We have:

$$(2.23) \quad \begin{aligned} \frac{[H^0, F]}{i\hbar} &= \frac{[H^0, \alpha(t) b_1 \dots b_l D_t^m]}{i\hbar} \\ &= \alpha(t) \sum_{i=1}^n \theta_i \frac{[P_i, b_1 \dots b_l]}{i\hbar} D_t^m + \frac{[D_t, \alpha(t)]}{i\hbar} b_1 \dots b_l D_t^m \\ &= \alpha(t) \sum_{i=1}^n \theta_i \frac{[P_i, b_1 \dots b_l]}{i\hbar} D_t^m + \alpha'(t) b_1 \dots b_l D_t^m \end{aligned}$$

If for  $i = 1 \dots n$ ,  $k_i = \#\{m \in \{1, \dots, l\}, b_m = a_i^*\}$  and  $j_i = \#\{m \in \{1, \dots, l\}, b_m = a_i\}$ , we deduce from (2.20) that:

$$(2.24) \quad \frac{[P_i, b_1 \dots b_l]}{i\hbar} = \sqrt{-1} (j_i - k_i) b_1 \dots b_l$$

Hence:

$$(2.25) \quad \frac{[H^0, F]}{\sqrt{-1}\hbar} = \sqrt{-1} \sum_{i=1}^n \theta_i (j_i - k_i) \alpha(t) b_1 \dots b_l D_t^m + \alpha'(t) b_1 \dots b_l D_t^m$$

The problem:  $\frac{[H^0, F]}{\sqrt{-1}\hbar} = G$  admits a solution if there exists  $\alpha$  such that:

$$(2.26) \quad \sqrt{-1} \sum_{i=1}^n \theta_i (j_i - k_i) \alpha(t) + \alpha'(t) = \beta(t)$$

If  $c_p(\alpha)$  and  $c_p(\beta)$  are the Fourier coefficients of  $\alpha$  and  $\beta$ , it is sufficient for the  $c_p(\alpha)$  to be solution of:

$$(2.27) \quad \sqrt{-1} \left( \sum_{i=1}^n \theta_i (j_i - k_i) + 2\pi p \right) c_p(\alpha) = c_p(\beta)$$

and

$$(2.28) \quad c_p(\alpha) \underset{p \rightarrow +\infty}{=} O\left(\frac{1}{|p|^\infty}\right)$$

If the  $n$ -tuples  $j$  and  $k$  are different, the non-degeneracy condition on the  $\theta_i$ 's together with the fact that  $c_p(\beta) \underset{p \rightarrow +\infty}{=} O\left(\frac{1}{|p|^\infty}\right)$  (because  $\beta$  is smooth), gives the existence of  $c_p(\alpha)$  satisfying (2.27) and (2.28).

If  $r$  is odd,  $j$  and  $k$  can't be equal, hence lemma 2.12 is proved in this case ( $r$  odd and  $G$  monomial)

If  $r$  is even, and  $j = k$ , there exists a family  $(c_p(\alpha))_{p \in \mathbb{Z}^*}$  satisfying (2.27) and (2.28). Hence, if  $\alpha$  is the smooth function with Fourier coefficients  $c_p(\alpha)$  for  $p \neq 0$  and  $c_0(\alpha) = 0$ , we get:

$$(2.29) \quad \frac{[H^0, F]}{\sqrt{-1}\hbar} = G + c_0(\beta)b_1 \dots b_l D_t^m$$

And from the proof of lemma 2.10, we know that  $c_0(\beta)b_1 \dots b_l D_t^m$  that is a linear combination of  $G_1(P, D_t, \hbar) := c_0(\beta) \sum_{2p+2|k|=l} a_{p,k} \hbar^p P^k D_t^m$ , and lemma 2.12 is proved in the case where  $r$  is even and  $G$  is monomial.

The general case is easily deduced from the case where  $G$  is monomial, since  $G$  is a finite sum of monomials of the same order.

Also, the form of  $F$  allows us to conclude immediately that  $F$  is symmetric if  $G$  is so.  $\square$

Now we have everything we need for the proof by induction of proposition 2.2.

*Proof of proposition 2.2.* Microlocally near  $x = \xi = \tau = 0$ ,  $H(x, \hbar D_x)$  satisfies

$$(2.30) \quad H := H(x, \hbar D_x) \sim H^0(P_1, \dots, P_n, \hbar D_t) + \sum_{q \geq 3} H_q$$

where:

$$(2.31) \quad H_q := [H(x, \hbar D_x)]_q$$

Let us look for  $\widetilde{W}_{\leq 3}$  under the form predicted in proposition 2.2, that is:

$$(2.32) \quad \widetilde{W}_{\leq 3} = W_3 + (|D_t|^2 + \sum_{i=1}^n P_i)^4$$

where  $W_3$  is PO(3).

$$\begin{aligned} e^{\frac{i\widetilde{W}_{\leq 3}}{\hbar}} H(x, \hbar D_x) e^{-\frac{i\widetilde{W}_{\leq 3}}{\hbar}} &\sim H(x, \hbar D_x) + \frac{i}{\hbar} [\widetilde{W}_{\leq 3}, H] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq 3}, \dots, \widetilde{W}_{\leq 3}, H]}^{l \text{ times}} \\ &\sim H^0 + H_3 + \frac{i}{\hbar} [W_3, H^0] \\ &\quad + \frac{i}{\hbar} [W_3, H - H^0] + \frac{i}{\hbar} [\widetilde{W}_{\leq 3} - W_3, H(x, \hbar D_x)] \\ &\quad + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq 3}, \dots, \widetilde{W}_{\leq 3}, H(x, \hbar D_x)]}^{l \text{ times}} + \sum_{q \geq 4} H_q \end{aligned}$$

Since  $H_3$  is polynomial of order 3, let us choose  $W_3$ , as in lemma 2.12, such that:

$$(2.33) \quad H_3 + \frac{i}{\hbar} [W_3, H^0] = H^1(P_1, \dots, P_n, D_t, \hbar) \equiv 0$$

Since  $W_3$  is PO(3) and the expansion of  $H - H^0$  in PO( $r$ ) contains no PO( $r$ ) of order less or equal to 2, the expansion of  $\widetilde{W}_{\leq 3} - W_3$  no term order less or equal to 3, and the

one of  $H(x, \hbar D_x)$  no term of order less or equal to 1, we know from lemma 2.11 that the expansion of:

$$(2.34) \quad \frac{i}{\hbar}[W_3, H - H^0] + \frac{i}{\hbar}[\widetilde{W}_{\leq 3} - W_3, H] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq 3}, \dots, \widetilde{W}_{\leq 3}, H]}^{l \text{ times}} + \sum_{q \geq 4} H_q$$

contains no term of order less or equal to 3.

Therefore, corollary 2.8 gives us:

$$(2.35) \quad \left\| \left( e^{\frac{i\widetilde{W}_{\leq 3}}{\hbar}} H e^{-\frac{i\widetilde{W}_{\leq 3}}{\hbar}} - H^0(P, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O(|\mu\hbar| + |\nu\hbar|)^2$$

We can construct by induction  $(W_q)_{q \geq 3}$  and  $(H^q)_{q \geq 1}$ , such that:

- for  $q \geq 3$ ,  $W_q$  is PO( $q$ ) and for  $H^{q-2}$  is zero if  $q$  is odd, an homogenous polynomial function of total order  $\frac{q}{2}$  if  $q$  is even.
- 

$$(2.36) \quad H_3 + \frac{i}{\hbar}[W_3, H^0] = H^1(P, D_t, \hbar)$$

- and for any  $q \geq 4$ :

$$\frac{i}{\hbar}[W_q, H^0] + H_q + \left[ \frac{i}{\hbar}[W_{\leq q-1}, H - H^0] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq q-1}, \dots, \widetilde{W}_{\leq q-1}, H]}^{l \text{ times}} \right]_q = H^{q-2}(P, D_t, \hbar)$$

Let us now set:  $\widetilde{W}_{\leq N} := \sum_{q=3}^N W_q + (|D_t|^2 + \sum_{i=1}^n P_i)^{\frac{N+1}{2}}$ . Also, as for any  $q \geq 0$ ,  $H^{2q}$  is an homogenous polynomial function of total order  $q+1$ , we can choose by Borel's lemma a smooth function  $h$  such that for any  $N \geq 1$ , in a neighborhood of  $p = \tau = 0$ .

$$(2.37) \quad \left| h(p, \tau, \hbar) - \sum_{q=0}^{N-1} H^{2q}(p, \tau, \hbar) \right| = O((|p| + |\tau| + |\hbar|)^{N+1})$$

Now, let us write, for any  $N \geq 4$

$$\begin{aligned} e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} &\sim H + \frac{i}{\hbar}[\widetilde{W}_{\leq N}, H] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, H]}^{l \text{ times}} \\ &\sim H + \frac{i}{\hbar}[W_{\leq N}, H^0] + \frac{i}{\hbar}[W_{\leq N}, H - H^0] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, H]}^{l \text{ times}} \\ &\quad + \frac{i}{\hbar}[\widetilde{W}_{\leq N} - W_{\leq N}, H] \end{aligned}$$

Let us also observe that lemma 2.9 gives us for  $q \leq N$ :

$$(2.38) \quad \left\{ \begin{aligned} \left[ \frac{i}{\hbar}[W_{\leq N}, H - H^0] \right]_q &= \left[ \frac{i}{\hbar}[W_{\leq q-1}, H - H^0] \right]_q \\ \left[ \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, H]}^{l \text{ times}} \right]_q &= \left[ \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq q-1}, \dots, \widetilde{W}_{\leq q-1}, H]}^{l \text{ times}} \right]_q \end{aligned} \right.$$

Therefore for any  $q \leq N$ :

$$(2.39) \quad \left[ e^{\frac{i\tilde{W} \leq N}{\hbar}} H e^{-\frac{i\tilde{W} \leq N}{\hbar}} \right]_q = H^{q-2}(P, D_t, \hbar) = [h(P, D_t, \hbar)]_q$$

And corollary 2.8 gives us:

$$(2.40) \quad \left\| \left( e^{\frac{i\tilde{W} \leq N}{\hbar}} H e^{-\frac{i\tilde{W} \leq N}{\hbar}} - h(P, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N+1}{2}}$$

which concludes the proof.  $\square$

The next result is the first inverse result needed for the proof of our main result.

**Proposition 2.14.** *Let  $O$  be a pseudodifferential operator, whose principal symbol vanishes on  $\gamma$ .*

(1) *There exists a smooth function  $f$  vanishing at  $(0, 0, 0)$  such that for any  $N \geq 3$ :*

$$(2.41) \quad \langle \mu, \nu | e^{\frac{i\tilde{W} \leq N}{\hbar}} O e^{-\frac{i\tilde{W} \leq N}{\hbar}} | \mu, \nu \rangle = f\left(\left(\mu + \frac{1}{2}\right)\hbar, 2\pi\nu\hbar, \hbar\right) + O\left(|\mu\hbar| + |\nu\hbar|\right)^{\frac{N}{2}}$$

Moreover let, for any integer  $l$ ,  $\phi_l$  be a Schwartz function whose Fourier transform is compactly supported in  $(l-1, l+1)$  and let  $(a_j^l(O))_{l \geq 0}$  provided by the trace formula:

$$(2.42) \quad \text{Tr}\left(O\phi_l\left(\frac{H-E}{\hbar}\right)\right) \sim \sum_{j=0}^{+\infty} a_j^l(O)\hbar^j$$

(2) *The Taylor expansion of  $f$  up to order  $N$  is entirely determined by the family  $(a_j^l(O))$ ,  $0 \leq j \leq N$ ,  $l \in \mathbb{N}$ .*

*Proof.* Let us first prove point 1.

Let us consider a monomial  $G = \alpha(t)b_1 \dots b_l D_t^m$  where:

- $\alpha$  is smooth
- $l + 2m = r$
- For  $j = 1 \dots l$ ,  $b_j \in \{a_1, a_1^*, \dots, a_n, a_n^*\}$

Let us set for  $i = 1 \dots n$ ,  $k_i = \#\{m \in \{1, \dots, l\}, b_m = a_i^*\}$   
and  $j_i = \#\{m \in \{1, \dots, l\}, b_m = a_i\}$ .

If  $j \neq k$  or  $\alpha \notin \mathbb{C}$ , then:  $\langle \mu, \nu | G | \mu, \nu \rangle = 0$  for any  $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$ .

If now  $j = k$  and  $\alpha \in \mathbb{C}$ , then there exists complex numbers  $\alpha_l$  ( $0 \leq l_i \leq j_i$  for  $i = 1 \dots n$ ), such that:

$$(2.43) \quad G = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} P_1^{j_1 - l_1} \dots P_n^{j_n - l_n} D_t^m$$

and:  $\alpha_0 = \alpha$ .

Therefore for any  $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$ :

$$(2.44) \quad \langle \mu, \nu | G | \mu, \nu \rangle = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} \left( \left( \mu + \frac{1}{2} \right) \hbar \right)^{j-l} (2\pi\nu\hbar)^m$$

Hence, if  $G$  is  $\text{PO}(r)$ , then for any  $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$ :

- $\langle \mu, \nu | G | \mu, \nu \rangle = 0$  if  $r$  is odd.
- If  $r$  is even, there exists an homogenous polynomial function  $g$  of order  $\frac{r}{2}$  such that:

$$(2.45) \quad \langle \mu, \nu | G | \mu, \nu \rangle = g \left( \left( \mu + \frac{1}{2} \right) \hbar, 2\pi\nu\hbar, \hbar \right)$$

From proposition 2.6, corollary 2.8 and Borel's lemma, we get that that for any operator  $A$ , there exists a function  $g$  such that for any  $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$ :

$$(2.46) \quad \langle \mu, \nu | A | \mu, \nu \rangle = g \left( \left( \mu + \frac{1}{2} \right) \hbar, 2\pi\nu\hbar, \hbar \right) + O \left( (|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right)$$

Hence, the only point remaining to prove, is that function  $f$  in point 1 does not depend on  $N$ . It is therefore sufficient to prove that for any  $q \leq N - 1$ ,

$$(2.47) \quad \left[ e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{-\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right]_q = \left[ e^{\frac{i\widetilde{W}_{\leq q+1}}{\hbar}} O e^{-\frac{-i\widetilde{W}_{\leq q+1}}{\hbar}} \right]_q$$

But (2.47) is a direct consequence of lemma 2.9. Indeed,

$$(2.48) \quad e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{-\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim O + \sum_{l \geq 1} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, O]}^{l \text{ times}}$$

and since the principal symbol of  $O$  vanishes on  $\gamma$ , lemma 2.9 gives us for any  $l \geq 1$  and any  $q \leq N - 1$ :

$$(2.49) \quad \left[ \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, O]}^{l \text{ times}} \right]_q = \left[ \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq q+1}, \dots, \widetilde{W}_{\leq q+1}, O]}^{l \text{ times}} \right]_q$$

Let us now move on to the proof of point 2

Since  $\hat{\phi}_l$  is supported near a single period of the flow, we know from the general theory of Fourier integral operators that one can microlocalize the trace formula with observables near  $\gamma$ :

$$(2.50) \quad \text{Tr} \left( O \phi_l \left( \frac{H - E}{\hbar} \right) \right) = \text{Tr} \left( O \int_{\mathbb{R}} \hat{\phi}_l(t) \rho(P_1 + \dots + P_n + |\zeta|) e^{it \frac{H - E}{\hbar}} dt \right) + O(\hbar^\infty)$$

where  $\rho \in C_0^\infty(\mathbb{R})$  is compactly supported and  $\rho = 1$  in a neighborhood of  $p = \tau = 0$ . Therefore we can conjugate (2.50) by the microlocally unitary operator  $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}$ :

$$\begin{aligned} & \text{Tr} \left( O \phi_l \left( \frac{H - E}{\hbar} \right) \right) = \\ & = \text{Tr} \left( \left( e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{-\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \int_{\mathbb{R}} \hat{\phi}_l(t) \rho(P_1 + \dots + P_n + |\zeta|) e^{it \frac{e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{-\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - E}} dt \right) \right) + O(\hbar^\infty) \end{aligned}$$

Thanks to proposition 2.2, we can lighten the r.h.s. for any  $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$

$$(2.51) \quad \begin{aligned} & \int_{\mathbb{R}} \hat{\phi}_l(t) \rho(P_1 + \dots + P_n + |\zeta|) e^{it \frac{i\tilde{W} \leq N}{\hbar} H e \frac{-i\tilde{W} \leq N}{\hbar} - E} dt |\mu, \nu\rangle \\ &= \left( \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left( (|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar \right) e^{it \frac{h((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar) - E + O(|\mu\hbar| + |\nu\hbar|) \frac{N+1}{2}}{\hbar}} dt \right) |\mu, \nu\rangle \end{aligned}$$

As  $\hat{\phi}_l$  is smooth and compactly supported, together with the non-degeneracy condition on the  $\theta_i$ 's, we can assure that if we choose a sufficiently small support for  $\rho$ , we have for any  $\eta > 0$ :

$$\begin{aligned} & \left( \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left( (|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar \right) e^{it \frac{h((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar) - E + O(|\mu\hbar| + |\nu\hbar|) \frac{N+1}{2}}{\hbar}} dt \right) |\mu, \nu\rangle \\ &= \left( \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left( (|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar^\eta \right) e^{it \frac{h((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar) - E + O(|\mu\hbar| + |\nu\hbar|) \frac{N+1}{2}}{\hbar}} dt \right) |\mu, \nu\rangle + O(\hbar^\infty) \end{aligned}$$

Hence, choosing  $\eta < \frac{1}{2}$ :

$$\begin{aligned} & Tr \left( O\phi_l \left( \frac{H - E}{\hbar} \right) \right) + O(\hbar^\infty) \\ &= \sum_{\mu, \nu} \langle \mu, \nu | e^{\frac{i\tilde{W} \leq N}{\hbar}} O e^{\frac{-i\tilde{W} \leq N}{\hbar}} | \mu, \nu \rangle \times \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left( (|\mu| + \frac{n}{2} + |\nu|) \hbar^\eta \right) e^{it(2\pi\nu + \theta \cdot (\mu + \frac{1}{2}))} \dots \\ & \quad \dots \exp \left( \frac{it}{\hbar} \sum_{1 \leq q \leq N-2} H^q \left( (\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar \right) + O \left( (|\mu| + |\nu|) \frac{N+1}{2} \hbar^{\frac{N-1}{2}} \right) \right) dt \\ &= \sum_{\mu, \nu} \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left( (|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar^\eta \right) e^{it(2\pi\nu + \theta \cdot (\mu + \frac{1}{2}))} \\ & \quad \left( 1 + \sum_{i \geq 1}^{\frac{N-1}{2}} \hbar^i Q_i \left( \mu + \frac{1}{2}, \nu, t \right) \right) \times \sum_{p \geq 1}^{\frac{N+1}{2}} \sum_{|k|+m \leq p} b_{k, m, p-|k|-m} \left( \mu + \frac{1}{2} \right)^k (2\pi\nu)^m \hbar^p dt + O(\hbar^{\frac{N+1}{2}}) \end{aligned}$$

where for any  $i \leq \frac{N-1}{2}$ ,  $Q_i$  is a determined polynomial function, of degree in  $(\mu + \frac{1}{2}, \nu)$  less or equal to  $i + 1$ , which depends on the  $H^q$ 's and the Taylor expansion of exp, and the  $b_{k, m, s}$  ( $(k, m, s) \in \mathbb{N}^{n+2} \setminus \{0\}$ ) come from the Taylor expansion at  $(0, 0, 0)$  of the function  $f$  defined in the first point of proposition 2.14, *i.e.* for any  $N \geq 1$ :

$$(2.52) \quad f(x, y, z) = \sum_{1 \leq |k|+m+s \leq N} b_{k, m, s} x^k y^m z^s + O(|x| + |y| + |z|)^{N+1}$$

Now, let us set:

$$(2.53) \quad \forall t \in \mathbb{R}^*, \forall \alpha \in \left( \mathbb{R} \setminus \frac{2\pi}{t} \mathbb{Z} \right)^n, g(t, \alpha) := \frac{e^{i\frac{1}{2}(\alpha_1 + \dots + \alpha_n)}}{\prod_i (1 - e^{it\alpha_i})}$$

By the non-degeneracy condition on the  $\theta_i$ 's,  $g$  is well defined on the compact support of  $\hat{\phi}_l$  around a single period, which is precisely  $l$ . It also implies that  $\theta_i \cdot \mu$  is bounded below by  $C|\mu|$  (where  $C > 0$ ) as  $|\mu|$  goes to  $\infty$ .

Therefore we get from the Poisson formula and the Riemann-Lebesgue lemma that the following quantity  $X_p(l)$  can be computed recursively on  $p \leq \frac{N+1}{2}$  from the  $a_j^l(O)$ ,  $j = 0, \dots, p$ :

$$(2.54) \quad \begin{aligned} X_p(l) &= \sum_{|k|+m \leq p} b_{k,m,p-|k|-m} \left[ \left( -i \frac{\partial}{\partial t} \right)^m \left( \hat{\phi}_l(t) \left( \frac{-i}{t} \right)^k \frac{\partial^k g}{\partial \alpha^k}(t, \alpha) \right) \right] (l, \theta) \\ &= \sum_{|k|+m \leq p} b_{k,m,p-|k|-m} \left[ \left( -i \frac{\partial}{\partial t} \right)^m \left( -i \frac{\partial}{t \partial \alpha} \right)^k g \right] (l, \theta) \end{aligned}$$

since  $\hat{\phi}_l$  is identically 1 around  $l$ .

Now, let us set, for any  $i \in \{1, \dots, n\}$ , any  $t \in \mathbb{R}$  and any  $\alpha \in (\mathbb{R} \setminus \frac{2\pi}{t}\mathbb{Z})^n$ ,  $x_i(t, \alpha) = e^{i \frac{t \alpha_i}{2}}$ . and also define holomorphic function  $h$  on  $\mathbb{C} \setminus \{-1, 1\}$  by  $h(z) = \frac{z}{1-z^2}$  for  $z \in \mathbb{C} \setminus \{-1, 1\}$ . We have for any  $k \in \mathbb{N}^n$  :

$$(2.55) \quad \left( -i \frac{\partial}{t \partial \alpha} \right)^k g = \prod_{i=1}^n \left( -i \frac{\partial}{t \partial \alpha_i} \right)^{k_i} (h \circ x_i)$$

For any  $i \in \{1, \dots, n\}$ , an easy induction on  $k_i \in \mathbb{N}$  leads to the following, since for any  $z \in \mathbb{C} \setminus \{-1, 1\}$ ,  $h(z) = \frac{1}{2} \left( \frac{1}{1-z} - \frac{1}{1+z} \right)$ , and  $-i \frac{\partial x_i}{t \partial \alpha_i} = \frac{1}{2} x_i$ :

$$(2.56) \quad \left( -i \frac{\partial}{t \partial \alpha_i} \right)^{k_i} (h \circ x_i) = \frac{k_i!}{2^{k_i+1}} \left( \frac{x_i}{(1-x_i)^{k_i+1}} + \frac{x_i}{(1+x_i)^{k_i+1}} \right)$$

Now, since  $-i \frac{\partial x_i}{\partial t} = \frac{\alpha_i}{2} x_i$ , an induction on  $s_i \in \mathbb{N}$  shows that:

$$(2.57) \quad \left( -i \frac{\partial}{\partial t} \right)^{s_i} \left( -i \frac{\partial}{t \partial \alpha_i} \right)^{k_i} (h \circ x_i) = \frac{(k_i + s_i)! \alpha_i^{s_i}}{2^{k_i+s_i+1}} \left( \frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}} \right)$$

Let us now introduce for any n-tuple  $s$  such that  $|s| = m$ , the multinomial coefficient:

$$\binom{m}{s} = \frac{m!}{s_1! \dots s_n!}$$

We have:

$$(2.58) \quad \left( -i \frac{\partial}{\partial t} \right)^m \left( -i \frac{\partial}{t \partial \alpha} \right)^k g = \sum_{|s|=m} \binom{m}{s} \prod_{i=1}^n \left( -i \frac{\partial}{\partial t} \right)^{s_i} \left( -i \frac{\partial}{t \partial \alpha_i} \right)^{k_i} (h \circ x_i)$$

Let us use Kronecker theorem, whose hypothesis is precisely the non-degeneracy condition on the  $\theta_i$ 's : for any  $n$ -tuple  $(x_1, \dots, x_n) \in \mathbb{S}_1^n$ , one can find a sequence of integers  $(l_p)_{p \in \mathbb{Z}}$ , such that:

$$\forall j \in \{1, \dots, n\}, x_j(l_p, \theta) \xrightarrow{p \rightarrow +\infty} x_j$$

Therefore, if one sets, for any  $(x_1, \dots, x_n) \in (\mathbb{S}_1 \setminus \{-1, 1\})^n$  and  $(k, m) \in \mathbb{N}^{n+1}$ :

$$u^{(k,m)} = \sum_{|s|=m} \binom{m}{s} \prod_{i=1}^n \frac{(k_i + s_i)! \theta_i^{s_i}}{2^{k_i+s_i+1}} \left( \frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}} \right)$$

Then (2.54), (2.57) and (2.58) together with Kronecker theorem allows us to conclude that the following quantity is determined by the  $a_j^l(O)$ ,  $j = 0, \dots, p$ :

$$(2.59) \quad X_p = \sum_{|k|+m \leq p} b_{k,m,p-|k|-m} u^{(k,m)}$$

Hence, the only thing remaining to prove is that, if one chooses the  $x_i$ 's tending to 1 in a way convenient to us, the  $|u^{(k,m)}|$ 's will tend to  $\infty$  to different orders.

Let us be more precise:

Let the  $x_i$ 's tend to 1 in a way such that:

$$(2.60) \quad \forall i \in \{1, \dots, n-1\}, |1 - x_i| \ll |1 - x_{i+1}|^p$$

If  $\simeq$  means that two functions are equivalent, as the  $x_i$ 's tend to 1 as in (2.60), up a multiplicative constant, we have for any  $(k, m) \in \mathbb{N}^{n+1}$ :

$$(2.61) \quad (1 - x_1)^m u^{(k,m)} \simeq \prod_{i=1}^n \frac{1}{(1 - x_i)^{k_i+1}}$$

Hence, if one sets  $\tilde{m} = (m, 0, \dots, 0)$ :

$$(2.62) \quad u^{(k,m)} \ll u^{(k',m')} \text{ si } k + \tilde{m} < k' + \tilde{m}'$$

where  $<$  is the lexicographical order on  $\mathbb{N}^n$ . Therefore, for any  $p \in \mathbb{N}$  and for any  $(k, m) \in \mathbb{N}^{n+1}$  such that  $|k_0| + m_0 \leq p$ , the following quantity can be recursively determined from  $X_p$ :

$$(2.63) \quad X_{k_0, m_0} = \sum_{k' + \tilde{m}' = k + \tilde{m}} b_{k', m', p - |k'| - m'} u^{(k', m')}$$

Reversing for example the roles of  $i = 1$  and  $i = 2$  in (2.60), and observing that  $k_2 + m \neq k'_2 + m'$  if  $k + \tilde{m} = k' + \tilde{m}'$  and  $(k, m) \neq (k', m')$ , one determines  $b_{k', m', p - |k'| - m'}$  from (2.63) recursively on  $m$ . Finally, each  $b_{k', m', s}$  with  $|k'| + m' + s \leq N$  is determined by the  $a_j^l(O)$ , with  $j = 0 \dots N$  and  $l \in \mathbb{N}$  and the point 2 is proved, which ends the proof of proposition 2.14. □

Our next result shows how the knowledge of the matrix elements of the conjugation of a given known selfadjoint operator by a unitary one determines the latter (in the framework of asymptotic expansions).

For any  $(m, n, d, s) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$ , and any  $(x, \xi, t, \tau) \in T^*(\mathbb{R}^n \times S^1)$ , let us define:

$$(2.64) \quad \mathcal{O}_{mnds}(x, \xi, t, \tau) = e^{i2\pi dt} \tau^s \prod_{j=1}^n (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}.$$

and let  $O_{mnds}$  be a pseudodifferential operator whose Weyl principal symbol is  $\mathcal{O}_{mnds}$ .

By proposition 2.14, there exists a smooth function  $f_{mnds}$  vanishing at  $(0, 0, 0)$  such that for any  $N \geq 3$ :

(2.65)

$$\langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle = f_{mnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, 2\pi\nu\hbar, \hbar \right) + O \left( (|\mu\hbar| + |\nu\hbar|)^{\frac{N}{2}} \right)$$

Theorem 2.1 will now be a direct consequence of proposition 2.14 and following proposition:

**Proposition 2.15.** *Let  $N \geq 3$ . The Taylor expansion of  $f_{mnds}$  up to order  $N - 1$  for any  $(m, n, d, s) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$  satisfying conditions*

- (1)  $|m| + |n| \leq N$
- (2)  $\forall j = 1 \dots n, m_j = 0$  **or**  $n_j = 0$
- (3)  $s = 1$  if  $m = n = 0$ , otherwise  $s = 0$

*determines completely  $W_{\leq N}$*

*Remark 2.16.* Let us remark, like it will be seen in the proof of proposition 2.15, that the only relevant information is the asymptotic expansion of  $\langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle$  as  $\hbar$  tends to 0 and  $\mu, \nu$  go to  $\infty$  slower than any negative power of  $\hbar$ .

*Proof of proposition 2.15.* Let  $N \geq 3$  and  $(m, n, d, s) \in (\mathbb{N}^n)^2 \times \mathbb{Z} \times \{0, 1\}$  satisfy conditions (1), (2) and (3).

Then, we have:

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} \sim O_{mnds} + \frac{i}{\hbar} [\widetilde{W}_{\leq N}, O_{mnds}] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, O_{mnds}]}^{l \text{ times}}$$

Therefore:

(2.66)

$$\begin{aligned} & \langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle - \langle \mu, \nu | O_{mnds} | \mu, \nu \rangle \\ &= \frac{i}{\hbar} \langle \mu, \nu | [\widetilde{W}_{\leq N}, O_{mnds}] | \mu, \nu \rangle + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \langle \mu, \nu | \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, O_{mnds}]}^{l \text{ times}} | \mu, \nu \rangle + O \left( (|\mu\hbar| + |\nu\hbar|)^\infty \right) \end{aligned}$$

Now, since  $\widetilde{W}_{\leq N}$  is a sum of polynomial operators of order greater than 3, we get from proposition 2.9 that for any  $l \geq 2$

$$(2.67) \quad \frac{i^l}{\hbar^l} \langle \mu, \nu | \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, \cdot]}^{l-1 \text{ times}} \rangle$$

maps a  $\text{PO}(r)$  into a sum of polynomial operators of order strictly larger than  $r$ . Therefore, if  $A$  is a  $\text{PO}(r)$ , we have:

$$(2.68) \quad \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \langle \mu, \nu | \overbrace{[\widetilde{W}_{\leq N}, \dots, \widetilde{W}_{\leq N}, A]}^{l-1 \text{ times}} | \mu, \nu \rangle = O \left( (|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}} \right) \langle \mu, \nu | A | \mu, \nu \rangle$$

Finally, let us recall that:

$$(2.69) \quad \begin{aligned} W_N &= \sum_{2p+|j|+|k|+2q=N} \alpha_{p,j,k,q}(t) \hbar^p \mathcal{O}p^W(z^j \bar{z}^k) D_t^q \\ &:= \sum_{2p+|j|+|k|+2q=N} \sum_{r \in \mathbb{Z}} \alpha_{p,j,k,q,r} \hbar^p e^{-i2\pi r t} \mathcal{O}p^W(z^j \bar{z}^k) D_t^q \end{aligned}$$

Let us also state the following lemma, whose proof will be given after the end of the present proof.

**Lemma 2.17.**

$$(2.70) \quad \langle \mu, \nu | [e^{-i2\pi dt} \mathcal{O}p^W(z^j \bar{z}^k) D_t^q, O_{mnds}] | \mu, \nu \rangle = \hbar g_{jkqrmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right) + O(\hbar^2)$$

where, if  $j + m = k + n$  and  $r = d$ :

$$(2.71) \quad g_{jkqrmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right) = (2\pi\nu\hbar)^{q+s} (\mu\hbar)^{\max(j,k)} \left( \sum_{\substack{i=1 \\ |j_i|+|k_i|>0}}^n \frac{j_i n_i - k_i m_i}{\mu_i \hbar} + \frac{d(q+s)}{\nu \hbar} \right)$$

and if  $j + m \neq k + n$  or  $r \neq d$ ,  $g_{jkqrmnds} \equiv 0$

Let us now proceed by induction on  $N \geq 3$ , and first assume  $N = 3$ .

Equation (2.65) gives us that the Taylor expansion up to order 2 of function  $f_{mnds}$  determines modulo  $O((|\mu\hbar| + |\nu\hbar|)^3)$ :

$$(2.72) \quad \langle \mu, \nu | e^{\frac{i\bar{W} \leq 6}{\hbar}} O_{mnds} e^{-\frac{-i\bar{W} \leq 6}{\hbar}} | \mu, \nu \rangle - \langle \mu, \nu | O_{mnds} | \mu, \nu \rangle$$

Thanks to (2.68), (2.72) is equal, modulo  $O((|\mu\hbar| + |\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}})$ , to:

$$(2.73) \quad \sum_{2p+|j|+|k|+2q=3} \sum_{r \in \mathbb{Z}} \alpha_{p,j,k,q,r} \hbar^p \left( 1 + O((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}) \right) \langle \mu, \nu | \frac{i}{\hbar} [e^{-i2\pi r t} \mathcal{O}p^W(z^j \bar{z}^k) D_t^q, O_{mnds}] | \mu, \nu \rangle$$

and with the lemma's notations modulo  $O((|\mu\hbar| + |\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}) + O(\hbar)$  to:

$$(2.74) \quad \sum_{\substack{|j|+|k|+2q=3 \\ j+m=k+n}} i \alpha_{0,j,k,q,d} \left( 1 + O((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}) \right) g_{jkqdmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right)$$

Let us assume we already proved (assertion  $(\star)$ ) that quantity (2.74) determines coefficients  $\alpha_{0,j,k,q,d}$  ( $|j| + |k| + 2q = 3$ ,  $j + m = k + n$ ).

We'll have determined every function  $\alpha_{0,j,k,q}$  ( $|j| + |k| + 2q = 3$ ). Indeed, for any  $(j, k, q) \in \mathbb{N}^{2n+1}$  such that  $|j| + |k| + 2q = 3$ , and for any  $i \in \{1, \dots, n\}$ , let us choose:

$$(2.75) \quad n_i = \max(j_i - k_i, 0) \text{ and } m_i = \max(k_i - j_i, 0)$$

$d \in \mathbb{Z}^*$  and  $s = 1$  if  $m = n = 0$ ,  $d \in \mathbb{Z}$  and  $s = 0$  otherwise.

We have for any  $i \in \{1, \dots, n\}$ ,  $m_i = 0$  or  $n_i = 0$ , and

$$|m| + |n| = \sum_{i=1}^n |j_i - k_i| \leq |j| + |k| \leq 3$$

Therefore,  $(m, n, d, s)$  verifies the three assumptions (1), (2), and (3): (2.74) will hence determine  $\alpha_{0,j,k,q,d}$  and letting  $d$  describe  $\mathbb{Z}$  if  $j \neq k$ ,  $\mathbb{Z}^*$  if  $j = k$ , we will have determined functions  $\alpha_{0,j,k,q}$  (thanks to remark 2.13 for the case  $j = k$ )

Let us prove assertion  $(\star)$  in the two cases :  $m \neq n$  and  $m = n$ .

Let us also define the set  $\Gamma$  of  $(j, k, q)$  such that:  $|j| + |k| + 2q = 3$  and  $j + m = k + n$ .

Let us first assume that  $m \neq n$ , and choose  $\mu_1(\hbar), \dots, \mu_n(\hbar), \nu(\hbar)$  such that, as  $\hbar$  tends to 0:

$$(2.76) \quad 1 \ll \mu_1, \mu_n^{2N} \ll \nu, \text{ and } \forall i \in \{1, \dots, n-1\}, \mu_i^{2N} \ll \mu_{i+1}$$

Let us also define  $i_0 := \min\{i \in \{1, \dots, n\}, m_i \neq n_i\}$ . We have, for  $(j, k, q) \in \Gamma$ :

$$(2.77) \quad g_{jkqdmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right) \underset{\hbar \rightarrow 0}{\sim} \frac{j_{i_0} n_{i_0} - k_{i_0} m_{i_0}}{\mu_{i_0} \hbar} (2\pi\nu\hbar)^q \prod_{i=1}^n (\mu_i \hbar)^{\max(j_i, k_i)}$$

and  $j_{i_0} n_{i_0} - k_{i_0} m_{i_0}$  never vanishes.

Also, (2.76) in addition to (2.77) gives us that :

$$(2.78) \quad g_{jkqdmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right) \ll g_{j'k'q'dmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right)$$

if  $(j, k, q) < (j', k', q')$ , where  $<$  is a strict total order on  $\Gamma$  defined by the lexicographical order of  $(\max(j_1, k_1), \dots, \max(j_n, k_n), q)$ . It is indeed asymmetric since for  $i = 1 \dots n$ , the sign of  $m_i - n_i$  determines whether  $\max(j_i, k_i)$  is equal to  $j_i$  or  $k_i$ .

Therefore, making additional assumption on function  $\mu_1(\hbar)$  that:  $\hbar = O(\mu_1(\hbar)^3 \hbar^3)$ , we get that quantity (2.74) is determined modulo  $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)$  and assertion  $(\star)$  easily follows by induction on  $(\Gamma, <)$  in the case  $m \neq n$ .

If now  $m = n$ , we may assume that  $d \neq 0$  like seen before. Also,  $s = 1$ , thus for any  $q$ ,  $(q+s)d \neq 0$ .

Hence,

$$(2.79) \quad g_{jjqdmnds} \left( \left( \mu + \frac{1}{2} \right) \hbar, \nu \hbar \right) = (2\pi\nu\hbar)^q (q+1)d \prod_{i=1}^n (\mu_i \hbar)^{j_i}$$

and assertion  $(\star)$  is proved just as before.

Finally, all functions  $\alpha_{0,j,k,q}$  are determined for  $(j, k, q)$  satisfying  $|j| + |k| + 2q = 3$ . Let  $(m, n, d, s)$  satisfy conditions (1), (2), and (3) with  $N = 1$ .

Therefore, we obtain from (2.73), that the Taylor expansion of  $f_{mnds}$  up to order 2 also determines, modulo  $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{2+|m|+|n|+2s}{2}}\right)$ :

$$(2.80) \quad \sum_{|j|+|k|+2q=1} \sum_{r \in \mathbb{Z}} \alpha_{1,j,k,q,r} \hbar \left( 1 + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}\right) \right) \langle \mu, \nu | \frac{\partial}{\hbar} [e^{-i2\pi r t} Op^W(z^j \bar{z}^k) D_t^q, O_{mnds}] | \mu, \nu \rangle$$

Just as before, with assumptions (2.76) and  $|\mu\hbar| + |\nu\hbar| \ll \hbar^{\frac{2}{3}}$ , we can determine every  $\alpha_{1,j,k,q,d}$  with  $|j| + |k| + 2q = 1$  and  $j + m = k + n$  (there is actually just one corresponding to  $q = 0$ , and  $(j, k) = (n, m)$ ), and finally, every function  $\alpha_{1,j,k,q}$  with  $|j| + |k| + 2q = 1$

This prove the statement for  $N = 3$ .

Let us now assume that we already every  $\alpha_{p,j,k,q}$  up to order  $2p + |j| + |k| + 2q = N \geq 3$ . Let  $(m, n, d, s)$  conditions (1) (with  $N + 1$ ), (2), and (3).

The Taylor expansion up to order  $N$  of function  $f_{mnds}$  determines modulo  $O((|\mu\hbar| + |\nu\hbar|)^{N+1})$ :

$$(2.81) \quad \langle \mu, \nu | e^{\frac{i\tilde{W} < 2N+2}{\hbar}} O_{mnds} e^{-\frac{-i\tilde{W} < 2N+2}{\hbar}} | \mu, \nu \rangle - \langle \mu, \nu | O_{mnds} | \mu, \nu \rangle$$

which is equal, thanks to (2.68) and lemma 2.17 modulo  $O((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}) + O(\hbar)$ , to:

$$\sum_{\substack{|j|+|k|+2q \leq N+1 \\ j+m=k+n}} i\alpha_{0,j,k,q,d} \left(1 + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}\right)\right) g_{jkqdmnds} \left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right)$$

and by induction hypothesis, the following quantity is determined modulo  $O((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}) + O(\hbar)$ :

$$(2.82) \quad \sum_{\substack{|j|+|k|+2q=N+1 \\ j+m=k+n}} i\alpha_{0,j,k,q,d} \left(1 + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}\right)\right) g_{jkqdmnds} \left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right)$$

Now, making assumptions (2.76) and  $\hbar = O((|\mu\hbar| + |\nu\hbar|)^N)$ , we determine every  $\alpha_{0,j,k,q,d}$  with  $|j| + |k| + 2q = N + 1$  and  $j + m = k + n$ , and like before, letting  $(m, n, d, s)$  run over all possible values (under conditions (1), (2), and (3)), we determine every function  $\alpha_{0,j,k,q}$ .

Functions  $\alpha_{p,j,k,q}$  ( $2p + |j| + |k| + 2q = N + 1$ ) will now be determined by induction on  $p$ . Let  $0 \leq p_0 \leq \frac{N-1}{2}$  and let us assume we determined functions  $\alpha_{p,j,k,q}$  ( $0 \leq p \leq p_0$  and  $|j| + |k| + 2q = N + 1 - 2p$ ).

Let  $(m, n, d, s)$  satisfy conditions (1) (with  $N + 1 - 2(p_0 + 1)$ ), (2), and (3). Thus, the Taylor expansion of  $f_{mnds}$  up to order  $N$  determines modulo  $O((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|+2s}{2}}) + O(\hbar^{p_0+2})$

$$(2.83) \quad \sum_{\substack{2p_0+2+|j|+|k|+2q=N+1 \\ j+m=k+n}} i\alpha_{p,j,k,q,d} \hbar^{p_0+1} \left(1 + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{1}{2}}\right)\right) g_{jkqdmnds} \left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right)$$

And with assumptions (2.76) and  $|\mu\hbar| + |\nu\hbar| \ll \hbar^{\frac{2(p_0+1)}{2p_0+3}}$ , heredity can be proved just as before, which concludes the proof.  $\square$

*Proof of lemma 2.17.* The principal symbol of  $\frac{1}{i\hbar}[e^{-i2\pi dt} \text{Op}^W z^j \bar{z}^k D_t^q, O_{mnds}]$  is:

$$(2.84) \quad \sigma_{jkdq}(z, \bar{z}, t, \tau) = \{e^{-i2\pi dt} z^j \bar{z}^k \tau^q, O_{mnds}\} = \{e^{-i2\pi dt} z^j \bar{z}^k \tau^q, e^{i2\pi dt} z^m \bar{z}^n \tau^s\}$$

where  $e^{-i2\pi dt} z^j \bar{z}^k \tau^q$  is meant for the function  $(z, \bar{z}, t, \tau) \mapsto e^{-i2\pi dt} z^j \bar{z}^k \tau^q$ .

Hence

(2.85)

$$\begin{aligned}
\sigma_{jkdq}(z, \bar{z}, t, \tau) &= -i \sum_{i=1}^n \frac{\partial}{\partial z_i} (e^{-i2\pi dt} z^j \bar{z}^k \tau^q) \frac{\partial}{\partial \bar{z}_i} (e^{i2\pi dt} z^m \bar{z}^n \tau^s) \\
&\quad + i \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} (e^{-i2\pi dt} z^j \bar{z}^k \tau^q) \frac{\partial}{\partial z_i} (e^{i2\pi dt} z^m \bar{z}^n \tau^s) \\
&\quad + \frac{\partial}{\partial t} (e^{-i2\pi dt} z^j \bar{z}^k \tau^q) \frac{\partial}{\partial \tau} (e^{i2\pi dt} z^m \bar{z}^n \tau^s) - \frac{\partial}{\partial \tau} (e^{-i2\pi dt} z^j \bar{z}^k \tau^q) \frac{\partial}{\partial t} (e^{i2\pi dt} z^m \bar{z}^n \tau^s) \\
&= -i z \bar{z}^{|\max(j,k)|} \tau^{q+s} \left( \sum_{i=1}^n \frac{j_i n_i - k_i m_i}{z_i \bar{z}_i} + 2\pi \frac{d(s+q)}{\tau} \right)
\end{aligned}$$

Which means that:

(2.86)

$$\begin{aligned}
\frac{1}{\hbar} [e^{-i2\pi dt} \text{Op}^W z^j \bar{z}^k D_t^q, O_{mnds}] &= D_t^{q+s} \sum_{\substack{i=1 \\ |j_i|+|k_i|>0}}^n (j_i n_i - k_i m_i) P_i^{\max(j_i, k_i)-1} \prod_{\substack{i'=1 \\ i' \neq i}}^n P_{i'}^{\max(j_{i'}, k_{i'})} \\
&\quad + 2\pi(q+s) D_t^{q+s-1} P^{\max(j,k)} + O(\hbar)
\end{aligned}$$

Hence,

(2.87)

$$\begin{aligned}
\frac{1}{\hbar} \langle \mu, \nu | [e^{-i2\pi dt} \text{Op}^W z^j \bar{z}^k D_t^q, O_{mnds}] | \mu, \nu \rangle &= (2\pi\nu\hbar)^{q+s} (\mu\hbar)^{\max(j,k)} \sum_{\substack{i=1 \\ |j_i|+|k_i|>0}}^n \frac{j_i n_i - k_i m_i}{\mu_i \hbar} \\
&\quad + 2\pi(q+s) (2\pi\nu\hbar)^{q+s-1} (\mu\hbar)^{\max(j,k)} + O(\hbar)
\end{aligned}$$

□

### 3. REDUCTION TO THE FLAT CASE

The aim of this section is to prove that Theorem 1.2 is a consequence of his analog in the flat case: Theorem 2.1.

Let  $H(x, \hbar D_x)$  be as in theorem 1.2: a self-adjoint semiclassical elliptic pseudodifferential operator, on a compact manifold  $X$  of dimension  $n+1$ , whose symbol,  $H(x, \xi)$ , is proper (as a map from  $T^*X$  into  $\mathbb{R}$ ). Let  $E$  be a regular value of  $H$  and  $\gamma$  a non-degenerate periodic trajectory of period  $T_\gamma$  lying on the energy surface  $H = E$ .

As in [5], thanks to [14], there exists a symplectomorphism  $\phi$  from a neighborhood of  $S^1$  in  $T^*(\mathbb{R}^n \times S^1)$  in a neighborhood of  $\gamma$  in  $T^*(X)$  such that in the standard symplectic coordinates of  $T^*(S^1 \times \mathbb{R}^n)$

$$(3.1) \quad H_0 \circ \phi(x, \xi, t, \tau) = H^0 + H_2 \text{ and } \phi \circ \gamma(t) = (0, 0, t, 0)$$

where  $H^0$  is defined as in (2.3):

$$H^0(x, \xi, t, \tau) = E + \sum_{i=1}^n \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau$$

and  $H_2$  satisfies condition (2.2):

$$H_2 = O(|x|^3 + |\xi^3| + |x\tau| + |\xi\tau|)$$

Moreover, one can assume that:

$$(3.2) \quad \phi(t, \tau, z, \bar{z}) = (t, \tau, (z, \bar{z})A(t))$$

where  $z_i = x_i + i\xi_i$  for  $i = 1 \dots n$  and  $A(t)$  is a complex symplectic matrix of size  $2n$ , which also satisfies

$$(3.3) \quad \forall t \in S^1, \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, n\}, \begin{cases} A_{i+n, j+n}(t) = \bar{A}_{ij}(t) \\ A_{i, j+n}(t) = \bar{A}_{i+n, j}(t) \end{cases}$$

Expressing our original symplectic coordinates in some Fermi normal coordinates  $(t, \tau, x, \xi)$  on  $T^*(X)$ , that is, coordinates in which the principal Hamiltonian can be written  $H^0 + H_2$  as before ((2.2) and (2.3)), determines matrix  $A$  satisfying the conditions above. Hence, identifying those Fermi coordinates with the canonical symplectic coordinates of  $T^*(\mathbb{R}^n \times S^1)$ , one can assume that  $X = \mathbb{R}^n \times S^1$  and it is sufficient to prove that Theorem 2.1 holds for operators  $O_{mnds}$  whose principal  $\mathcal{O}_{mnds}$  can be written as in (2.4) in some symplectic coordinates:

$$\mathcal{O}_{mnds}(x, \xi, t, \tau) = e^{i2\pi dt} \tau^s \Pi_j (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}.$$

with, given  $N \geq 3$ , conditions (1), (2) and (3) on the index:

- (1)  $|m| + |n| \leq N$
- (2)  $\forall j = 1 \dots n, m_j = 0$  **or**  $n_j = 0$
- (3)  $s = 1$  if  $m = n = 0$ , otherwise  $s = 0$

Let us therefore consider any symplectic coordinates on  $T^*(\mathbb{R}^n \times S^1)$ , operators  $O_{mnds}$  satisfying conditions above for a given  $N \geq 3$  and a microlocally unitary Fourier integral operator  $A_\phi : C_0^\infty(\mathbb{R}^n \times S^1) \rightarrow C^\infty(\mathbb{R}^n \times S^1)$  implementing symplectomorphism  $\phi$ .

Let us finally assume that the coefficients intervening in the trace formula associated to our Hamiltonian  $H(x, \hbar D_x)$  and the  $O_{mnds}$  are known up to order  $N$ , or equivalently, the coefficients of the trace formula associated to  $A_\phi^{-1}H(x, \hbar D_x)A_\phi$  and  $\widehat{O}_{mnds} = A_\phi^{-1}O_{mnds}A_\phi$ .

According to proposition 2.14, one can determine the asymptotic expansion up to order  $N$  of following matrix elements:

$$(3.4) \quad \langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} \widehat{O}_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle$$

where  $\widetilde{W}_{\leq N}$  is defined in proposition 2.2:  $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} A_\phi^{-1}H(x, \hbar D_x)A_\phi e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}}$  and the quantum Birkhoff normal form have the same expansion in PO (2.13) up to order  $N$ . And thanks to proposition 2.15, it is enough to determine the asymptotic expansion up to order  $N$  of following matrix elements:

$$(3.5) \quad \langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} \widetilde{O}_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle$$

for operators  $\widetilde{O}_{mnds}$  whose principal symbol in the standard symplectic coordinates are precisely functions  $\mathcal{O}_{mnds}$  (with conditions (1), (2) and (3) on the index) in order to determine  $\widetilde{W}_{\leq N}$ , and hence conclude the proof.

Now it is enough to remark that the principal symbol of  $\widehat{O}_{mnds}$  in the standard symplectic coordinates is  $\mathcal{O}_{mnds} \circ \phi$ . Therefore, the linearized form we have chose for  $\phi$  allows

to conclude that  $\mathcal{O}_{mnds}$  can be expressed as a infinite (due to Fourier coefficients of matrix  $A$ ) sum of functions  $\mathcal{O}_{m'n'd's'} \circ \phi$  times polynomials in the  $z_i \bar{z}_i$ ,  $i = 1 \dots n$ , where  $|m'| + |n'| \leq |m| + |n|$ . Since the  $P_i$ 's each commute with  $e^{\frac{i\tilde{W} \leq N}{\hbar}}$ , we determined

$$(3.6) \quad \langle \mu, \nu | e^{\frac{i\tilde{W} \leq N}{\hbar}} \tilde{\mathcal{O}}_{mnds} e^{-\frac{i\tilde{W} \leq N}{\hbar}} | \mu, \nu \rangle$$

for operators  $\tilde{\mathcal{O}}_{mnds}$  satisfying the conditions above, which leads to the conclusion of the proof.

#### 4. A CLASSICAL ANALOG

In this section we want to prove a classical analog to proposition 2.15. It is well known that matrix elements of quantum observables between eigenvectors of integrable Hamiltonians are given at the classical limit by Fourier coefficients in action-angle variables of the classical Hamiltonian. More precisely in the case of diagonal matrix elements the result states that, with the notation of section 2, for any bounded pseudodifferential operator  $O$  on  $L^2(\mathbb{R}^n \times S^1)$ ,

$$(4.1) \quad \langle \mu, \nu | O | \mu, \nu \rangle \sim \int_{\mathbb{T}^n \times S^1} \mathcal{O}'(\mu\hbar, \nu\hbar; \varphi, s) d\varphi ds,$$

where  $\mathcal{O}'(p, \tau : \varphi, s)$  is the principal symbol of  $O$  expressed in the action angles variables  $(p_i, \varphi_i)$  such that  $x_l + i\xi_l = \sqrt{p_l} e^{i\varphi_l}$ . Therefore it is natural to ask if angle-averages of observables expressed in Birkhoff coordinates determine the original Hamiltonian. Our result is the following.

**Theorem 4.1.** *Let  $(x, \xi, t, \tau) \in T^*(\mathbb{R}^n \times S^1)$  be **any** system of local coordinates near  $\gamma$ , non degenerate elliptic periodic orbit of the Hamiltonian flow generated by the Hamiltonian  $H$ . Let us define, for  $(m, d, s, n) \in \mathbb{N}^n \times \mathbb{Z} \times \{0, 1\}$  the functions*

$$(4.2) \quad \mathcal{O}_{mnds}(x, \xi; s, \tau) := e^{i2\pi dt} \tau^s \prod_j (x_j + i\xi_j)^{m_j} (x_j - i\xi_j)^{n_j}.$$

*Let us denote by  $\Phi : T^*(\mathbb{R}^n \times S^1) \rightarrow T^*X$  the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form and  $(p, \varphi; \tau_0, s)$  the corresponding Birkhoff coordinates. Let us define*

$$(4.3) \quad \mathcal{O}_{mnds}^0(p, \tau_0) := \int_{\mathbb{T}^n \times S^1} \mathcal{O} \circ \Phi(p, \tau_0; \varphi, s) d\varphi ds.$$

*Then the knowledge of the Taylor expansion of the averages  $\mathcal{O}_{mnds}^0$  for*

- (1)  $|m| + |n| \leq N$
- (2)  $\forall j = 1 \dots n, m_j = 0$  **or**  $n_j = 0$
- (3)  $s = 1$  if  $m = n = 0$ , otherwise  $s = 0$

*determines the Taylor expansion of  $\Phi$  near  $\gamma$  up to order  $N$ . Therefore the knowledge of these quantities together with the normal form up to order  $N$  determine the Taylor expansion of the "true" Hamiltonian  $H$  up to the same order.*

*Proof.* We saw in the preceding sections that the diagonal matrix elements of the quantum observables  $\mathcal{O}_{mnds}$  determine the full semiclassical expansion of the Taylor expansion of the total symbol of the Hamiltonian. What's left to be done is, roughly speaking, to check that the classical limit of the matrix elements determine the one of the symbol. We will need the following lemma (see [12] for a proof)

**Lemma 4.2.** *Let  $O$  be an pseudodifferential operator on  $L^2(\mathbb{R}^n \times S^1)$  whose Weyl symbol, expressed in polar and cylindrical coordinates is the function  $\mathcal{O}(p, \tau; \varphi, s)$ . Then*

$$(4.4) \quad \langle \mu, \nu | O | \mu, \nu \rangle = \int_{\mathbb{T}^n \times S^1} \mathcal{O}(\mu \hbar, \nu \hbar; \varphi, s) d\varphi ds + O(\hbar).$$

Let  $O_{mnds}$  be the pseudodifferential operator on  $L^2(\mathbb{R}^n \times S^1)$  whose Weyl symbol is the function  $\mathcal{O}_{mnds}$ . In order to prove theorem 4.1, it is enough to see that one can recover from the Taylor expansion of the averages  $O_{mnds}^0$  up to order  $N$  the principal symbol of  $\widetilde{W}_{\leq N}$  up to order  $N$ . We will proceed by induction on  $N$  just as in the proof of proposition 2.15.

Let us first remark that the principal symbols of  $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O_{mnds} e^{-\frac{i\widetilde{W}_{\leq N}}{\hbar}}$ , and  $\mathcal{O}_{mnds} \circ \Phi$  have the same Taylor expansion up to order  $N$ .

Hence, using Lemma 4.2 we get:

$$\begin{aligned} \mathcal{O}_{mnds}^0(\mu \hbar, \nu \hbar) &= \langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N+1}}{\hbar}} O_{mnds} e^{-\frac{i\widetilde{W}_{\leq N+1}}{\hbar}} | \mu, \nu \rangle + O\left(\left(|\mu| + |\nu|\right)\hbar^{N/2+1}\right) + O(\hbar) \\ &= \frac{i}{\hbar} \langle \mu, \nu | [\widetilde{W}_{\leq N+1}, O_{mnds}] | \mu, \nu \rangle + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \langle \mu, \nu | \overbrace{[\widetilde{W}_{\leq N+1}, \dots, \widetilde{W}_{\leq N+1}, O_{mnds}]}^{l \text{ times}} | \mu, \nu \rangle \\ &\quad + O\left(\left(|\mu| + |\nu|\right)\hbar^{N/2+1}\right) + O(\hbar) \\ &= \sum_{\substack{|j|+|k|+2q=N+1 \\ j+m=k+n}} \alpha_{0,j,k,q,d} i \hbar^{-1} \langle \mu, \nu | [e^{-i2\pi dt} a^j (a^*)^k D_t^q, O_{mnds}] | \mu, \nu \rangle \\ &\quad + \frac{i}{\hbar} \langle \mu, \nu | [W_{\leq N}, O_{mnds}] | \mu, \nu \rangle + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \langle \mu, \nu | \overbrace{[W_{\leq N}, \dots, W_{\leq N}, O_{mnds}]}^{l \text{ times}} | \mu, \nu \rangle \\ &\quad + O\left(\left(|\mu| + |\nu|\right)\hbar^{N/2+1}\right) + O(\hbar) \end{aligned}$$

We now remark that the Taylor expansion of the principal symbol  $\sigma_N(z, \bar{z}, t, \tau)$  of  $\widetilde{W}_{\leq N}$  up to order  $N$  is exactly  $\sigma_N(z, \bar{z}, t, \tau) = \sum_{|j|+|k|+2q \leq N} \alpha_{0,j,k,q}(t) z^j \bar{z}^k \tau^q$  up to  $(|z|^2 + |\tau|)^{\frac{N+1}{2}}$ .

We therefore have the

**Lemma 4.3.**

$$(4.5) \quad \frac{i}{\hbar} \langle \mu, \nu | [W_{\leq N}, O_{mnds}] | \mu, \nu \rangle + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \langle \mu, \nu | \overbrace{[W_{\leq N}, \dots, W_{\leq N}, O_{mnds}]}^{l \text{ times}} | \mu, \nu \rangle$$

depends only on  $\alpha_{0,j,k,q}(t)$ ,  $|j| + |k| + 2q \leq N$ , up to  $O\left(\left(|\mu| + |\nu|\right)\hbar^{N/2+1}\right) + O(\hbar)$ .

For  $N = 2$  we have that  $W_{\leq N} = 0$ . Therefore

$$\begin{aligned} \mathcal{O}_{mnds}^0(\mu \hbar, \nu \hbar) &= \sum_{\substack{|j|+|k|+2q=3 \\ j+m=k+n}} \alpha_{0,j,k,q,d} i \hbar^{-1} \langle \mu, \nu | [e^{-i2\pi dt} a^j (a^*)^k D_t^q, O_{mnds}] | \mu, \nu \rangle \\ &\quad + O\left(\left(|\mu| + |\nu|\right)\hbar^2\right) + O(\hbar). \end{aligned}$$

So the same argument that the one in the proof of proposition 2.15, in particular using lemma 2.17, allows to conclude the case  $N = 2$ .

Moreover lemma 4.3 shows clearly that we can conclude by induction again just like in proposition 2.15. □

Let us remark to finish this section that theorem 4.1, though probably provable by strictly classical methods, was naturally derived and proved out of quantum considerations.

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