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Abstract

By combining variational convergence with ergodic theory of subadditive processes, we study the macroscopic behavior of a randomly fibered medium. The cross sections of the fibers are randomly distributed according to a stationary point process, their size is of order ε while the stiffness of the material in the matrix is of order ε^p. The variational limit functional energy obtained when ε tends to 0 is deterministic and non local.

Résumé

En combinant convergence variationnelle et théorie ergodique des processus sous-additifs, nous étudions le comportement macroscopique d'un milieu aléatoirement fibré. Les sections des fibres sont réparties aléatoirement selon un processus ponctuel stationnaire, leur taille est de l'ordre de ε alors que la rigidité du matériau dans la matrice est d'ordre ε^p. La fonctionnelle énergie limite obtenue lorsque ε tend vers 0 est déterministe et non locale.

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*I3M and MIPA, UMR-CNRS 5149, Université Montpellier II and Université de Nîmes, Case courier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, micha@math.univ-montp2.fr

†LMGC, UMR-CNRS 5508, Université Montpellier II, Case courier 048, Place Eugène Bataillon, 34095 Montpellier Cedex 5, azdine.nait-ali@univ-montp2.fr, stephane.pagano@lmgc.univ-montp2.fr

1 Introduction

We are interested in the determination of the macroscopic behavior of a randomly fibered mechanical structure whose reference configuration is the open subset $\mathcal{O} := \widehat{\mathcal{O}} \times (0,h)$ of $\mathbb{R}^3$, with basis $\widehat{\mathcal{O}} := (0,l_1)\times(0,l_2) \subset \mathbb{R}^2$. More precisely for $\varepsilon = \frac{1}{n}$ we consider the union of fibers $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$ where $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$ and $D(\omega_i)$ are disks distributed at random in $\mathbb{R}^2$ following a stochastic point process $\omega = (\omega_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^2$ associated with a suitable probability space $(\Omega, \mathcal{A}, \mathbf{P})$. The random fibered structure is then given by $\mathcal{O} = (\mathcal{O} \setminus T_\varepsilon(\omega)) \cup \mathcal{O} \cap T_\varepsilon(\omega)$ (see Figure 1 and Figures 3, 4 in Section 2), and

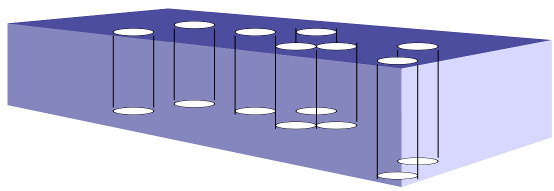

Figure 1: The random fibered structure

we aim to supply a deterministic equivalent variational limit when ε tends to zero, of the sequence of random integral functionals H_ε mapping $\Omega \times L^p(\mathcal{O},\mathbb{R}^3)$ into $\mathbb{R}^+ \cup \{+\infty\}$, defined for every ω in $(\Omega,\mathcal{A},\mathbf{P})$ by

$$H_\varepsilon(\omega,u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon(\omega)} f(\nabla u)\,dx + \int_{\mathcal{O}\cap T_\varepsilon(\omega)} g(\nabla u)\,dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

The space $W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3)$ is made up of the functions u in $W^{1,p}(\mathcal{O},\mathbb{R}^3)$ such that $u=0$ on $\Gamma_0 := \widehat{\mathcal{O}} \times \{0\}$ in the trace sense. For more precision on the stochastic point process $(\omega_i)_{i\in\mathbb{N}}$ and for all question of measurability relating to the considered random maps we refer the reader to the next section. For short we sometimes write T_ε instead of $T_\varepsilon(\omega)$.

We assume that f and g are two quasiconvex functions defined on the set $\mathbf{M}^{3\times3}$ of 3×3-matrices and satisfy the standard growth condition of order $p>1$: there exist two positive constants α, β, such that $\forall M, M' \in \mathbf{M}^{3\times3}$

$$\alpha|M|^p \leq f(M) \leq \beta(1+|M|^p), \tag{1}$$

idem for g. Note that f satisfies automatically the Lipschitz property

$$|f(M) - f(M')| \leq L|M-M'|(1+|M|^{p-1}+|M|^{p-1}) \tag{2}$$

for some positive constant L, idem for g. Furthermore, we assume that there exists $\beta'>0$, $0<\gamma<p$ and a p-positively homogeneous function $f^{\infty,p}$ (the p-recession function of f) such that for all $M \in \mathbf{M}^{3\times3}$

$$|f(M) - f^{\infty,p}(M)| \leq \beta'(1+|M|^{p-\gamma}). \tag{3}$$

From (3) we infer $\displaystyle\lim_{t\to+\infty} \frac{f(tM)}{t^p} = f^{\infty,p}(M)$ so that from (1), $f^{\infty,p}$ satisfies for all $M \in \mathbf{M}^{3\times3}$

$$\alpha|M|^p \leq f^{\infty,p}(M) \leq \beta|M|^p. \tag{4}$$

and

$$|f^{\infty,p}(M) - f^{\infty,p}(M')| \leq L|M-M'|(|M|^{p-1}+|M|^{p-1}) \tag{5}$$

for all $(M,M') \in \mathbf{M}^{3\times3} \times \mathbf{M}^{3\times3}$.

As a consequence of the variational convergences we will provide an equivalent deterministic problem of

$$(\mathcal{P}_{H_\varepsilon}) \qquad \inf\left\{H_\varepsilon(\omega,u)-\int_{\mathcal{O}}\mathcal{L}.u\ dx : u\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

where $\mathcal{L}\in L^q(\mathcal{O},\mathbb{R}^3)$, $q=\frac{p}{p-1}$.

The functional H_ε models the internal energy of a mechanical structure made up of the union T_ε of thin parallel cylinders which represent the rigid fibers and a soft elastic material matrix occupying $\mathcal{O}\setminus T_\varepsilon$. We only have a statistical knowledge of the cross sections of the fibers in the sense that their positions are statistically homogeneous. From the mathematical point of view, this means that they are placed at random according to a stationary point process. The stiffness of the elastic material occupying $\mathcal{O}\setminus T_\varepsilon$ is of order ε^p. The functions u represent the displacements of the mechanical structure subjected to a given load $\mathcal{L}$ and clamped on the plane $\Gamma_0=[x_3=0]$. We assume large deformations in the matrix and the fibers so that the strong and soft materials are hyperelastic. Our objective is to analyze the behavior of $(\mathcal{P}_{H_\varepsilon})$ in a variational way when ε tends to 0 while the fillling ratio of the fibers is kept constant and, consequently, to provide a simplified but accurate model for the behavior of the slices of the geomaterial TexSol^{TM} ([12, 14, 15]). It is a soil reinforcement process created in 1984 by Leflaive, Khay and Blivet from the LCPC (Laboratoire Central des Ponts et Chaussées) which mixes the soil (sand) with a wire. The obtained reinforced material has a better mechanical resistance than the sand without wire. The wire is randomly distributed on the free surface and is covered with sand simultaneously to create a TexSol^{TM} layer. In our simplified model we assume the wire to cut the surface perpendicularly (the size h is small) so that the thin parallel cylinders, randomly distributed, represent the pieces of the wire which are perfectly stuck with a hyperelastic matrix which represent the sand (cf. Figure 2).

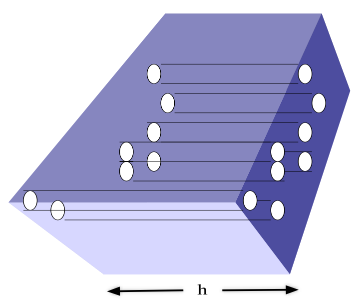

Figure 2: A slice of real material

From the mathematical point of view we reexamine the work of [5, 6, 17] in a stochastic setting and in the scope of nonlinear elasticity. We establish the almost sure convergence of $(\mathcal{P}_{H_\varepsilon})$ when $\varepsilon\to 0$ to the deterministic and homogeneous problem

$$(\mathcal{P}_H) \qquad \min\left\{H(u)-\int_{\mathcal{O}}\mathcal{L}.u\ dx : v\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

where the energy functional H is of non local nature. More precisely we establish the almost sure Γ-convergence of the sequence $(H_\varepsilon)_{\varepsilon>0}$ to the infimum convolution $F_0\bigtriangledown G_0$ defined for every $u\in L^p(\mathcal{O},\mathbb{R}^3)$ by

$$F_0\bigtriangledown G_0\ (u):=\inf_{v\in L^p(\mathcal{O},\mathbb{R}^3)}\Big(F_0(u-v)+G_0(v)\Big)$$

(Theorem 5.1 and Corollary 5.1) where F_0 and G_0 are the functionals energy Γ-limits of the functionals

$u \mapsto \varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u)\ dx$ and $u \mapsto \int_{\mathcal{O}\cap T_\varepsilon} g(\nabla u)\ dx$ respectively, which are defined in $L^p(\mathcal{O},\mathbb{R}^3)$ by

$$F_0(u) = \int_{\mathcal{O}} f_0^{**}(u)\ dx,$$

$$G_0(u) = \begin{cases} \theta \int_{\mathcal{O}} (g^\perp)^{**}(\frac{\partial u}{\partial x_3})dx \text{ if } \ u \in V_0 \\ +\infty \text{ otherwise,} \end{cases}$$

where $V_0 := \left\{ u \in L^p(\mathcal{O},\mathbb{R}^3) : \frac{\partial u}{\partial x_3} \in L^p(\mathcal{O},\mathbb{R}^3),\ u(\hat{x},0) = 0 \text{ on } \widehat{\mathcal{O}} \right\}$. The densities f_0 and $g^\perp$ are defined by

$$\begin{aligned} f_0(a) &= \inf_{n\in\mathbb{N}^*} \left\{ \int_\Omega \frac{\widehat{\mathcal{S}}_{[0,n[^2}(\omega,a)}{n^2} d\mathbf{P}(\omega) \right\},\ a \in \mathbb{R}^3, \\ g^\perp(a) &:= \inf_{\xi\in\mathbf{M}^{3\times 2}} g\left(\xi|a\right),\ \mathbf{M}^{3\times 2} \text{ is the set of } 3\times 2 \text{ matrices,} \end{aligned}$$

where $\hat{A} \mapsto \widehat{\mathcal{S}}_{\hat{A}}$ is a suitable discrete subadditive process on subsets of $\mathbb{R}^2$, and $\theta \in (0,1)$ is the asymptotic volume fraction $\int_\Omega |\hat{Y} \cap D(\omega)|\ d\mathbf{P}(\omega)$, $\hat{Y} = (0,1)^2$ of the fibers. In our probabilistic model the random set $D(\omega)$ is statistically not too sparse so that $\theta > 0$ (Remark 2.1). In the deterministic case, i.e., when the fibers are periodically distributed, θ reduces to $|\hat{Y} \cap D|$, and the density f_0^{**} to

$$f_0^{**}(a) = \inf\left\{ \int_{\hat{Y}} (f^{\infty,p})^{**}(\hat{\nabla} w, 0)\ d\hat{y} : w \in W^{1,p}_{\#}(\hat{Y},\mathbb{R}^3),\ \int_{\hat{Y}} w\ d\hat{y} = a,\ w = 0 \text{ in } D \right\}$$

where $W^{1,p}_{\#}(\hat{Y},\mathbb{R}^3)$ denotes the subset of $W^{1,p}(\hat{Y},\mathbb{R}^3)$ made up of $\hat{Y}$-periodic functions (Corollary 2.1).

2 The probabilistic framework

No difference is made between $\mathbb{R}^3$ and the three dimensional euclidean physical space equipped with an orthogonal basis denoted by (e_1,e_2,e_3). For all $x = (x_1,x_2,x_3)$ of $\mathbb{R}^3$, $\hat{x}$ stands for (x_1,x_2) and $\mathbf{M}^{3\times 3}$, $\mathbf{M}^{3\times 2}$ denotes the sets of 3×3 and 3×2 matrices. We denote by $\hat{Y}$ the unit cell $(0,1)^2$ of $\mathbb{R}^2$ and by Y the unit cell $(0,1)^3$ of $\mathbb{R}^3$.

For any $\delta > 0$ and any non empty bounded set $\hat{A}$ of $\mathbb{R}^2$, we make use of the following notation: $\hat{A}_\delta := \left\{ x \in \hat{A} : d(x,\mathbb{R}^2 \setminus \hat{A}) > \delta \right\}$. For any bounded Borel set A of $\mathbb{R}^2$ or $\mathbb{R}^3$, $|A|$ denotes its Lebesgue measure and $\#(A)$ its cardinal when it is finite.

Let d be a given number satisfying $0 < d \leq 1$ and consider the set

$$\Omega = \left\{ (\omega_i)_{i\in\mathbb{N}} : \omega_i \in \mathbb{R}^2,\ |\omega_i - \omega_j| \geq d \text{ for } i \neq j \right\}$$

equipped with the trace σ-algebra $\mathcal{A}$ of the standard product σ-algebra on Ω. Let $\hat{B}_{d/2}(0)$ be the open ball of $\mathbb{R}^2$ centered at 0 with radius $d/2$, then for every $\omega = (\omega_i)_{(i)\in\mathbb{N}}$ we form the disk $D(\omega_i) := \omega_i + \hat{B}_{d/2}(0)$ and consider $D(\omega) := \bigcup_{i\in\mathbb{N}} D(\omega_i)$. Therefore $\omega \mapsto T(\omega) = D(\omega) \times \mathbb{R}$ is a random set in $\mathbb{R}^3$, union of random cylinders, whose basis is the union of the pairwise disjoint disks $D(\omega_i)$ of $\mathbb{R}^2$ centered at ω_i. We set $T_\varepsilon(\omega) := \varepsilon D(\omega) \times \mathbb{R}$.

For every $z \in \mathbf{Z}^2$ we define the operator $\tau_z : \Omega \to \Omega$ by $\tau_z\omega = \omega - z$. Note that $D(\tau_z\omega) = D(\omega) - z$. Furthermore we assume that there exists a probability measure on $(\Omega,\mathcal{A})$ which satisfies the system of three following axioms:

(A_1) *Non sparsely distribution*: $\mathbf{P}\Big(\left\{ \omega \in \Omega : |\hat{Y} \cap D(\omega)| > 0 \right\} \Big) = 1$;

(A_2) *Stationary condition*: $\forall z \in \mathbf{Z}^2$, $\tau_z\#\mathbf{P} = \mathbf{P}$ where $\tau_z\#\mathbf{P}$ denotes the probability image of $\mathbf{P}$ by τ_z;

(A_3) *Asymptotic mixing property*: for all sets E and F of $\mathcal{A}$, $\lim_{|z|\to+\infty} \mathbf{P}(\tau_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$.

Remark 2.1. *i) It would be more natural to consider stationary condition (A_2) with respect to the continuous group $(\tau_t)_{t\in\mathbb{R}^2}$ defined in the same way by $\tau_t\omega = \omega - t$. Actually the discrete group $(\tau_z)_{z\in\mathbf{Z}^2}$ suffices for the mathematical analysis. The size of the cell $\hat{Y}$ is chosen in such a way to fix the generator of the group $(\tau_z)_{z\in\mathbf{Z}^2}$. Condition (A_2) then says that every random function X taking its source in Ω is statistically homogeneous in the sense that X and $X\circ\tau_z$ have the same law (i.e. $X\#\mathbf{P} = X\circ\tau_z\#\mathbf{P}$). Roughly speaking, moving a window $\hat{A}$ in $\mathbb{R}^2$ following the translations in $\mathbb{R}^2$, the distributions of cross sections in the window are statistically the same.*

ii) Condition (A_1) together with condition (A_2) yield that the random set $D(\omega)$ is statistically not too sparse in $\mathbb{R}^2$. Indeed for every $\mathbf{Z}^2$-translated $\hat{A} = \hat{Y} + z$ of $\hat{Y}$

$$\begin{aligned}\mathbf{P}\Big(\Big\{\omega : |\hat{A}\cap D(\omega)| > 0\Big\}\Big) &= \mathbf{P}\Big(\Big\{\omega : |\hat{Y}\cap (D(\omega)-z)| > 0\Big\}\Big)\\ &= \mathbf{P}\Big(\Big\{\omega : |\hat{Y}\cap (D(\tau_z\omega))| > 0\Big\}\Big)\\ &= \mathbf{P}\Big(\Big\{\omega : |\hat{Y}\cap (D(\omega))| > 0\Big\}\Big) = 1.\end{aligned}$$

Note that from (A_1), the asymptotic volume fraction satisfies $\int_\Omega |\hat{Y}\cap D(\omega)|\, d\mathbf{P}(\omega) > 0$.

iii) Condition (A_3) says that the events $\tau_z E$ and F are independent provided that z be large enough.

iv) Consider $\bar{\omega} = (\bar{\omega}_i)_{i\in\mathbb{N}}$ where $\bar{\omega}_i$ are the centers of the hexagonal close-packing of disks in $\mathbb{R}^2$. Then $\bar{\omega}$ is a "maximal" distribution in the sense that $|\hat{Y}\cap D(\omega)| \le |\hat{Y}\cap D(\bar{\omega})|$ for a.s. ω in Ω.

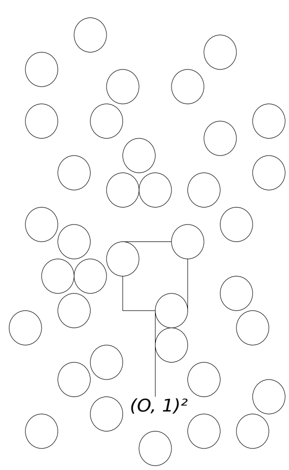

Figure 3: Random cross sections at scale $\varepsilon = 1$

A simple specimen of probability space which fulfills all the conditions above is the generalized random chessboard described below.

Example 2.1 (Random chessboard-like)**.** Given $0 < d < 1$, let us consider a countable set of points $\Omega_0 = \{x_k : k \in \mathbb{N}\}$ in $\hat{Y}_{d/2}$ and set $\Omega := \Pi_{z \in \mathbf{Z}^2} \Omega_z$ where $\Omega_z = \Omega_0 + z$ for all $z \in \mathbf{Z}^2$. We equip Ω with the σ-algebra $\mathcal{A}$ generated by the cylinders of Ω. For a given family $(\alpha_k)_{k \in \mathbb{N}}$ of non negative numbers satisfying $\sum_{k \in \mathbb{N}} \alpha_k = 1$ we consider the probability measure $\mu_0 = \sum_{k \in \mathbb{N}} \alpha_k \delta_{x_k}$ on Ω_0 and the product probability measure $\mathbf{P} = \Pi_{z \in \mathbf{Z}} \mu_z$ on $(\Omega, \mathcal{A})$ where $\mu_z = \mu_0$ for all $z \in \mathbf{Z}$. Then it is easy to check that $\mathbf{P}$ satisfies axioms (A_1)-(A_3).

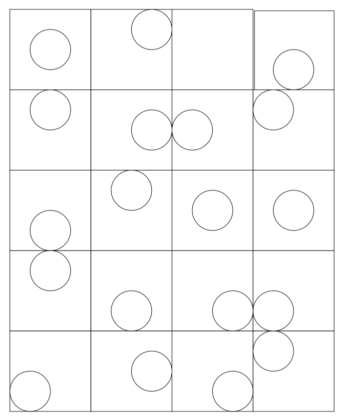

Figure 4: A piece of a random chessboard of cross sections at scale $\varepsilon = 1$ with $\#(\Omega_0) = 9$

Remark 2.2. *All the results of the paper remain valid if we substitute for the disk $\hat{B}_{d/2}(0)$, any connex compact set of $\mathbb{R}^2$ included in $\hat{B}_{d/2}(0)$ and chosen at random.*

Let us recall the following general basic notion of discrete subadditive process. We consider a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and a group $(\tau_z)_{z \in \mathbf{Z}^N}$ of $\mathbf{P}$-preserving transformations on $(\Omega, \mathcal{A})$. The group $(\tau_z)_{z \in \mathbf{Z}^N}$ is said to be ergodic if every set E in $\mathcal{A}$, such that $\tau_z E = E$ for every $z \in \mathbf{Z}^N$, satisfies $\mathbf{P}(E) = 0$ or $\mathbf{P}(E) = 1$. A sufficient condition to ensure ergodicity of $(\tau_z)_{z \in \mathbf{Z}^N}$ is the mixing condition (A_3): for every E and F in $\mathcal{A}$

$$\lim_{|z| \to +\infty} \mathbf{P}(\tau_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$$

which expresses an asymptotic independence.

Let $\mathcal{I}$ denote the set of half open intervals $[a, b)$ of the lattice spanned by $(0, 1)^N$. A discrete subadditive process with respect to $(\tau_z)_{z \in \mathbf{Z}^N}$ is a set function $\mathcal{S} : \mathcal{I} \longrightarrow L^1(\Omega, \mathcal{A}, \mathbf{P})$ satisfying

(i) for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j \in J}$ of disjoint intervals in $\mathcal{I}$ with $I = \bigcup_{j \in J} I_j$,

$$\mathcal{S}_I(\cdot) \leq \sum_{j \in J} \mathcal{S}_{I_j}(\cdot),$$

(ii) $\forall I \in \mathcal{I},\ \forall z \in \mathbf{Z}^N,\ \mathcal{S}_I \circ \tau_z = \mathcal{S}_{z+I}$

A family $(I_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{I}$ is called regular if there exists another family $(I'_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{I}$ such that

(i) $I_n \subset I'_n$ for all $n \in \mathbb{N}$;

(ii) (I'_n) is non decreasing;

(iii) there exists a constant $C > 0$ such that $0 < |I'_n| \le C|I_n|$ for all $n \in \mathbb{N}$,

(iv) $\mathbb{R}^N_+ = \bigcup I'_n$.

The following subadditive ergodic theorem is due to Ackoglu-Krengel.

Theorem 2.1. *Let $\mathcal{S}$ be a discrete subadditive process with respect to an ergodic group $(\tau_z)_{z\in\mathbf{Z}^N}$ satisfying*

$$\inf\left\{\int_\Omega \frac{\mathcal{S}_I(\omega)}{|I|}\,\mathbf{P}(d\omega) I \in \mathcal{I},\ |I| \neq 0\right\} > -\infty$$

and let $(I_n)_{n\in\mathbb{N}}$ be a regular family of sets in $\mathcal{I}$. Then almost surely

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathcal{S}_{I_n}}{|I_n|} &= \lim_{n\to\infty} \frac{\mathcal{S}_{[0,n[^N}}}{n^N} \\ &= \inf_{n\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\mathcal{S}_{[0,n[^N}}}{n^N}\right\} = \lim_{n\to\infty} \mathbf{E}\frac{\mathcal{S}_{[0,n[^N}}}{n^N}\end{aligned}$$

where $\mathbf{E}$ denotes the expectation operator.

For a proof see [1] and, for some extensions, see [13, 16].

We are going to define the limit density energy associated with the random integral functional $u \mapsto \int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u)\,dx$ by applying Theorem 2.1 with $N = 2$ to a suitable set function $\widehat{\mathcal{S}}$ on subsets of $\mathcal{I}$, which ranges over the space $L^1(\Omega, \mathcal{A}, \mathbf{P})$ governed by axioms (A_1)-(A_3). More precisely, for all $\hat{A} \in \mathcal{I}$ and all $a \in \mathbb{R}^3$ set

$$\begin{aligned}\widehat{\mathcal{S}}_{\hat{A}}(\omega, a) &:= \inf\left\{\int_{\overset{\circ}{\hat{A}}\times(0,1)} f^{\infty,p}(\nabla w)\,dx : w \in \mathrm{Adm}_A(\omega, a)\right\}, \\ \mathrm{Adm}_{\hat{A}}(\omega, a) &:= \left\{w \in W^{1,p}_0\big(\overset{\circ}{\hat{A}}\times(0,1)\setminus T(\omega), \mathbb{R}^3\big) : \fint_{\overset{\circ}{\hat{A}}\times(0,1)} w\,dx = a\right\},\end{aligned}$$

where we still denote by w the extension by zero on $T(\omega) \cap \big(\overset{\circ}{\hat{A}}\times(0,1)\big)$ of every function w in $\mathrm{Adm}_{\hat{A}}(\omega, a)$. Since the Lebesgue measure does not charge the boundary of the elements of $\mathcal{I}$, one can take as $\mathcal{I}$ the set of all open intervals (a, b) of the lattice spanned by $\hat{Y}$ that we still denote by $\mathcal{I}$. Subsequently the subadditivity condition (i) becomes: for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j\in J}$ of disjoint intervals in $\mathcal{I}$ with $|I \setminus \bigcup_{j\in J} I_j| = 0$,

$$\widehat{\mathcal{S}}_I(\cdot) \le \sum_{j\in J} \widehat{\mathcal{S}}_{I_j}(\cdot).$$

It is standard to see that the random functionals defined in the introduction are measurable when $\Omega \times L^p(\mathcal{O}, \mathbb{R}^3)$ is equipped with the product σ-algebra $\mathcal{A}\otimes\mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra associated with the normed space $L^p(\mathcal{O}, \mathbb{R}^3)$. Consequently, for all fixed $\hat{A}$ in $\mathcal{I}$ and all fixed a in $\mathbb{R}^3$, the map $\omega \mapsto \widehat{\mathcal{S}}_{\hat{A}}(\omega, a)$ is measurable. Actually we have

Theorem 2.2. *For all fixed $a \in \mathbb{R}^3$, the map*

$$\begin{aligned}\widehat{\mathcal{S}}(., a) : \qquad & \mathcal{I} \longrightarrow L^1(\Omega, \mathcal{A}, \mathbf{P}) \\ & \hat{A} \longmapsto \widehat{\mathcal{S}}_{\hat{A}}(., a)\end{aligned}$$

is a subadditive process with respect to the group $(\tau_z)_{z\in\mathbf{Z}^2}$ defined by $\tau_z(\omega)=\omega-z$. It satisfies for all $a\in\mathbb{R}^3$, all $\hat{A}\in\mathcal{I}$ and all $\delta>0$ small enough

$$\widehat{\mathcal{S}}_{\hat{A}}(\omega,a)\leq\frac{C(p)}{\delta^p\left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|}|a|^p|\hat{A}| \tag{6}$$

where $C(p)$ is a non negative constant depending only of p.

Therefore for any regular family $(I_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$, the limit $\lim_{n\to\infty}\frac{\widehat{\mathcal{S}}_{I_n}(\omega,a)}{|I_n|}$ exists for $\mathbf{P}$ almost every $\omega\in\Omega$ and

$$\lim_{n\to\infty}\frac{\widehat{\mathcal{S}}_{I_n}(a,\omega)}{|I_n|}=\lim_{n\to\infty}\frac{\widehat{\mathcal{S}}_{[0,n[^2}(.,a)}{n^2}=\inf_{m\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\widehat{\mathcal{S}}_{[0,m[^2}(.,a)}{m^2}\right\}.$$

We denote by f_0 the common value above.

Proof. We establish that $\mathrm{Adm}_{\hat{A}}(\omega,a)$ is non empty and that $\widehat{\mathcal{S}}_{\hat{A}}\in L^1(\Omega,\mathcal{A},\mathbf{P})$ by establishing (6). The rest of the proof consists in checking each condition (i) and (ii) and is straightforward. Fix $\hat{A}\in\mathcal{I}$. For $0<\delta$ small enough consider $\phi_\delta=\rho_\delta*\mathbb{1}_{(\hat{A}\setminus D(\omega))_\delta}$ where ρ_δ is a standard mollifier. Clearly

$$\phi_\delta(\hat{x})=\begin{cases}1 \text{ if } \hat{x}\in(\hat{A}\setminus D(\omega))_{2\delta},\\ 0 \text{ if } \hat{x}\in\mathbb{R}^2\setminus(\hat{A}\setminus D(\omega)).\end{cases}$$

Therefore

$$\fint_{\hat{A}}\phi_\delta\ d\hat{x}\geq\frac{\left|(\hat{A}\setminus D(\omega))_{2\delta}\right|}{|\hat{A}|}.$$

Take $\bar{\omega}$ the close-packing distribution in $\mathbb{R}^2$ (Remark 2.1). According to (A_2), (A_3) we infer

$$\begin{aligned}\fint_{\hat{A}}\phi_\delta\ d\hat{x}&\geq\frac{\left|\sum_{z\in\hat{A}\cap\mathbf{Z}^2}(\hat{Y}+z\setminus D(\omega))_{2\delta}\right|}{|\hat{A}|}\\&=\frac{\left|\sum_{z\in\hat{A}\cap\mathbf{Z}^2}(\hat{Y}\setminus D(\tau_z\omega))_{2\delta}\right|}{|\hat{A}|}\\&\geq\frac{\#(\hat{A})}{|\hat{A}|}\left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|=\left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|.\end{aligned} \tag{7}$$

Take now $\theta\in\mathcal{C}_0^1(0,1)$ satisfying $\int_0^1\theta(t)\ dt=1$. The random function defined by $w_\delta(\hat{x},x_3)=a\frac{\phi_\delta(\hat{x})\theta(x_3)}{\fint_A\phi_\delta\ d\hat{x}}$ clearly belongs to $\mathrm{Adm}_{\hat{A}}(\omega,a)$ (for short we do not indicate the dependance on ω). Moreover from (7) and the growth condition satisfied by $f^{\infty,p}$,

$$\begin{aligned}\widehat{\mathcal{S}}_{\hat{A}}(\omega,a)&\leq\int_{\hat{A}\times(0,1)\setminus T(\omega)}f^{\infty,p}(\nabla w_\delta)\ d\hat{x}\\&\leq\frac{C(p)}{\delta^p\left|(\hat{Y}\setminus D(\bar{\omega}))_{2\delta}\right|}|a|^p||\hat{A}|.\end{aligned}$$

where $C(p)$ is a non negative constant which depends only on p. □

We define the elastic density associated with the limit internal energy of the material occupying $\mathcal{O}\setminus T_\varepsilon(\omega)$ by:

$$\begin{aligned}\forall a\in\mathbb{R}^3,\ f_0^{**}(a)&=\left[\lim_{n\to\infty}\left\{\frac{\widehat{\mathcal{S}}_{[0,n[^2}(\omega,a)}{n^2}\right\}\right]^{**}(a)\ \omega\ a.s.\\&=\left[\inf_{m\in\mathbb{N}^*}\left\{\mathbf{E}\frac{\widehat{\mathcal{S}}_{[0,m[^2}(.,a)}{m^2}\right\}\right]^{**}(a)\end{aligned}$$

where, for any function $h : \mathbb{R}^3 \to \mathbb{R}$, h^{**} stands for its convexification, i.e., the greatest convex function less than h.

In order to provide some flexibility in the proofs of Section 3, it is convenient to introduce a new subadditive process $A \mapsto \mathcal{S}_A$ where now A runs over half open cubes of $\mathbb{R}^3$, converging toward the same limit $f_0(a)$. Precisely, let us still denote by $\mathcal{I}$ the set of all open intervals (a, b) of the lattice spanned by Y, we apply Theorem 2.1 with $N = 3$ to the set function defined for all $A \in \mathcal{I}$ and all $a \in \mathbb{R}^3$ by

$$\begin{aligned}
\mathcal{S}_A(\omega, a) &:= \inf \left\{ \int_{\overset{\circ}{A}} f^{\infty,p}(\nabla w)\, dx : w \in \mathrm{Adm}_A(\omega, a) \right\}, \\
\mathrm{Adm}_A(\omega, a) &:= \left\{ w \in W_0^{1,p}(\overset{\circ}{A} \setminus \overline{T(\omega)}, \mathbb{R}^3) : \fint_{\overset{\circ}{A}} w\, dx = a \right\}.
\end{aligned}$$

Theorem 2.3. *For all fixed $a \in \mathbb{R}^3$, the map*

$$\begin{aligned}
\mathcal{S}(.,a) : \qquad \mathcal{I} &\longrightarrow L^1(\Omega, \mathcal{A}, \mathbf{P}) \\
A &\longmapsto \mathcal{S}_A(.,a)
\end{aligned}$$

is a subadditive process with respect to the group $(\tau_z)_{z \in \mathbf{Z}^3}$ defined by $\tau_z(\omega) = \omega - \hat{z}$ where $z = (\hat{z}, z_3)$. Therefore for any regular family $(I_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ the limit $\lim_{n\to\infty} \dfrac{\mathcal{S}_{I_n}(\omega, a)}{|I_n|}$ exists for $\mathbf{P}$ almost every $\omega \in \Omega$ and $\lim_{n\to\infty} \dfrac{\mathcal{S}_{I_n}(a, \omega)}{|I_n|} = f_0(a)$.

Proof. By repeating the proof of Theorem 2.2 with minor changes, we establish in the same way the existence of the limit $\lim_{n\to\infty} \dfrac{\mathcal{S}_{I_n}(\omega, a)}{|I_n|}$. Take now $I_n = (0, n^2)^2 \times (0, n)$. Clearly $(I_n)_{n\in\mathbb{N}}$ is a regular family in $\mathcal{I}$ ($I'_n = I_n$ is suitable), and a change of scale yields

$$\begin{aligned}
\frac{\mathcal{S}_{I_n}(\omega, a)}{|I_n|} &= \frac{\mathcal{S}_{(0,n^2)^2\times(0,n)}(\omega, a)}{n^5} \\
&= \frac{\mathcal{S}_{(0,n)^2\times(0,1)}(\omega, a)}{n^2} \\
&= \frac{\widehat{\mathcal{S}}_{(0,n)^2}(\omega, a)}{n^2},
\end{aligned}$$

so that for a.s. ω in Ω, $\lim_{n\to\infty} \dfrac{\mathcal{S}_{I_n}(\omega, a)}{|I_n|} = f_0(a)$. □

Corollary 2.1. *Assume that the fibers are periodically distributed, i.e., in the chessboard-like example above, Ω_0 and Ω_1 are reduced to a single point, then for all $a \in \mathbb{R}^3$,*

$$f_0(a) = \inf_{n\in\mathbb{N}^*} \frac{\widehat{\mathcal{S}}_{(0,n)^2}(a)}{n^2}$$

where

$$\begin{aligned}
\mathcal{S}_{\hat{A}}(a) &:= \inf \left\{ \int_{\overset{\circ}{\hat{A}}\times(0,1)} f^{\infty,p}(\nabla w)\, dx : w \in Adm_{\hat{A}}(a) \right\}, \\
Adm_{\hat{A}}(a) &:= \left\{ w \in W_0^{1,p}(\overset{\circ}{\hat{A}} \times (0,1) \setminus \overline{T}, \mathbb{R}^3) : \fint_{\overset{\circ}{\hat{A}}\times(0,1)} w\, dx = a \right\}.
\end{aligned}$$

Furthermore $f_0^{**}(a)$ reduces to

$$f_0^{**}(a) = \inf \left\{ \int_{\hat{Y}} (f^{\infty,p})^{**}(\hat{\nabla} w, 0)\, d\hat{y} : w \in W_{\#}^{1,p}(\hat{Y}, \mathbb{R}^3),\ \int_{\hat{Y}} w\, d\hat{y} = a,\ w = 0 \text{ in } D \right\}$$

where $W_{\#}^{1,p}(\hat{Y}, \mathbb{R}^3)$ denotes the subset of $W^{1,p}(\hat{Y}, \mathbb{R}^3)$ made up of $\hat{Y}$-periodic functions.

Proof. Clearly $f_0(a) = \inf_{n\in\mathbb{N}^*} \dfrac{\widehat{\mathcal{S}}_{(0,n)^2}(a)}{n^2}$. Thus for all $n \in \mathbb{N}^*$

$$f_0^{**}(a) \le \Big(\frac{\widehat{\mathcal{S}}_{(0,n)^2}(.)}{n^2}\Big)^{**}(a)$$

so that

$$f_0^{**}(a) \le \inf_{n\in\mathbb{N}^*} \Big(\frac{\widehat{\mathcal{S}}_{(0,n)^2}(.)}{n^2}\Big)^{**}(a).$$

Since the converse inequality is obviously satisfied, we conclude to

$$f_0^{**}(a) = \inf_{n\in\mathbb{N}^*} \Big(\frac{\widehat{\mathcal{S}}_{(0,n)^2}(.)}{n^2}\Big)^{**}(a). \tag{8}$$

But from standard arguments using Fenchel's Duality,

$$\Big(\frac{\widehat{\mathcal{S}}_{(0,n)^2}(.)}{n^2}\Big)^{**}(a) = \frac{1}{n^2}\inf\left\{\int_{n\hat{Y}\times(0,1)} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in \mathrm{Adm}_{n\hat{Y}}(a)\right\}. \tag{9}$$

Let $w_\#$ be a minimizer of

$$\inf\left\{\int_{Y\setminus T} (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T\right\},$$

extended by $\hat{Y}$-periodicity on $\mathbb{R}^2\times(0,1)$ and fix $n \in \mathbb{N}^*$. Clearly $\partial(f^{\infty,p})^{**}(\nabla w_\#(x))$ is non empty and for short, we assume that it is single valued. Note that $-\mathrm{div}\partial(f^{\infty,p})^{**}(\nabla w_\#) = 0$ a.e. in $n\hat{Y}\times(0,1)$ and $\partial(f^{\infty,p})^{**}(\nabla w_\#).\nu$ is anti-periodic, ν denoting the unit normal to the boundary of $n\hat{Y}\times(0,1)$. Take any $w \in \mathrm{Adm}_{n\hat{Y}}(a)$. According to the subdifferential inequality we have

$$\int_{nY} (f^{\infty,p})^{**}(\nabla w)\ dy \ge \int_{n\hat{Y}\times(0,1)} (f^{\infty,p})^{**}(\nabla w_\#)\ dy + \int_{n\hat{Y}\times(0,1)} \partial(f^{\infty,p})^{**}(\nabla w_\#).\nabla(w - w_\#)\ dy.$$

Integrating by parts, we infer

$$\int_{n\hat{Y}\times(0,1)} (f^{\infty,p})^{**}(\nabla w)\ dy \ge \int_{n\hat{Y}\times(0,1)} (f^{\infty,p})^{**}(\nabla w_\#)\ dy = n^2\int_Y (f^{\infty,p})^{**}(\nabla w_\#)\ dy$$

so that from (9)

$$\Big(\frac{\widehat{\mathcal{S}}_{(0,n)^2}(.)}{n^2}\Big)^{**}(a) \ge \inf\left\{\int_Y (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T\right\}.$$

Thus, from (8) and since the converse inequality clearly holds

$$f_0^{**}(a) = \inf\left\{\int_Y (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T\right\}.$$

The conclusion then follows by noticing that

$$\inf\left\{\int_Y (f^{\infty,p})^{**}(\nabla w)\ dy : w \in W^{1,p}_\#(Y,\mathbb{R}^3),\ \int_Y w\ dy = a,\ w = 0 \text{ in } T\right\}$$

is equal to

$$\inf\left\{\int_{\hat{Y}} (f^{\infty,p})^{**}(\hat{\nabla} w, 0)\ d\hat{y} : w \in W^{1,p}_\#(\hat{Y},\mathbb{R}^3),\ \int_{\hat{Y}} w\ d\hat{y} = a,\ w = 0 \text{ in } D\right\}$$

which is a straightforward consequence of Jensen's inequality. $\square$

The following proposition is a straightforward consequence of estimate (6).

Proposition 2.1. *The function f_0^{**} is a positively homogeneous convex function of degree p, satisfies the growth conditions (4) with the same constant α, with a constant β possibly different, and satisfies the Lipschitz condition (5) with a constant L possibly different.*

Proof. Clearly, f_0^{**} is positively homogeneous of degree p. The upper bound in (4) follows straightforwardly from (6), and (5) will be deduced by using standard argument of convex analysis provided that we establish: $f^{**}(a) \geq \alpha\,|a|^p$ for all $a \in \mathbb{R}^3$. The assertion follows by noticing that for every function w in $\mathrm{Adm}_{n\hat{Y}}(\omega, a)$ we have

$$\begin{aligned}
\alpha\,|a|^p &= \alpha \left| \fint_{n\hat{Y}\times(0,1)} w\ dx \right|^p \\
&\leq \alpha \fint_{n\hat{Y}\times(0,1)} |w|^p\ dx \\
&\leq \alpha \fint_{n\hat{Y}\times(0,1)} |\nabla w|^p\ dx \\
&\leq \fint_{n\hat{Y}\times(0,1)} f^{\infty,p}(\nabla w)\ dx
\end{aligned}$$

where we have used Poincaré inequality in the second inequality. □

We end this section by the following proposition which is a consequence of Theorem 2.1 when $\mathcal{S}$ is additive. It extends the Birkoff ergodic theorem.

Proposition 2.2. *Let $n \in \mathbb{N}^*$, and $\psi : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^2)$-measurable function satisfying the three following conditions:*

i) for $\mathbf{P}$-almost every $\omega \in \Omega$, $\hat{y} \mapsto \psi(\omega, \hat{y})$ belongs to $L^1_{loc}(\mathbb{R}^2)$;

ii) for all bounded Borel set $\hat{A}$ of $\mathbb{R}^2$ the map $\hat{A} \mapsto \int_{\hat{A}} \psi(\omega, \hat{y})\ d\hat{y}$ belongs to $L^1(\Omega, \mathcal{A}, \mathbf{P})$;

iii) for all $z \in n\mathbf{Z}^2$, for all $\hat{y} \in \mathbb{R}^2$, $\psi(\omega, \hat{y} + z) = \psi(\tau_z\omega, \hat{y})$ for $\mathbf{P}$-almost every $\omega \in \Omega$.

Then

$$\psi(\omega, \frac{\cdot}{\varepsilon}) \rightharpoonup x \mapsto \mathbf{E} \fint_{(0,n)^2} \psi(., \hat{y})\ d\hat{y}$$

for the $\sigma(L^1(\mathcal{O}), L^\infty(\mathcal{O}))$ topology.

Proof. See Theorem 4.2 and Proposition 5.3 in [8]. □

3 The limit problem associated with the soft material structure

This section is devoted to the asymptotic analysis of the functional $F_\varepsilon^v(\omega, .) : L^p(\mathcal{O}, \mathbb{R}^3) \to \mathbb{R}^+ \cup \{+\infty\}$

$$F_\varepsilon^v(\omega, u) = \begin{cases} \varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u)\ dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3),\ u = v \text{ on } \mathcal{O} \cap T_\varepsilon \\ +\infty \text{ otherwise,} \end{cases}$$

where v is a given function in $W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3)$. Before to establish the almost sure Γ-convergence of the functional F_ε^v we start by establishing a compactness result which explains why we equip $L^p(\mathcal{O}, \mathbb{R}^3)$ with its weak convergence. Note that the choice of the topology is crucial in the Γ-convergence process (see [2, 3, 9]). All along the paper we denote by $\to$ and $\rightharpoonup$ the strong and weak convergences in the various topological spaces, we do not relabel the subsequences and C will denote various nonnegative constants independent of ε and ω which may vary from line to line.

Lemma 3.1 (compactness). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence satisfying $\sup_{\varepsilon>0} F^v_\varepsilon(\omega, u_\varepsilon) < +\infty$ for $\mathbf{P}$ a.s. $\omega \in \Omega$. Then for $\mathbf{P}$ a.s. $\omega \in \Omega$, there exists a subsequence possibly depending on ω and $u \in L^p(\mathcal{O}, \mathbb{R}^3)$ possibly depending on ω such that $u_\varepsilon \rightharpoonup u$ in $L^p(\mathcal{O}, \mathbb{R}^3)$.*

Proof. Fix $\omega \in \Omega$ such that (A_1) holds and such that $\sup_{\varepsilon>0} F^v_\varepsilon(\omega, u_\varepsilon) < +\infty$. Consider $w \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^3)$. According to the Poincaré-Wirtinger inequality, there exists a constant $C(\omega)$ such that

$$\int_{\hat{Y}} \left| w - \fint_{\hat{Y}\cap D(\omega)} w \; d\hat{y} \right|^p d\hat{x} \leq C(\omega) \int_{\hat{Y}} |\nabla w|^p \, d\hat{x}$$

from which we easily deduce

$$\int_{\varepsilon\hat{Y}} \left| w - \fint_{\varepsilon\hat{Y}\cap \varepsilon D(\omega)} w \; d\hat{y} \right|^p d\hat{x} \leq C(\omega) \int_{\varepsilon\hat{Y}} |\varepsilon\nabla w|^p \, d\hat{x}$$

and finally

$$\int_{\varepsilon\hat{Y}} |w|^p \;\, d\hat{x} \leq 2^p \Big(\varepsilon^2 \fint_{\varepsilon\hat{Y}\cap \varepsilon D(\omega)} |w|^p \, d\hat{x} + C(\omega) \int_{\varepsilon\hat{Y}} |\varepsilon\nabla w|^p \, d\hat{x} \Big). \tag{10}$$

Applying (10) to the function $\tau_{\varepsilon z} w$ defined by $\tau_{\varepsilon z} w(\hat{x}) := w(\hat{x} + \varepsilon z)$ we infer

$$\begin{aligned}
\int_{\varepsilon(\hat{Y}+z)} |w|^p \;\, d\hat{x} &= \int_{\varepsilon\hat{Y}} |\tau_{\varepsilon z} w|^p \, d\hat{x} \\
&\leq 2^p \Big(\varepsilon^2 \fint_{\varepsilon\hat{Y}\cap \varepsilon D(\omega)} |\tau_{\varepsilon z} w|^p \, d\hat{x} + C(\omega) \int_{\varepsilon\hat{Y}} |\varepsilon\nabla \tau_{\varepsilon z} w|^p \, d\hat{x} \Big) \\
&= 2^p \Big(\varepsilon^2 \fint_{\varepsilon(\hat{Y}+z)\cap \varepsilon D(\tau_{-z}\omega)} |w|^p \, d\hat{x} + C(\omega) \int_{\varepsilon(\hat{Y}+z)} |\varepsilon\nabla w|^p \, d\hat{x} \Big).
\end{aligned} \tag{11}$$

Noticing that $\left| \widehat{\mathcal{O}} \setminus \bigcup_{z\in I_\varepsilon} \varepsilon(\hat{Y}+z) \right| = 0$ where I_ε is a finite subset of $\mathbf{Z}^2$ and $(\hat{Y}+z)_{z\in\mathbf{Z}^2}$ are pairwise disjoint, from (11), and since $u_\varepsilon = v$ on $\varepsilon D(\tau_{-z}\omega) \times (0,h)$, we obtain

$$\begin{aligned}
\int_{\mathcal{O}} |u_\varepsilon|^p \, dx &\leq 2^p \Big(\varepsilon^2 \sum_{z\in I_\varepsilon} \int_0^h \fint_{\varepsilon(\hat{Y}+z)\cap \varepsilon D(\tau_{-z}\omega)} |v|^p \, dx + C(\omega) \int_{\mathcal{O}} |\varepsilon \nabla u_\varepsilon|^p \, dx \Big) \\
&= C \Big(\sum_{z\in I_\varepsilon} \frac{1}{|(\hat{Y}+z)\cap D(\tau_{-z}\omega)|} \int_0^h \int_{\varepsilon(\hat{Y}+z)\cap \varepsilon D(\tau_{-z}\omega)} |v|^p \, dx + C(\omega) \int_{\mathcal{O}} |\varepsilon \nabla u_\varepsilon|^p \, dx \Big) \\
&\leq C \Big(\frac{1}{|\hat{Y}\cap D(\omega)|} \|v\|^p_{L^p(\mathcal{O},\mathbb{R}^3)} + C(\omega) \int_{\mathcal{O}} |\varepsilon \nabla u_\varepsilon|^p \, dx \Big)
\end{aligned}$$

and the conclusion follows from $\varepsilon^p \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} |\nabla u_\varepsilon|^p \, dx \leq \frac{1}{\alpha} F^v_\varepsilon(\omega, u_\varepsilon)$. ☐

Remark 3.1. *The same compactness result clearly holds when we substitute any function v_ε for v provided that $\sup\limits_{\varepsilon>0} \|v_\varepsilon\|_{L^p(\mathcal{O}\cap T_\varepsilon,\mathbb{R}^3)} < +\infty$ and $\sup\limits_{\varepsilon>0} \|\nabla v_\varepsilon\|_{L^p(\mathcal{O}\cap T_\varepsilon,\mathbf{M}^{3\times 3})} < +\infty$.*

Let us define the functional $F^v_0 : L^p(\mathcal{O}, \mathbb{R}^3) \to \mathbb{R}^+$ by

$$F^v_0(u) = \int_{\mathcal{O}} f^{**}_0(u - v) \; dx$$

where f_0 is the function defined in Section 2. The following theorem is a consequence of the two bounds established in the next two subsections.

Theorem 3.1. *The sequence of random functionals $(F^v_\varepsilon)_{\varepsilon>0}$ almost surely sequentially Γ-converges to the deterministic functional F^v_0 when $L^p(\mathcal{O}, \mathbb{R}^3)$ is equipped with its weak topology.*

3.1 The upper bound

Proposition 3.1. *There exists a set $\Omega' \in \mathcal{A}$ of full probability such that for all $u \in L^p(\mathcal{O}, \mathbb{R}^3)$ and all $\omega \in \Omega'$ there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O}, \mathbb{R}^3)$ satisfying*

$$\begin{array}{l} u_\varepsilon(\omega) \rightharpoonup u \text{ in } L^p(\mathcal{O}, \mathbb{R}^3) \\ F_0^v(u) \geq \limsup_{\varepsilon\to 0} F_\varepsilon^v(\omega, u_\varepsilon(\omega)) \end{array}$$

or, equivalently, for all $\omega \in \Omega'$, $\Gamma - \limsup F_\varepsilon^v(\omega, .) \leq F_0^v$.

Proof. We proceed into two steps.

Step 1. For every $v \in W^{1,p}(\mathcal{O}, \mathbb{R}^3)$, let us consider the function $\tilde{F}_0^v : L^p(\mathcal{O}, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{F}_0^v(u) = \begin{cases} \displaystyle\int_{\mathcal{O}} f_0(u - v)\, dx \text{ if } u \in \mathcal{C}_c^1(\mathcal{O}, \mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

We establish $\Gamma - \limsup F_\varepsilon(\omega, .) \leq \tilde{F}_0^v$ for $\mathbf{P}$ a.s. $\omega \in \Omega''$.

Let $\eta \in \mathbf{Q}^+$ intended to go to 0 and let $(Q_{i,\eta})_{i\in I_\eta}$ be a finite family of pairwise disjoint cubes of size η included in $\mathcal{O}$, such that

$$\left| \mathcal{O} \setminus \bigcup_{i\in I_\eta} Q_{i,\eta} \right| = 0.$$

Let $v_\delta \in \mathcal{C}_c^1(\mathcal{O}, \mathbb{R}^3)$ be a regular approximation of v in $L^p(\mathcal{O}, \mathbb{R}^3)$, i.e., satisfying $v_\delta \to v$ strongly in $L^p(\mathcal{O}, \mathbb{R}^3)$, set $z_\delta := u - v_\delta$ and $z_{\delta,\eta} := \displaystyle\sum_{i\in I_\eta} z_\delta(x_{i,\eta}) \mathbb{1}_{Q_{i,\eta}}$ where $x_{i,\eta}$ is arbitrarily chosen in $Q_{i,\eta}$. Since z_δ is a Lipschitz function on $\mathcal{O}$, clearly $z_{\delta,\eta} \to z_\delta = u - v_\delta$ in $L^p(\mathcal{O}, \mathbb{R}^3)$ when $\eta \to 0$.

Consider the greater open cube $C_{i,\eta,\varepsilon}$ in $\mathcal{I}$ included in $\frac{1}{\varepsilon} Q_{i,\eta}$ and let $w_{i,\eta,\varepsilon} \in \mathrm{Adm}_{C_{i,\eta,\varepsilon}}(\omega, z_\delta(x_{i,\eta}))$ be a minimizer of $\mathcal{S}_{C_{i,\eta,\varepsilon}}(\omega, z_\delta(x_{i,\eta}))$ extended by zero outside $C_{i,\eta,\varepsilon} \setminus T(\omega)$ (for shorten notation, we do not indicate the dependance on δ). The family $(C_{i,\eta,\varepsilon})_\varepsilon$ is regular. Indeed for every cube $Q =]a, b[$ in $\mathbb{R}^3$, let denote by Q' the associated cube $]0, b[$ and consider the family $(C'_{i,\eta,\varepsilon})_\varepsilon$. One has

$$\frac{|C_{i,\eta,\varepsilon}|}{|C'_{i,\eta,\varepsilon}|} = \frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon} Q_{i,\eta}\right|} \times \frac{|Q_{i,\eta}|}{|Q'_{i,\eta}|} \times \frac{\left|\frac{1}{\varepsilon} Q'_{i,\eta}\right|}{|C'_{i,\eta,\varepsilon}|}.$$

But one can easily check that $\displaystyle\lim_{\varepsilon\to 0} \frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon} Q_{i,\eta}\right|} = \lim_{\varepsilon\to 0} \frac{|C'_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon} Q'_{i,\eta}\right|} = 1$ so that, for ε small enough (depending on fixed η) $\displaystyle\frac{|C_{i,\eta,\varepsilon}|}{|C'_{i,\eta,\varepsilon}|} \leq 2 \frac{|Q_{i,\eta}|}{|Q'_{i,\eta}|}$. The family $(C'_{i,\eta,\varepsilon})_\varepsilon$ then clearly satisfies regularity conditions (i)-(iv).

Therefore, according to Theorem 2.3

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{\mathcal{S}_{C_{i,\eta,\varepsilon}}(\omega, z(x_{i,\delta,\eta}))}{|C_{i,\eta,\varepsilon}|} &= \lim_{\varepsilon\to 0} \frac{1}{|C_{i,\eta,\varepsilon}|} \int_{C_{i,\eta,\varepsilon}\setminus T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega, y))\, dy \\ &= f_0(z_\delta(x_{i,\eta})) \end{aligned} \tag{12}$$

for all $\omega \in \Omega_{i,\eta}$ satisfying $\mathbf{P}(\Omega_{i,\eta}) = 1$. In what follows we denote the set of full probability $\bigcap_{\eta\in\mathbf{Q}^+} \bigcap_{i\in I_\eta} \Omega_{i,\eta}$

by Ω' and we fix $\omega \in \Omega'$. From (12) we infer

$$\begin{aligned}
\int_{\mathcal{O}} f_0(z_{\delta,\eta})\ dx &= \sum_{i\in I_\eta} \int_{Q_{i,\eta}} f_0(z_{\delta,\eta})\ dx \\
&= \sum_{i\in I_\eta} |Q_{i,\eta}|\, f_0(z(x_{i,\eta}))\ dx \\
&= \lim_{\varepsilon\to 0} \sum_{i\in I_\eta} |Q_{i,\eta}| \frac{1}{|C_{i,\eta,\varepsilon}|} \int_{C_{i,\eta,\varepsilon}\setminus T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,y))\ dy \\
&= \lim_{\varepsilon\to 0} \sum_{i\in I_\eta} |Q_{i,\eta}| \frac{1}{|\varepsilon C_{i,\eta,\varepsilon}|} \int_{\varepsilon C_{i,\eta,\varepsilon}\setminus \varepsilon T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))dy \\
&= \lim_{\varepsilon\to 0} \sum_{i\in I_\eta} \frac{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}{|C_{i,\eta,\varepsilon}|} \int_{Q_{i,\eta}\setminus \varepsilon T(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))dy \\
&= \lim_{\varepsilon\to 0} \sum_{i\in I_\eta} \int_{Q_{i,\eta}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\nabla w_{i,\eta,\varepsilon}(\omega,\frac{y}{\varepsilon}))\ dy. \qquad (13)
\end{aligned}$$

We have used the fact that $\lim_{\varepsilon\to 0} \frac{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|}{|C_{i,\eta,\varepsilon}|} = 1$ and that $w_{i,\eta,\varepsilon} = 0$ outside $C_{i,\varepsilon,\eta} \setminus T(\omega)$.

Let us define the function $u_{\delta,\eta,\varepsilon}$ on $\mathcal{O}$ by :

$$u_{\delta,\eta,\varepsilon}(\omega,x) = v(x) + \sum_{i\in I_\eta} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\mathbb{1}_{Q_{i,\eta}}(x).$$

According to the boundary condition satisfied by $w_{i,\eta,\varepsilon}$, clearly $u_{\delta,\eta,\varepsilon} \in W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3)$ and $u_{\delta,\eta,\varepsilon} = v$ on $\mathcal{O} \cap T_\varepsilon(\omega)$. Furthermore, from (13), (5) and (3) we deduce

$$\begin{aligned}
\int_{\mathcal{O}} f_0(z_\eta)\ dx &= \lim_{\varepsilon\to 0} \sum_{i\in I_\eta} \int_{Q_{i,\eta}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\varepsilon\nabla u_{\delta,\eta,\varepsilon})\ dx \\
&= \lim_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_\varepsilon(\omega)} f^{\infty,p}(\varepsilon\nabla u_{\delta,\eta,\varepsilon})\ dx \\
&= \lim_{\varepsilon\to 0} \varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon(\omega)} f(\nabla u_{\delta,\eta,\varepsilon})\ dx = \lim_{\varepsilon\to 0} F^v_\varepsilon(\omega, u_{\delta,\eta,\varepsilon}(\omega,.)). \qquad (14)
\end{aligned}$$

Letting $\eta \to 0$, then $\delta \to 0$ in (14) and since $w \mapsto \int_{\mathcal{O}} f_0(w)\ dx$ is clearly strongly continuous in $L^p(\mathcal{O},\mathbb{R}^3)$ we finally obtain

$$\int_{\mathcal{O}} f_0(z)\ dx = \lim_{\delta\to 0}\lim_{\eta\to 0}\lim_{\varepsilon\to 0} F^v_\varepsilon(\omega, u_{\delta,\eta,\varepsilon}(\omega,.)). \qquad (15)$$

On the other hand, since $w_{i,\eta,\varepsilon} \in \mathrm{Adm}_{C_{i,\eta,\varepsilon}}(\omega, z(x_{i,\eta}))$, one has

$$\begin{aligned}
⨍_{Q_{i,\eta}} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\ dx &= \frac{1}{|Q_{i,\eta}|} \int_{\varepsilon C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\ dx \\
&= \frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|} ⨍_{\varepsilon C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega,\frac{x}{\varepsilon})\ dx \\
&= \frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|} ⨍_{C_{i,\eta,\varepsilon}} w_{i,\eta,\varepsilon}(\omega,x)\ dx \\
&= \frac{|C_{i,\eta,\varepsilon}|}{\left|\frac{1}{\varepsilon}Q_{i,\eta}\right|} z_\delta(x_{i,\eta})
\end{aligned}$$

so that letting successively $\varepsilon \to 0$ and $\eta \to 0$ we easily infer

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0} u_{\delta,\eta,\varepsilon}(\omega,.) = v + (u - v_\delta) \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3).$$

Then letting $\delta \to 0$,

$$\lim_{\delta\to 0}\lim_{\eta\to 0}\lim_{\varepsilon\to 0} u_{\delta,\eta,\varepsilon}(\omega,.) = u \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3). \tag{16}$$

Collecting (15) and (16), a standard diagonalization argument[1] furnishes a map $\varepsilon \mapsto (\delta(\varepsilon),\eta(\varepsilon))$ such that

$$\begin{aligned} &u_\varepsilon(\omega,.) := u_{\delta(\varepsilon),\eta(\varepsilon),\varepsilon}(\omega,.) \rightharpoonup u \text{ weakly in } L^p(\mathcal{O},\mathbb{R}^3);\\ &\lim_{\varepsilon\to 0} F^v_\varepsilon(\omega,u_\varepsilon(\omega,.)) = \tilde{F}^v_0(u) \end{aligned}$$

and the conclusion of step 1 follows straightforwardly.

Step 2. We end the proof by a relaxation argument. According to the first step

$$\Gamma - \limsup F^v_\varepsilon(\omega,.) \le \tilde{F}^v_0.$$

Taking the lower semicontinuity envelope of each two members in the space $L^p(\mathcal{O},\mathbb{R}^3)$ equipped with its weak topology, we infer

$$\Gamma - \limsup F^v_\varepsilon(\omega,.) \le (\tilde{F}^v_0)^{**}.$$

But, from standard relaxation result $(\tilde{F}^v)^{**} = F^v_0$ and the conclusion follows. $\square$

3.2 The lower bound

Proposition 3.2. *For all u_ε weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$ and for $\mathbf{P}$ a.s. $\omega \in \Omega$ one has*

$$F^v_0(u) \le \liminf_{\varepsilon\to 0} F^v_\varepsilon(\omega,u_\varepsilon).$$

Proof. From (3) and since $u_\varepsilon = v$ on $T_\varepsilon(\omega)\cap\mathcal{O}$ we easily deduce

$$\liminf_{\varepsilon\to 0} \varepsilon^p \int_{\mathcal{O}\setminus T_\varepsilon} f(\nabla u_\varepsilon)dx = \liminf_{\varepsilon\to 0}\int_{\mathcal{O}} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)dx.$$

Fix x_0 in $\mathcal{O}$ and set $Q_\rho(x_0) := S_\rho(\hat{x}_0)\times I_\rho(x_{0,3})$ (to shorten notation we sometimes do not indicate the fixed argument x_0). By using a blow up argument, it is enough to prove that for a.e. x_0 one has

$$\lim_{\rho\to 0}\liminf_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\, dx \ge f^{**}_0(u(x_0)-v(x_0)).$$

According to the decomposition lemma (cf [3, 11]), there exists w_ε in $W^{1,p}_0(Q_\rho,\mathbb{R}^3)$ such that $(|\nabla w_\varepsilon|^p)_{\varepsilon>0}$ is uniformly integrable and such that the sequences $(\nabla w_\varepsilon)_{\varepsilon>0}$ and $(\nabla \varepsilon u_\varepsilon)_{\varepsilon>0}$ generate the same Young measure μ (for shorten notation we do not indicate the dependance on ρ for w). Therefore applying standard lower semicontinuity and continuity properties for Young measures (see Proposition 4.3.4 and Theorem 4.3.3 in [3]) we infer

$$\liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} f^{\infty,p}(\nabla w_\varepsilon)\, dx = \int_{Q_\rho(x_0)}\int_{\mathbf{M}^{3\times 3}_S} f^{\infty,p}(M)\, d\mu_x\, dx \le \liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} f^{\infty,p}(\varepsilon\nabla u_\varepsilon)\, dx$$

so that it suffices to establish

$$\liminf_{\rho\to 0}\liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} f^{\infty,p}(\nabla w_\varepsilon)\, dx \ge f^{**}_0(u(x_0)-v(x_0)). \tag{17}$$

Note that since

$$\begin{aligned} &\varepsilon u_\varepsilon \rightharpoonup 0 \text{ in } W^{1,p}(\mathcal{O},\mathbb{R}^3)\\ &\varepsilon u_\varepsilon \to 0 \text{ in } L^p(\mathcal{O},\mathbb{R}^3) \end{aligned}$$

[1] One can easily check that $u_{\delta,\eta,\varepsilon}(\omega,.)$ belongs to a fixed ball $\mathcal{B}(0,r)$ of $L^p(\mathcal{O},\mathbb{R}^3)$. Since the weak topology of $L^p(\mathcal{O},\mathbb{R}^3)$ induces a metric on bounded sets, the diagonalization argument holds.

we infer

$$w_\varepsilon \rightharpoonup 0 \text{ in } W^{1,p}(Q_\rho,\mathbb{R}^3) \tag{18}$$

$$w_\varepsilon \to 0 \text{ in } L^p(Q_\rho,\mathbb{R}^3) \tag{19}$$

when $\varepsilon \to 0$.

Let $C_{\varepsilon,\rho}$ be the smallest cube in $\mathcal{I}$ containing $\frac{1}{\varepsilon}Q_\rho$. Our strategy is to change the function w_ε in order to obtain a function in $\mathrm{Adm}_{C_{\varepsilon,\rho}}(\omega, u(x_0)-v(x_0))$ whose gradient decreases the left hand side of (17).

First change. For $\eta > 0$ intended to go to 0 set $\hat{A}_\eta := (S_\rho \setminus \varepsilon D(\omega))_\eta$ (we do not indicate the dependence on ε and ω) and $w_{\varepsilon,\eta}(x) := \phi_{\varepsilon,\eta}(\hat{x})w_\varepsilon(x) + \varepsilon v(x_0)$ where $\phi_{\varepsilon,\eta} := \rho_\eta * 1_{\hat{A}_\eta}$. Note that $\phi_{\varepsilon,\eta}$ satisfies

$$|grad(\phi_{\varepsilon,\eta})| \le \frac{C}{\eta}$$

and

$$\phi_{\varepsilon,\eta} = \begin{cases} 0 \text{ in } \partial(S_\rho \setminus \varepsilon D(\omega)), \\ 1 \text{ in } \hat{A}_{2\eta}, \end{cases}$$

thus $w_{\varepsilon,\eta} := \varepsilon v(x_0)$ on $\partial(Q_\rho \setminus \varepsilon T(\omega))$. We extend $w_{\varepsilon,\eta}$ by $\varepsilon v(x_0)$ on the complementary set of $Q_\rho \setminus \varepsilon T(\omega)$. Set $A_{2\eta} := \hat{A}_{2\eta} \times I_\rho$. From the growth condition (3) we deduce

$$\begin{aligned} &\int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx \\ = \;& \int_{A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx + \int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx \\ \le \;& \int_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\, dx \; + \int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx. \end{aligned} \tag{20}$$

On the other hand

$$\begin{aligned} &\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx \\ \le \;& C\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} \big(||grad\ \phi||^p|w_\varepsilon|^p + |\phi_{\varepsilon,\eta}|^p|\nabla w_\varepsilon|^p\big)\, dx \\ \le \;& C\Big(\frac{1}{\eta^p}\int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} |w_\varepsilon|^p dx + \sup_\varepsilon \int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} |\nabla w_\varepsilon|^p dx\Big). \end{aligned}$$

Letting successively $\varepsilon \to 0$ and $\eta \to 0$ and since $(|\nabla w_\varepsilon|^p)_{\varepsilon>0}$ is uniformly integrable we finally deduce

$$\limsup_\eta \limsup_\varepsilon \int_{(Q_\rho\setminus\varepsilon T(\omega))\setminus A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx = 0. \tag{21}$$

Consequently, combining (20) and (21)

$$\liminf_\eta \liminf_\varepsilon \int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx \quad \le \quad \liminf_\varepsilon ⨍_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\, dx. \tag{22}$$

Second change. With the function $\phi_{\varepsilon,\eta}$ defined previously, set $\psi_{\varepsilon,\eta} := \frac{\phi_{\varepsilon,\eta}}{⨍_{S_\rho} \phi_{\varepsilon,\eta}\, d\hat{x}}$ and consider the random function

$$z_{\varepsilon,\eta} := w_{\varepsilon,\eta} + \psi_{\varepsilon,\eta}[\varepsilon u(x_0) - ⨍_{Q_\rho} w_{\varepsilon,\eta}\, dx].$$

The function $\psi_{\varepsilon,\eta}$ fulfills the following conditions

$$\psi_{\varepsilon,\eta}(\hat{x}) = \begin{cases} 0 \quad \text{on } \partial(S_\rho \setminus \varepsilon D(\omega)), \\ \dfrac{1}{⨍_{S_\rho} \phi_{\varepsilon,\eta}\, d\hat{x}} \quad \text{on } \hat{A}_{2\eta}, \end{cases}$$

$⨍_{S_\rho} \psi_{\varepsilon,\rho}\, d\hat{x} = 1$, and $|grad(\psi_{\varepsilon,\eta})| \le \dfrac{1}{C(\eta)\eta}$ where $C(\eta)$ is a nonnegative constants depending only of ρ and η (change of scale and argue as in (7) by reasoning on $\frac{1}{\varepsilon}S_\rho \setminus D(\omega)$). Thus

$$\begin{cases} ⨍_{Q_\rho} z_{\varepsilon,\eta}\, dx = \varepsilon u(x_0) \\ \text{and} \\ z_{\varepsilon,\eta} = \varepsilon v(x_0) \text{ on } \partial(Q_\rho \setminus \varepsilon T(\omega)). \end{cases}$$

From the definition of $z_{\varepsilon,\eta}$ we derive

$$\begin{aligned} \int_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\, dx &= \int_{A_{2\eta}} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx + R_{\varepsilon,\eta,\rho} \\ &\le \int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dxdx + R_{\varepsilon,\eta,\rho} \end{aligned} \tag{23}$$

where

$$R_{\varepsilon,\eta,\rho} := \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\, dx.$$

From the growth condition (4)

$$|R_{\varepsilon,\eta,\rho}| \le \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} |\nabla w_{\varepsilon,\eta}|^p\ dx + C\frac{|(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}|^p}{\eta^p C(\eta)^p}\Big[\varepsilon^p |u(x_0)|^p + \Big|⨍_{Q_\rho} w_{\varepsilon,\eta}\, d\hat{x}\Big|^p\ \Big].$$

But applying estimate (21) with the function $|\ .\ |^p$ substituted for $f^{\infty,p}$

$$\limsup_{\eta\to 0} \limsup_{\varepsilon\to 0} \int_{(Q_\rho \setminus \varepsilon T(\omega)) \setminus A_{2\eta}} |\nabla w_{\varepsilon,\eta}|^p\ dx = 0.$$

On the other hand

$$\Big[\varepsilon^p |u(x_0)|^p + \Big|⨍_{Q_\rho} w_{\varepsilon,\eta}\, d\hat{x}\Big|^p\ \Big]$$

clearly tends to 0 when $\varepsilon \to 0$ (recall that w_ε strongly converges to 0 in $L^p(Q_\rho, \mathbb{R}^3)$). Thus, letting successively $\varepsilon \to 0$ and $\eta \to 0$ in (23) we obtain

$$\liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \int_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\, dx \le \liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \int_{Q_\rho} f^{\infty,p}(\nabla w_{\varepsilon,\eta})\, dx$$

and finally from (22)

$$\liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \int_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\, dx \le \liminf_{\varepsilon} ⨍_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\, dx. \tag{24}$$

Let us define the Sobolev function $z'_{\varepsilon,\eta}$ on $\frac{1}{\varepsilon}Q_\rho$ by $z'_{\varepsilon,\eta}(y) := \dfrac{1}{\varepsilon} z_{\varepsilon,\eta}(\varepsilon y)$ which satisfies the following conditions:

$$\begin{aligned} z'_{\varepsilon,\eta}(y) &= v(x_0) \quad \text{on } \partial(\frac{1}{\varepsilon}Q_\rho \backslash T(\omega)), \\ ⨍_{Q_\rho} z'_{\varepsilon,\eta}(y)dy &= \frac{1}{\varepsilon} ⨍_{\frac{1}{\varepsilon}Q_\rho} z_{\varepsilon,\eta}(\varepsilon y)dy \\ &= \frac{1}{\varepsilon} ⨍_{Q_\rho} z_{\varepsilon,\eta}(x)dx \\ &= u(x_0). \end{aligned}$$

Extend $z'_{\varepsilon,\eta}$ by $v(x_0)$ on $C_{\varepsilon,\rho} \setminus \frac{1}{\varepsilon}Q_\rho$, and set

$$z"_{\varepsilon,\eta} := \frac{|C_{\varepsilon,\eta}|}{\left|\frac{1}{\varepsilon}S_\rho\right|}\left(z'_{\varepsilon,\eta} - v(x_0)\right).$$

It is easy to check that $z"_{\varepsilon,\eta}$ belongs to $\mathrm{Adm}_{C_{\varepsilon,\eta}}(u(x_0) - v(x_0))$. Changing of scale at the left hand side of (24), using the facts that $f^{\infty,p}$ is positively p-homogeneous and that $\lim_{\varepsilon\to 0} \frac{|C_{\varepsilon,\eta}|}{\left|\frac{1}{\varepsilon}Q_\rho\right|} = 1$, we obtain for $\mathbf{P}$ almost every ω in Ω:

$$\begin{aligned}
f_0(u(x_0) - v(x_0)) &= \lim_{\varepsilon\to 0} \frac{1}{|C_{\varepsilon,\rho}|}\mathcal{S}_{C_{\varepsilon,\rho}}(\omega, u(x_0) - v(x_0)) \\
&\leq \limsup_{\eta\to 0}\limsup_{\varepsilon\to 0} ⨍_{C_{\varepsilon,\rho}} f^{\infty,p}(\nabla z''_{\varepsilon,\eta})\, dx \\
&= \limsup_{\eta\to 0}\limsup_{\varepsilon\to 0} ⨍_{\frac{1}{\varepsilon}Q_\rho} f^{\infty,p}(\nabla z'_{\varepsilon,\eta})\, dx \\
&= \liminf_{\eta\to 0}\liminf_{\varepsilon\to 0} ⨍_{Q_\rho} f^{\infty,p}(\nabla z_{\varepsilon,\eta})\, dx \\
&\leq \liminf_{\varepsilon} ⨍_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\, dx.
\end{aligned}$$

Therefore $f_0^{**}(u(x_0) - v(x_0)) \leq \liminf_{\varepsilon} ⨍_{Q_\rho} f^{\infty,p}(\nabla w_\varepsilon)\, dx$ which completes the proof. $\square$

Remark 3.2. *A carefully analysis of the proof above lead us to the following generalization of the lover bound: for all* $u_\varepsilon \rightharpoonup u$ *in* $L^p(\mathcal{O}, \mathbb{R}^3)$, *for all function* v_ε *in* $W^{1,p}(\mathcal{O}, \mathbb{R}^3)$ *satisfying* $\sup_{\varepsilon>0} \|\nabla v_\varepsilon)\|_{L^p(\mathcal{O}\cap T_\varepsilon, \mathbf{M}^{3\times 3})} < +\infty$, *and for all function* $\zeta \in L^p(\mathcal{O}, \mathbb{R}^3)$

$$F^\zeta(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon^{v_\varepsilon}(\omega, u_\varepsilon)$$

where, for every $u \in L^p(\mathcal{O}, \mathbb{R}^3)$,

$$F_\varepsilon^{v_\varepsilon}(\omega, u) = \begin{cases} \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} f(\varepsilon\nabla u)\, dx \ \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O}, \mathbb{R}^3), u = v_\varepsilon \text{ on } \mathcal{O}\cap T_\varepsilon \\ +\infty \text{ otherwise,} \end{cases}$$

$$F^\zeta(u) = \int_{\mathcal{O}} f_0^{**}(u - \zeta).$$

4 The limit problem associated with the fibers

4.1 The limit functional

In the following we denote by $a(\omega, .)$ the characteristic function of the random set $D(\omega)$ so that $1_{D_\varepsilon}(\omega)(\hat{x}) = 1_{D_r(\omega)}(\frac{\hat{x}}{\varepsilon}) := a(\omega, \frac{\hat{x}}{\varepsilon})\ \ \forall \hat{x} \in \widehat{\mathcal{O}}$. According to this notation we consider the random integral functional $G_\varepsilon(\omega, .)$ defined in $L^p(\mathcal{O}, \mathbb{R}^3)$ by

$$G_\varepsilon(\omega, u) = \begin{cases} \displaystyle\int_{\mathcal{O}} a(\omega, \frac{\hat{x}}{\varepsilon}) g(\nabla u) dx \text{ if } u\lfloor(\mathcal{O}\setminus T_\varepsilon) = 0,\ u\lfloor(\mathcal{O}\cap T_\varepsilon) \in W^{1,p}_{\Gamma_0}(\mathcal{O}\cap T_\varepsilon, \mathbf{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

According to Proposition 2.2 of Section 2 we have

Proposition 4.1. *There exists a* $\mathbf{P}$*-mesurable set* $\Omega'' \subset \Omega$*, with* $P(\Omega") = 1$ *such that*

$$\forall \omega \in \Omega'', \quad a(\omega, \frac{\cdot}{\varepsilon}) \rightharpoonup \mathbf{E}(\int_{\hat{Y}} a(\omega, \hat{y}) d\hat{y}) := \theta \quad \textit{for the topology} \;\; \sigma(L^\infty, L^1).$$

Now let us consider the function $g^\perp : \mathbb{R}^3 \to \mathbb{R}$ defined for every $a \in \mathbb{R}$ by

$$g^\perp(a) := \inf_{\xi \in \mathbf{M}^{3\times 2}} g\left(\xi | a\right),$$

and we define the deterministic functional $\widetilde{G}_0 : L^p(\mathcal{O}, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ by

$$\widetilde{G}_0(u) = \begin{cases} \theta \displaystyle\int_{\mathcal{O}} (g^\perp)^{**}(\frac{1}{\theta}\frac{\partial u}{\partial x_3}) dx \text{ if } \; u \in V_0 \\ +\infty \text{ otherwise} \end{cases}$$

where $V_0 := \left\{ u \in L^p(\mathcal{O}, \mathbb{R}^3) : \dfrac{\partial u}{\partial x_3} \in L^p(\mathcal{O}, \mathbb{R}^3), \; u(\hat{x}, 0) = 0 \text{ on } \widehat{\mathcal{O}} \right\}$. Note that the function G_0 considered in the introduction is nothing but the function defined by $G_0(v) = \tilde{G}_0(\theta v)$. In the next sections we are going to establish

Theorem 4.1. *The sequence* $(G_\varepsilon)_{\varepsilon > 0}$ *almost surely sequentially* Γ*-converges to the functional* $\widetilde{G}_0$ *when* $L^p(\mathcal{O}, \mathbb{R}^3)$ *is equipped with its weak topology.*

The use of the weak topology in $L^p(\mathcal{O}, \mathbb{R}^3)$ comes from the next compactness result.

Lemma 4.1. *Let* $(u_\varepsilon)_{\varepsilon > 0}$ *be a sequence satisfying* $\sup_\varepsilon G_\varepsilon(\omega, u_\varepsilon) < +\infty$ *for all* $\omega \in \Omega"$*. Then for all* $\omega \in \Omega"$ *there exist a subsequence possibly depending on* ω *and* $u \in V_0$ *possibly depending on* ω *such that*

$$a(\omega, \frac{\cdot}{\varepsilon}) u_\varepsilon \rightharpoonup u \;\; \textit{in} \;\; L^p(\mathcal{O}, \mathbb{R}^3) \tag{25}$$

$$a(\omega, \frac{\cdot}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} \;\; \textit{in} \;\; L^p(\mathcal{O}, \mathbb{R}^3). \tag{26}$$

Proof. From the coercivity of g, and since $u_\varepsilon = 0$ on Γ_0, we infer

$$\begin{aligned} \int_{\mathcal{O} \cap T_\varepsilon} |u_\varepsilon(\hat{x}, x_3)|^p dx &\leq C \int_{\mathcal{O} \cap T_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^p dx \\ &\leq \frac{C}{\alpha} G_\varepsilon(\omega, u_\varepsilon) \end{aligned}$$

which gives (25). Weak convergence (26) is obvious and $u(\hat{x}, 0) = 0$ on $\widehat{\mathcal{O}}$ is easily checked. Note that since $V_0 \subset W^{1,p}((0,h), L^p(\mathcal{O}, \mathbb{R}^3)) \subset \mathcal{C}([0,h], L^p(\mathcal{O}, \mathbb{R}^3))$ equality $u(.,0) = 0$ may be understood in a classical sense. □

4.2 The lower bound

Proposition 4.2. *For all* u_ε *such that* $a(\omega, \frac{\cdot}{\varepsilon}) u_\varepsilon$ *weakly converges to* u *in* $L^p(\mathcal{O}, \mathbb{R}^3)$ *and all* $\omega \in \Omega"$

$$\widetilde{G}_0(u) \leq \liminf_{\varepsilon \to 0} G_\varepsilon(\omega, u_\varepsilon)$$

Proof. Fix $\omega \in \Omega"$ and assume that $\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) < +\infty$. From inequalities $g \geq g^\perp \geq (g^\perp)^{**}$ and the Moreau-Rockafellar duality principle we infer that for all ϕ in $L^q(\mathcal{O}, \mathbb{R}^3)$ where $q = \frac{p}{p-1}$ is the conjugate exponent of p:

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\ G_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon\to 0}\int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon})(g^\perp)^{**}(\frac{\partial u_\varepsilon}{\partial x_3})dx\\
&\geq \liminf_{\varepsilon\to 0}\Big(\int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon})\phi.\frac{\partial u_\varepsilon}{\partial x_3}dx-\int_{\mathcal{O}} a(\omega,\frac{\hat{x}}{\varepsilon})(g^\perp)^{*}(\phi)dx\Big)\\
&= \int_{\mathcal{O}}\phi.\frac{\partial u}{\partial x_3}dx-\theta\int_{\mathcal{O}}(g^\perp)^*(\phi)dx\\
&= \theta\ \left[\int_{\mathcal{O}}\frac{1}{\theta}\phi\frac{\partial u}{\partial x_3}dx-\int_{\mathcal{O}}(g^\perp)^*(\phi)dx\right].
\end{aligned}$$

By taking the the supremum over all functions ϕ in $\phi\in L^q(\mathcal{O},\mathbb{R}^3)$ we finally obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) &\geq \theta\sup_{\phi\in L^q(\mathcal{O},\mathbb{R}^3)}\left[\int_{\mathcal{O}}\frac{1}{\theta}\phi\frac{\partial u}{\partial x_3}dx-\int_{\mathcal{O}}(g^\perp)^*(\phi)dx\right]\\
&= \theta\int_{\mathcal{O}}(g^\perp)^{**}(\frac{1}{\theta}\frac{\partial u}{\partial x_3})dx
\end{aligned}$$

which completes the proof. □

4.3 The upper bound

Proposition 4.3. *For all $u\in V_0$ and all $\omega\in\Omega$" there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$ such that*

$$\widetilde{G}_0(u)\geq\limsup_{\varepsilon\to 0}\ G_\varepsilon(\omega,u_\varepsilon(\omega)).$$

Proof. In all the proof we fix ω in Ω". We proceed into two steps.

First step. let $u\in C^1(\overline{\mathcal{O}},\mathbb{R}^3)$, with $u=0$ on Γ_0. We construct a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}\in L^p(\mathcal{O},\mathbb{R}^3)$ such that $u_\varepsilon\lfloor(\mathcal{O}\backslash T_\varepsilon)=0$, $u_\varepsilon\lfloor(\mathcal{O}\cap T_\varepsilon)\in W^{1,p}(\mathcal{O}\cap T_\varepsilon,\mathbb{R}^3)$ and satisfying

$$\begin{aligned}
&a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon(\omega)\rightharpoonup u \text{ in } L^p(\mathcal{O},\mathbb{R}^3),\\
&\lim_{\varepsilon\to 0}\ G_\varepsilon(\omega,u_\varepsilon(\omega))=\theta\int_{\mathcal{O}} g^\perp(\frac{1}{\theta}\frac{\partial u}{\partial x_3})dx.
\end{aligned}$$

For $\eta>0$ intended to tend to zero, consider ξ^η in $L^p(\mathcal{O},\mathbf{M}^{3\times 2})$ such that

$$\begin{aligned}
\theta\int_{\mathcal{O}} g^\perp(\frac{1}{\theta}\frac{\partial u}{\partial x_3})dx &= \theta\int_{\mathcal{O}}\inf_{\xi\in\mathbf{M}^{3\times 2}} g\left(\xi+\frac{1}{\theta}\hat{\nabla}u,\ \frac{1}{\theta}\frac{\partial u}{\partial x_3}\right)dx\\
&\geq \theta\int_{\mathcal{O}} g\left(\xi^\eta+\hat{\nabla}u,\frac{1}{\theta}\frac{\partial u}{\partial x_3}\right)dx-\eta. \qquad (27)
\end{aligned}$$

The measurability of the matrix valued function $x\mapsto\xi^\eta(x)$ comes from the coercivity and the growth condition fulfilled by g and may be proven thanks to the measurable selection theorem (see [7]). Since $C^1_c(\mathcal{O},\mathbf{M}^{3\times 2})$ is dense in $L^p(\mathcal{O},\mathbf{M}^{3\times 2})$, according to the Lipschitz property of the convex function g one may assume that $\xi^\eta\in C^1_c(\mathcal{O},\mathbf{M}^{3\times 2})$.

Let us consider a random function $\phi(\omega,.)=(\phi_1(\omega,.),\phi_2(\omega,.))$ in $\mathcal{C}^1_c(\mathbb{R}^2,\mathbb{R}^2)$ satisfying $\phi(\omega,\hat{y})=\hat{y}$ whenever $\hat{y}\in D(\omega)$ and set

$$u_{\varepsilon,\eta}=a(\omega,\frac{\hat{x}}{\varepsilon})\big[\frac{1}{\theta}u(x)+\varepsilon\phi_1(\omega,\frac{\widehat{x}}{\varepsilon})\xi^\eta_1+\varepsilon\phi_2(\omega,\frac{\widehat{x}}{\varepsilon})\xi^\eta_2\big]. \qquad (28)$$

Clearly $u_{\varepsilon,\eta}\in W^{1,p}_{\Gamma_0}(\mathcal{O}\cap T_\varepsilon,\mathbb{R}^3)$ and $u_{\varepsilon,\eta}\lfloor(\mathcal{O}\backslash T_\varepsilon)=0$. Furthermore, from Proposition 4.1, $u_{\varepsilon,\eta}(\omega)\rightharpoonup u$ in $L^p(\mathcal{O},\mathbb{R}^3)$. On the other hand a straightforward calculation yields

$$\begin{aligned}
&\hat{\nabla}u_{\varepsilon,\eta}=\tfrac{1}{\theta}\hat{\nabla}u+\xi^\eta+O_\eta(\varepsilon),\\
&\tfrac{\partial u_{\varepsilon,\eta}}{\partial x_3}=\tfrac{1}{\theta}\tfrac{\partial u}{\partial x_3}+O_\eta(\varepsilon)
\end{aligned}$$

on $T_\varepsilon \cap \mathcal{O}$, where $\lim_{\varepsilon\to 0} O_\eta(\varepsilon) = 0$. From (27) and the Lebesgue dominated convergence theorem we infer

$$\begin{aligned} \lim_{\eta\to 0}\lim_{\varepsilon\to 0} G_\varepsilon(u_{\varepsilon,\eta}(\omega)) &= \lim_{\eta\to 0}\theta\int_{\mathcal{O}} g\left(\xi^\eta + \frac{1}{\theta}\hat{\nabla} u,\ \frac{1}{\theta}\frac{\partial u}{\partial x_3}\right) dx \\ &= \theta\int_{\mathcal{O}} (g^\perp)(\frac{1}{\theta}\frac{\partial u}{\partial x_3}) dx. \end{aligned}$$

By using a standard diagonalization argument, there exists a map $\varepsilon \mapsto \eta(\varepsilon)$, $\eta(\varepsilon) \to 0$ when $\varepsilon \to 0$ so that, setting $u_\varepsilon = u_{\varepsilon,\eta(\varepsilon)}$

$$\begin{cases} u_\varepsilon(\omega) \rightharpoonup u \qquad \text{in } L^p(\mathcal{O},\mathbb{R}^3) \\ \lim\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon(\omega)) = \theta\displaystyle\int_{\mathcal{O}} (g^\perp)(\frac{1}{\theta}\frac{\partial u}{\partial x_3}) dx. \end{cases}$$

Second step (Relaxation). let $u \in V_0$. Thus $\widetilde{G}_0(u) = \theta\int_{\mathcal{O}} (g^\perp)^{**}(\frac{1}{\theta}\frac{\partial u}{\partial x_3})\ dx$. According to standard relaxation results there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^1_c(\mathcal{O},\mathbb{R}^3)$ weakly converging to $\dfrac{\partial u}{\partial x_3}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_{\mathcal{O}} g^\perp(\frac{1}{\theta}\zeta_n) = \int_{\mathcal{O}} (g^\perp)^{**}(\frac{1}{\theta}\frac{\partial u}{\partial x_3}) dx. \tag{29}$$

For all $x \in \mathcal{O}$, consider the function $v_n \in V_0$ defined by

$$v_n(x) := \int_0^{x_3} \zeta_n(\hat{x}, s)\ ds.$$

Then $\dfrac{\partial v_n}{\partial x_3} \rightharpoonup \dfrac{\partial u}{\partial x_3}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ so that $v_n \rightharpoonup u$ in $L^p(\mathcal{O},\mathbb{R}^3)$. From (29) we infer that $(v_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}^1(\overline{\mathcal{O}},\mathbb{R}^3)$-functions in V_0 weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$ and satisfying

$$\lim_{n\to+\infty} \theta\int_{\mathcal{O}} g^\perp(\frac{1}{\theta}\frac{\partial v_n}{\partial x_3}) = \widetilde{G}_0(u).$$

Last step. With the notation of the previous step, according to the first step there exists a sequence $(u_{\varepsilon,n}(\omega))_{\varepsilon>0}$ satisfying

$$\begin{cases} u_{\varepsilon,n}(\omega) \rightharpoonup v_n \qquad \text{in } L^p(\mathcal{O},\mathbb{R}^3) \text{ when } \varepsilon\to 0, \\ \lim\limits_{\varepsilon\to 0} G_{\varepsilon,n}(u_{\varepsilon,n}(\omega)) = \theta\displaystyle\int_{\mathcal{O}} (g^\perp)(\frac{1}{\theta}\frac{\partial v_n}{\partial x_3}) dx. \end{cases}$$

Letting $n \to +\infty$ in the two estimates above and using again a standard diagonalization argument, we deduce that there exists a map $\varepsilon \mapsto n(\varepsilon)$ such that

$$\begin{cases} u_{\varepsilon,n(\varepsilon)}(\omega) \rightharpoonup u \qquad \text{in } L^p(\mathcal{O},\mathbb{R}^3) \\ \lim\limits_{\varepsilon\to 0} G_\varepsilon(\omega, u_{\varepsilon,n(\varepsilon)}(\omega)) = \widetilde{G}_0(u). \end{cases} \tag{30}$$

We end the proof by setting $u_\varepsilon(\omega) := u_{\varepsilon,n(\varepsilon)}(\omega)$. □

5 The limit problem associated with the complete structure

Now, we deal with the asymptotic behavior of the complete structure. Let us recall that the functional energy H_ε is defined in $L^p(\mathcal{O},\mathbb{R}^3)$ by :

$$H_\varepsilon(\omega,u) = \begin{cases} \displaystyle\int_{\mathcal{O}\setminus T_\varepsilon} \varepsilon^p f(\nabla u)\ dx + \int_{\mathcal{O}\cap T_\varepsilon} g(\nabla u)\ dx \text{ if } u \in W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3) \\ +\infty \text{ otherwise.} \end{cases}$$

It is worth noticing that for u in $W^{1,p}_{\Gamma_0}(\mathcal{O},\mathbb{R}^3)$, one has

$$H_\varepsilon(\omega,u)=F^u_\varepsilon(\omega,u)+G_\varepsilon(\omega,1_{T_\varepsilon\cap\mathcal{O}}u).$$

We define in $L^p(\mathcal{O},\mathbb{R}^3)$ the deterministic functional G_0 by $G_0(v)=\widetilde{G}_0(\theta v)$, i.e.,

$$G_0(v)=\begin{cases}\theta\displaystyle\int_{\mathcal{O}}(g^\perp)^{**}(\frac{\partial v}{\partial x_3})dx \text{ if } \ v\in V_0\\ +\infty \text{ otherwise.}\end{cases}$$

We equip $L^p(\mathcal{O},\mathbb{R}^3)$ with its weak topology and establish the following main theorem of the paper:

Theorem 5.1. *The sequence $(H_\varepsilon)_{\varepsilon>0}$ almost surely sequentially Γ-converges to the infimum convolution $F_0\bigtriangledown G_0$ defined for every $u\in L^p(\mathcal{O},\mathbb{R}^3)$ by*

$$F_0\bigtriangledown G_0\ (u):=\inf_{v\in L^p(\mathcal{O},\mathbb{R}^3)}\Big(F_0(u-v)+G_0(v)\Big).$$

Consequently $(H_\varepsilon+\mathcal{L})_{\varepsilon>0}$ almost surely sequentially Γ-converges to the functional $F_0\bigtriangledown G_0+\mathcal{L}$.

The choice of the weak topology which equips $L^p(\mathcal{O},\mathbb{R}^3)$ is suggested by the following compactness result. The proof is very similar to that of Lemma 3.1 and 4.1 and left to the reader (see Remark 3.1).

Lemma 5.1. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\mathcal{O},\mathbb{R}^3)$ such that $\sup_{\varepsilon>0}H_\varepsilon(\omega,u_\varepsilon)<+\infty$ and set $v_\varepsilon=a(\omega,\frac{\cdot}{\varepsilon})u_\varepsilon$. Then, there exist (u,v) in $L^p(\mathcal{O},\mathbb{R}^3)$ and a subsequence possibly depending on ω such that for* **P** *almost every ω*

$$(u_\varepsilon,v_\varepsilon)\rightharpoonup(u,v) \text{ in } L^p(\mathcal{O},\mathbb{R}^3)\times L^p(\mathcal{O},\mathbb{R}^3)$$

$$\frac{\partial v_\varepsilon}{\partial x_3}\rightharpoonup\frac{\partial v}{\partial x_3} \text{ in } L^p(\mathcal{O},\mathbb{R}^3).$$

5.1 The lower bound

In this section, we establish the lower bound in the definition of the Γ-convergence of H_ε to H:

Proposition 5.1. *For every u_ε weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$, and for* **P***-almost every ω in Ω*

$$\mathcal{H}(u)\le\liminf_{\varepsilon\to0}H_\varepsilon(\omega,u_\varepsilon).$$

Proof. One may assume $\liminf_{\varepsilon\to0}H_\varepsilon(\omega,u_\varepsilon)<+\infty$, so that, for a non relabeled subsequence,

$$H_\varepsilon(\omega,u)=F^{u_\varepsilon}_\varepsilon(\omega,u_\varepsilon)+G_\varepsilon(\omega,1_{T_\varepsilon\cap\mathcal{O}}u_\varepsilon)$$

and, from Lemma 5.1, there exists $v\in V_0$ such that

$$(u_\varepsilon,v_\varepsilon)\rightharpoonup(u,v) \text{ in } L^p(\mathcal{O},\mathbb{R}^3)\times L^p(\mathcal{O},\mathbb{R}^3).$$

According to Proposition 3.2, Remark 3.2 and Proposition 4.2, we infer for **P** a.s. $\omega\in\Omega$

$$\liminf_{\varepsilon\to0}H_\varepsilon(\omega,u_\varepsilon)\ge F_0^{(1/\theta)v}(u)+\widetilde{G}_0(v)=F_0(u-\frac{1}{\theta}v)+G_0(\frac{1}{\theta}v),$$

thus

$$\liminf_{\varepsilon\to0}H_\varepsilon(\omega,u_\varepsilon)\ge\inf_{w\in L^p(\mathcal{O},\mathbb{R}^3)}\Big(F_0(u-w)+G_0(w)\Big)$$

which ends the proof. $\square$

5.2 The upper bound

Now we establish the upper bound in the definition of Γ-convergence:

Proposition 5.2. *For every u in $L^p(\mathcal{O},\mathbb{R}^3)$, there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $L^p(\mathcal{O},\mathbb{R}^3)$ such that for* **P***-almost every $\omega\in\Omega$, $u_\varepsilon(\omega)\rightharpoonup u$ and*

$$\limsup_{\varepsilon\to 0} H_\varepsilon(\omega,u_\varepsilon(\omega))\le H(u). \tag{31}$$

Proof. One may assume $H(u)<+\infty$. For $\eta>0$ intended to go to zero, let v_η be a η-minimizer in the definition of $H(u)$:

$$H(u)\ge F_0^{v_\eta}(u)+G_0(v_\eta)-\eta.$$

It is easily seen that one may assume that $v_\eta\in\mathcal{C}(\overline{\mathcal{O}},\mathbb{R}^3)$. According to Propositions 3.1, 3.2 there exists $u_{\eta,\varepsilon}(\omega)$ almost surely weakly converging to u in $L^p(\mathcal{O},\mathbb{R}^3)$ with $1_{T_\varepsilon\cap\mathcal{O}}u_{\eta,\varepsilon}(\omega)=v_\eta$, such that for **P**-almost every ω in Ω

$$\lim_{\varepsilon\to 0} F_\varepsilon^{v_\eta}(\omega,u_{\eta,\varepsilon}(\omega))=F_0^{v_\eta}(u). \tag{32}$$

On the other hand combining (28) and (30), there exists $v_{\eta,\varepsilon}(\omega)$ in $W^{1,p}_{\Gamma_0}(\mathcal{O}\cap T_\varepsilon,\mathbb{R}^3)$ almost surely weakly converging to θv_η in $L^p(\mathcal{O},\mathbb{R}^3)$ of the form

$$v_{\eta,\varepsilon}(\omega)=a(\omega,\frac{\hat{x}}{\varepsilon})\big[v_\eta(x)+\varepsilon\phi_1(\omega,\frac{\widehat{x}}{\varepsilon})\xi_1^\eta+\varepsilon\phi_2(\omega,\frac{\widehat{x}}{\varepsilon})\xi_2^\eta\big].$$

which satisfies for **P**-almost every ω

$$\lim_{\varepsilon\to 0} G_\varepsilon(\omega,v_{\eta,\varepsilon}(\omega))=\widetilde{G}_0(\theta v_\eta)=G_0(v_\eta). \tag{33}$$

From now on, we do not indicate the dependence of the functions $u_{\eta,\varepsilon}$ and $v_{\eta,\varepsilon}$ on ω. Combining (32) and (33) we infer

$$F_0^{v_\eta}(u)+G_0(v_\eta)=\lim_{\varepsilon\to 0} F_\varepsilon^{v_\eta}(\omega,u_{\eta,\varepsilon})+\lim_{\varepsilon\to 0} G_\varepsilon(\omega,v_{\eta,\varepsilon}). \tag{34}$$

let us set

$$\tilde{u}_{\eta,\varepsilon}=u_{\eta,\varepsilon}+\varepsilon\phi_1(\omega,\frac{\widehat{x}}{\varepsilon})\xi_1^\eta+\varepsilon\phi_2(\omega,\frac{\widehat{x}}{\varepsilon})\xi_2^\eta.$$

Note that $1_{T_\varepsilon\cap\mathcal{O}}\tilde{u}_{\eta,\varepsilon}=v_{\eta,\varepsilon}$ and that, from the Lipschitz condition satisfied by f,

$$\lim_{\varepsilon\to 0} F_\varepsilon^{\tilde{u}_{\eta,\varepsilon}}(\omega,\tilde{u}_{\eta,\varepsilon})=\lim_{\varepsilon\to 0} F_\varepsilon^{v_\eta}(\omega,u_{\eta,\varepsilon}).$$

Thus (34) yields

$$\begin{aligned} F_0^{v_\eta}(u)+G_0(v_\eta) &= \lim_{\varepsilon\to 0}\Big(F_\varepsilon^{\tilde{u}_{\eta,\varepsilon}}(\omega,\tilde{u}_{\eta,\varepsilon})+G_\varepsilon(\omega,v_{\eta,\varepsilon})\Big)\\ &= \lim_{\varepsilon\to 0} H_\varepsilon(\omega,\tilde{u}_{\eta,\varepsilon}) \end{aligned} \tag{35}$$

Clearly $\tilde{u}_{\eta,\varepsilon}\rightharpoonup u$ in $L^p(\mathcal{O},\mathbb{R}^3)$. We end the proof by letting $\eta\to 0$ and using a standard diagonalization argument. □

Collecting Lemma 5.1 and Theorem 5.1, and according to the variational nature of the Γ-convergence we obtain

Corollary 5.1. *The problem*

$$(\mathcal{P}_{H_\varepsilon})\qquad \inf\left\{H_\varepsilon(\omega,u)-\int_{\mathcal{O}} L.u\ dx : v\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

almost surely converges to the problem

$$(\mathcal{P}_H)\qquad \min\left\{H(u)-\int_{\mathcal{O}} L.u\ dx : v\in L^p(\mathcal{O},\mathbb{R}^3)\right\}$$

in the sense of the Γ-convergence and, up to a subsequence, every sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ of ε-minimizers of $(\mathcal{P}_{H_\varepsilon})$ almost surely weakly converges in $L^p(\mathcal{O},\mathbb{R}^3)$ to a minimizer of $(\mathcal{P}_H)$.

6 References

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