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Marianne Bessemoulin-Chatard, Claire Chainais-Hillairet, Francis Filbet. On discrete functional inequalities for some finite volume schemes. 2012. hal-00672591v2

**HAL Id: hal-00672591**

**<https://hal.science/hal-00672591v2>**

Preprint submitted on 21 Feb 2012 (v2), last revised 15 Jan 2014 (v3)

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# On discrete functional inequalities for some finite volume schemes

Marianne BESSEMOULIN-CHATARD, Claire CHAINAIS-HILLAIRET  
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February 21, 2012

## Abstract

We prove several discrete Gagliardo-Nirenberg-Sobolev and Sobolev-Poincaré inequalities for some approximations with arbitrary boundary values on finite volume admissible meshes. The keypoint of our approach is to use the continuous embedding of the space  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$  for a Lipschitz domain  $\Omega \subset \mathbb{R}^N$ , with  $N \geq 2$ . Finally, we give several applications to discrete duality finite volume (DDFV) schemes which are used for the approximation of nonlinear and non isotropic elliptic and parabolic problems.

**Keywords.** Discrete functional inequality, finite volume scheme, DDFV.

**AMS Subject Classification.** 65N08, 65M12.

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# 1 Introduction

In this paper, we establish some discrete functional inequalities which are sometimes useful for the convergence analysis of finite volume schemes. In the continuous framework, the Gagliardo-Nirenberg-Sobolev and Sobolev-Poincaré inequalities are fundamental for the analysis of partial differential equations. They are a standard tool in existence and regularity theories for solutions. The  $L^2$  framework is generally used for linear elliptic problems, more precisely it is a classical way to prove the coercivity of bilinear forms in  $H_0^1$ , which then allows to apply the Lax-Milgram theorem to prove existence of weak solutions. More generally, the  $L^p$  framework is crucial for the study of nonlinear elliptic or parabolic equations, to obtain some energy estimates which are useful to prove existence of weak solutions. Poincaré-type inequalities are also one of the step in the study of convergence to equilibrium for kinetic equations.

## 1.1 Gagliardo-Nirenberg-Sobolev and Sobolev-Poincaré inequalities

In the continuous situation, the Gagliardo-Nirenberg-Sobolev inequality writes as follows. Let assume  $N \geq 2$  and  $\Omega$  be an open domain of  $\mathbb{R}^N$ . Then for  $1 < p \leq N$  and  $q \geq 1$ , there exists a constant  $C > 0$  such that for all  $u \in W^{1,p}(\Omega) \cap L^q(\Omega)$ ,

$$\|u\|_{L^m(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}, \quad (1)$$

where

$$0 \leq \theta \leq \frac{p}{p+q(p-1)} \leq 1 \quad \text{and} \quad \frac{1}{m} = \frac{1-\theta}{q} + \frac{\theta}{p} - \frac{\theta}{N}.$$

We refer to [19, 31] for a proof of this result. We also remind the well-known Sobolev-Poincaré inequality [1, 8]:

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \quad (2)$$

for

$$1 \leq q \leq \frac{pN}{N-p} \quad \text{if} \quad 1 \leq p < N,$$

$$\text{or} \quad 1 \leq q < +\infty \quad \text{if} \quad p \geq N.$$

The mathematical analysis of convergence and error estimates for numerical methods are performed using functional analysis tools, such as discrete Sobolev inequalities. Several Poincaré-Sobolev inequalities have been established for the finite volume schemes as well as for the finite element methods. Concerning the finite volume framework, the first estimates were obtained in the particular case  $N = 2$ ,  $p = q = 2$  (which is the standard Poincaré inequality) for Dirichlet boundary conditions Y. Coudière and G. Manzini [11]. The idea of the proof in these papers is to use some geometrical properties of the mesh. More precisely, given an oriented direction  $\mathcal{D}$ , any cell center of the mesh is connected to an upstream (with respect to  $\mathcal{D}$ ) center of an edge of the boundary  $\partial\Omega$  by a straight line of direction  $\mathcal{D}$ . This connection crosses a certain number of cells and their interfaces, and this argument allows to link a norm of the piecewise constant function considered with a norm of a discrete version of its gradient. This result was later generalized to the case of dimension  $N = 3$  by R. Eymard, R. Herbin and Th. Gallouët [16]. Also the same method has been applied to get more general Sobolev-Poincaré inequalities (2) for  $1 \leq p = q \leq 2$  by J. Droniou, Th. Gallouët and R. Herbin [15], and for  $p = 2$ ,  $1 \leq q < \infty$  if  $N = 2$ ,  $1 \leq q \leq 6$  if  $N = 3$  by Y. Coudière Th. Gallouët and R. Herbin [9], still in the case of Dirichlet boundary conditions. Concerning the case of Neumann boundary conditions, a discrete Poincaré-Wirtinger inequality ( $p = q = 2$ ) was established in [16, 20] for  $N = 2$  or 3 by using the same method.

More recently, another idea was used to prove this type of discrete inequalities: the continuous embedding of  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$  for a Lipschitz domain  $\Omega$ . This argument was first exploited in [18] to prove a discrete Sobolev-Poincaré inequality (2) in dimension  $N = 2$  with  $q = 2$  and  $p = 1$ , in the case of Neumann boundary conditions. Then this method was used in [17] to prove general Sobolev-Poincaré inequalities (2) in any dimension  $N \geq 1$  in the particular case of homogeneous Dirichlet boundary conditions. We also mention [5] where the continuous embedding of  $BV(\mathbb{R}^N)$  into  $L^{N/(N-1)}(\mathbb{R}^N)$  is used to establish an improved discrete Gagliardo-Nirenberg-Sobolev inequality in the whole space  $\mathbb{R}^N$ ,  $N \geq 1$ .

Finally for  $p = 2$ , general discrete Sobolev-Poincaré inequalities are obtained in [21] for Voronoi finite volume approximations in the case of arbitrary boundary conditions by using an adaptation of Sobolev's integral representation and the Voronoi property of the mesh. Concerning the finite element framework, a variant of a Poincaré-type inequality ( $p = q = 2$ ) for functions in broken Sobolev spaces was derived in [4] for  $N = 2$  and in [7, 32] for  $N = 2, 3$ . Then a generalised result was proposed in [28], providing bounds on the  $L^q$  norms in terms of a broken  $H^1$  norm ( $p = 2$ ,  $1 \leq q < \infty$  if  $N = 2$  and  $1 \leq q \leq 2N/(N - 2)$  if  $N \geq 3$ ). The proof is based on elliptic regularity results and nonconforming finite element interpolants. Finally, a result in non-Hilbertian setting ( $p \neq 2$ ) was obtained in [13], taking inspiration from the technique used by F. Filbet [18] and also R. Eymard, R. Herbin and Th. Gallouët [17], namely the continuous embedding of  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$ .

## 1.2 Aim of the paper and outline

In this paper our aim is to provide a simple proof to discrete versions of Gagliardo-Nirenberg-Sobolev (1) and Sobolev-Poincaré (2) inequalities for functions coming from finite volume schemes with arbitrary boundary values. Several Sobolev-Poincaré inequalities are already proved as mentioned above but here we propose a unified result. It includes in particular the case of mixed boundary conditions. Concerning Gagliardo-Nirenberg-Sobolev inequalities, the result of F. Bouchut, R. Eymard and A. Prignet [5] is to our knowledge the only available, and it deals with the case of the whole space  $\mathbb{R}^N$ .

Our starting point to prove these discrete estimates is the continuous embedding of  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$ , as in [18, 17, 13, 5]. The main difficulty appears when boundary conditions must be taken into account. In the papers mentioned previously [18, 17, 13], the boundary conditions are either homogeneous Dirichlet or Neumann on the whole boundary. In [5], the problem is considered in the whole space  $\mathbb{R}^N$ . In the case where the function satisfies homogeneous Dirichlet boundary conditions only on a part  $\Gamma^0 \subsetneq \partial\Omega$  of the boundary, we cannot use the same strategy as in [17], which consists of extending the function considered to  $\mathbb{R}^N$  by zero. Our idea is to thicken the boundary of  $\Omega$  to take the mixed boundary conditions into account in this case.

The outline of the paper is as follows. In Section 2, we first define the functional spaces: the space of finite volume approximations and the space  $BV(\Omega)$ . We will see that  $BV(\Omega)$  is a natural space to study piecewise constant functions as finite volume approximations. In Section 3, we do not take into account any boundary conditions and prove the discrete Gagliardo-Nirenberg-Sobolev inequalities (Theorem 3) and the discrete Sobolev-Poincaré inequalities (Theorem 4) in this case. These results are the discrete counterpart of (1) and (2). They may be used for instance in the convergence analysis of finite volume schemes in the case with Neumann boundary conditions. Then, in Section 4, we consider the case where the discrete function is given by a finite volume scheme with homogeneous boundary conditions on a part of the boundary. In this case, the discrete space (for the finite volume approximations) is unchanged. However, the discrete  $W^{1,p}$  seminorm will take into account some jumps on the boundary. We prove discrete

Gagliardo-Nirenberg-Sobolev inequalities (Theorem 5) and discrete Sobolev-Poincaré inequalities (Theorem 6), similar to (1) and (2) but with the  $W^{1,p}$ -seminorm instead of the full  $W^{1,p}$ -norm. Finally, in Section 5, we show how to extend the results from Sections 3 and 4 to finite volume approximations coming from discrete dual finite volume (DDFV) schemes. This family of schemes is mainly applied to elliptic and parabolic problems. This method can be applied to a wide class of 2D meshes (but also 3D [10]) and inherits the main qualitative properties of the continuous problem: monotonicity, coercivity, variational formulation, etc...

## 2 Functional spaces

### 2.1 The space of finite volume approximations

We now introduce the discrete settings, including notations and assumptions on the meshes and definitions of the discrete norms. Let  $\Omega$  be an open bounded polyhedral subset (Lipschitz domain) of  $\mathbb{R}^N$ ,  $N \geq 2$ , and  $\Gamma := \partial\Omega$  its boundary. An admissible mesh of  $\Omega$  is given by a family  $\mathfrak{M}$  of control volumes, a family  $\mathcal{E}$  of relatively open parts of hyperplans in  $\mathbb{R}^N$  (which represent the faces of the control volumes) and a family of points  $(x_K)_{K \in \mathfrak{M}}$  which satisfy the Definition 9.1 in [16]. It implies that the straight line between two neighboring centers of cells  $(x_K, x_L)$  is orthogonal to the face  $\sigma = K|L$ . In the set of faces  $\mathcal{E}$ , we distinguish the interior faces  $\sigma \in \mathcal{E}_{int}$  and the boundary faces  $\sigma \in \mathcal{E}_{ext}$ . For a control volume  $K \in \mathfrak{M}$ , we denote by  $\mathcal{E}_K$  the set of its faces,  $\mathcal{E}_{int,K}$  the set of its interior faces and  $\mathcal{E}_{ext,K}$  the set of faces of  $K$  included in the boundary  $\Gamma$ .

In the sequel we denote by  $d$  the distance in  $\mathbb{R}^N$ ,  $m$  the Lebesgue measure in  $\mathbb{R}^N$  or  $\mathbb{R}^{N-1}$ . For all  $\sigma \in \mathcal{E}$ , we define

$$d_\sigma = \begin{cases} d(x_K, x_L) & \text{for } \sigma = K|L \in \mathcal{E}_{int}, \\ d(x_K, \sigma) & \text{for } \sigma \in \mathcal{E}_{ext,K}. \end{cases}$$

We assume that the family of meshes considered satisfies the following regularity constraint: there exists  $\xi > 0$  such that

$$d(x_K, \sigma) \geq \xi d_\sigma, \quad \text{for } K \in \mathfrak{M}, \quad \text{for } \sigma \in \mathcal{E}_K. \quad (3)$$

The size of the mesh is defined by

$$h = \max_{K \in \mathfrak{M}} (\text{diam}(K)). \quad (4)$$

In general, finite volume methods lead to the computation of one discrete unknown by control volume. The corresponding finite volume approximation is a piecewise constant function. Therefore, we define the set  $X(\mathfrak{M})$  of the finite volume approximation:

$$X(\mathfrak{M}) = \left\{ u \in L^1(\Omega) / \exists (u_K)_{K \in \mathfrak{M}} \text{ such that } u = \sum_{K \in \mathfrak{M}} u_K \mathbf{1}_K \right\}.$$

Let us now define some discrete norms and seminorms on  $X(\mathfrak{M})$ .

**Definition 1.** *Let  $\Omega$  be a bounded polyhedral subset of  $\mathbb{R}^N$ ,  $\mathfrak{M}$  an admissible mesh of  $\Omega$ .*

1. *For  $p \in [1, +\infty)$ , the discrete  $L^p$  norm is defined by*

$$\|u\|_{0,p,\mathfrak{M}} = \left( \sum_{K \in \mathfrak{M}} m(K) |u_K|^p \right)^{\frac{1}{p}}, \quad \forall u \in X(\mathfrak{M}).$$

2. In the general case, for  $p \in [1, +\infty)$ , the discrete  $W^{1,p}$ -seminorm is defined by:

$$|u|_{1,p,\mathfrak{M}} = \left( \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{m(\sigma)}{d_\sigma^{p-1}} |u_L - u_K|^p \right)^{\frac{1}{p}}, \forall u \in X(\mathfrak{M})$$

and the discrete  $W^{1,p}$ -norm is defined by

$$\|u\|_{1,p,\mathfrak{M}} = \|u\|_{0,p,\mathfrak{M}} + |u|_{1,p,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}). \quad (5)$$

3. In the case where homogeneous Dirichlet boundary conditions are underlying (because the piecewise constant function comes from a finite volume scheme), we need to take into account jumps on the boundary in the discrete  $W^{1,p}$ -seminorm. Let  $\Gamma^0 \subset \Gamma$  be a part of the boundary. In the set of exterior faces  $\mathcal{E}_{ext}$ , we distinguish  $\mathcal{E}_{ext}^0$  the set of boundary faces included in  $\Gamma^0$ . For  $p \in [1, +\infty)$ , we define the discrete  $W^{1,p}$ -seminorm (which depends on  $\Gamma^0$ ) by

$$|u|_{1,p,\Gamma^0,\mathfrak{M}} = \left( \sum_{\sigma \in \mathcal{E}} \frac{m(\sigma)}{d_\sigma^{p-1}} (D_\sigma u)^p \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty, \quad (6)$$

where

$$D_\sigma u = \begin{cases} |u_K - u_L| & \text{if } \sigma = K|L \in \mathcal{E}_{int}, \\ |u_K| & \text{if } \sigma \in \mathcal{E}_{ext}^0 \cap \mathcal{E}_K, \\ 0 & \text{if } \sigma \in \mathcal{E}_{ext} \setminus \mathcal{E}_{ext}^0. \end{cases} \quad (7)$$

We then define the discrete  $W^{1,p}$  norm by

$$\|u\|_{1,p,\mathfrak{M}} = \|u\|_{0,p,\mathfrak{M}} + |u|_{1,p,\Gamma^0,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}). \quad (8)$$

## 2.2 The space $BV(\Omega)$

Let us first recall some results concerning functions of bounded variation (we refer to [2, 33] for a thorough presentation  $BV(\Omega)$ ). Let  $\Omega$  be an open set of  $\mathbb{R}^N$  and  $u \in L^1(\Omega)$ . The *total variation* of  $u$  in  $\Omega$ , denoted by  $TV_\Omega(u)$ , is defined by

$$TV_\Omega(u) = \sup \left\{ \int_\Omega u(x) \operatorname{div}(\phi(x)) \, dx, \quad \phi \in \mathcal{C}_c^1(\Omega), \quad |\phi(x)| \leq 1, \quad \forall x \in \Omega \right\} \quad (9)$$

and the function  $u \in L^1(\Omega)$  belongs to  $BV(\Omega)$  if and only if  $TV_\Omega(u) < +\infty$ . The space  $BV(\Omega)$  is endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + TV_\Omega(u).$$

The space  $BV(\Omega)$  is a natural space to study finite volume approximations. Indeed, as it is proved for instance in [18], for  $u \in X(\mathfrak{M})$ , we have

$$TV_\Omega(u) \leq \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} m(\sigma) |u_L - u_K| = |u|_{1,1,\mathfrak{M}} < +\infty.$$

The discrete space  $X(\mathfrak{M})$  is included in  $L^1 \cap BV(\Omega)$ . Moreover,  $\|u\|_{BV(\Omega)} \leq \|u\|_{1,1,\mathfrak{M}}$ .

Our starting point for the discrete functional inequalities is the continuous embedding of  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$  for a Lipschitz domain  $\Omega$ , recalled in Theorem 1.

**Theorem 1.** *Let  $\Omega$  be a Lipschitz bounded domain of  $\mathbb{R}^N$ ,  $N \geq 2$ . Then there exists a constant  $c(\Omega)$  only depending on  $\Omega$  such that:*

$$\left( \int_{\Omega} |u|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \leq c(\Omega) \|u\|_{BV(\Omega)}, \quad \forall u \in BV(\Omega). \quad (10)$$

There are also more precise results involving only the seminorm  $TV_{\Omega}(u)$  instead of the norm  $\|u\|_{BV(\Omega)}$ . Indeed, the seminorm  $TV_{\Omega}$  becomes a norm on the space of BV functions vanishing on a part of the boundary and also on the space of BV functions with a zero mean value. In these cases, the continuous embedding of  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$  rewrites as in Theorem 2.

**Theorem 2.** *Let  $\Omega$  be a Lipschitz bounded domain of  $\mathbb{R}^N$ ,  $N \geq 2$ .*

1. *There exists a constant  $c(\Omega) > 0$  only depending on  $\Omega$  such that, for all  $u \in BV(\Omega)$ ,*

$$\left( \int_{\Omega} |u - \bar{u}|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \leq c(\Omega) TV_{\Omega}(u), \quad (11)$$

where  $\bar{u}$  is the mean value of  $u$ :

$$\bar{u} = \frac{1}{m(\Omega)} \int_{\Omega} u(x) dx.$$

2. *Let  $\Gamma^0 \subset \partial\Omega$ ,  $\Gamma^0 \neq \emptyset$ . There exists a constant  $c(\Omega) > 0$  only depending on  $\Omega$  such that, for all  $u \in BV(\Omega)$  satisfying  $u = 0$  on  $\Gamma^0$ ,*

$$\left( \int_{\Omega} |u|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} \leq c(\Omega) TV_{\Omega}(u). \quad (12)$$

Actually, the constant  $c(\Omega)$  involved in Theorems 1 and 2 depends only on  $\theta$  and  $r$  such that the domain  $\Omega$  has the cone property for these parameters (see [1, Lemma 4-24], [30, Theorem 8-8]).

### 3 Discrete functional inequalities in the general case

We first consider the general case  $u \in X(\mathfrak{M})$  with the discrete  $W^{1,p}$  norm defined by (5). The discrete functional inequalities we will prove may be useful in the convergence analysis of finite volume methods for problems with homogeneous Neumann boundary conditions.

#### 3.1 General discrete Gagliardo-Nirenberg-Sobolev inequality

We start with the discrete Gagliardo-Nirenberg-Sobolev inequalities which are the discrete counterpart of (1).

**Theorem 3** (General discrete Gagliardo-Nirenberg-Sobolev inequality). *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^N$ ,  $N \geq 2$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4).*

*Then for  $1 < p \leq N$  and  $q \geq 1$ , there exists a constant  $C > 0$  only depending on  $p$ ,  $q$ ,  $N$  and  $\Omega$  such that*

$$\|u\|_{0,m,\mathfrak{M}} \leq \frac{C}{\xi^{(p-1)\theta/p}} \|u\|_{1,p,\mathfrak{M}}^{\theta} \|u\|_{0,q,\mathfrak{M}}^{1-\theta}, \quad \forall u \in X(\mathfrak{M}), \quad (13)$$

where

$$0 \leq \theta \leq \frac{p}{p+q(p-1)} \leq 1 \quad (14)$$

and

$$\frac{1}{m} = \frac{1-\theta}{q} + \frac{\theta}{p} - \frac{\theta}{N}. \quad (15)$$

*Proof.* Throughout this proof,  $C$  denotes constants which depend only on  $\Omega$ ,  $N$ ,  $p$  and  $q$ .

As seen in Section 2.2, we have  $\|v\|_{BV(\Omega)} \leq \|v\|_{1,1,\mathfrak{M}}$  for all  $v \in X(\mathfrak{M})$ . Therefore, applying Theorem 1, we get

$$\|v\|_{0,N/(N-1),\mathfrak{M}} \leq c(\Omega) (\|v\|_{1,1,\mathfrak{M}} + \|v\|_{0,1,\mathfrak{M}}) \quad \forall v \in X(\mathfrak{M}). \quad (16)$$

Let  $s \geq 1$ . For  $u \in X(\mathfrak{M})$ , we now define  $v \in X(\mathfrak{M})$  by  $v_K = |u_K|^s$  for all  $K \in \mathfrak{M}$ . We note that

$$\begin{aligned} \|v\|_{0,N/(N-1),\mathfrak{M}} &= \left( \sum_{K \in \mathfrak{M}} m(K) |u_K|^{\frac{sN}{N-1}} \right)^{\frac{N-1}{N}} = \|u\|_{0,sN/(N-1),\mathfrak{M}}^s \\ \|v\|_{0,1,\mathfrak{M}} &= \sum_{K \in \mathfrak{M}} m(K) |u_K|^s = \|u\|_{0,s,\mathfrak{M}}^s \end{aligned}$$

and

$$|v|_{1,1,\mathfrak{M}} = \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} m(\sigma) \left| |u_K|^s - |u_L|^s \right|.$$

But, for all  $\sigma = K|L$ , we have

$$\left| |u_K|^s - |u_L|^s \right| \leq s (|u_K|^{s-1} + |u_L|^{s-1}) |u_K - u_L|.$$

Applying a discrete integration by parts and Hölder's inequality, we get, for any  $1 < p \leq N$  and  $s \geq 1$ :

$$\begin{aligned} \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} m(\sigma) \left| |u_K|^s - |u_L|^s \right| &\leq s \sum_{K \in \mathfrak{M}} \sum_{\sigma = K|L} m(\sigma) |u_K|^{s-1} |u_K - u_L| \\ &\leq s \left( \sum_{K \in \mathfrak{M}} \sum_{\sigma = K|L} \frac{m(\sigma)}{d_\sigma^{p-1}} |u_L - u_K|^p \right)^{\frac{1}{p}} \left( \sum_{K \in \mathfrak{M}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_\sigma |u_K|^{\frac{(s-1)p}{p-1}} \right)^{\frac{p-1}{p}}. \end{aligned}$$

But, the regularity constraint (3) on the mesh ensures that for all  $K \in \mathfrak{M}$ :

$$\sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_\sigma \leq \frac{1}{\xi} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d(x_K, \sigma) = \frac{N}{\xi} m(K), \quad (17)$$

and then, for any  $1 < p \leq N$ ,  $s \geq 1$ , we get:

$$\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} m(\sigma) \left| |u_K|^s - |u_L|^s \right| \leq \frac{C_s}{\xi^{(p-1)/p}} |u|_{1,p,\mathfrak{M}} \|u\|_{0,(s-1)p/(p-1),\mathfrak{M}}^{(s-1)}$$



with  $1 < p \leq N$  and  $s \geq 1$ . Therefore, from (16), we obtain that for  $u \in X(\mathfrak{M})$ ,  $1 < p \leq N$ ,  $s \geq 1$ , and  $q := (s-1)p/(p-1) \geq 1$ :

$$\|u\|_{0, sN/(N-1), \mathfrak{M}}^s \leq C \left( \frac{1}{\xi^{(p-1)/p}} \|u\|_{1, p, \mathfrak{M}} \|u\|_{0, q, \mathfrak{M}}^{s-1} + \|u\|_{0, s, \mathfrak{M}}^s \right).$$

However, since

$$\frac{1}{s} = \frac{1/s}{p} + \frac{(s-1)/s}{q} \leq 1,$$

we obtain by interpolation that

$$\|u\|_{0, s, \mathfrak{M}} \leq \|u\|_{0, p, \mathfrak{M}}^{1/s} \|u\|_{0, q, \mathfrak{M}}^{(s-1)/s}, \quad (18)$$

which yields

$$\|u\|_{0, sN/(N-1), \mathfrak{M}}^s \leq C \left( \frac{1}{\xi^{(p-1)/p}} \|u\|_{1, p, \mathfrak{M}} \|u\|_{0, q, \mathfrak{M}}^{s-1} + \|u\|_{0, p, \mathfrak{M}} \|u\|_{0, q, \mathfrak{M}}^{s-1} \right),$$

and finally

$$\|u\|_{0, sN/(N-1), \mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/(ps)}} \|u\|_{0, q, \mathfrak{M}}^{(s-1)/s} \|u\|_{1, p, \mathfrak{M}}^{1/s}. \quad (19)$$

Using the following interpolation inequality

$$\|u\|_{0, m, \mathfrak{M}} \leq \|u\|_{0, r, \mathfrak{M}}^\alpha \|u\|_{0, q, \mathfrak{M}}^{1-\alpha} \quad \text{when} \quad \frac{1}{m} = \frac{1-\alpha}{q} + \frac{\alpha}{r} \quad \text{with} \quad 0 \leq \alpha \leq 1, \quad (20)$$

with  $r = sN/(N-1)$ , we get

$$\|u\|_{0, m, \mathfrak{M}} \leq \frac{C}{\xi^{\alpha(p-1)/(ps)}} \|u\|_{1, p, \mathfrak{M}}^{\alpha/s} \|u\|_{0, q, \mathfrak{M}}^{1-\alpha/s}, \quad \forall 0 \leq \alpha \leq 1.$$

Setting  $\theta = \alpha/s$ , with  $s = (p-1)q/p + 1$ , we get the expected inequality (13) under the conditions (14) and (15).  $\square$

### 3.2 General discrete Sobolev-Poincaré inequality

We now give the discrete counterpart of the Sobolev-Poincaré inequalities (2).

**Theorem 4** (General discrete Sobolev-Poincaré inequality). *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^N$ ,  $N \geq 2$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4).*

*Then there exists a constant  $C > 0$  only depending on  $p$ ,  $q$ ,  $N$  and  $\Omega$  such that:*

- if  $1 \leq p < N$ , for all  $1 \leq q \leq p^* := \frac{pN}{N-p}$ ,

$$\|u\|_{0, q, \mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} \|u\|_{1, p, \mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}), \quad (21)$$

- if  $p \geq N$ , for all  $1 \leq q < +\infty$ ,

$$\|u\|_{0, q, \mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} \|u\|_{1, p, \mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}). \quad (22)$$

*Proof.* Throughout this proof,  $C$  denotes constants which depend only on  $\Omega$ ,  $N$ ,  $p$  and  $q$ . The case  $p = 1$ ,  $q = p^* = N/(N-1)$  corresponds to estimate (16). Then we obtain (21) for  $p = 1$  and for all  $1 \leq q \leq p^*$  by using the fact that  $L^{p^*}(\Omega) \subset L^q(\Omega)$  since  $\Omega$  is bounded. Let us first consider the case where  $1 < p < N$ . We get from (19) that

$$\|u\|_{0, sN/(N-1), \mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/(ps)}} \|u\|_{0, q, \mathfrak{M}}^{(s-1)/s} \|u\|_{1, p, \mathfrak{M}}^{1/s} \quad (23)$$

with

$$q = \frac{(s-1)p}{p-1} \leq 1.$$

Then we choose  $s \geq 1$  such that  $\frac{sN}{N-1} = \frac{(s-1)p}{p-1}$ , that is  $s = \frac{N-1}{N-p}p$ , which implies

$$\frac{sN}{N-1} = \frac{pN}{N-p}.$$

Therefore, we get

$$\|u\|_{0, pN/(N-p), \mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} \|u\|_{1, p, \mathfrak{M}} \quad \forall 1 < p < N,$$

and since  $L^{pN/(N-p)}(\Omega) \subset L^q(\Omega)$  for all  $1 \leq q \leq pN/(N-p)$ , it yields

$$\|u\|_{0, q, \mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} \|u\|_{1, p, \mathfrak{M}} \quad \forall q \in \left[1, \frac{pN}{N-p}\right],$$

and the proof is complete for  $1 \leq p < N$ .

Now for  $p \geq N$ , we begin with the case  $p = N$ . Using the Young's inequality in (19), we obtain

$$\|u\|_{0, sN/(N-1), \mathfrak{M}} \leq C_\star \left( \frac{1}{\xi^{(N-1)/N}} \|u\|_{1, N, \mathfrak{M}} + \|u\|_{0, (s-1)N/(N-1), \mathfrak{M}} \right), \quad (24)$$

with  $C_\star$  independant of  $s$ . Then we proceed by induction on  $s \geq N-1$  to prove that there exists a constant  $C_s$  depending on  $s$  such that

$$\|u\|_{0, sN/(N-1), \mathfrak{M}} \leq \frac{C_s}{\xi^{(N-1)/N}} \|u\|_{1, N, \mathfrak{M}}. \quad (25)$$

For  $s = N-1$ , the result is given by (23). Then let  $s \geq N$  such that (25) is true for  $s-1$ . Using (24) and (25), we get

$$\|u\|_{0, sN/(N-1), \mathfrak{M}} \leq \frac{C_\star(1 + C_{s-1})}{\xi^{(N-1)/N}} \|u\|_{1, N, \mathfrak{M}}.$$

Then (25) is true for all  $s \geq N-1$ , which finally yields

$$\|u\|_{0, q, \mathfrak{M}} \leq \frac{C_q}{\xi^{(N-1)/N}} \|u\|_{1, N, \mathfrak{M}} \quad \forall q \in [1, +\infty[, \quad (26)$$

which is the result for  $p = N$ . We emphasize that  $C_q \rightarrow +\infty$  as  $q \rightarrow +\infty$ .

Finally for  $p > N$ , we obtain the result using the fact that

$$\|u\|_{1, N, \mathfrak{M}} \leq \frac{C}{\xi^{(p-N)/pN}} \|u\|_{1, p, \mathfrak{M}} \quad \forall p \geq N. \quad (27)$$

Gathering (26) and (27) we get

$$\|u\|_{0,q,\mathfrak{M}} \leq \frac{C}{\xi^{(N-1)/N}} \|u\|_{1,N,\mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} \|u\|_{1,p,\mathfrak{M}} \quad \forall q \in [1, +\infty[,$$

which completes the proof of Theorem 4.  $\square$

### 3.3 Other discrete functional inequalities

From Theorems 3 and 4, we can deduce a discrete Nash inequality:

**Corollary 1** (Discrete Nash inequality). *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^N$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4). Then there exists a constant  $C > 0$  only depending on  $\Omega$  and  $N$  such that*

$$\|u\|_{0,2,\mathfrak{M}}^{1+\frac{2}{N}} \leq \frac{C}{\sqrt{\xi}} \|u\|_{1,2,\mathfrak{M}} \|u\|_{0,1,\mathfrak{M}}^{\frac{2}{N}}, \quad \forall u \in X(\mathfrak{M}). \quad (28)$$

*Proof.* For  $N = 2$ , the result is directly given by the application of Theorem 3 with  $p = 2$ ,  $q = 1$ ,  $\theta = 1/N = 1/2$  and  $m = 2$ . For  $N \geq 3$ , let us first apply Hölder's inequality:

$$\|u\|_{0,2,\mathfrak{M}}^2 = \sum_{K \in \mathfrak{M}} m(K) |u_K|^{4/(N+2)} |u_K|^{2N/(N+2)} \leq \|u\|_{0,1,\mathfrak{M}}^{4/(N+2)} \|u\|_{0,2N/(N-2),\mathfrak{M}}^{2N/(N+2)}. \quad (29)$$

Then we apply Theorem 4 with  $1 \leq p = 2 < N$  and  $q = p^* = 2N/(N-2)$ :

$$\|u\|_{0,2N/(N-2),\mathfrak{M}} \leq \frac{C}{\sqrt{\xi}} \|u\|_{1,2,\mathfrak{M}}. \quad (30)$$

Gathering (29) and (30), it yields the result.  $\square$

In the proofs of Theorem 3 and Theorem 4, we have used the continuous embedding of  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$  as it is written in Theorem 1. But, starting with (11) instead of (10) leads to a discrete Poincaré-Wirtinger inequality given in Proposition 1. This result has already been proved in [18].

**Proposition 1.** *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^N$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4). Then there exists a constant  $C > 0$  only depending on  $\Omega$  and  $N$  such that*

$$\|u - \bar{u}\|_{0,N/(N-1),\mathfrak{M}} \leq \frac{c(\Omega)}{\sqrt{\xi}} |u|_{1,1,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}).$$

*It yields estimates only involving the discrete  $W^{1,2}$ -seminorm in some particular cases:*

$$\|u - \bar{u}\|_{0,q,\mathfrak{M}} \leq \frac{C}{\sqrt{\xi}} |u|_{1,2,\mathfrak{M}} \quad \forall q \in \left[1, \frac{N}{N-1}\right], \forall u \in X(\mathfrak{M}).$$

(We recall that  $\bar{u} = \frac{1}{m(\Omega)} \int_{\Omega} u(x) dx = \frac{1}{m(\Omega)} \sum_{K \in \mathfrak{M}} m(K) u_K$  for  $u \in X(\mathfrak{M})$ .)

## 4 Discrete functional inequalities in the case of Dirichlet boundary conditions

In this Section, we consider the case where the finite volume approximation  $u \in X(\mathfrak{M})$  is coming from a finite volume scheme where homogeneous boundary conditions are prescribed on a part of the boundary. This part of the boundary is denoted by  $\Gamma^0 \subset \Gamma$ ,  $\Gamma^0 \neq \emptyset$ . In this case, the natural discrete counterparts of the  $W^{1,p}$ -seminorm and  $W^{1,p}$ -norm are defined by (6) and (8). Moreover, the  $W^{1,p}$ -seminorm becomes a norm on the space of  $W^{1,p}$  functions vanishing on a part of the boundary and the Gagliardo-Nirenberg-Sobolev inequalities and the Sobolev-Poincaré inequalities may be rewritten with the  $W^{1,p}$ -seminorm instead of the  $W^{1,p}$ -norm. Our aim in this Section is to prove the discrete counterpart of such inequalities (see Theorem 5 and Theorem 6).

As in the general case, the starting point will be the continuous embedding from  $BV(\Omega)$  into  $L^{N/(N-1)}(\Omega)$ , which rewrites as (12) with homogeneous Dirichlet boundary conditions on the part of the boundary. However, (12) can not be directly applied to  $u \in X(\mathfrak{M})$ . Indeed,  $u \in X(\mathfrak{M})$  belongs to  $BV(\Omega)$  and therefore its trace on the boundary is well defined; but it does not necessarily vanish on  $\Gamma^0$ . Some adaptations must be done in order to apply (12) and get its discrete counterpart. It will be done in Section 4.1 and yield the discrete functional inequalities presented in Section 4.2 and Section 4.3.

In this section, we assume the open set  $\Omega$  is also convex. This will be particularly crucial in Lemma 1.

### 4.1 Preliminary Lemma

We begin with a Lemma which gives the discrete counterpart of (12). This Lemma is crucial to prove Theorems 5 and 6.

**Lemma 1.** *Let  $\Omega$  be an open convex bounded polyhedral domain of  $\mathbb{R}^N$  and  $\Gamma^0 \neq \emptyset$  be a part of the boundary  $\Gamma$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4). Then there exists a constant  $c(\Omega)$  only depending on  $\Omega$  such that*

$$\|u\|_{0,N/(N-1),\mathfrak{M}} \leq c(\Omega) |u|_{1,1,\Gamma^0,\mathfrak{M}}, \quad \forall u \in X(\mathfrak{M}).$$

*Proof.* Let us consider  $u \in X(\mathfrak{M})$ ; since  $u$  is piecewise constant,  $u$  belongs to  $BV(\Omega)$ . Then we can define the trace  $Tu$  of  $u$  by: for almost every  $x \in \Gamma$ ,

$$\lim_{r \rightarrow 0} \frac{1}{\mathfrak{m}(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega} |v - Tv(x)| dy = 0.$$

Thus in general  $Tu|_{\Gamma^0} \neq 0$  and in this framework we cannot take into account the prescribed homogeneous Dirichlet boundary conditions  $u_\sigma = 0$  for  $\sigma \in \mathcal{E}_{ext}^0$ . Therefore the idea is to thicken the domain  $\Omega$  into a larger domain  $\Omega_\varepsilon$  with  $\Omega \subset \Omega_\varepsilon$  and define an extension  $u_\varepsilon$  of  $u$  to  $\Omega_\varepsilon$  such that  $u_\varepsilon \in BV(\Omega_\varepsilon)$  and  $Tu_\varepsilon = 0$  on a non empty part of the boundary  $\partial\Omega_\varepsilon$ , which allows to apply Theorem 2 to  $u_\varepsilon$ .

Let  $\sigma \in \mathcal{E}_{ext}$  be a face included in the boundary  $\Gamma$ . Then  $\sigma$  is a part of an hyperplane  $\mathcal{H}$  in  $\mathbb{R}^N$ . We denote by  $\mathbf{n}_\sigma$  the unit vector normal to  $\mathcal{H}$  outward to  $\Omega$ . For every  $x \in \mathbb{R}^N$ , there exists a unique  $(y, y') \in \mathcal{H} \times \mathbb{R}$  such that  $x = y + y' \mathbf{n}_\sigma$ . For  $\varepsilon > 0$ , we define (see Figure 1)

$$K_\sigma^\varepsilon := \{x = y + y' \mathbf{n}_\sigma \in \mathbb{R}^N : y \in \sigma \text{ and } 0 < y' < \varepsilon\}.$$

Since  $\Omega$  is convex,  $K_\sigma^\varepsilon \cap K_{\sigma'}^\varepsilon$  is empty for all  $\sigma, \sigma' \in \mathcal{E}_{ext}$  with  $\sigma \neq \sigma'$ . Now we can define (see Figure 1)

$$\Omega_\varepsilon := \Omega \cup \left( \bigcup_{\sigma \in \mathcal{E}_{ext}} K_\sigma^\varepsilon \right)$$

The subset  $\Omega_\varepsilon$  is polyhedral, then it is a Lipschitz domain. We point out that for all  $x \in \Omega_\varepsilon \setminus \Omega$ ,  $d(x, \Omega) \leq \varepsilon$  and then if we consider a face between two new cells  $K_\sigma^\varepsilon$  and  $K_{\sigma'}^\varepsilon$ , we have:

$$m(\overline{K_\sigma^\varepsilon} \cap \overline{K_{\sigma'}^\varepsilon}) \leq C \varepsilon^{N-1}.$$

Then we define

$$u_{K_\sigma^\varepsilon} := \begin{cases} 0 & \text{if } \sigma \in \mathcal{E}_{ext}^0, \\ u_K & \text{if } \sigma \in \mathcal{E}_{ext} \setminus \mathcal{E}_{ext}^0, \end{cases} \quad \sigma \in \mathcal{E}_K,$$

and the function  $u_\varepsilon$  in the following way:

$$u_\varepsilon := \sum_{K \in \mathfrak{M}} u_K \mathbf{1}_K + \sum_{\sigma \in \mathcal{E}_{ext}} u_{K_\sigma^\varepsilon} \mathbf{1}_{K_\sigma^\varepsilon}.$$

We have obviously

$$\|u\|_{0, N/(N-1), \mathfrak{M}} \leq \|u_\varepsilon\|_{L^{N/(N-1)}(\Omega_\varepsilon)}.$$

Moreover, since the function  $u_\varepsilon$  is piecewise constant and has a finite number of jumps (which corresponds to the number of faces  $\sigma \in \mathcal{E}$  added to the number of faces between the new cells  $K_\sigma^\varepsilon$ ) we get that  $u_\varepsilon$  belongs to  $BV(\Omega_\varepsilon)$ , and

$$TV_{\Omega_\varepsilon}(u_\varepsilon) \leq \sum_{\sigma \in \mathcal{E}} m(\sigma) |D_\sigma u| + \sum_{\overline{K_\sigma^\varepsilon} \cap \overline{K_{\sigma'}^\varepsilon} \neq \emptyset} C \varepsilon^{N-1} |u_{K_\sigma^\varepsilon} - u_{K_{\sigma'}^\varepsilon}|.$$

Furthermore, since  $u_\varepsilon = 0$  on a non empty part of the boundary  $\partial\Omega_\varepsilon$ , we can apply the result (12) of Theorem 2 to  $u_\varepsilon$ . We obtain that there exists a constant  $c(\Omega_\varepsilon)$  such that

$$\|u\|_{0, N/(N-1), \mathfrak{M}} \leq c(\Omega_\varepsilon) \left( |u|_{1,1, \Gamma^0, \mathfrak{M}} + C \varepsilon^{N-1} \sum_{\overline{K_\sigma^\varepsilon} \cap \overline{K_{\sigma'}^\varepsilon} \neq \emptyset} |u_{K_\sigma^\varepsilon} - u_{K_{\sigma'}^\varepsilon}| \right).$$

Now since the open set  $\Omega$  is polyhedral, it is a Lipschitz domain then it satisfies the cone condition [1, Definition 4-6] for some cone  $C$  and by construction the open set  $\Omega_\varepsilon$  also satisfies the cone condition with the same cone. Therefore, applying Lemma 4-24 in [1], the constant  $c(\Omega_\varepsilon)$  only depends on the dimension of this cone, and not on  $\varepsilon > 0$ . Then passing to the limit  $\varepsilon \rightarrow 0$  we finally get that

$$\|u\|_{0, N/(N-1), \mathfrak{M}} \leq c(\Omega) |u|_{1,1, \mathfrak{M}}.$$

□

Now using this Lemma we can prove the discrete Gagliardo-Nirenberg-Sobolev and Sobolev-Poincaré inequalities in the case with some homogeneous Dirichlet boundary conditions.

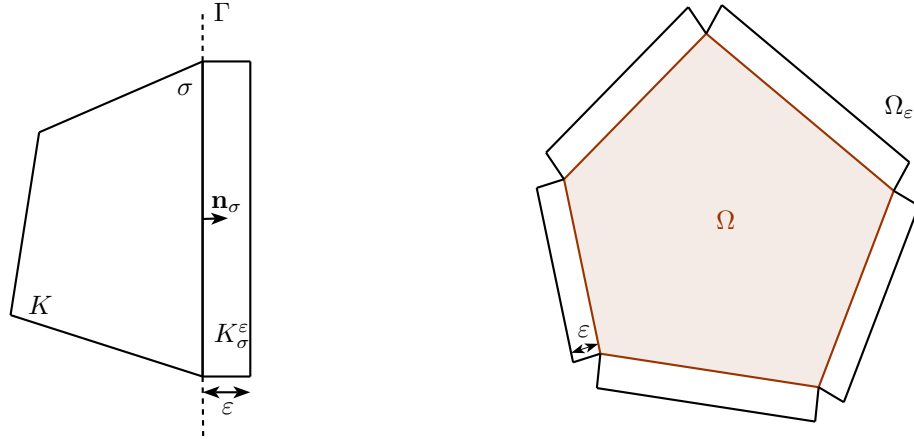


Figure 1: Construction of the cell  $K_\sigma^\epsilon$  (left) and of the domain  $\Omega_\epsilon$  (right).

## 4.2 Discrete Gagliardo-Nirenberg-Sobolev inequalities

**Theorem 5** (Discrete Gagliardo-Nirenberg-Sobolev inequalities). *Let  $\Omega$  be an open convex bounded polyhedral domain of  $\mathbb{R}^N$  and  $\Gamma^0 \neq \emptyset$  be a part of the boundary. Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4). Then for any  $1 < p \leq N$  and  $q \geq 1$ , there exists a constant  $C > 0$  only depending on  $p, q, N$  and  $\Omega$  such that*

$$\|u\|_{0,m,\mathfrak{M}} \leq \frac{C_1}{\xi^{(p-1)\theta/p}} |u|_{1,p,\Gamma^0,\mathfrak{M}}^\theta \|u\|_{0,q,\mathfrak{M}}^{1-\theta}, \quad \forall u \in X(\mathfrak{M}), \quad (31)$$

where  $\theta$  and  $m$  satisfy (14) and (15).

*Proof.* The proof is similar to the proof of Theorem 3. Let  $1 < p \leq N$  and  $s \geq 1$ . For  $u \in X(\mathfrak{M})$ , we apply Lemma 1 to  $v \in X(\mathfrak{M})$  defined by  $v_K = u_K^s$  for all  $K \in \mathfrak{M}$ . It yields

$$\|u\|_{0,sN/(N-1),\mathfrak{M}}^s \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\Gamma^0,\mathfrak{M}} \|u\|_{0,(s-1)p/(p-1),\mathfrak{M}}^{(s-1)}$$

with  $1 < p \leq N$  and  $s \geq 1$ .

Choosing  $q = (s-1)p/(p-1)$  ( $s = 1 + q - \frac{q}{p} \geq 1$ ), we obtain

$$\|u\|_{0,sN/(N-1),\mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/ps}} |u|_{1,p,\Gamma^0,\mathfrak{M}}^{1/s} \|u\|_{0,q,\mathfrak{M}}^{(s-1)/s}. \quad (32)$$

Then, using an interpolation inequality as in the proof of Theorem 3, we get

$$\|u\|_{0,m,\mathfrak{M}} \leq \|u\|_{0,sN/(N-1),\mathfrak{M}}^\alpha \|u\|_{0,q,\mathfrak{M}}^{1-\alpha} \leq \frac{C}{\xi^{\alpha(p-1)/ps}} |u|_{1,p,\Gamma^0,\mathfrak{M}}^{\alpha/s} \|u\|_{0,q,\mathfrak{M}}^{1-(\alpha/s)},$$

Taking  $\theta = \alpha/s$  concludes the proof.  $\square$

### 4.3 Discrete Sobolev-Poincaré and Nash inequalities

In the case with some homogeneous Dirichlet boundary conditions, the discrete Sobolev-Poincaré inequalities rewrite as follows.

**Theorem 6** (Discrete Sobolev-Poincaré inequality). *Let  $\Omega$  be an open convex bounded polyhedral domain of  $\mathbb{R}^N$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4). Let  $\Gamma^0 \subset \Gamma$ , with  $\Gamma^0 \neq \emptyset$ . Then there exists a constant  $C > 0$  which only depends on  $p, q, N$  and  $\Omega$  such that:*

- if  $1 \leq p < N$ , for all  $1 \leq q \leq p^* := \frac{pN}{N-p}$

$$\|u\|_{0,q,\mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\Gamma^0,\mathfrak{M}} \quad \forall u \in X(\mathfrak{M}), \quad (33)$$

- if  $p \geq N$ , for all  $1 \leq q < +\infty$ ,

$$\|u\|_{0,q,\mathfrak{M}} \leq \frac{C}{\xi^{(p-1)/p}} |u|_{1,p,\Gamma^0,\mathfrak{M}} \quad \forall u \in X(\mathfrak{M}). \quad (34)$$

*Proof.* The proof is similar to the proof of Theorem 4, starting from (32) instead of (19).  $\square$

Now using Theorems 5 and 6, we easily get a discrete version of Nash inequality:

**Corollary 2** (Discrete Nash inequality). *Let  $\Omega$  be an open convex bounded polyhedral domain of  $\mathbb{R}^N$ . Let  $\mathfrak{M}$  be an admissible mesh satisfying (3)-(4). Let  $\Gamma^0 \subset \Gamma$ , with  $\Gamma^0 \neq \emptyset$ . Then there exists a constant  $C > 0$  only depending on  $\Omega$  and  $N$  such that*

$$\|u\|_{0,2,\mathfrak{M}}^{1+\frac{2}{N}} \leq \frac{C}{\sqrt{\xi}} |u|_{1,2,\mathfrak{M}} \|u\|_{0,1,\mathfrak{M}}^{\frac{2}{N}} \quad \forall u \in X(\mathfrak{M}). \quad (35)$$

## 5 Application to finite volume approximations coming from DDFV schemes

The discrete duality finite volume methods have been developed for ten years for the approximation of anisotropic elliptic problems on almost general meshes in 2D and 3D. They are based on some discrete operators (divergence and gradient), satisfying a discrete Green formula (the “discrete duality”). The DDFV approximations were first proposed for the discretization of anisotropic and/or nonlinear diffusion problems on rather general meshes. We refer to the pioneer work of F. Hermeline [22, 23, 24, 25, 26] who proposed a new approach dealing with primal and dual meshes and Y. Coudière, J.-P. Vila and Ph. Villedieu [11] who proposed a method of reconstruction for the discrete gradients. Next, K. Domelevo and P. Omnès [14], S. Delcourte, K. Domelevo and P. Omnès [12] presented the discrete duality finite volume approach (DDFV) for the Laplace operator. Then, B. Andreianov, F. Boyer and F. Hubert [3] gave a general background of DDFV methods for anisotropic and nonlinear elliptic problems. Most of these works treat 2D linear anisotropic, heterogeneous diffusion problems, while the case of discontinuous diffusion operators have been treated later by F. Boyer and F. Hubert in [6]. F. Hermeline [25, 26] treats the analogous 3D problems, S. Krell [27] treats the Stokes problem in 2D and in 3D whereas Y. Coudière and G. Manzini [10] treat linear elliptic convection-diffusion equations.

The construction of DDFV schemes needs the definition of three meshes: a primal mesh, a dual mesh and a diamond mesh. Then, the approximate solutions are defined both on the primal and the dual meshes, while the approximate gradients are defined on the diamond meshes. Therefore, we need to adapt the definition of the spaces of approximate solutions and the definition of the discrete norms. It will be done in Section 5.1. Then, we will be able to establish some discrete Gagliardo-Nirenberg-Sobolev and Sobolev-Poincaré inequalities, in the general case (Section 5.2) as in the case with Dirichlet boundary conditions (Section 5.3).

## 5.1 Meshes and functional spaces

**Meshes.** Let  $\Omega$  be an open bounded polygonal domain of  $\mathbb{R}^2$ . The mesh construction starts with the partition of  $\Omega$  with disjoint open polygonal control volumes. This partition, denoted by  $\mathfrak{M}$ , is called the interior primal mesh. We then denote by  $\partial\mathfrak{M}$  the set of boundary edges, which are considered as degenerate control volumes. Then, the primal mesh is defined by  $\overline{\mathfrak{M}} = \mathfrak{M} \cup \partial\mathfrak{M}$ . To each primal cell  $\mathcal{K} \in \overline{\mathfrak{M}}$ , we associate a point  $x_{\mathcal{K}} \in \mathcal{K}$ , called the center of the primal cell. Notice that for a degenerate control volume  $\mathcal{K}$ , the point  $x_{\mathcal{K}}$  is necessarily the midpoint of  $\mathcal{K}$ . This family of centers is denoted by  $\mathcal{X} = \{x_{\mathcal{K}}, \mathcal{K} \in \overline{\mathfrak{M}}\}$ .

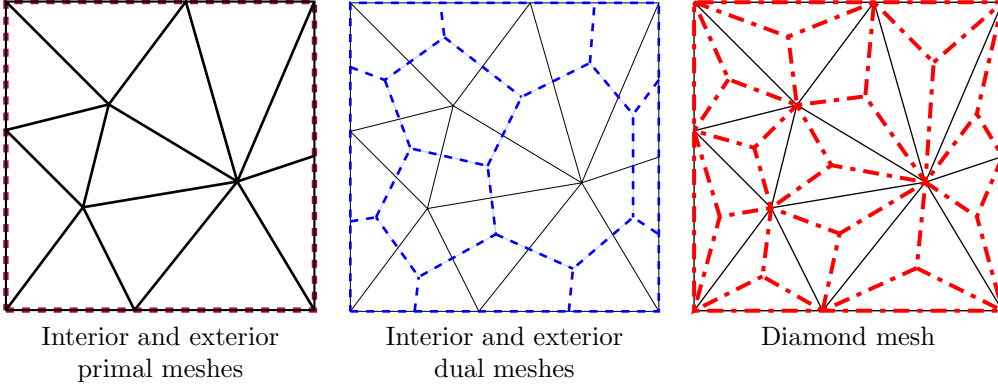


Figure 2: Presentation of the meshes

Let  $\mathcal{X}^*$  denote the set of the vertices of the primal control volumes in  $\overline{\mathfrak{M}}$ . Distinguishing the interior vertices from the vertices lying on the boundary, we split  $\mathcal{X}^*$  into  $\mathcal{X}^* = \mathcal{X}_{int}^* \cup \mathcal{X}_{ext}^*$ . To any point  $x_{\mathcal{K}^*} \in \mathcal{X}_{int}^*$ , we associate the polygon  $\mathcal{K}^*$  obtained by joining the centers of the primal cells whose  $x_{\mathcal{K}^*}$  is a vertex. The set of such polygons defines the interior dual mesh denoted by  $\mathfrak{M}^*$ . To any point  $x_{\mathcal{K}^*} \in \mathcal{X}_{ext}^*$ , we then associate the polygon  $\mathcal{K}^*$ , whose vertices are  $\{x_{\mathcal{K}^*}\} \cup \{x_{\mathcal{K}} \in \mathcal{X} / x_{\mathcal{K}^*} \in \bar{\mathcal{K}}, \mathcal{K} \in \overline{\mathfrak{M}}\}$ . It defines the boundary dual mesh  $\partial\mathfrak{M}^*$  and the dual mesh is defined by  $\overline{\mathfrak{M}^*} = \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ .

In the sequel, we will assume that each primal cell  $\mathcal{K} \in \mathfrak{M}$  is star-shaped with respect to  $x_{\mathcal{K}}$  and each dual cell  $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$  is star-shaped with respect to  $x_{\mathcal{K}^*}$ .

For all neighboring primal cells  $\mathcal{K}$  and  $\mathcal{L}$ , we assume that  $\partial\mathcal{K} \cap \partial\mathcal{L}$  is a segment, corresponding to an edge of the mesh  $\mathfrak{M}$ , denoted by  $\sigma = \mathcal{K}|\mathcal{L}$ . Let  $\mathcal{E}$  be the set of such edges. We similarly define the edges  $\mathcal{E}^*$  of the dual mesh  $\overline{\mathfrak{M}^*}$ :  $\sigma^* = \mathcal{K}^*|\mathcal{L}^*$ . For each couple  $(\sigma, \sigma^*) \in \mathcal{E} \times \mathcal{E}^*$  such that  $\sigma = \mathcal{K}|\mathcal{L} = (x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$  and  $\sigma^* = \mathcal{K}^*|\mathcal{L}^* = (x_{\mathcal{K}}, x_{\mathcal{L}})$ , we define the quadrilateral diamond cell  $\mathcal{D}_{\sigma, \sigma^*}$  whose diagonals are  $\sigma$  and  $\sigma^*$ . If  $\sigma \in \mathcal{E} \cap \partial\Omega$ , we note that the diamond degenerates into a triangle. The set of the diamond cells defines a partition of  $\Omega$ , which is called the diamond mesh



and is denoted by  $\mathfrak{D}$ . Let us note that  $\mathfrak{D}$  can be splitted into  $\mathfrak{D} = \mathfrak{D}_{int} \cup \mathfrak{D}_{ext}$  where  $\mathfrak{D}_{int}$  is the set of interior (non degenerate) diamond cells and  $\mathfrak{D}_{ext}$  is the set of degenerate diamond cells.

Finally, the DDFV mesh is made of the triple  $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}}^*, \mathfrak{D})$ . See Figure 2 for an example of DDFV mesh.

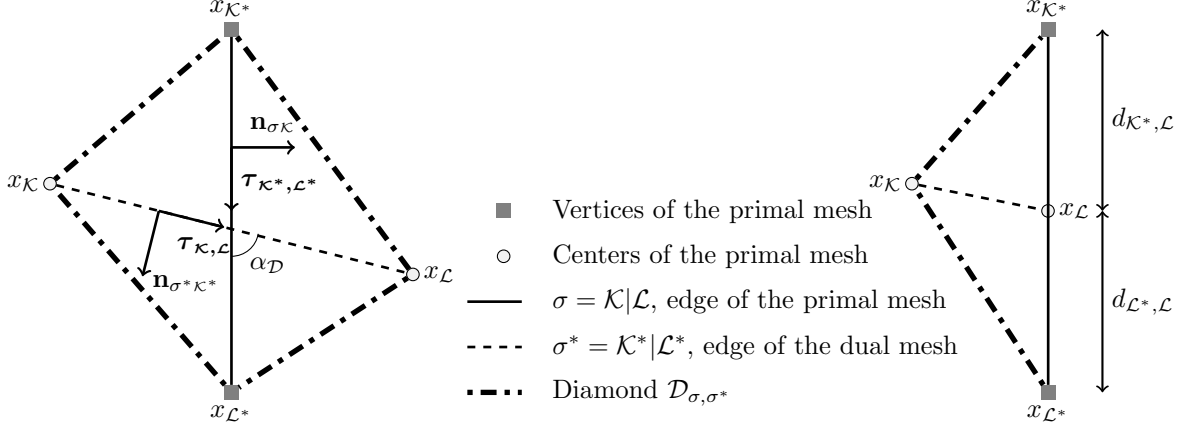


Figure 3: Definition of the diamonds  $\mathcal{D}_{\sigma, \sigma^*}$

Let us now introduce some notations associated to the mesh  $\mathcal{T}$ . For each primal cell or dual cell  $V$  in  $\overline{\mathfrak{M}}$  or  $\overline{\mathfrak{M}}^*$ , we define  $m_V$ , the measure of the cell  $V$ ,  $\mathcal{E}_V$ , the set of edges of  $V$ ,  $\mathfrak{D}_V = \{\mathcal{D}_{\sigma, \sigma^*} \in \mathfrak{D}, \sigma \in \mathcal{E}_V\}$ ,  $d_V$ , the diameter of  $V$ . For a diamond  $\mathcal{D}_{\sigma, \sigma^*}$ , whose vertices are  $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$ , we define :  $m_{\sigma}$  and  $m_{\sigma^*}$  the lengths of the primal edge  $\sigma$  and the dual edge  $\sigma^*$ ,  $m_{\mathfrak{D}}$ , the measure of  $\mathfrak{D}$ ,  $d_{\mathfrak{D}}$  its diameter and  $\alpha_{\mathfrak{D}}$  the angle between  $(x_{\mathcal{K}}, x_{\mathcal{L}})$  and  $(x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$ . As shown on Figure 3, we will also use two direct basis  $(\tau_{\mathcal{K}^*, \mathcal{L}^*}, \mathbf{n}_{\sigma\mathcal{K}})$  and  $(\mathbf{n}_{\sigma^*\mathcal{K}^*}, \tau_{\mathcal{K}, \mathcal{L}})$ , where  $\mathbf{n}_{\sigma\mathcal{K}}$  is the unit normal to  $\sigma$ , outward  $\mathcal{K}$ ,  $\mathbf{n}_{\sigma^*\mathcal{K}^*}$  is the unit normal to  $\sigma^*$ , outward  $\mathcal{K}^*$ ,  $\tau_{\mathcal{K}^*, \mathcal{L}^*}$  is the unit tangent vector to  $\sigma$ , oriented from  $\mathcal{K}^*$  to  $\mathcal{L}^*$ ,  $\tau_{\mathcal{K}, \mathcal{L}}$  is the unit tangent vector to  $\sigma^*$ , oriented from  $\mathcal{K}$  to  $\mathcal{L}$ . For a boundary edge  $\sigma = [x_{\mathcal{K}^*}, x_{\mathcal{L}}] \in \partial\overline{\mathfrak{M}}$ , we define  $d_{\mathcal{K}^*, \mathcal{L}}$  the length of the segment  $[x_{\mathcal{K}^*}, x_{\mathcal{L}}]$  and  $d_{\mathcal{L}^*, \mathcal{L}}$  the length of the segment  $[x_{\mathcal{L}^*}, x_{\mathcal{L}}]$ .

In all the sequel, we will assume that the diamonds cannot be flat. It means :

$$\exists \alpha_{\mathcal{T}} \in ]0, \frac{\pi}{2}] \text{ such that } |\sin(\alpha_{\mathfrak{D}})| \geq \sin(\alpha_{\mathcal{T}}) \quad \forall \mathfrak{D} \in \mathfrak{D}. \quad (36)$$

As for all  $\mathcal{D}_{\sigma, \sigma^*} \in \mathfrak{D}$ , we have  $2m_{\mathfrak{D}} = m_{\sigma}m_{\sigma^*} \sin(\alpha_{\mathfrak{D}})$ , the hypothesis (36) implies

$$m_{\sigma}m_{\sigma^*} \leq \frac{2m_{\mathfrak{D}}}{\sin(\alpha_{\mathcal{T}})}. \quad (37)$$

We also assume some regularity of the mesh, as in [3], which implies

$$\begin{aligned} \exists \zeta > 0, \quad \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma}m_{\sigma^*} &\leq \frac{m_{\mathcal{K}}}{\zeta} \quad \forall \mathcal{K} \in \mathfrak{M}, \\ \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma}m_{\sigma^*} &\leq \frac{m_{\mathcal{K}^*}}{\zeta} \quad \forall \mathcal{K}^* \in \overline{\mathfrak{M}}^*. \end{aligned} \quad (38)$$

**Definition of the approximate solution.** A discrete duality finite volume scheme leads to the computation of discrete unknowns on the primal and the dual meshes :  $(u_{\mathcal{K}})_{\mathcal{K} \in \overline{\mathfrak{M}}}$  and  $(u_{\mathcal{K}^*})_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}}$ . From these discrete unknowns, we can reconstruct two different approximate solutions :

$$u_{\mathfrak{M}} = \sum_{\mathcal{K} \in \overline{\mathfrak{M}}} u_{\mathcal{K}} \mathbf{1}_{\mathcal{K}} \text{ and } u_{\overline{\mathfrak{M}^*}} = \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} u_{\mathcal{K}^*} \mathbf{1}_{\mathcal{K}^*}.$$

But, in order to use simultaneously the discrete unknowns computed defined on the primal and the dual meshes, we prefer to define the approximate solution as

$$u = \frac{1}{2}(u_{\mathfrak{M}} + u_{\overline{\mathfrak{M}^*}}).$$

Therefore, the space of approximate solutions  $Z(\mathcal{T})$  is defined by:

$$Z(\mathcal{T}) = \left\{ u \in L^1(\Omega) / \exists u_{\mathcal{T}} = ((u_{\mathcal{K}})_{\mathcal{K} \in \overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}}) \right. \\ \left. \text{such that } u = \frac{1}{2} \left( \sum_{\mathcal{K} \in \overline{\mathfrak{M}}} u_{\mathcal{K}} \mathbf{1}_{\mathcal{K}} + \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} u_{\mathcal{K}^*} \mathbf{1}_{\mathcal{K}^*} \right) \right\}.$$

For a given function  $u \in Z(\mathcal{T})$ , we define the discrete  $L^p$  norm by

$$\|u\|_{0,p,\mathcal{T}} = \left( \frac{1}{2} \sum_{\mathcal{K} \in \overline{\mathfrak{M}}} m_{\mathcal{K}} |u_{\mathcal{K}}|^p + \frac{1}{2} \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} m_{\mathcal{K}^*} |u_{\mathcal{K}^*}|^p \right)^{1/p}.$$

**Discrete gradient.** A key point in the construction of the DDFV schemes is the definition of the discrete operators (divergence and gradient). We just focus here on the definition of the discrete gradient, which will be useful for the definition of the discrete  $W^{1,p}$ -seminorms.

Let  $u \in Z(\mathcal{T})$ . The discrete gradient of  $u$ ,  $\nabla^d u$  is defined as a piecewise constant function on each diamond cell :

$$\nabla^d u = \sum_{\mathcal{D} \in \mathfrak{D}} \nabla^{\mathcal{D}} u \mathbf{1}_{\mathcal{D}},$$

where, for  $\mathcal{D} \in \mathfrak{D}$ ,

$$\nabla^{\mathcal{D}} u = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left( \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \mathbf{n}_{\sigma_{\mathcal{K}}} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \mathbf{n}_{\sigma^*_{\mathcal{K}^*}} \right).$$

This discrete gradient has been introduced in [11]. It verifies:

$$\nabla^{\mathcal{D}} u \cdot \boldsymbol{\tau}_{\mathcal{K}^*, \mathcal{L}^*} = \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \text{ and } \nabla^{\mathcal{D}} u \cdot \boldsymbol{\tau}_{\mathcal{K}, \mathcal{L}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}}.$$

Using this discrete gradient, we may now define the discrete  $W^{1,p}$ -seminorm and norm of a given function  $u \in Z(\mathcal{T})$ :

$$|u|_{1,p,\mathcal{T}} = \left( \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |\nabla^{\mathcal{D}} u|^p \right)^{1/p} \\ \|u\|_{1,p,\mathcal{T}} = \|u\|_{0,p,\mathcal{T}} + |u|_{1,p,\mathcal{T}}$$

## 5.2 Discrete functional inequalities in the general case

Our aim is now to extend the results of Section 3 to the case of finite volume approximations coming from some DDFV schemes:  $u \in Z(\mathcal{T})$ . We will use that such functions are defined as  $u = \frac{1}{2}(u_{\mathfrak{M}} + u_{\overline{\mathfrak{M}}^*})$  with  $u_{\mathfrak{M}} \in X(\mathfrak{M})$  and  $u_{\overline{\mathfrak{M}}^*} \in X(\overline{\mathfrak{M}}^*)$ . Nevertheless, we may take care because the primal and the dual meshes  $\mathfrak{M}$  and  $\overline{\mathfrak{M}}^*$  does not satisfy the admissibility condition required in Theorems 3 and 4.

**Theorem 7** (General discrete Gagliardo-Nirenberg-Sobolev inequality in the DDFV framework). *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^2$ . Let  $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}}^*, \mathfrak{D})$  be a DDFV mesh satisfying (36) and (38).*

*Then for  $1 < p \leq 2$  and  $q \geq 1$ , there exists a constant  $C > 0$  only depending on  $p, q$  and  $\Omega$  such that*

$$\|u\|_{0,m,\mathcal{T}} \leq \frac{C}{(\sin(\alpha_{\mathcal{T}}))^{\theta/p} \zeta^{\theta(p-1)/p}} \|u\|_{1,p,\mathcal{T}}^{\theta} \|u\|_{0,q,\mathcal{T}}^{1-\theta}, \quad \forall u \in Z(\mathcal{T}), \quad (39)$$

where

$$0 \leq \theta \leq \frac{p}{p+q(p-1)} \leq 1$$

and

$$\frac{1}{m} = \frac{1-\theta}{q} + \frac{\theta}{p} - \frac{\theta}{2}.$$

*Proof.* We start as in the proof of Theorem 3. Let  $s \geq 1$  and set  $q = (s-1)p/(p-1) \geq 1$ . For  $u \in Z(\mathcal{T})$ , as  $u_{\mathfrak{M}} \in X(\mathfrak{M})$  and  $u_{\overline{\mathfrak{M}}^*} \in X(\overline{\mathfrak{M}}^*)$ , we may write:

$$\|u_{\mathfrak{M}}\|_{0,2s,\mathfrak{M}}^s \leq c(\Omega) \left( \| |u_{\mathfrak{M}}|^s \|_{1,1,\mathfrak{M}} + \|u_{\mathfrak{M}}\|_{0,s,\mathfrak{M}}^s \right) \quad (40)$$

$$\|u_{\overline{\mathfrak{M}}^*}\|_{0,2s,\overline{\mathfrak{M}}^*}^s \leq c(\Omega) \left( \| |u_{\overline{\mathfrak{M}}^*}|^s \|_{1,1,\overline{\mathfrak{M}}^*} + \|u_{\overline{\mathfrak{M}}^*}\|_{0,s,\overline{\mathfrak{M}}^*}^s \right) \quad (41)$$

But, following the same computations as in the proof of Theorem 3, we get

$$\begin{aligned} \| |u_{\mathfrak{M}}|^s \|_{1,1,\mathfrak{M}} &= \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{int}} m_{\sigma} \left| |u_{\mathcal{K}}|^s - |u_{\mathcal{L}}|^s \right| \\ &\leq s \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{int}} m_{\sigma} m_{\sigma^*} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{m_{\sigma^*}} \right| (|u_{\mathcal{K}}|^{s-1} + |u_{\mathcal{L}}|^{s-1}) \\ &\leq s \left( \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{int}} m_{\sigma} m_{\sigma^*} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{m_{\sigma^*}} \right|^p \right)^{\frac{1}{p}} \left( \sum_{\mathcal{K} \in \mathfrak{M}} \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} m_{\sigma^*} |u_{\mathcal{K}}|^{\frac{(s-1)p}{p-1}} \right)^{\frac{p-1}{p}} \end{aligned}$$

Using the regularity hypotheses on the mesh, we get

$$\| |u_{\mathfrak{M}}|^s \|_{1,1,\mathfrak{M}} \leq \frac{C}{(\sin(\alpha_{\mathcal{T}}))^{1/p} \zeta^{(p-1)/p}} \left( \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{int}} m_{\mathcal{D}} \left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{m_{\sigma^*}} \right|^p \right)^{\frac{1}{p}} \|u_{\mathfrak{M}}\|_{0,q,\mathfrak{M}}^{s-1}.$$

But, by definition,  $\frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{m_{\sigma^*}} = \nabla^{\mathcal{D}} u \cdot \boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}$  and therefore  $\left| \frac{u_{\mathcal{K}} - u_{\mathcal{L}}}{m_{\sigma^*}} \right| \leq |\nabla^{\mathcal{D}} u|$ . It yields :

$$\| |u_{\mathfrak{M}}|^s \|_{1,1,\mathfrak{M}} \leq \frac{C}{(\sin(\alpha_{\mathcal{T}}))^{1/p} \zeta^{(p-1)/p}} \|u\|_{1,p,\mathcal{T}} \|u_{\mathfrak{M}}\|_{0,q,\mathfrak{M}}^{s-1}.$$

Injecting this last inequality in (40), and using the interpolation inequality (18), we get:

$$\begin{aligned} \|u_{\mathfrak{M}}\|_{0,2s,\mathfrak{M}}^s &\leq \frac{C}{(\sin(\alpha_\tau))^{1/p}\zeta^{(p-1)/p}} \|u_{\mathfrak{M}}\|_{0,q,\mathfrak{M}}^{s-1} (|u|_{1,p,\mathcal{T}} + \|u_{\mathfrak{M}}\|_{0,p,\mathfrak{M}}) \\ &\leq \frac{C}{(\sin(\alpha_\tau))^{1/p}\zeta^{(p-1)/p}} \|u_{\mathfrak{M}}\|_{0,q,\mathfrak{M}}^{s-1} \|u\|_{1,p,\mathcal{T}} \end{aligned}$$

because  $\|u_{\mathfrak{M}}\|_{0,p,\mathfrak{M}} \leq 2 \|u\|_{1,p,\mathcal{T}}$  by definition. Then, using the interpolation inequality (20) with  $r = 2s$  and  $\frac{1}{m} = \frac{1-\alpha}{q} + \frac{\alpha}{r}$  with  $0 \leq \alpha \leq 1$ , as in the proof of Theorem 3, we obtain:

$$\|u_{\mathfrak{M}}\|_{0,m,\mathfrak{M}} \leq \frac{C}{(\sin(\alpha_\tau))^{\alpha/(sp)}\zeta^{\alpha(p-1)/(sp)}} \|u\|_{1,p,\mathcal{T}}^{\alpha/s} \|u\|_{0,q,\mathcal{T}}^{1-\alpha/s}.$$

With similar computations on the dual mesh, from (41), we get

$$\|u_{\overline{\mathfrak{M}}^*}\|_{0,m,\overline{\mathfrak{M}}^*} \leq \frac{C}{(\sin(\alpha_\tau))^{\alpha/(sp)}\zeta^{\alpha(p-1)/(sp)}} \|u\|_{1,p,\mathcal{T}}^{\alpha/s} \|u\|_{0,q,\mathcal{T}}^{1-\alpha/s}.$$

Finally, setting  $\theta = \alpha/s$  with  $s = (p-1)q/p + 1$ , it yields the expected inequality (39).  $\square$

As in the classical finite volume framework, we can now prove discrete Sobolev-Poincaré inequalities. The proof is similar to the proof of Theorem 4; it will not be detailed here.

**Theorem 8** (General discrete Sobolev-Poincaré inequality in the DDFV framework). *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^2$ . Let  $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}}^*, \mathfrak{D})$  be a DDFV mesh satisfying (36) and (38).*

*Then there exists a constant  $C > 0$  only depending on  $p, q$  and  $\Omega$  such that:*

- if  $1 \leq p < 2$ , for all  $1 \leq q \leq \frac{2p}{2-p}$ ,

$$\|u\|_{0,q,\mathcal{T}} \leq \frac{C}{(\sin(\alpha_\tau))^{1/p}\zeta^{(p-1)/p}} \|u\|_{1,p,\mathcal{T}}, \quad \forall u \in Z(\mathcal{T}), \quad (42)$$

- if  $p \geq 2$ , for all  $1 \leq q < +\infty$ ,

$$\|u\|_{0,q,\mathcal{T}} \leq \frac{C}{(\sin(\alpha_\tau))^{1/p}\zeta^{(p-1)/p}} \|u\|_{1,p,\mathcal{T}}, \quad \forall u \in Z(\mathcal{T}). \quad (43)$$

Let us now focus on the Poincaré-Wirtinger inequality in the DDFV case. This result has been proved recently in [29]. We will give here a proof using the embedding of  $BV(\Omega)$  into  $L^2(\Omega)$  (11) recalled in Theorem 2.

**Theorem 9** (Discrete Poincaré-Wirtinger inequality in the DDFV framework). *Let  $\Omega$  be an open bounded polyhedral domain of  $\mathbb{R}^2$ . Let  $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}}^*, \mathfrak{D})$  be a DDFV mesh satisfying (36). There exists a constant  $C > 0$  depending only on  $\Omega$ , such that for all  $u \in Z(\mathcal{T})$  satisfying*

$$\sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}} = \sum_{\mathcal{K} \in \overline{\mathfrak{M}}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*} = 0, \quad (44)$$

*we have*

$$\|u\|_{0,2,\mathcal{T}} \leq \frac{C}{\sin(\alpha_\tau)} |u|_{1,2,\mathcal{T}}. \quad (45)$$

*Proof.* Let  $u \in Z(\mathcal{T})$ . Applying (11) to  $u_{\mathfrak{M}} \in X(\mathfrak{M})$  and  $u_{\overline{\mathfrak{M}}^*} \in X(\overline{\mathfrak{M}}^*)$ , we get, under the hypothesis (44),

$$\begin{aligned} \|u_{\mathfrak{M}}\|_{0,2,\mathfrak{M}} &\leq c(\Omega)TV_{\Omega}(u_{\mathfrak{M}}) \leq c(\Omega)|u_{\mathfrak{M}}|_{1,1,\mathfrak{M}} \\ \|u_{\overline{\mathfrak{M}}^*}\|_{0,2,\overline{\mathfrak{M}}^*} &\leq c(\Omega)TV_{\Omega}(u_{\overline{\mathfrak{M}}^*}) \leq c(\Omega)|u_{\overline{\mathfrak{M}}^*}|_{1,1,\overline{\mathfrak{M}}^*} \end{aligned}$$

But,

$$\begin{aligned} |u_{\mathfrak{M}}|_{1,1,\mathfrak{M}} &\leq \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{int}} m_{\sigma}m_{\sigma^*} \frac{|u_{\mathcal{K}} - u_{\mathcal{L}}|}{m_{\sigma^*}} \\ &\leq \frac{2}{\sin(\alpha_{\mathcal{T}})} \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{int}} m_{\mathcal{D}} \frac{|u_{\mathcal{K}} - u_{\mathcal{L}}|}{m_{\sigma^*}} \\ &\leq \frac{2}{\sin(\alpha_{\mathcal{T}})} m(\Omega)^{1/2} |u|_{1,2,\mathcal{T}}, \end{aligned}$$

thanks to Cauchy-Schwarz inequality. By the same way, we get the same bound for  $|u_{\overline{\mathfrak{M}}^*}|_{1,1,\overline{\mathfrak{M}}^*}$  and it finally yields  $\|u\|_{0,2,\mathcal{T}} \leq \frac{2}{\sin(\alpha_{\mathcal{T}})} m(\Omega)^{1/2} c(\Omega) |u|_{1,2,\mathcal{T}}$ .  $\square$

### 5.3 Discrete functional inequalities in the case with Dirichlet boundary conditions

In this Section, we want to extend the discrete Gagliardo-Nirenberg-Sobolev inequalities of Section 4.2 to finite volume approximations obtained from a DDFV scheme. We first recall how Dirichlet boundary conditions are taken into account in DDFV methods. Let  $\Gamma_0$  be a non empty part of the boundary. At the discrete level, homogeneous Dirichlet boundary conditions on  $\Gamma_0$  will be written:

$$u_{\mathcal{K}} = 0, \forall \mathcal{K} \in \partial\mathfrak{M}, \mathcal{K} \subset \Gamma^0 \text{ and } u_{\mathcal{K}^*} = 0, \forall \mathcal{K}^* \in \partial\mathfrak{M}^*, \overline{\mathcal{K}^*} \cap \Gamma^0 \neq \emptyset. \quad (46)$$

Therefore, we consider the corresponding set of finite volume approximations,  $Z_{\Gamma^0}(\mathcal{T})$  defined by:

$$Z_{\Gamma^0}(\mathcal{T}) = \{u \in Z(\mathcal{T}) \text{ satisfying (46)}\}.$$

Let us note that the definition of the discrete  $W^{1,p}$  seminorm is the same on  $Z_{\Gamma^0}(\mathcal{T})$  as on  $Z(\mathcal{T})$ . Indeed, the fact that the approximate solution vanishes at the boundary is taken into account in the definition of the discrete gradient  $\nabla^{\mathcal{D}}u$  for  $\mathcal{D} \in \mathfrak{D}_{ext}$ , and therefore in  $|u|_{1,p,\mathcal{T}}$ .

Finally, combining the techniques of proof of Theorem 5 (using Lemma 1) and Theorem 7, we establish the following Theorem.

**Theorem 10** (Discrete Gagliardo-Nirenberg-Sobolev inequality in the DDFV framework). *Let  $\Omega$  be an open convex bounded polyhedral domain of  $\mathbb{R}^2$  and  $\Gamma^0$  be a part of the boundary. Let  $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}}^*, \mathfrak{D})$  be a DDFV mesh satisfying (36) and (38).*

*Then for  $1 < p \leq 2$  and  $q \geq 1$ , there exists a constant  $C > 0$  only depending on  $p, q$  and  $\Omega$  such that*

$$\|u\|_{0,m,\mathcal{T}} \leq \frac{C}{(\sin(\alpha_{\mathcal{T}}))^{\theta/p} \zeta^{\theta(p-1)/(p)}} |u|_{1,p,\mathcal{T}}^{\theta} \|u\|_{0,q,\mathcal{T}}^{1-\theta}, \quad \forall u \in Z_{\Gamma^0}(\mathcal{T}), \quad (47)$$

where

$$0 \leq \theta \leq \frac{p}{p+q(p-1)} \leq 1$$

and

$$\frac{1}{m} = \frac{1-\theta}{q} + \frac{\theta}{p} - \frac{\theta}{2}.$$

**Acknowledgement:** This work was partially supported by the European Research Council ERC Starting Grant 2009, project 239983-NuSiKiMo.

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