Structural properties of recursively partitionable graphs with connectivity 2

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Abstract

A connected graph G is said to be *arbitrarily partitionable* (AP for short) if for every partition $(n_1, ..., n_p)$ of $|V(G)|$ there exists a partition $(V_1, ..., V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G on n_i vertices. Some stronger versions of this property were introduced, namely the ones of being *online arbitrarily partitionable* and *recursively arbitrarily partitionable* (OL-AP and R-AP for short, respectively), in which the subgraphs induced by a partition of G must not only be connected but also fulfil additional conditions. In this paper, we point out some structural properties of OL-AP and R-AP graphs with connectivity 2. In particular, we show that deleting a cut pair of these graphs results in a graph with a bounded number of components, some of whom have a small number of vertices. We obtain these results by studying a simple class of 2-connected graphs called *balloons*.

1 Introduction

Let G be a graph with order n. A sequence $\tau = (n_1, ..., n_p)$ of positive integers is said to be *admissible for* G if it performs a partition of n, that is if $\sum_{i=1}^{p} n_i = n$. When, for such an admissible sequence for G, there exists a partition $(V_1, ..., V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G on n_i vertices, then τ is called *realizable in* G. If every admissible sequence for G is also realizable in G, then G is said to be *arbitrarily partitionable* (AP for short).

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The notion of AP graphs was introduced in [1] to deal with the following problem. Suppose that we want to share a network of n computing resources among p users, where the i^{th} user needs n_i resources, and that, for the sake of performance, we do not want the sharing to be performed arbitrarily but in such a way that the following two conditions are met:

- a resource must be allocated to exactly one user,
- the subnetwork attributed to a user must be connected[1].

We can use the previously introduced notions to deduce an optimal sharing of the resources. Indeed, let G be the graph modelling the network; then we can satisfy the users with this specific resource demand if the sequence $(n_1, ..., n_p)$ is realizable in G. Moreover, observe that, regarding this allocation problem, the networks which are of most interest are those which can be shared among an arbitrary number of users no matter how many resources they need. Clearly, these networks are the ones that have an AP graph topology.

It has to be known that the problem of deciding whether a sequence is realizable in a graph in computationally hard, even when restricted to trees [2], although some exceptions can be pointed out. As an illustration, let us mention the early result by Győri and Lovász independently, proved long before the introduction of AP graphs, that states that k-connected graphs are always partitionable into k connected subgraphs no matter what their requested orders are [9, 11]. For a deeper overview of the background of AP graphs, the interested reader is referred to [1, 2, 12].

Observe that the definition of AP graphs is quite static and thus not representative of the difficulties we can encounter while actually partitioning a network; notably, one could point out the following two issues:

1. In the definition, a graph is fully partitioned at once; from the network sharing point of view, this constraint is like waiting for every single resource of the network to be needed before eventually satisfying the users. This is, of course, not satisfying since we would like to satisfy them as soon as possible (immediately, ideally).
2. When a sequence is realized in a graph, the induced subgraphs must only meet the connectivity constraint. But according to our network analogy, it would be more convenient to make sure that the allocated subnetworks themselves have the property of being shareable at will. This can be quite useful if, for example, a user wants himself to share his subnetwork between several other users or if he wants to delimit it to accomplish many different tasks simultaneously.

Motivated by these deficiencies, the following augmented definitions have been introduced:

Definition 1. *[10] A graph G is said to be online arbitrarily partitionable (OL-AP for short) if and only if one of the following two conditions holds.*

- *G is isomorphic to K_1.*

[1] In the sense that two resources of a subnetwork must be able to communicate within it.

- *G is connected and for every $\lambda \in \{1, ..., n-1\}$, there exists a partition $(S_\lambda, S_{n-\lambda})$ of $V(G)$ such that $G[S_\lambda]$ is connected on λ vertices and $G[S_{n-\lambda}]$ is OL-AP with order $n - \lambda$.*

Definition 2. *[5] A graph G is said to be recursively arbitrarily partitionable (R-AP for short) if and only if one of the following two conditions holds.*

- *G is isomorphic to K_1.*
- *G is connected and for every sequence $\tau = (n_1, ..., n_p)$ admissible for it, there exists a partition $(V_1, ..., V_p)$ of $V(G)$ into p parts such that each V_i induces a R-AP subgraph of G on n_i vertices.*

Observe that the notion of OL-AP graphs (resp. R-AP graphs) can be used to deal with our network sharing problem taking issue 1 (resp. issue 2) pointed out above into account. It has to be known that the two properties of being OL-AP and R-AP are actually quite similar to each other. Indeed, previous investigations have shown that every R-AP graph is also OL-AP [5] and that, in the context of some classes of graphs (like *trees* and so-called *suns*), there only exist a few OL-AP graphs which are not R-AP [5, 4].

Theorem 1. *[5] Every R-AP graph is also OL-AP, but the contrary does not necessarily hold.*

We here focus on the class of *balloon graphs* introduced in [5].

Definition 3. *Let $b_1, ..., b_k$ be $k \geq 1$ positive integers and $B_1, ..., B_k$ be paths on $b_1 + 2, ..., b_k + 2$ vertices, respectively, where the endvertices of the i^{th} path are denoted by u_i and v_i. The balloon with k branches $B(b_1, ..., b_k)$ (sometimes called k-balloon for short) is the graph obtained by identifying all the u_i's together and all the v_i's together.*

Two examples of such graphs are given in Figure 1. Balloons are interesting regarding our problem because their structure is closely related to the one of partitionable graphs with connectivity 2.

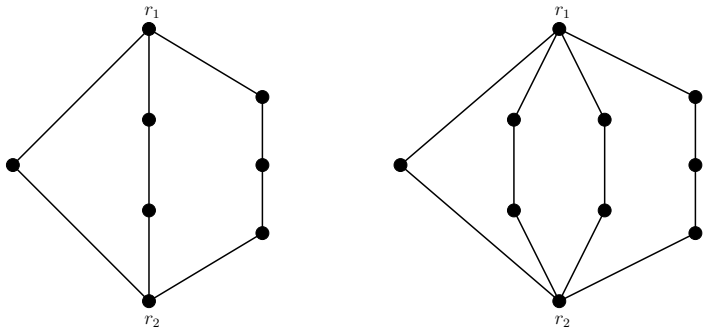

Figure 1: The balloons $B(1, 2, 3)$ and $B(1, 2, 2, 3)$

Observation 1. *Let G be a graph with connectivity 2, u and v be two vertices forming a cut pair of G, and $b_1, ..., b_k$ be the numbers of vertices of the $k \geq 2$ connected components of $G - \{u, v\}$. If G is AP, OL-AP or R-AP, then $B(b_1, ..., b_k)$ is AP, OL-AP or R-AP, respectively.*

Indeed, observe that no graph with order n is easier to partition than the path on n vertices. Hence, a realization of a sequence τ in $B(b_1, ..., b_k)$ can be directly deduced from a realization of τ in G. By Observation 1, we get that some properties holding for AP, OL-AP or R-AP balloons also hold for AP, OL-AP or R-AP graphs with connectivity 2, respectively. In particular, if $c \geq 1$ is an upper bound on the number of branches in an AP balloon, then c is also an upper bound on the number of components resulting from the deletion of a cut pair in an AP graph. One can also deduce an upper bound on the number of vertices in some of these components from a bound on the order of the corresponding branches in an AP balloon. Of course, these statements also hold for OL-AP and R-AP balloons and OL-AP and R-AP graphs with connectivity 2. This approach for exhibiting structural properties of AP graphs with given connectivity was already used in [3].

In this article are exhibited such upper bounds on the structure of OL-AP and R-AP balloons. After having provided in Section 2 some preliminary definitions and tools necessary to understand our results, we then show, in Section 3, that an OL-AP or R-AP balloon has at most 5 branches. In Section 4 is exhibited an infinite family of OL-AP and R-AP 5-balloons showing that the number of these graphs is not bounded and that our previous bound is tight. We then give constant upper bounds on the orders of the smallest branches in an OL-AP or R-AP 4- or 5-balloon in Section 5. Finally, concluding Section 6 outlines some structural properties of OL-AP or R-AP graphs with connectivity 2 that can be derived from our results on OL-AP and R-AP balloons.

2 Terminology and preliminary results

2.1 Terminology and notation

Let $x \geq 1$ be an integer. Throughout this paper, we denote by x^+ (resp. x^-) an arbitrary integer y such that $y \geq x$ (resp. $y \leq x$).

We deal with connected, non-oriented and simple graphs, using mainly the terminology of [8]. The vertex and edges sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively, or simply by V and E when no ambiguity is possible. The order of G, commonly denoted by n, is its number of vertices. Given a subset of vertices $S \subseteq V$ of G, we denote by $G[S]$ the subgraph of G induced by the vertices of S. We additionally denote by $G - S$ the subgraph of G induced by all the vertices of $V - S$. If $F \subseteq E$ is a subset of edges of G, we denote by $G - F$ the partial graph of G obtained by removing all the edges of F from G.

We denote by P_n the path of order n. Given $k \geq 1$ positive integers $a_1, ..., a_k$, the *k-pode* $P(a_1, ..., a_k)$ is the tree obtained by linking one central node to one extremity of each one of k disjoint paths on $a_1, ..., a_k$ vertices, respectively. Since previous investigations [5, 7], a 3-pode $P(1, a_2, a_3)$ is often referenced as a *caterpillar* and is denoted by $Cat(a_2+1, a_3+1)$ for convenience[2]. Hence, throughout this paper, every mention to caterpillars actually refers of caterpillars of the form $Cat(a, b)$.

We now give more notations associated with the notion of balloon graphs. Let $B = B(b_1, ..., b_k)$ be a k-balloon. The vertices r_1 and r_2 of degree k in B are called the *roots of* B, while the path of order b_i connecting them is said to

[2]Observe that $Cat(a, b)$ has order $a + b$.

be the i^{th} *branch of* B. For every $i \in \{1, ..., k\}$, the vertices of the i^{th} branch of B are denoted by $v_1^i, ..., v_{b_i}^i$ in such a way that $v_1^i r_1, v_{b_i}^i r_2 \in E$ and $v_j^i v_{j+1}^i \in E$ for every $j \in \{1, ..., b_i - 1\}$. Finally, we denote by $b_i(B)$ the number of vertices composing the i^{th} branch of B.

We denote by $PB(b_1, ..., b_x, \overline{b_{x+1}}, ..., \overline{b_{x+y}}, \underline{b_{x+y+1}}, ..., \underline{b_{x+y+z}})$ the graph $B(b_1, ..., b_{x+y+z}) - (\bigcup_{i=x+1}^{x+y} \{v_{b_i}^i r_2\}, \bigcup_{i=x+y+1}^{x+y+z} \{v_1^i r_1\})$ obtained by removing $y+z$ edges from a $(x+y+z)$-balloon. Such a graph is called a *partial balloon with* $(x+y+z)$ *branches*, or *partial* $(x+y+z)$*-balloon* for short. In this paper, the notations introduced above for balloons are used in an analogous way for partial balloons. For convenience, the vertices of a *hanging branch* with order b_i of a partial balloon B (that is, a branch linked to only one root of B) are successively denoted by $v_1^i, ..., v_{b_i}^i$ where v_1^i is the degree-1 vertex of the branch and $v_{b_i}^i$ is the vertex adjacent to one root of B. Please refer to Figure 2 for two examples of partial balloons and their associated notations.

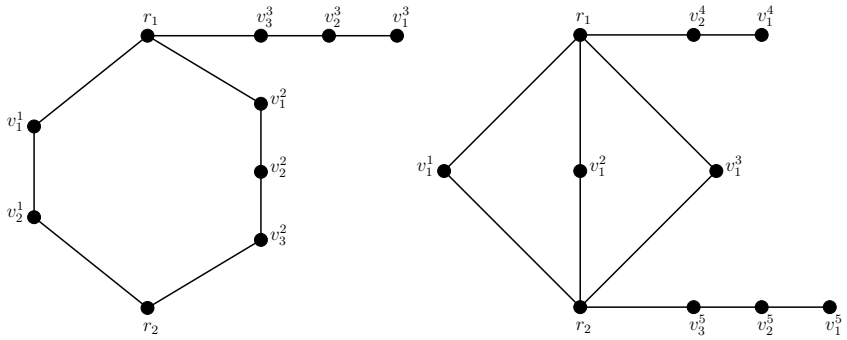

Figure 2: The partial balloons $PB(2, 3, \overline{3})$ and $PB(1, 1, 1, \overline{2}, \underline{3})$

2.2 Some properties of OL-AP and R-AP graphs

The next two theorems give a complete characterization of OL-AP and R-AP trees.

Theorem 2. *[10] A tree is OL-AP if and only if it is either isomorphic to a path, to a caterpillar* $Cat(a, b)$ *with* a *and* b *given in Table 1, or to the 3-pode* $P(2, 4, 6)$*.*

Theorem 3. *[5] A tree is R-AP if and only if it is either isomorphic to a path, to a caterpillar* $Cat(a, b)$ *with* a *and* b *given in Table 2, or to the 3-pode* $P(2, 4, 6)$*.*

Theorems 2 and 3 were proved using the following two observations, which provide two alternative methods to check whether a graph is OL-AP or R-AP.

Observation 2. *A graph* G *is R-AP if and only if for every* $\lambda \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$ *there exists a partition* $(S_\lambda, S_{n-\lambda})$ *of* V *into two parts such that* $G[S_\lambda]$ *and* $G[S_{n-\lambda}]$ *are connected R-AP graphs on* λ *and* $n - \lambda$ *vertices, respectively.*

Observation 3. *The property of being OL-AP (resp. R-AP) is closed under edge-additions: if a graph has an OL-AP (resp. a R-AP) spanning subgraph, then it is OL-AP (resp. R-AP).*

a	b
2, 4	$\equiv 1 \bmod 2$
3	$\equiv 1, 2 \bmod 3$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \bmod 6$
7	8, 9, 11, 13, 15
8	11, 19
9, 10	11
11	12

Table 1: Values a and b, with $b \geq a$, such that $Cat(a, b)$ is OL-AP.

a	b
2, 4	$\equiv 1 \bmod 2$
3	$\equiv 1, 2 \bmod 3$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 2: Values a and b, with $b \geq a$, such that $Cat(a, b)$ is R-AP.

2.3 Some properties of AP, OL-AP and R-AP balloons

First, notice that every path is AP, OL-AP and R-AP. For this reason, it follows that every *traceable graph*[3] is AP, OL-AP and R-AP too. Since 1-, 2- and 3-balloons are traceable, we are mainly interested in balloons with at least 4 branches in this work.

We now give some properties on the number of vertices of the branches in a partitionable balloon. It was previously proved that an AP balloon can always be partitioned in such a way that its branches with order 0 (that is, its branches with no vertices) are not used by the partition, and thus that $B(0, b_1, ..., b_k)$ is AP if and only if $B(b_1, ..., b_k)$ is AP [5]. We show that this result also holds when considering OL-AP or R-AP balloons.

Lemma 1. *Let $b_1 \leq ... \leq b_k$ be positive integers with $k \geq 1$. The balloon $B(0, b_1, ..., b_k)$ is OL-AP (resp. R-AP) if and only if $B(b_1, ..., b_k)$ is OL-AP (resp. R-AP).*

Proof. The necessary condition follows directly from Observation 3. Let us now prove the sufficient condition. Whenever $k \leq 3$, the lemma is true since both $B(0, b_1, ..., b_k)$ and $B(b_1, ..., b_k)$ are traceable for every $b_1, ..., b_k \geq 1$.

Suppose now that there exist OL-AP balloons with at least 4 branches composed by at least 1 vertex contradicting the claim, and let us denote by $B = B(0, b_1, ..., b_k)$ one of those with minimum order and $k \geq 4$. In particular, it means, according to Definition 1, that there exists $\lambda \in \{1, ..., n-1\}$ such that the following two conditions are met.

- The vertex set of B admits a partition $(S_\lambda, S_{n-\lambda})$ such that $B[S_\lambda]$ is connected on λ vertices and $B[S_{n-\lambda}]$ is OL-AP with order $n - \lambda$.

[3] A graph is *traceable* if it has a Hamiltonian path.

- The partition $(S_\lambda, S_{n-\lambda})$ is not a satisfying partition of the vertex set of $B' = B(b_1, ..., b_k)$ regarding the conditions above.

It follows that the edge r_1r_2 is necessary for the partition, and thus that either S_λ or $S_{n-\lambda}$ contains both r_1 and r_2. This implies that the part not containing r_1r_2 only contains some consecutive vertices from one branch of B, say the one with order b_i where $b_i \geq 1$.

On the one hand, if $r_1, r_2 \in S_\lambda$, then $B[S_\lambda]$ remains connected even if we remove r_1r_2 from it since r_1 and r_2 are also connected in $B[S_\lambda]$ thanks to at least 3 branches of B. In this case, the branch with order 0 of B is useless for the partition, a contradiction.

On the other hand, if $r_1, r_2 \in S_{n-\lambda}$, then $B[S_{n-\lambda}]$ is a partial balloon. Using Observation 3, we can deduce an OL-AP balloon B'' with less vertices than B and a branch with no vertex. By the minimality of B, it follows that B'' remains OL-AP even when the edge linking its two roots is removed from it. Hence, $B[S_{n-\lambda}]$ remains OL-AP even if we remove the edge r_1r_2 from it, and we could deduce a partitioning of B' respecting the conditions above. Again, a contradiction.

This concludes the proof for OL-AP balloons. A similar proof can be led for R-AP balloons using the fact that the parts S_λ and $S_{n-\lambda}$ of the partition of B must induce R-AP subgraphs of B (see Observation 2).

□

Hence, throughout this paper, we will only consider balloons having their smallest branch composed of at least 1 vertex. We additionally introduce the following results on the orders of the branches in an AP balloon.

Observation 4. *[5] Let B be an AP balloon. If n is odd, then B has at most 3 branches of odd order. If n is even, then B has at most two branches of odd order.*

Lemma 2. *[3] Let $B(b_1, ..., b_k)$ be an AP balloon with $b_1 \leq ... \leq b_k$. For every $i \in \{2, ..., k\}$, we have $2b_i \geq \sum_{j=1}^{i-1} b_j$.*

Since every OL-AP or R-AP graph is also AP, the previous two results naturally also hold when considering OL-AP and R-AP balloons.

3 OL-AP balloons cannot have more than 5 branches

It is already known that a R-AP ballon has at most 5 branches [5]. In this section, we prove a more general statement by showing the same upper bound on the number of branches in an OL-AP balloon. By Theorem 1, this result also holds for R-AP balloons.

Theorem 4. *An OL-AP balloon cannot have more than 5 branches.*

Proof. The proof is by contradiction. Let $\mathcal{B}$ be the set of OL-AP balloons with at least 6 branches and B denote a k-balloon of $\mathcal{B}$ with the least order.

By Definition 1, B is OL-AP if and only if for every $\lambda \in \{1, ..., n-1\}$ we can partition V into two parts S_λ and $S_{n-\lambda}$ such that $B[S_\lambda]$ is connected on λ vertices and $B[S_{n-\lambda}]$ is OL-AP on $n - \lambda$ vertices. Observe that, because of

the minimality of B, the subgraph $B[S_{n-\lambda}]$ cannot be a partial k'-balloon with $k' \geq 6$ since otherwise there would exist a balloon of $\mathcal{B}$ with less vertices than B (Observation 3). It follows that $B[S_{n-\lambda}]$ is either an OL-AP 5-balloon or an OL-AP tree (see Theorem 2).

We claim that B has branches with 1, 2, 3, 4, 5 and 6 vertices. Let us suppose that $\lambda \in \{1, ..., 6\}$ and that B does not have a branch with order λ. We show that it is not possible to partition V into two parts with cardinal λ and $n - \lambda$, respectively, satisfying the above conditions.

- $\lambda \in \{1, 2, 3\}$: for every choice of S_λ, the subgraph $B[S_{n-\lambda}]$ is either a partial k-balloon having less vertices than B or a tree with maximum degree at least 4. In both cases, the subgraph $B[S_{n-\lambda}]$ is not OL-AP.

- $\lambda = 4$: so far, we have shown that B necessarily has branches with order 1, 2 and 3. Similarly as in the previous case, observe that for every choice of S_λ, the subgraph $B[S_{n-\lambda}]$ is either a partial k-balloon or partial $(k+1)$-balloon having less vertices than B or a tree having maximum degree at least 3. Hence, the only possibility here is to choose S_λ in such a way that $B[S_{n-\lambda}]$ is a tree with maximum degree 3, but this is only possible when $B = B(1, 1, 1, 2, 3, ...)$. According to Observation 4, such a balloon is not AP, and thus is not OL-AP.

- $\lambda = 5$: by the previous cases, we know that B has branches composed of 1, 2, 3 and 4 vertices. For the same reasons as before, S_λ must be chosen in such a way that $B[S_{n-\lambda}]$ is either a path or an OL-AP 3-pode. Hence, since B has at least 6 branches, S_λ must contain one root of B and all the vertices of at least 3 of its branches. Observe that S_λ can only be chosen in this way when $k = 6$ and $B = B(1, 1, 1, 2, 3, 4)$. It follows that B has 4 branches of odd order, and thus that it is not AP according to Observation 4. It cannot be OL-AP.

- $\lambda = 6$: we know that B has branches with 1, 2, 3, 4 and 5 vertices. Moreover, since $k \geq 6$, the balloon B has one additional branch of order b_i. If $b_i \leq 7$, then B is not AP by Lemma 2, and thus is not OL-AP. Hence, $b_i \geq 8$ but, again, we cannot exhibit a correct subset S_λ for the same reasons as before. Hence B is not OL-AP.

Finally, B is isomorphic to $B(1, 2, 3, 4, 5, 6, ...)$ which is not AP following Lemma 2; it thus cannot be OL-AP. □

4 There are infinitely many OL-AP and R-AP 4- or 5-balloons

By Theorem 4, we know that the number of branches in an OL-AP or R-AP balloon is bounded by 5. In what follows, we prove that this bound is tight by exhibiting two infinite families of R-AP balloons with 4 and 5 branches, respectively. We use the following two lemmas for this purpose.

Lemma 3. *The partial balloon $PB(1, 1, 2, \overline{k})$ is R-AP for every $k \geq 1$.*

Proof. Observe that this claim is true whenever $k = 1$, $k = 2$ or $k = 3$ since the corresponding partial balloons are spanned by $Cat(2, 5)$, $Cat(3, 5)$ and $Cat(4, 5)$, respectively.

Suppose now that this claim holds for every k up to $i - 1$ and consider the partial balloon $B = PB(1, 1, 2, \overline{i})$. By Observation 2, we know that B is R-AP if it can be partitioned, for every $\lambda \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$, into two R-AP subgraphs on λ and $n - \lambda$ vertices, respectively. One can consider the following partitions:

- $\lambda = 1$: P_1 and $PB(1, 2, \overline{i})$ (traceable).
- $\lambda = 2$: P_2 and $PB(1, 1, \overline{i})$ (traceable).
- $\lambda \in \{3, 4\}$: P_λ and $P_{n-\lambda}$.
- $\lambda = 5$: $Cat(2, 3)$ and P_{i+1}.
- $\lambda = 6$: $B(1, 1, 2)$ (traceable) and P_i.
- $\lambda \geq 7$: $PB(1, 1, 2, \overline{\lambda - 6})$ (induction hypothesis) and $P_{i-\lambda+6}$.

□

Lemma 4. *The partial balloon $PB(1, 2, 3, \overline{k})$ is R-AP for every $k \geq 1$.*

Proof. The proof is by induction on k. If we first suppose that $k = 1$, $k = 2$, or $k = 3$, then observe that the corresponding partial 4-balloons are R-AP since they are spanned by the R-AP caterpillars $Cat(2, 7)$, $Cat(3, 7)$ and $Cat(4, 7)$, respectively.

Let us secondly suppose that this lemma holds for every $k \leq i - 1$, and consider the partial balloon $B = PB(1, 2, 3, \overline{i})$. Once again, by Observation 2, it is sufficient to show, to prove that B is R-AP, that we can partition it into two R-AP subgraphs on λ and $n - \lambda$ vertices, respectively, for every $\lambda \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$. We show that these partitions exist for every λ:

- $\lambda = 1$: P_1 and $PB(2, 3, \overline{i})$ (traceable).
- $\lambda = 2$: P_2 and $PB(1, 3, \overline{i})$ (traceable).
- $\lambda = 3$: P_3 and $PB(1, 2, \overline{i})$ (traceable).
- $\lambda \in \{4, 5, 6\}$: P_λ and $P_{n-\lambda}$.
- $\lambda = 7$: $Cat(3, 4)$ and P_{i+1}.
- $\lambda = 8$: $B(1, 2, 3)$ (traceable) and P_i.
- $\lambda \geq 9$: $PB(1, 2, 3, \overline{\lambda - 8})$ (induction hypothesis) and $P_{n-\lambda+8}$.

□

Observe that, according to Observation 3, Lemmas 3 and 4 directly imply that there exist infinitely many R-AP 4-balloons (and, thus, infinitely many OL-AP 4-balloons, see Theorem 1).

Corollary 1. *The 4-balloons $B(1, 1, 2, k)$ and $B(1, 2, 3, k)$ are OL-AP and R-AP for every $k \geq 1$.*

We now prove that there exists an unbounded family of R-AP 5-balloons.

Theorem 5. *The partial balloon $PB(1,1,2,3,\overline{2k})$ is R-AP for every $k \geq 1$.*

Proof. Let $B = PB(1,1,2,3,\overline{2k})$ with $k \geq 1$. Recall that, according to Observation 2, the partial balloon B is R-AP if we can partition it into two R-AP subgraphs $B[S_\lambda]$ and $B[S_{n-\lambda}]$ having order λ and $n-\lambda$, respectively, for every $\lambda \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$. One can consider the following partitions for the first values of λ:

- $\lambda = 1$: P_1 and $B(1,2,3,\overline{2k})$ (Lemma 4).
- $\lambda = 2$: P_2 and $B(1,1,3,\overline{2k})$ (spanned by $Cat(2, 5+2k)$).
- $\lambda = 3$: P_3 and $B(1,1,2,\overline{2k})$ (Lemma 3).
- $\lambda = 4$: P_4 and $Cat(4, 2k+1)$.
- $\lambda = 5$: $Cat(2,3)$ and P_{2k+4}.
- $\lambda = 6$: P_6 and $Cat(2, 2k+1)$.
- $\lambda = 7$: $Cat(3,4)$ and P_{2k+2}.

By now, it should be clear that the proposition holds for every partial balloon $PB(1,1,2,3,\overline{2k})$ such that $n \leq 15$ (that is, for each $k \in \{1,2,3\}$). Let us suppose, as an induction hypothesis, that the claim is true for every $k \leq i-1$, and consider the partition of $B = PB(1,1,2,3,\overline{2i})$ into two R-AP subgraphs for the remaining values of λ, that is for every $\lambda \in \{8, ..., \lfloor \frac{n}{2} \rfloor\}$.

- $\lambda \geq 8$, λ even: observe that $\lambda \leq 2i$ since $\lambda \leq \lfloor \frac{n}{2} \rfloor$ and we handled the cases where $k \leq 3$. We can thus partition B into P_λ and either $B(1,1,2,3)$ (when $i = 4$) or $PB(1,1,2,3,\overline{2i-\lambda})$ (when $i > 4$). These graphs are R-AP according to Lemma 4 and by the induction hypothesis (since $2i - \lambda$ is even), respectively.
- $\lambda = 9$: $B(1,1,2,3)$ (spanned by $Cat(2,7)$) and P_{2i}.
- $\lambda > 9$, λ odd: $PB(1,1,2,3,\overline{\lambda-9})$ (induction hypothesis since $\lambda - 9$ is even) and $P_{n-\lambda+9}$.

□

Combining Observation 3 and Theorem 5 we get that the 5-balloon $B(1,1,2,3,2k)$ is R-AP for every $k \geq 1$. Since every R-AP graph is also OL-AP (Theorem 1), we deduce the following:

Corollary 2. *The 5-balloon $B(1,1,2,3,2k)$ is OL-AP and R-AP for every $k \geq 1$.*

5 Constant upper bounds on the orders of the smallest branches in OL-AP or R-AP 4- or 5-balloons

Let $B = (b_1, ..., b_k)$ be a k-balloon on n vertices with $b_1 \leq ... \leq b_k$. In this section, we prove the following result.

Theorem 6. *If B is an OL-AP 4- or 5-balloon, then $b_1 \leq 11$.*

The proof of this claim reads as follows. When considering the partition of B according to Definition 1 for $\lambda = 1$ (that is, a partition of B into two connected subgraphs $B[S_1]$ and $B[S_{n-1}]$ that are an isolated vertex and an OL-AP graph on $n-1$ vertices, respectively), the possible ways for choosing $B[S_{n-1}]$ are actually quite limited. Moreover, it turns out that, under the assumption that $b_1 \geq 12$, none of these possibilities leads to a correct partition of B. Hence, B cannot be OL-AP for such a value of b_1. Lemmas 5 to 8 below list these graph structures that can be obtained while partitioning B.

Lemma 5. *The graph $PB(12^+, 12^+, \overline{x}, \underline{y})$ is not OL-AP for every $x, y \geq 1$.*

Proof. We prove this claim by induction on $x + y$. As a base case, consider $x = y = 1$ and the partial balloon $B = PB(12^+, 12^+, \overline{1}, \underline{1})$. By Definition 1, recall that B is OL-AP if and only if, for every $\lambda \in \{1, ..., n-1\}$, there exists a partition $(S_\lambda, S_{n-\lambda})$ of V such that $B[S_\lambda]$ and $B[S_{n-\lambda}]$ are connected on λ vertices and OL-AP on $n - \lambda$ vertices, respectively. In particular, observe here that B cannot be partitioned in this way for $\lambda = 2$. Indeed, every possible choice of S_2 makes $B[S_{n-2}]$ being either disconnected, a caterpillar $Cat(13^+, 13^+)$ or $Cat(11^+, 15^+)$, or a tree with two degree-3 vertices. Since none of these graphs is OL-AP (Theorem 2), B is not OL-AP.

To complete the base case, let us now suppose that $x+y = 3$ and denote by B the partial balloon $PB(12^+, 12^+, \overline{1}, \underline{2})$. As in the previous base case, one has to observe that B is not OL-AP since it cannot be partitioned in the way specified by Definition 1 for $\lambda = 3$. In particular, observe that for every possible choice of S_3, the graph $B[S_{n-3}]$ is not OL-AP for it is disconnected, a non-caterpillar 3-pode different from $P(2, 4, 6)$, a caterpillar $Cat(10^+, 16^+)$ or $Cat(13^+, 13^+)$, or a tree with two degree-3 vertices.

Consider now that the claim holds for every partial balloon $PB(12^+, 12^+, \overline{x}, \underline{y})$ such that $x + y \leq k - 1$ for some $k > 4$. We now prove that it is also true for a partial balloon $B = PB(12^+, 12^+, \overline{x}, \underline{y})$ when $x + y = k$. There are two cases to consider:

- $x > 1$ and $y > 1$: B is not OL-AP since we cannot partition its vertex set as explained above for $\lambda = 1$. Indeed, we must consider $S_1 = \{v_1^3\}$ or $S_1 = \{v_1^4\}$ since otherwise $B[S_{n-1}]$ would be either disconnected, or isomorphic to a large 3-pode or a tree with two degree-3 vertices. But for these two choices of S_1, the remaining graph $B[S_{n-1}]$ is isomorphic to a partial balloon $PB(12^+, 12^+, \overline{x'}, \underline{y'})$ with $x' + y' = x + y - 1 \leq k - 1$, which is not OL-AP by the induction hypothesis.

- $x = 1$ and $y > 2$: consider we want to partition B as previously for $\lambda = 2$. For the same reasons as above, we have to consider $S_2 = \{v_1^4, v_2^4\}$. But

then $B[S_{n-2}]$ is isomorphic to $PB(12^+, 12^+, \overline{x}, \underline{y-2})$ which is not OL-AP according to the induction hypothesis. Hence, B is not OL-AP. These arguments hold analogously when $x > 2$ and $y = 1$.

□

Since the proofs of Lemmas 6 to 8 are quite similar to the one of Lemma 5, the reader is referred to Appendix A for in-depth proofs on these statements.

Lemma 6. *The graph* $PB(12^+, 12^+, 12^+, \overline{x}, \underline{y})$ *is not OL-AP for every* $x, y \geq 1$.

Lemma 7. *The graph* $PB(12^+, 12^+, 12^+, \overline{x})$ *is not OL-AP for every* $x \geq 1$.

Lemma 8. *The graph* $PB(12^+, 12^+, 12^+, 12^+, \overline{x})$ *is not OL-AP for every* $x \geq 1$.

Using Lemmas 5 to 8, we now proof the statement of Theorem 6.

Proof of Theorem 6. Let $B = B(12^+, 12^+, 12^+, 12^+)$ be a 4-balloon. B is not OL-AP since its vertex set cannot be partitioned in the way specified by Definition 1 for $\lambda = 1$. Indeed, for every choice of S_1, the graph $B[S_{n-1}]$ is not OL-AP since it is either a tree with maximum degree 4 or a partial balloon which is not OL-AP by Lemma 6 or 7. It follows that an OL-AP 4-balloon must have a branch of order at most 11.

Now let $B = B(12^+, 12^+, 12^+, 12^+, 12^+)$ be a 5-balloon. Similarly as before, B is not OL-AP since there does not exist a partition of its vertex set respecting Definition 1 for $\lambda = 1$. Indeed, every possible choice of S_1 makes $B[S_{n-1}]$ being either a tree with maximum degree 5, a partial balloon which is not OL-AP according to Lemma 8, or a partial 6-balloon. In the latter case, observe that $B[S_{n-1}]$ cannot be OL-AP since otherwise there would exist, by Observation 3, an OL-AP 6-balloon contradicting Theorem 4. Hence, a 5-balloon cannot be OL-AP when its smallest branch has order at least 12. □

Since every OL-AP graph is also R-AP (Theorem 1), Theorem 6 directly implies that R-AP balloons with 4 or 5 branches have their smallest branch of order at most 11 too. However, using the fact that R-AP caterpillars are generally smaller than OL-AP caterpillars (see Theorems 3 and 2), one can easily derive Lemmas 5 to 8 above for R-AP partial balloons to get a better upper bound on the order of the smallest branch of a R-AP 4- or 5-balloon. The proof of this statement is omitted in this work since it is very similar to the proof of Theorem 6.

Theorem 7. *If* B *is a R-AP 4- or 5-balloon, then* $b_1 \leq 7$.

One can also get a similar constant upper bound on the order of the second smallest branch in a R-AP 4- or 5-balloon B, that is that $b_2 \leq 39$. The main argument in our proof of Theorem 7 is that partial 4-balloons of the form $PB(8^+, 8^+, \overline{x}, \underline{y})$ are generally not R-AP. Because of that fact, plenty of other partial balloons cannot be R-AP too, and the result follows. Such a statement is also true when considering that $b_1 \leq 7$ and $b_2 \geq 40$. Indeed, most of partial balloons $PB(7^-, 40^+, \overline{x}, \underline{y})$ cannot be R-AP because they cannot be partitioned into two R-AP subgraphs with order $\lfloor \frac{n}{2} \rfloor$ according to Observation 2: in general, one of these two subgraphs necessarily has to be isomorphic to a caterpillar $Cat(a, b)$ with a and b being greater than the values given by Theorem 3. Once again, the proof of the following claim is omitted here, but it can be obtained by deriving Lemmas 5 to 8 adequately.

Theorem 8. *If B is a R-AP 4- or 5-balloon, then $b_2 \leq 39$.*

Unfortunately, we did not manage to derive a similar constant upper bound on b_2 when B is an OL-AP 4- or 5-balloon. Indeed, when partitioning such a graph regarding Definition 1, the recursive property only concerns one of the two induced subgraphs. Hence, it appears tricky to find a constant value $k \geq 1$ such that partial balloons of the form $PB(11^-, k^+, \overline{x}, y)$ are generally not OL-AP. Because we cannot find such a class of non OL-AP partial balloons, we cannot derive our proof scheme as it was done before and get a bound on b_2 for OL-AP 4- or 5-balloons, if such a bound exists. So we leave the following question unanswered.

Question 1. *Is there a positive constant $c \geq 1$ such that if B is an OL-AP 4- or 5-balloon, then $b_2 \leq c$?*

Something more can be deduced from the previous bounds. In what follows, let us denote by $LP(G)$ the length of the longest simple path in a given graph G. Observe that if $B = B(b_1, ..., b_k)$ is a balloon with $k \geq 4$ such that $b_1 \leq ... \leq b_k$, then $LP(B) = b_k + b_{k-1} + b_{k-2} + 1$. Therefore, thanks to Theorems 6, 7 and 8, we get the following result on OL-AP or R-AP balloons.

Corollary 3. *Let B be a k-balloon with order n.*

- *If B is OL-AP and $k = 4$, then $LP(B) \geq n - 12$.*
- *If B is R-AP and $k = 4$, then $LP(B) \geq n - 8$.*
- *If B is R-AP and $k = 5$, then $LP(B) \geq n - 47$.*

Such a result meets the following question.

Question 2. *What are the classes of OL-AP or R-AP graphs $\mathcal{C}$ for which there exists a constant $c \geq 1$ such that, for every graph $G \in \mathcal{C}$, we have $LP(G) \geq n - c$ where n is the order of G?*

Note that not all families of graphs have this property since an infinite family of R-AP graphs that does not meet Question 2 was exhibited in [6]. According to Theorems 2 and 3, OL-AP and R-AP trees are such classes of graphs. By Corollary 3, we know that OL-AP 4-balloons and R-AP 4- or 5-balloons are such classes too. If the bound c on b_2 mentioned in Question 1 exists, then it would imply that OL-AP 5-balloons perform a satisfying class for Question 2.

6 Structural properties of OL-AP or R-AP graphs with connectivity 2

As mentioned in the introduction section, properties of OL-AP or R-AP balloons can be derived to properties of OL-AP or R-AP graphs with connectivity 2, respectively. In particular, the results on OL-AP or R-AP balloons we pointed out along Sections 3 to 5 are extendable in the following way.

Corollary 4. *Let G be a graph with connectivity 2, u and v be two vertices forming a cut pair of G, and $b_1 \leq ... \leq b_k$ be the numbers of vertices of the $k \geq 2$ connected components of $G - \{u, v\}$. If G is OL-AP or R-AP, then the following conditions hold:*

- $k \leq 5$,
- b_k *can be arbitrarily large,*
- *if* G *is R-AP and* $k \in \{4, 5\}$*, then* $b_1 \leq 7$ *and* $b_2 \leq 39$,
- *if* G *is OL-AP and* $k \in \{4, 5\}$*, then* $b_1 \leq 11$.

The first item of Corollary 4 follows directly from Theorem 4, the second item is derived from Corollaries 1 and 2, while the third and fourth items result from Theorems 6, 7 and 8.

Since 2- and 3-balloons are always OL-AP and R-AP because they are traceable, the only structural property we can derive is that if G is disconnected into only 2 or 3 components after the removal of u and v, then these components can all be arbitrarily large.

Notice that Corollary 3 cannot be extended to OL-AP or R-AP graphs with connectivity 2 as previously. Indeed, there is no direct analogy between the longest path in a balloon $B = B(b_1, ..., b_k)$, with $b_1 \leq ... \leq b_k$, and the longest path in a graph G that has a cut pair $\{u, v\}$ whose removal leads to k components with orders $b_1, ..., b_k$, respectively. As an illustration, suppose that the component with order b_k in $G - \{u, v\}$ is a $(b_k - 2)$-balloon $B(1, ..., 1)$ whose roots are connected to u and v in G. Depending on the structure of G, its longest path may not pass through its component with order b_k. In comparison, the longest path of B generally has to go along its k^{th} branch.

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A Appendix

This appendix gathers all the proofs of Lemmas 6 to 8 of Section 5, as well as some intermediate lemmas needed to prove these statements. Notice that these proofs often make implicit use of the full characterization of OL-AP trees (Theorem 2) and the two sufficient conditions for a graph to be OL-AP given by Observations 2 and 3. In all these proofs, it is assumed that $x \leq y$ or $x \leq y \leq z$ generally holds when the corresponding elements have been introduced. Given a graph G and an integer $\lambda \in \{1, ..., n-1\}$, an *OL-AP-partition of G for λ* is a partition $(S_\lambda, S_{n-\lambda})$ of V such that $G[S_\lambda]$ and $G[S_{n-\lambda}]$ are connected on λ vertices and OL-AP on $n-\lambda$ vertices, respectively. According to Definition 1, the graph G is OL-AP if and only if either G is an isolated vertex or G admits an OL-AP-partition for every $\lambda \in \{1, ..., n-1\}$.

Lemma 9. *The graph $PB(12^+, 12^+, \overline{x}, \overline{y})$ is not OL-AP for every $x \geq 1$ and $y \geq 10$.*

Proof. Let us prove this claim by induction on $x+y$ as we did to prove Lemma 5. As a base case, let us consider the graph $B = PB(12^+, 12^+, \overline{1}, \overline{10})$. Observe that B is not OL-AP since it does not admit an OL-AP-partition for 11. Indeed, every possible choice of 11 vertices inducing a connected subgraph of B makes $B[S_{n-11}]$ being either disconnected, a caterpillar $Cat(11, 13^+)$ or a tree with maximum degree 4. For similar reasons, observe that neither $PB(12^+, 12^+, \overline{1}, \overline{11})$ nor $PB(12^+, 12^+, \overline{2}, \overline{10})$ are OL-AP since they do not admit an OL-AP-partition for 12 and 11, respectively.

Let us now suppose that this lemma holds whenever $x + y \leq k - 1$ for some $k > 13$, and consider a graph $B = PB(12^+, 12^+, \overline{x}, \overline{y})$ such that $x + y = k$. We claim that there exists a $\lambda \in \{1, ..., n-1\}$ such that B does not admit an OL-AP-partition for λ, and thus that B is not OL-AP:

- $x > 1$ and $y > 10$: under these conditions, there does not exist an OL-AP-partition of B for 1. Indeed, every possible choice for S_1 which does not make $B[S_{n-1}]$ being disconnected makes this subgraph being isomorphic to either a non-caterpillar 3-pode different from $P(2, 4, 6)$, a tree with maximum degree 4, or a graph not OL-AP according to the induction hypothesis.

- $x = 1$ and $y > 11$: observe that there does not exist an OL-AP-partition of B for 2, since every coherent choice for S_2 makes $B[S_{n-2}]$ being disconnected or isomorphic to either a caterpillar $Cat(13^+, 13^+)$, a tree with maximum degree 4, or a partial balloon which is not OL-AP by the induction hypothesis.

- $x > 2$ and $y = 10$: once again, B does not admit an OL-AP-partition for 11 since every choice of 11 vertices inducing a connected subgraph of B makes $B[S_{n-11}]$ being either disconnected, a tree with maximum degree 4, or a non-OL-AP 3-pode.

□

Lemma 10. *The graph $PB(12^+, 12^+, \overline{x}, \overline{y}, \underline{z})$ is not OL-AP for every $x, y, z \geq 1$.*

Proof. We prove this claim by induction on $x+y+z$. Let us first suppose that $x = y = z = 1$ and consider the associated graph $B = PB(12^+, 12^+, \overline{1}, \overline{1}, \underline{1})$. Once again, B is not OL-AP since there does not exist an OL-AP-partition of B for 2. Indeed, every possible for S_2 makes $B[S_{n-2}]$ being either disconnected, or isomorphic to either a tree with maximum degree 4 or a tree having two degree-3 vertices.

To complete the base case, observe that $PB(12^+, 12^+, \overline{1}, \overline{2}, \underline{1})$ and $PB(12^+, 12^+, \overline{1}, \overline{1}, \underline{2})$ are not OL-AP for there do not admit an OL-AP-partition for 3: for every coherent choice of S_3, the subgraph $B[S_{n-3}]$ is disconnected, or isomorphic to either a tree with maximum degree 4, a tree having two degree-3 vertices, or a non-caterpillar 3-pode different from $P(2, 4, 6)$.

Suppose now that this claim holds whenever $x + y + z \leq k - 1$ for some $k > 5$, and consider a balloon $B = PB(12^+, 12^+, \overline{x}, \overline{y}, \underline{z})$ where $x + y + z = k$. Once again, we consider two main cases:

- $z > 1$: in this case, B is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, observe that removing one vertex from B makes the remaining subgraph being disconnected, isomorphic to a tree with maximum degree 4 or two degree-3 vertices, or isomorphic to a partial balloon which is not OL-AP according to the induction hypothesis or Lemma 5.

- $z = 1$: once again, B is not OL-AP under this condition since it cannot be OL-AP-partitioned for 2: for every coherent choice as S_2, the remaining graph $B[S_{n-2}]$ is indeed not connected, a tree with maximum degree 4 or two degree-3 vertices, or a partial balloon which is not OL-AP according to our induction hypothesis or previous Lemma 5.

□

Proof of Lemma 6. Once more, let us prove this claim by induction on $x + y$. Consider first that $x = y = 1$ and let $B = PB(12^+, 12^+, 12^+, \overline{1}, \underline{1})$. Observe that B is not OL-AP for it cannot be OL-AP-partitioned for 2. Indeed, every possible choice for S_2 makes $B[S_{n-2}]$ being disconnected, isomorphic to a tree with maximum degree 4, to a partial balloon which is not OL-AP by Lemma 10, or to a partial 6-balloon. In the latter case, such a graph cannot be OL-AP since otherwise there would exist an OL-AP 6-balloon contradicting Theorem 4.

Additionally, observe that $B = PB(12^+, 12^+, 12^+, \overline{1}, \underline{2})$ cannot be OL-AP-partitioned for 3: for every coherent choice for S_3, the subgraph $B[S_{n-3}]$ is not OL-AP for the same reasons as in the previous case. Hence, B is not OL-AP.

We now suppose that this claim holds for every $x+y \leq k-1$ for some $k > 4$, and consider a partial balloon $B = PB(12^+, 12^+, 12^+, \overline{x}, \underline{y})$ where $x+y = k$. Let us take the following two cases in consideration to show that B is not OL-AP.

- $x > 1$ and $y > 1$: notice that, in this situation, B cannot be OL-AP-partitioned for 1. Indeed, for some similar reasons as the ones we used for the base cases, we have to consider $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$. But in both cases, $B[S_{n-1}]$ cannot be OL-AP by the induction hypothesis.

- $x = 1$ and $y > 2$: once again, observe that B cannot be OL-AP-partitioned for 2. Indeed, observe that we must consider $S_2 = \{v_1^5, v_2^5\}$ since otherwise there would exist an OL-AP 6-balloon, an OL-AP tree having maximum

degree 4, or a graph contradicting Lemma 10. But for this choice of S_2, we have $B[S_{n-2}] = PB(12^+, 12^+, 12^+, \overline{1}, \underline{y-2})$ which is not OL-AP according to our induction hypothesis.

□

Proof of Lemma 7. Once again, this claim is proved by induction on x. Let us first suppose that $x = 1$ and let B be the partial balloon $PB(12^+, 12^+, 12^+, \overline{1})$. This time, B is not OL-AP since it cannot be OL-AP-partitioned for 2: for every possible choice of S_2, the remaining graph $B[S_{n-2}]$ is not OL-AP since it is disconnected, isomorphic to a tree with maximum degree 4, to a non-caterpillar 3-pode different from $P(2, 4, 6)$ or to a partial balloon which is not OL-AP by Lemma 5, 9 or 10.

Let us now suppose that this claim holds for every $x \leq k-1$ and some $k > 3$. To complete the proof, observe that a graph $B = PB(12^+, 12^+, 12^+, \overline{k})$ is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every choice of S_1, the subgraph $B[S_{n-1}]$ is not OL-AP according to the induction hypothesis, or because of one reason used for the base case.

□

Lemma 11. *The graph $PB(12^+, 12^+, \overline{x}, \overline{y}, \overline{z})$ is not OL-AP for every $x, y, z \geq 1$.*

Proof. We prove this claim by induction on $x + y + z$. First, let us suppose that $x = y = z = 1$ and consider the partial balloon $B = PB(12^+, 12^+, \overline{1}, \overline{1}, \overline{1})$. Notice that B is not OL-AP since it cannot be OL-AP-partitioned for 2. Indeed, every choice of S_2 implies that $B[S_{n-2}]$ is either disconnected or isomorphic to a tree with maximum degree at least 4. Analogously, observe that neither $PB(12^+, 12^+, \overline{1}, \overline{1}, \overline{2})$ nor $PB(12^+, 12^+, \overline{1}, \overline{2}, \overline{2})$ are OL-AP since they cannot be OL-AP-partitioned for 3.

Suppose now that this claim holds by induction whenever $x + y + z \leq k - 1$ for a $k > 6$, and consider a partial balloon $B = PB(12^+, 12^+, \overline{x}, \overline{y}, \overline{z})$ where $x + y + z = k$. We distinguish the following two main cases depending on x, y and z:

- $x > 1$, $y > 1$ and $z > 1$: suppose we want to OL-AP-partition B for 1. Then, we must consider $S_1 = \{v_1^3\}$, $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$ since, for every other choice of S_1, the remaining graph $B[S_{n-1}]$ is either disconnected or isomorphic to a tree having maximum degree at least 4. But in any of these three choices for S_1, the subgraph $B[S_{n-1}]$ is not OL-AP by the induction hypothesis. Thus, B is not OL-AP.

- $x = 1$: let $\alpha = \min(\{2, 3, 4\} - \{y, z\})$. In this situation, B cannot be OL-AP for the same reason as above but for an OL-AP-partition of B for α. Indeed, for every coherent choice of S_α, the remaining graph $B[S_{n-\alpha}]$ is not OL-AP either according to the induction hypothesis, or because it is isomorphic to a non-connected graph or a tree with maximum degree at least 4.

□

Lemma 12. *The graph $PB(12^+, 12^+, 12^+, \overline{x}, \overline{y})$ is not OL-AP for every $x, y \geq 1$.*

Proof. Once again, we prove this claim by induction on $x+y$. We first suppose that $x = y = 1$ and let $B = PB(12^+, 12^+, 12^+, \overline{1}, \overline{1})$. Similarly as in the proofs of the previous lemmas, B is not OL-AP for it cannot be OL-AP-partitioned for 2. Indeed, for every possible choice as S_2, the remaining graph $B[S_{n-2}]$ is not connected, a tree with maximum degree 5, a partial balloon not OL-AP according to Lemma 10 or 11, or a partial 6-balloon. For the latter case, recall that a partial 6-balloon cannot be OL-AP since otherwise there would exist a 6-balloon contradicting Theorem 4. Similarly, observe that $PB(12^+, 12^+, 12^+, \overline{1}, \overline{2})$ is not OL-AP since it cannot be OL-AP-partitioned for 3.

We finally suppose that the induction hypothesis is true whenever $x + y \leq k-1$ for some $k > 4$, and consider a partial balloon $B = PB(12^+, 12^+, 12^+, \overline{x}, \overline{y})$ where $x + y = k$. We distinguish two main cases, depending on the values of x and y, to prove that B is not OL-AP.

- $x > 1$ and $y > 1$: in this situation, B is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every choice of S_1, the remaining graph $B[S_{n-1}]$ is not OL-AP either for one of the reasons used for the base cases or according to the induction hypothesis.

- $x = 1$ and $y > 2$: the above arguments hold to prove that B cannot be OL-AP-partitioned for 2. Thus, B is not OL-AP.

□

Proof of Lemma 8. Let us prove this claim by induction on x. We first suppose that $x = 1$ and consider the OL-AP-partition of $B = PB(12^+, 12^+, 12^+, 12^+, \overline{1})$ for 2. Such a partition does not exist since for every choice of S_2, the remaining graph $B[S_{n-2}]$ cannot be OL-AP: indeed, this subgraph is either not connected, a tree with maximum degree at least 4, a partial balloon which cannot be OL-AP according to Lemma 6 or 12, or a partial 6-balloon. In the latter case, such a graph cannot be OL-AP since otherwise there would exist a graph contradicting Theorem 4.

Suppose now that $PB(12^+, 12^+, 12^+, 12^+, \overline{x})$ is not OL-AP for every $x \leq k-1$ and some $k > 3$, and consider a graph $B = PB(12^+, 12^+, 12^+, 12^+, \overline{k})$. Once again, B cannot be OL-AP since it cannot OL-AP-partitioned for 1. Indeed, for every possible choice for S_1, the graph $B[S_{n-1}]$ cannot be OL-AP either according to the induction hypothesis or because of one of the reasons used for the base case.

□