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On the length of the longest path in partitionable balloons

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Abstract A connected graph G is said to be *arbitrarily partitionable* (AP for short) if for every partition (τ_1, \dots, τ_k) of $|V(G)|$ there exists a partition (V_1, \dots, V_k) of $V(G)$ such that each V_i induces a connected subgraph of G on τ_i vertices. Some stronger versions of this property were introduced, namely the ones of being *online arbitrarily partitionable* and *recursively arbitrarily partitionable* (OL-AP and R-AP for short, respectively), in which the subgraphs induced by a partition of G must not only be connected but also fulfill some additional conditions. A *balloon* is a 2-connected graph obtained by connecting two distinct vertices by means of several *branches*, that is by vertex-disjoint paths. In this paper, we investigate the existence of a constant c such that every AP, OL-AP or R-AP balloons with order n has an elementary path whose order is at least $n - c$ for arbitrary n . From our results, we then deduce some properties of the graph resulting from the removal of an articulation pair in a 2-connected OL-AP or R-AP graph.

Keywords online arbitrarily partitionable graph · recursively arbitrarily partitionable graph · balloon · longest path

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1 Introduction

Let G be a graph with order n . A sequence $\tau = (\tau_1, \dots, \tau_k)$ of positive integers is said to be *admissible for G* if it performs a partition of n , that is if $\sum_{i=1}^k \tau_i = n$. When, for such an admissible sequence for G , there exists a partition (V_1, \dots, V_k) of $V(G)$ such that each V_i induces a connected subgraph of G on τ_i vertices, then τ is called *realizable in G* . If every admissible sequence for G is also realizable in G , this graph is said to be *arbitrarily partitionable* (AP for short).

The notion of AP graphs was recently introduced in [1] to deal with the following problem. Suppose that we want to share a network of n computing resources between k users, where the i^{th} user needs τ_i resources, and that, for the sake of performance, we do not want the sharing to be performed arbitrarily but in such a way that the following two conditions are met:

- a resource must be allocated to exactly one user,
- the subnetwork attributed to a user must be connected¹.

We can use the previously introduced notions to deduce an optimal sharing of our resources. Indeed, let G be the graph modelling our network; then we can satisfy our users with this specific resource demand if the sequence (τ_1, \dots, τ_k) is realizable in G . Moreover, observe that, regarding this allocation problem, the networks which are of most interest are those which can be shared between an arbitrary number of users no matter how many resources they need. Clearly, these networks are the ones having an AP graph topology.

The problem of deciding whether a graph can be partitioned following a given sequence is computationally hard, even when restricted to trees [2]. The interested reader is referred to [1, 2, 14] for a review of some of the most important results about AP graphs. It has to be known that partitioning graphs into connected subgraphs was studied long before the introduction of AP graphs. See, for example, [9, 13] for two interesting results by Györi and Lovász on the partition of a k -connected graph into at most k connected parts.

Observe that the above definition of AP graphs is quite static and thus not representative of the difficulties we can encounter while actually partitioning a network; notably, one could point out the following two issues:

1. In our definition, a graph is fully partitioned at once; from the network sharing point of view, this constraint is like waiting for every single resource of our network to be needed before eventually satisfying our users. This is, of course, not satisfying since we would like to satisfy them as soon as possible (immediately, ideally).
2. When a sequence is realized in a graph, the induced subgraphs must only meet the connectivity constraint. But according to our network analogy, it would be more convenient to make sure that the allocated subnetworks themselves have the property of being shareable at will. This can be quite useful if, for example, a user wants himself to share his subnetwork between

¹ In the sense that two resources of a subnetwork must be able to communicate within it.

several other users or if he wants to delimit it to accomplish many different tasks simultaneously.

Motivated by these deficiencies, the following augmented definitions have been introduced:

Definition 1 [10] A graph G is said to be *online arbitrarily partitionable* (OL-AP for short) if and only if one of the following two conditions holds.

- The graph G is isomorphic to K_1 .
- For every $\lambda \in \{1, \dots, n-1\}$, there exists a partition $(S_\lambda, S_{n-\lambda})$ of $V(G)$ such that $G[S_\lambda]$ is connected on λ vertices and $G[S_{n-\lambda}]$ is OL-AP with order $n-\lambda$.

Definition 2 [4] A graph G is said to be *recursively arbitrarily partitionable* (R-AP for short) if and only if one of the following two conditions holds.

- The graph G is isomorphic to K_1 .
- For every sequence $\tau = (\tau_1, \dots, \tau_k)$ admissible for G , there exists a partition (V_1, \dots, V_k) of $V(G)$ into k parts such that each V_i induces a R-AP subgraph of G on τ_i vertices.

Observe that the notion of OL-AP graphs (resp. R-AP graphs) can be used to deal with our network sharing problem taking issue 1 (resp. issue 2) pointed out above into account. It appears that the properties of being OL-AP and R-AP are quite similar to each other. Indeed, previous investigations have shown that every R-AP graph is also OL-AP [4] and that, in the context of some classes of graphs (like *trees* and so-called *suns*), there only exist a few OL-AP graphs which are not R-AP [4,5].

Theorem 3 [4] *Every R-AP graph is also OL-AP, but the contrary does not necessarily hold.*

We here focus on the class of *balloon* graphs introduced in [5].

Definition 4 Let b_1, \dots, b_k be $k \geq 1$ positive integers. The *balloon with k branches* $B(b_1, \dots, b_k)$ (sometimes called *k -balloon* for short) is the 2-connected graph obtained by linking two distinct vertices r_1 and r_2 by means of k vertex-disjoint paths with b_1, \dots, b_k vertices, respectively. By "linking" we mean that for each of these paths we add an edge between one of its degree-1 vertices and r_1 , and similarly for its other degree-1 vertex and r_2 .

Two examples of balloons are given in Figure 1. Balloons are interesting in the context of arbitrarily partitionable graphs for their structure is closely related to the one of 2-connected partitionable graphs:

Observation 5 *Let G be a 2-connected graph, u and v be two vertices forming an articulation pair of G , and n_1, \dots, n_k be the numbers of vertices of the $k \geq 2$ connected components of $G - \{u, v\}$. If G is AP, OL-AP or R-AP, then $B(n_1, \dots, n_k)$ is AP, OL-AP or R-AP, respectively.*

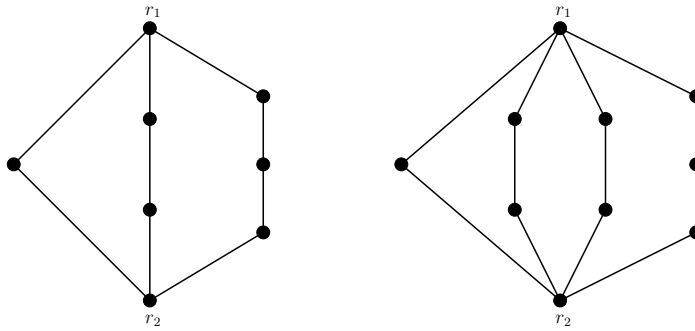


Fig. 1 The balloons $B(1, 2, 3)$ and $B(1, 2, 2, 3)$

Indeed, observe that no graph with order n is easier to partition than the path on n vertices. Hence, a realization of a sequence τ in $B(n_1, \dots, n_k)$ can be directly deduced from a realization of τ in G . Using this analogy between partitionable balloons and partitionable 2-connected graphs, it follows that any property of AP, OL-AP or R-AP balloons can be generalized into a property of AP, OL-AP or R-AP 2-connected graphs, respectively.

A graph is said to be *traceable* if it admits a Hamiltonian path. It was pointed out that every traceable graph is also AP, OL-AP and R-AP [4]. Besides, all non-traceable OL-AP or R-AP graphs with order n exhibited during previous studies are actually "nearly traceable", in the sense that they admit an elementary path whose order is almost n [4, 5]. Hence, in this paper, we investigate AP, OL-AP, and R-AP balloons regarding the well-known Longest Path Problem which, given a graph G , asks what is the maximum length of an elementary path in G (the reader is referred to [8, 11, 12] for further information). We introduce the following definition for this purpose.

Definition 6 Let \mathcal{G} be a family of graphs. We define $P_{\mathcal{G}}(n)$ as the greatest integer such that every graph on n vertices in \mathcal{G} has an elementary path of order $P_{\mathcal{G}}(n)$, that is $P_{\mathcal{G}}(n) = \max\{x \mid \text{every graph in } \mathcal{G} \text{ with order } n \text{ has an elementary path with order at least } x\}$.

If \mathcal{H} is the set of all traceable graphs, then clearly $P_{\mathcal{H}}(n) = n$. If we denote by \mathcal{T}_{AP} , \mathcal{T}_{OL-AP} and \mathcal{T}_{R-AP} the sets of all AP, OL-AP and R-AP trees, respectively, then we know that there exists a constant c such that $P_{\mathcal{T}_{OL-AP}}(n), P_{\mathcal{T}_{R-AP}}(n) \geq n - c$ for every n (see upcoming Corollary 9 of Section 2). However, there does not exist such a lower bound on $P_{\mathcal{T}_{AP}}(n)$ for arbitrary n since there exist AP trees with arbitrarily small diameter [15].

Let us denote by \mathcal{B}_{AP} , \mathcal{B}_{OL-AP} and \mathcal{B}_{R-AP} the sets of all AP, OL-AP and R-AP balloons, respectively. In this paper, we investigate the existence of such a $n - c$ lower bound on $P_{\mathcal{B}_{AP}}(n)$, $P_{\mathcal{B}_{OL-AP}}(n)$ and $P_{\mathcal{B}_{R-AP}}(n)$ for arbitrary n where c is a constant. For this purpose, we first begin by providing, in Section 2, some definitions and tools necessary to understand our results. We next explain, in Section 3, why such a constant c cannot exist regarding $P_{\mathcal{B}_{AP}}(n)$,

and how it could be deduced for $P_{\mathcal{B}_{OL-AP}}(n)$ and $P_{\mathcal{B}_{R-AP}}(n)$. Then, we show in Section 4 that OL-AP balloons cannot have more than five branches, this upper bound meeting the maximum number of branches of a R-AP balloon. We afterwards prove, in Section 5, that there exists an unbounded number of OL-AP and R-AP balloons with four or five branches. These results are finally used to deduce a first upper bound on the order of the smallest branch of an OL-AP or R-AP balloon with four or five branches in Sections 6 and 7. With these results, we exhibit an interesting property of 2-connected OL-AP or R-AP graphs and show that an OL-AP balloon (resp. a R-AP balloon) with four branches is "almost traceable", that is, it has a path of at least $n - 11$ vertices (resp. at least $n - 7$ vertices).

2 Terminology and preliminary results

2.1 Terminology and notation

Let $x \geq 1$ be an integer. Throughout this paper, we denote by x^+ an arbitrary integer y such that $y \geq x$.

We deal with connected, non-oriented and simple graphs, using mainly the terminology of [7]. The vertex and edges sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively, or simply by V and E when no ambiguity is possible. The order of G , commonly denoted by n , is its number of vertices. Given a subset of vertices $S \subseteq V$ of G , we denote by $G[S]$ the subgraph of G induced by the vertices of S . We additionally denote by $G - S$ the subgraph of G induced by all the vertices of $V - S$. If $F \subseteq E$ is a subset of edges of G , we denote by $G - F$ the partial graph of G obtained by removing all the edges of F from G .

We denote by P_n the path of order n . Given $k \geq 1$ positive integers a_1, \dots, a_k , the k -pode $P(a_1, \dots, a_k)$ is the tree obtained by linking one central node to one extremity of each one of k disjoint paths on a_1, \dots, a_k vertices, respectively. Since previous investigations [4, 6], a 3-pode $P(1, a_2, a_3)$ is often referenced as a *caterpillar* and is also denoted by $Cat(a_2 + 1, a_3 + 1)$ for convenience².

We now give more notations associated with the notion of balloon graphs. Let $B = B(b_1, \dots, b_k)$ be a k -balloon. The vertices r_1 and r_2 of degree k in B are called the *roots of B* , while the path of order b_i connecting them is said to be the i^{th} *branch of B* . For every $i \in \{1, \dots, k\}$, the vertices of the i^{th} branch of B are denoted by $v_1^i, \dots, v_{b_i}^i$ in such a way that $v_1^i r_1, v_{b_i}^i r_2 \in E$ and $v_j^i v_{j+1}^i \in E$ for every $j \in \{1, \dots, b_i - 1\}$. Finally, for every $i \in \{1, \dots, k\}$, we denote by $b_i(B)$ the number of vertices composing the i^{th} branch of B .

We denote by $PB(b_1, \dots, b_x, \overline{b_{x+1}}, \dots, \overline{b_{x+y}}, \overline{b_{x+y+1}}, \dots, \overline{b_{x+y+z}})$ the graph $B(b_1, \dots, b_{x+y+z}) - (\bigcup_{i=x+1}^{x+y} \{v_{b_i}^i r_2\}, \bigcup_{i=x+y+1}^{x+y+z} \{v_1^i r_1\})$ obtained by removing $y + z$ edges from a $(x + y + z)$ -balloon. Such a graph is called a *partial balloon with*

² Observe that $Cat(a, b)$ has order $a + b$.

$(x + y + z)$ branches, or *partial $(x + y + z)$ -balloon* for short. In this paper, the notations introduced above for balloons are used in an analogous way for partial balloons. For convenience, the vertices of a *hanging branch* with order b_i of a partial balloon B (that is, a branch linked to only one root of B) are successively denoted by $v_1^i, \dots, v_{b_i}^i$ where v_1^i is the degree-1 vertex of the branch and $v_{b_i}^i$ is the vertex adjacent to one root of B . Please refer to Figure 2 for two examples of partial balloons.

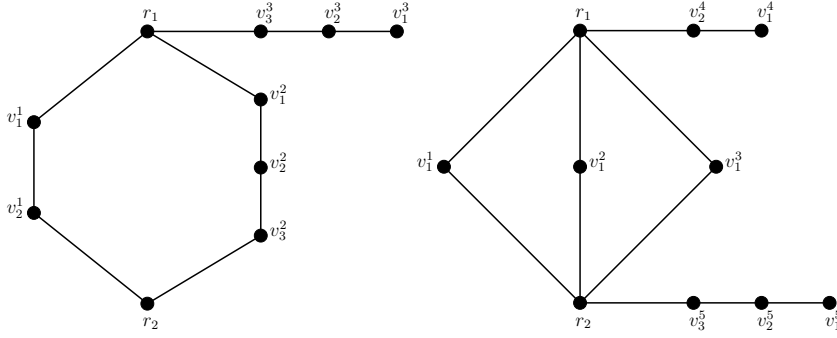


Fig. 2 The partial balloons $PB(2, 3, \bar{3})$ and $PB(1, 1, 1, \bar{2}, 3)$

2.2 Some properties of OL-AP and R-AP graphs

The next two theorems give a complete characterization of OL-AP and R-AP trees.

Theorem 7 [10] *A tree is OL-AP if and only if it is either isomorphic to a path, to a caterpillar $Cat(a, b)$ with a and b given in Table 1, or to the 3-pode $P(2, 4, 6)$.*

a	b
2, 4	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod{6}$
7	8, 9, 11, 13, 15
8	11, 19
9, 10	11
11	12

Table 1 Values a and b , with $b \geq a$, such that $Cat(a, b)$ is OL-AP.

Theorem 8 [4] *A tree is R-AP if and only if it is either isomorphic to a path, to a caterpillar $Cat(a, b)$ with a and b given in Table 2, or to the 3-pode $P(2, 4, 6)$.*

a	b
2, 4	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 2 Values a and b , with $b \geq a$, such that $Cat(a, b)$ is R-AP.

Recall that \mathcal{T}_{OL-AP} and \mathcal{T}_{R-AP} are the sets of all OL-AP and R-AP trees, respectively. Theorems 7 and 8 imply the following.

Corollary 9 *Let n be an arbitrary integer. We have:*

$$P_{\mathcal{T}_{OL-AP}}(n), P_{\mathcal{T}_{R-AP}}(n) = \begin{cases} n - 2 & \text{if } n = 13, \\ n - 1 & \text{if } n \in \{4, \dots, 30\} - \{6, 13\} \\ & \text{or } (n > 30 \text{ and } n \not\equiv 0 \pmod{6}), \\ n & \text{otherwise.} \end{cases}$$

Proof We have $P_{\mathcal{T}_{OL-AP}}(13) = P_{\mathcal{T}_{R-AP}}(13) = 11$ because $P(2, 4, 6)$ is OL-AP. Recall that a caterpillar $Cat(a, b)$ with order n has an elementary path on $n - 1$ vertices. One can check from Tables 1 and 2 that every caterpillar $Cat(a, b)$ with $n \leq 30$ is OL-AP and R-AP unless $n \leq 4$ or $n = 6$. When $n > 30$, the only potentially OL-AP and R-AP caterpillars $Cat(a, b)$ are mainly $Cat(2, b)$ and $Cat(3, b)$ which are both not OL-AP and R-AP when $n \equiv 0 \pmod{6}$. For every other value of n , the only OL-AP and R-AP tree is the path on n vertices. \square

Theorems 7 and 8 were proved using the following two observations, which provide two alternative methods to check whether a graph is OL-AP or R-AP.

Observation 10 *A graph G is R-AP if and only if for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ there exists a partition $(S_\lambda, S_{n-\lambda})$ of V into two parts such that $G[S_\lambda]$ and $G[S_{n-\lambda}]$ are R-AP on λ and $n - \lambda$ vertices, respectively.*

Observation 11 *The property of being OL-AP (resp. R-AP) is closed under edge-additions: if a graph has an OL-AP (resp. a R-AP) spanning subgraph, then it is OL-AP (resp. R-AP).*

2.3 Some properties of AP, OL-AP and R-AP balloons

We now give some properties on the number of vertices of the branches in a partitionable balloon. It was previously proved that an AP balloon can always

be partitioned in such a way that its branches with order 0 (that is, its branches with no vertices) are not used by the partition, and thus that $B(0, b_1, \dots, b_k)$ is AP if and only if $B(b_1, \dots, b_k)$ is AP [4]. We show that this result also holds when considering OL-AP or R-AP balloons.

Lemma 12 *Let $b_1 \leq \dots \leq b_k$ be positive integers with $k \geq 1$. The balloon $B(0, b_1, \dots, b_k)$ is OL-AP (resp. R-AP) if and only if $B(b_1, \dots, b_k)$ is OL-AP (resp. R-AP).*

Proof The necessary condition follows directly from Observation 11. Let us now prove the sufficient condition. Whenever $k \leq 3$, the lemma is true since both $B(0, b_1, \dots, b_k)$ and $B(b_1, \dots, b_k)$ are traceable for every $b_1, \dots, b_k \geq 1$.

Suppose now that there exist OL-AP balloons with at least four branches composed by at least 1 vertex contradicting the claim, and let us denote by $B = B(0, b_1, \dots, b_k)$ one of those with minimum order and $k \geq 4$. In particular, it means, according to Definition 1, that there exists $\lambda \in \{1, \dots, n-1\}$ such that the following two conditions are met.

- The vertex set of B admits a partition $(S_\lambda, S_{n-\lambda})$ such that $B[S_\lambda]$ is connected on λ vertices and $B[S_{n-\lambda}]$ is OL-AP with order $n-\lambda$.
- The partition $(S_\lambda, S_{n-\lambda})$ is not a satisfying partition of the vertex set of $B' = B(b_1, \dots, b_k)$ regarding the conditions above.

It follows that the edge $r_1 r_2$ is necessary for the partition, and thus that either S_λ or $S_{n-\lambda}$ contains both r_1 and r_2 . This implies that the part not containing $r_1 r_2$ only contains some consecutive vertices from one branch of B , say the one with order b_i where $b_i \geq 1$.

On the one hand, if $r_1, r_2 \in S_\lambda$, then $B[S_\lambda]$ remains connected even if we remove $r_1 r_2$ from it since r_1 and r_2 are also connected in $B[S_\lambda]$ thanks to at least three branches of B . In this case, the branch with order 0 of B is useless for the partition, a contradiction.

On the other hand, if $r_1, r_2 \in S_{n-\lambda}$, then $B[S_{n-\lambda}]$ is a partial balloon. Using Observation 11, we can deduce an OL-AP balloon B'' with less vertices than B and a branch with no vertex. By the minimality of B , it follows that B'' remains OL-AP even when the edge linking its two roots is removed from it. Hence, $B[S_{n-\lambda}]$ remains OL-AP even if we remove the edge $r_1 r_2$ from it, and we could deduce a partitioning of B' respecting the conditions above. Again, a contradiction.

This concludes the proof for OL-AP balloons. A similar proof can be led for R-AP balloons using the fact that the parts S_λ and $S_{n-\lambda}$ of the partition of B must induce R-AP subgraphs of B (see Observation 10). \square

Hence, throughout this paper, we will only consider balloons having their smallest branch composed by at least 1 vertex. We additionally introduce the following results on the orders of the branches of an AP balloon.

Observation 13 [4] *Let B be an AP balloon. If n is odd, then B has at most three branches of odd order. If n is even, then B has at most two branches of odd order.*

Lemma 14 [3] *Let $B(b_1, \dots, b_k)$ be an AP balloon with $b_1 \leq \dots \leq b_k$. For every $i \in \{2, \dots, k\}$, we have $2b_i \geq \sum_{j=1}^{i-1} b_j$.*

Since every OL-AP or R-AP graph is also AP, the previous two results naturally also hold when considering OL-AP and R-AP balloons.

3 On the order of the longest path in a partitionable balloon

Observe that every balloon with at most three branches is traceable, and, thus, has its longest elementary path of order n . Besides, it is clear that a balloon $B(b_1, \dots, b_k)$ with $k \geq 4$ and $b_1 \leq \dots \leq b_k$ has its longest path of order $b_{k-2} + b_{k-1} + b_k + 2$. Using Lemma 14, we deduce the following lower bound on $P_{\mathcal{B}_{AP}}(n)$.

Corollary 15 *We have $P_{\mathcal{B}_{AP}}(n) \geq \frac{19n}{27}$ for every n .*

Proof Recall that $P_{\mathcal{B}_{AP}}(n) = n - x$, where x is the maximum sum of the $k - 3$ smallest branches of an AP balloon on n vertices. Let $B = B(b_1, \dots, b_k)$ be an AP balloon on n vertices with $b_1 \leq \dots \leq b_k$ and such that $\sum_{i=1}^{k-3} b_i = x$. By Lemma 14, we know that $b_{k-2} \geq \frac{x}{2}$, $b_{k-1} \geq \frac{3x}{4}$ and $b_k \geq \frac{9x}{8}$. Since $n = 2 + \sum_{i=1}^k b_i$, it follows that $n \geq \frac{27x}{8} + 2$ and thus that $x \leq \frac{8n}{27}$. By the choice of B , we deduce the bound on $P_{\mathcal{B}_{AP}}(n)$. \square

Since OL-AP and R-AP graphs are also AP, the lower bound of Corollary 15 is also a lower bound on $P_{\mathcal{B}_{OL-AP}}(n)$ and $P_{\mathcal{B}_{R-AP}}(n)$. However, there does not exist a constant c such that every AP balloon on n vertices has a path with order at least $n - c$ for arbitrary n . Indeed, it was previously shown that there exist AP balloons having arbitrarily many branches and an arbitrarily long smallest branch.

Theorem 16 ([3]) *For arbitrary k and b_1 , there exists an AP k -balloon $B(b_1, \dots, b_k)$ with $b_1 \leq \dots \leq b_k$.*

Corollary 17 *There does not exist a constant c such that $P_{\mathcal{B}_{AP}}(n) \geq n - c$ for arbitrary n .*

Although, the question of the existence of such a constant c regarding R-AP balloons on n vertices seems to be more relevant since these balloons cannot have too many branches.

Theorem 18 [4] *A R-AP balloon cannot have more than five branches.*

According to Theorem 18, we only have to investigate R-AP balloons with at most five branches to determine $P_{\mathcal{B}_{R-AP}}(n)$. More precisely, if $k\mathcal{B}_{R-AP}$ (resp. $k\mathcal{B}_{OL-AP}$) denotes the set of all R-AP k -balloons (resp. OL-AP k -balloons) for any $k \geq 1$, then the following holds.

Observation 19 *We have $P_{\mathcal{B}_{R-AP}}(n) = \min\{P_{k\mathcal{B}_{R-AP}} \mid k \leq 5\}$ for every n .*

Since k -balloons with $k \in \{1, 2, 3\}$ are traceable, it follows that $P_{1\mathcal{B}_{R-AP}}(n) = P_{2\mathcal{B}_{R-AP}}(n) = P_{3\mathcal{B}_{R-AP}}(n) = n$ and thus that the value of $P_{\mathcal{B}_{R-AP}}(n)$ only depends on $P_{4\mathcal{B}_{R-AP}}(n)$ and $P_{5\mathcal{B}_{R-AP}}(n)$. Because the longest elementary path of any 4-balloon $B(b_1, \dots, b_4)$ (resp. of any 5-balloon $B(b_1, \dots, b_5)$) has order $b_2 + b_3 + b_4 + 2$ when $b_1 \leq \dots \leq b_4$ (resp. has order $b_3 + b_4 + b_5 + 2$ when $b_1 \leq \dots \leq b_5$), it follows that if there exists an upper bound c on the order of the smallest branch of a R-AP 4-balloon (resp. on the sum of the orders of the two smallest branches of a R-AP 5-balloon) on n vertices, then $n - c$ will be a lower bound on $P_{\mathcal{B}_{R-AP}}(n)$.

Notice that all these considerations on R-AP balloons naturally also hold for OL-AP balloons since OL-AP balloons cannot have more than five branches too (see upcoming Theorem 20 of Section 4).

4 An OL-AP balloon cannot have more than five branches

In this section, we prove that the bound of Theorem 18 also holds for OL-AP balloons.

Theorem 20 *An OL-AP balloon cannot have more than five branches.*

Proof The proof is by contradiction. Let \mathcal{B} be the set of OL-AP balloons with at least six branches and B denote a k -balloon of \mathcal{B} with the least order.

By Definition 1, B is OL-AP if and only if for every $\lambda \in \{1, \dots, n-1\}$ we can partition V into two parts S_λ and $S_{n-\lambda}$ such that $B[S_\lambda]$ is connected on λ vertices and $B[S_{n-\lambda}]$ is OL-AP on $n-\lambda$ vertices. Observe that, because of the minimality of B , the subgraph $B[S_{n-\lambda}]$ cannot be a partial k' -balloon with $k' \geq 6$ since otherwise there would exist a balloon of \mathcal{B} with less vertices than B (Observation 11). It follows that $B[S_{n-\lambda}]$ is either an OL-AP 5-balloon or an OL-AP tree (see Theorem 7).

We claim that B has branches with 1, 2, 3, 4, 5 and 6 vertices. Let us suppose that $\lambda \in \{1, \dots, 6\}$ and that B does not have a branch with order λ . We show that it is not possible to partition V into two parts with cardinal λ and $n-\lambda$, respectively, satisfying the above conditions.

- $\lambda \in \{1, 2, 3\}$: for every choice of S_λ , the subgraph $B[S_{n-\lambda}]$ is either a partial k -balloon having less vertices than B or a tree with maximum degree at least 4. In both cases, the subgraph $B[S_{n-\lambda}]$ is not OL-AP.
- $\lambda = 4$: so far, we have shown that B necessarily has branches with order 1, 2 and 3. Similarly as in the previous case, observe that for every choice of S_λ , the subgraph $B[S_{n-\lambda}]$ is either a partial k -balloon or partial $(k+1)$ -balloon having less vertices than B or a tree having maximum degree at least 3. Hence, the only possibility here is to choose S_λ in such a way that $B[S_{n-\lambda}]$ is a tree with maximum degree 3, but this is only possible when $B = B(1, 1, 1, 2, 3, \dots)$. According to Observation 13, such a balloon is not AP, and thus is not OL-AP.

- $\lambda = 5$: by the previous cases, we know that B has branches composed of 1, 2, 3 and 4 vertices. For the same reasons as before, S_λ must be chosen in such a way that $B[S_{n-\lambda}]$ is either a path or an OL-AP 3-pode. Hence, since B has at least six branches, S_λ must contain one root of B and all the vertices of at least three of its branches. Observe that S_λ can only be chosen in this way when $k = 6$ and $B = B(1, 1, 1, 2, 3, 4)$. It follows that B has four branches of odd order, and thus that it is not AP according to Observation 13. It cannot be OL-AP.
- $\lambda = 6$: we know that B has branches with 1, 2, 3, 4 and 5 vertices. Moreover, since $k \geq 6$, the balloon B has one additional branch of order b_i . If $b_i \leq 7$, then B is not AP by Lemma 14, and thus is not OL-AP. Hence, $b_i \geq 8$ but, again, we cannot exhibit a correct subset S_λ for the same reasons as before. Hence B is not OL-AP.

Finally, B is isomorphic to $B(1, 2, 3, 4, 5, 6, \dots)$ which is not AP following Lemma 14; it thus cannot be OL-AP. \square

5 Infinite families of OL-AP and R-AP balloons with four or five branches

The number of branches of OL-AP and R-AP balloons being upper bounded (Theorems 18 and 20), it would be interesting to obtain a complete characterization of these graphs (as it was done for trees, see Theorems 7 and 8). In what follows, we prove that the number of OL-AP and R-AP balloons with four or five branches is unbounded. We introduce the following two lemmas for this purpose.

Lemma 21 *The partial balloon $PB(1, 1, 2, \bar{k})$ is R-AP for every $k \geq 1$.*

Proof Observe that this claim is true whenever $k = 1$, $k = 2$ or $k = 3$ since the corresponding partial balloons are spanned by $Cat(2, 5)$, $Cat(3, 5)$ and $Cat(4, 5)$, respectively.

Suppose now that this claim holds for every k up to $i - 1$ and consider the partial balloon $B = PB(1, 1, 2, \bar{i})$. By Observation 10, we know that B is R-AP if it can be partitioned, for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$, into two R-AP subgraphs on λ and $n - \lambda$ vertices, respectively. One can consider the following partitions:

- $\lambda = 1$: P_1 and $PB(1, 2, \bar{i})$ (traceable).
- $\lambda = 2$: P_2 and $PB(1, 1, \bar{i})$ (traceable).
- $\lambda \in \{3, 4\}$: P_λ and $P_{n-\lambda}$.
- $\lambda = 5$: $Cat(2, 3)$ and P_{i+1} .
- $\lambda = 6$: $B(1, 1, 2)$ (traceable) and P_i .
- $\lambda \geq 7$: $PB(1, 1, 2, \bar{\lambda} - 6)$ (induction hypothesis) and $P_{i-\lambda+6}$.

\square

Lemma 22 *The partial balloon $PB(1, 2, 3, \bar{k})$ is R-AP for every $k \geq 1$.*

Proof The proof is by induction on k . If we first suppose that $k = 1$, $k = 2$, or $k = 3$, then observe that the corresponding partial 4-balloons are R-AP since they are spanned by the R-AP caterpillars $Cat(2, 7)$, $Cat(3, 7)$ and $Cat(4, 7)$, respectively.

Let us secondly suppose that this lemma holds for every $k \leq i - 1$, and consider the partial balloon $B = PB(1, 2, 3, \bar{i})$. Once again, by Observation 10, it is sufficient to show, to prove that B is R-AP, that we can partition it into two R-AP subgraphs on λ and $n - \lambda$ vertices, respectively, for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. We show that these partitions exist for every λ :

- $\lambda = 1$: P_1 and $PB(2, 3, \bar{i})$ (traceable).
- $\lambda = 2$: P_2 and $PB(1, 3, \bar{i})$ (traceable).
- $\lambda = 3$: P_3 and $PB(1, 2, \bar{i})$ (traceable).
- $\lambda \in \{4, 5, 6\}$: P_λ and $P_{n-\lambda}$.
- $\lambda = 7$: $Cat(3, 4)$ and P_{i+1} .
- $\lambda = 8$: $B(1, 2, 3)$ (traceable) and P_i .
- $\lambda \geq 9$: $PB(1, 2, 3, \overline{\lambda - 8})$ (induction hypothesis) and $P_{n-\lambda+8}$.

□

Observe that, according to Observation 11, Lemmas 21 and 22 directly imply that there exist infinitely many R-AP 4-balloons (and, thus, infinitely many OL-AP 4-balloons, see Theorem 3).

Corollary 23 *The 4-balloons $B(1, 1, 2, k)$ and $B(1, 2, 3, k)$ are OL-AP and R-AP for every $k \geq 1$.*

We now prove that there exists an unbounded family of R-AP 5-balloons.

Theorem 24 *The partial balloon $PB(1, 1, 2, 3, \overline{2k})$ is R-AP for every $k \geq 1$.*

Proof Let $B = PB(1, 1, 2, 3, \overline{2k})$ with $k \geq 1$. Recall that, according to Observation 10, B is R-AP if we can partition it into two R-AP subgraphs $B[S_\lambda]$ and $B[S_{n-\lambda}]$ having order λ and $n - \lambda$, respectively, for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. One can consider the following partitions for the first values of λ :

- $\lambda = 1$: P_1 and $B(1, 2, 3, \overline{2k})$ (Lemma 22).
- $\lambda = 2$: P_2 and $B(1, 1, 3, \overline{2k})$ (spanned by $Cat(2, 5 + 2k)$).
- $\lambda = 3$: P_3 and $B(1, 1, 2, \overline{2k})$ (Lemma 21).
- $\lambda = 4$: P_4 and $Cat(4, 2k + 1)$.
- $\lambda = 5$: $Cat(2, 3)$ and P_{2k+4} .
- $\lambda = 6$: P_6 and $Cat(2, 2k + 1)$.
- $\lambda = 7$: $Cat(3, 4)$ and P_{2k+2} .

By now, it should be clear that the proposition holds for every partial balloon $PB(1, 1, 2, 3, \overline{2k})$ such that $n \leq 15$ (that is, for each $k \in \{1, 2, 3\}$). Let us suppose, as an induction hypothesis, that the claim is true for every $k \leq i - 1$, and consider the partition of $B = PB(1, 1, 2, 3, \overline{2i})$ into two R-AP subgraphs for the remaining values of λ , that is for every $\lambda \in \{8, \dots, \lfloor \frac{n}{2} \rfloor\}$.

- $\lambda \geq 8$, λ even: observe that $\lambda \leq 2i$ since $\lambda \leq \lfloor \frac{n}{2} \rfloor$ and we handled the cases where $k \leq 3$. We can thus partition B into P_λ and either $B(1, 1, 2, 3)$ (when $i = 4$) or $PB(1, 1, 2, 3, \overline{2i - \lambda})$ (when $i > 4$). These graphs are R-AP according to Lemma 22 and by the induction hypothesis (since $2i - \lambda$ is even), respectively.
- $\lambda = 9$: $B(1, 1, 2, 3)$ (spanned by $Cat(2, 7)$) and P_{2i} .
- $\lambda > 9$, λ odd: $PB(1, 1, 2, 3, \overline{\lambda - 9})$ (induction hypothesis since $\lambda - 9$ is even) and $P_{n-\lambda+9}$.

□

Combining Observation 11 and Theorem 24 we get that the 5-balloon $B(1, 1, 2, 3, 2k)$ is R-AP for every $k \geq 1$. Since every R-AP graph is also OL-AP (Theorem 3), we deduce the following:

Corollary 25 *The 5-balloon $B(1, 1, 2, 3, 2k)$ is OL-AP and R-AP for every $k \geq 1$.*

6 The smallest branch of an OL-AP (resp. R-AP) balloon with four or five branches has at most 11 (resp. 7) vertices

In this section, we give an upper bound on the order of the smallest branch of an OL-AP or R-AP balloon with four or five branches. For this purpose, we first show, in Lemmas 26 to 29 below, that some partial balloons cannot be OL-AP.

Lemma 26 *The graph $PB(12^+, 12^+, \bar{x}, y)$ is not OL-AP for every $x, y \geq 1$.*

Proof We prove this claim by induction on $x + y$. As a base case, consider $x = y = 1$ and the partial balloon $B = PB(12^+, 12^+, \bar{1}, \underline{1})$. By Definition 1, recall that B is OL-AP if and only if, for every $\lambda \in \{1, \dots, n - 1\}$, there exists a partition $(S_\lambda, S_{n-\lambda})$ of V such that $B[S_\lambda]$ and $B[S_{n-\lambda}]$ are connected on λ vertices and OL-AP on $n - \lambda$ vertices, respectively. In particular, observe here that B cannot be partitioned in this way for $\lambda = 2$. Indeed, every possible choice of S_2 makes $B[S_{n-2}]$ being either disconnected, a caterpillar $Cat(13^+, 13^+)$ or $Cat(11^+, 15^+)$, or a tree with two degree-3 vertices. Since none of these graphs is OL-AP (Theorem 7), B is not OL-AP.

To complete the base case, let us now suppose that $x + y = 3$ and denote by B the partial balloon $PB(12^+, 12^+, \bar{1}, \underline{2})$. As in the previous base case, one has to observe that B is not OL-AP since it cannot be partitioned in the way specified by Definition 1 for $\lambda = 3$. In particular, observe that for every possible choice of S_3 , the graph $B[S_{n-3}]$ is not OL-AP for it is disconnected, a non-caterpillar 3-pode different from $P(2, 4, 6)$, a caterpillar $Cat(10^+, 16^+)$ or $Cat(13^+, 13^+)$, or a tree with two degree-3 vertices.

Consider now that the claim holds for every partial balloon $PB(12^+, 12^+, \bar{x}, y)$ such that $x + y \leq k - 1$ for some $k > 4$. We now prove that it is also true for a partial balloon $B = PB(12^+, 12^+, \bar{x}, y)$ when $x + y = k$. There are two cases to consider:

- $x > 1$ and $y > 1$: B is not OL-AP since we cannot partition its vertex set as explained above for $\lambda = 1$. Indeed, we must consider $S_1 = \{v_1^3\}$ or $S_1 = \{v_1^4\}$ since otherwise $B[S_{n-1}]$ would be either disconnected, or isomorphic to a large 3-pode or a tree with two degree-3 vertices. But for these two choices of S_1 , the remaining graph $B[S_{n-1}]$ is isomorphic to a partial balloon $PB(12^+, 12^+, \bar{x}', y')$ with $x' + y' = x + y - 1 \leq k - 1$, which is not OL-AP by the induction hypothesis.
- $x = 1$ and $y > 2$: consider we want to partition B as previously for $\lambda = 2$. For the same reasons as above, we have to consider $S_2 = \{v_1^4, v_2^4\}$. But then $B[S_{n-2}]$ is isomorphic to $PB(12^+, 12^+, \bar{x}, y - 2)$ which is not OL-AP according to the induction hypothesis. Hence, B is not OL-AP. These arguments hold analogously when $x > 2$ and $y = 1$.

□

Since the proofs of Lemmas 27 to 29 are quite similar to the one of Lemma 26, the reader is referred to Appendix A for in-depth proofs on these statements.

Lemma 27 *The graph $PB(12^+, 12^+, 12^+, \bar{x}, y)$ is not OL-AP for every $x, y \geq 1$.*

Lemma 28 *The graph $PB(12^+, 12^+, 12^+, \bar{x})$ is not OL-AP for every $x \geq 1$.*

Lemma 29 *The graph $PB(12^+, 12^+, 12^+, 12^+, \bar{x})$ is not OL-AP for every $x \geq 1$.*

Using Lemmas 26 to 29, we now prove that the smallest branch of an OL-AP balloon with four or five branches is at most 11.

Theorem 30 *Let $B = B(b_1, b_2, b_3, b_3)$ be a 4-balloon with $b_1 \leq \dots \leq b_4$. If B is OL-AP, then $b_1 \leq 11$.*

Proof Let $B = B(12^+, 12^+, 12^+, 12^+)$ be a 4-balloon. B is not OL-AP since its vertex set cannot be partitioned in the way specified by Definition 1 for $\lambda = 1$. Indeed, for every choice of S_1 , the graph $B[S_{n-\lambda}]$ is not OL-AP since it is either a tree with maximum degree 4 or a partial balloon which is not OL-AP by Lemma 27 or 28. It follows that an OL-AP 4-balloon must have a branch of order at most 11. □

Theorem 31 *Let $B = B(b_1, b_2, b_3, b_4, b_5)$ be a 5-balloon with $b_1 \leq \dots \leq b_5$. If B is OL-AP, then $b_1 \leq 11$.*

Proof Let $B = B(12^+, 12^+, 12^+, 12^+, 12^+)$ be a 5-balloon. As in the proof of Theorem 30, the balloon B is not OL-AP since there does not exist a partition of its vertex set respecting Definition 1 for $\lambda = 1$. Indeed, every possible choice of S_1 makes $B[S_{n-1}]$ being either a tree with maximum degree 5, a partial balloon which is not OL-AP according to Lemma 29, or a partial 6-balloon. In the latter case, observe that $B[S_{n-1}]$ cannot be OL-AP since otherwise there would exist, by Observation 11, an OL-AP 6-balloon contradicting Theorem 20. Hence, a 5-balloon cannot be OL-AP when its smallest branch has order at least 12. □

Since every OL-AP graph is also R-AP (Theorem 3), Theorems 30 and 31 directly imply that R-AP balloons with four or five branches have their smallest branch of order at most 11 too. However, using the fact that R-AP caterpillars have their smallest branch of order at most 7 (Theorem 8), one can easily derive Lemmas 26 to 29 above for R-AP partial balloons to get a better upper bound on the order of the smallest branch of a R-AP balloon with four or five branches.

Theorem 32 *Let B be a balloon with four or five branches. If B is R-AP, then B has its smallest branch of order at most 7.*

7 Conclusions and questions

By Corollaries 23 and 25, we know that there exists an infinite family \mathcal{F} of R-AP balloons with four or five branches. Since, for every n , we have $P_{\mathcal{F}}(n) = n - 2$, the following holds by Theorem 3.

Corollary 33 *We have $P_{\mathcal{B}_{OL-AP}}(n), P_{\mathcal{B}_{R-AP}}(n) \leq n - 2$ for every n .*

Notice that we do not know if the bounds of Theorems 30, 31 and 32 are sharp since we do not know OL-AP or R-AP balloons having the order of their smallest branch meeting the corresponding bounds. However, these results allow us to deduce the following lower bounds.

Corollary 34 *We have $P_{4\mathcal{B}_{OL-AP}}(n) \geq n - 11$ and $P_{4\mathcal{B}_{R-AP}}(n) \geq n - 7$ for every n .*

With a constant upper bound c on the order of the second smallest branch of an OL-AP or R-AP 5-balloon of order n , we could use Theorems 31 and 32 to deduce a better lower bound on $P_{5\mathcal{B}_{OL-AP}}(n)$ and $P_{5\mathcal{B}_{R-AP}}(n)$, respectively, than the one exhibited by Corollary 15. According to Observation 19, this would next lead to a lower bound on $P_{\mathcal{B}_{OL-AP}}(n)$ and $P_{\mathcal{B}_{R-AP}}(n)$. But we still do not know if c actually exists; hence, we ask the following:

Question 35 *Does there exist a constant c such that the second smallest branch of an OL-AP or R-AP 5-balloon has order at most c ?*

In a more general context, the combination of Observation 5 and Theorems 30, 31 and 32 imply the following result on the structure of 2-connected OL-AP and R-AP graphs.

Corollary 36 *Let G be a 2-connected OL-AP graph (resp. 2-connected R-AP graph), and u and v be two vertices forming an articulation pair of G . If the removal of $\{u, v\}$ from G disconnects G into exactly four or five components, then one of these components has order at most 11 (resp. at most 7).*

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A Appendix

This appendix gathers all the proofs of Lemmas 27 to 29 of Section 6, as well as some intermediate lemmas needed to prove these statements. Notice that these proofs often make implicit use of the full characterization of OL-AP trees (Theorem 7) and the two sufficient conditions for a graph to be OL-AP given by Observations 10 and 11. In all these proofs, it is assumed that $x \leq y$ or $x \leq y \leq z$ generally holds when the corresponding elements have been introduced. Given a graph G and an integer $\lambda \in \{1, \dots, n-1\}$, an *OL-AP-partition of G for λ* is a partition $(S_\lambda, S_{n-\lambda})$ of V such that $G[S_\lambda]$ and $G[S_{n-\lambda}]$ are connected on λ vertices and OL-AP on $n-\lambda$ vertices, respectively. According to Definition 1, the graph G is OL-AP if and only if either G is an isolated vertex or G admits an OL-AP-partition for every $\lambda \in \{1, \dots, n-1\}$.

Lemma 37 *The graph $PB(12^+, 12^+, \bar{x}, \bar{y})$ is not OL-AP for every $x \geq 1$ and $y \geq 10$.*

Proof Let us prove this claim by induction on $x+y$ as we did to prove Lemma 26. As a base case, let us consider the graph $B = PB(12^+, 12^+, \bar{1}, \bar{10})$. Observe that B is not OL-AP since it does not admit an OL-AP-partition for 11. Indeed, every possible choice of 11 vertices inducing a connected subgraph of B makes $B[S_{n-11}]$ being either disconnected, a caterpillar $Cat(11, 13^+)$ or a tree with maximum degree 4. For similar reasons, observe that neither $PB(12^+, 12^+, \bar{1}, \bar{11})$ nor $PB(12^+, 12^+, 2, 10)$ are OL-AP since they do not admit an OL-AP-partition for 12 and 11, respectively.

Let us now suppose that this lemma holds whenever $x+y \leq k-1$ for some $k > 13$, and consider a graph $B = PB(12^+, 12^+, \bar{x}, \bar{y})$ such that $x+y = k$. We claim that there exists a $\lambda \in \{1, \dots, n-1\}$ such that B does not admit an OL-AP-partition for λ , and thus that B is not OL-AP:

- $x > 1$ and $y > 10$: under these conditions, there does not exist an OL-AP-partition of B for 1. Indeed, every possible choice for S_1 which does not make $B[S_{n-1}]$ being disconnected makes this subgraph being isomorphic to either a non-caterpillar 3-pode different from $P(2, 4, 6)$, a tree with maximum degree 4, or a graph not OL-AP according to the induction hypothesis.
- $x = 1$ and $y > 11$: observe that there does not exist an OL-AP-partition of B for 2, since every coherent choice for S_2 makes $B[S_{n-2}]$ being disconnected or isomorphic to either a caterpillar $Cat(13^+, 13^+)$, a tree with maximum degree 4, or a partial balloon which is not OL-AP by the induction hypothesis.
- $x > 2$ and $y = 10$: once again, B does not admit an OL-AP-partition for 11 since every choice of 11 vertices inducing a connected subgraph of B makes $B[S_{n-11}]$ being either disconnected, a tree with maximum degree 4, or a non-OL-AP 3-pode.

□

Lemma 38 *The graph $PB(12^+, 12^+, \bar{x}, \bar{y}, \bar{z})$ is not OL-AP for every $x, y, z \geq 1$.*

Proof We prove this claim by induction on $x+y+z$. Let us first suppose that $x = y = z = 1$ and consider the associated graph $B = PB(12^+, 12^+, \bar{1}, \bar{1}, \bar{1})$. Once again, B is not OL-AP since there does not exist an OL-AP-partition of B for 2. Indeed, every possible for S_2 makes $B[S_{n-2}]$ being either disconnected, or isomorphic to either a tree with maximum degree 4 or a tree having two degree-3 vertices.

To complete the base case, observe that $PB(12^+, 12^+, \bar{1}, \bar{2}, \bar{1})$ and $PB(12^+, 12^+, \bar{1}, \bar{1}, \bar{2})$ are not OL-AP for there do not admit an OL-AP-partition for 3: for every coherent choice of S_3 , the subgraph $B[S_{n-3}]$ is disconnected, or isomorphic to either a tree with maximum degree 4, a tree having two degree-3 vertices, or a non-caterpillar 3-pode different from $P(2, 4, 6)$.

Suppose now that this claim holds whenever $x+y+z \leq k-1$ for some $k > 5$, and consider a balloon $B = PB(12^+, 12^+, \bar{x}, \bar{y}, \bar{z})$ where $x+y+z = k$. Once again, we consider two main cases:

- $z > 1$: in this case, B is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, observe that removing one vertex from B makes the remaining subgraph being

disconnected, isomorphic to a tree with maximum degree 4 or two degree-3 vertices, or isomorphic to a partial balloon which is not OL-AP according to the induction hypothesis or Lemma 26.

- $z = 1$: once again, B is not OL-AP under this condition since it cannot be OL-AP-partitioned for 2: for every coherent choice as S_2 , the remaining graph $B[S_{n-2}]$ is indeed not connected, a tree with maximum degree 4 or two degree-3 vertices, or a partial balloon which is not OL-AP according to our induction hypothesis or previous Lemma 26.

□

Lemma 27 (Section 5) *The graph $PB(12^+, 12^+, 12^+, \bar{x}, \underline{y})$ is not OL-AP for every $x, y \geq 1$.*

Proof Once more, let us prove this claim by induction on $x+y$. Consider first that $x = y = 1$ and let $B = PB(12^+, 12^+, 12^+, \bar{1}, \underline{1})$. Observe that B is not OL-AP for it cannot be OL-AP-partitioned for 2. Indeed, every possible choice for S_2 makes $B[S_{n-2}]$ being disconnected, isomorphic to a tree with maximum degree 4, to a partial balloon which is not OL-AP by Lemma 38, or to a partial 6-balloon. In the latter case, such a graph cannot be OL-AP since otherwise there would exist an OL-AP 6-balloon contradicting Theorem 20.

Additionally, observe that $B = PB(12^+, 12^+, 12^+, \bar{1}, \underline{2})$ cannot be OL-AP-partitioned for 3: for every coherent choice for S_3 , the subgraph $B[S_{n-3}]$ is not OL-AP for the same reasons as in the previous case. Hence, B is not OL-AP.

We now suppose that this claim holds for every $x + y \leq k - 1$ for some $k > 4$, and consider a partial balloon $B = PB(12^+, 12^+, 12^+, \bar{x}, \underline{y})$ where $x + y = k$. Let us take the following two cases in consideration to show that B is not OL-AP.

- $x > 1$ and $y > 1$: notice that, in this situation, B cannot be OL-AP-partitioned for 1. Indeed, for some similar reasons as the ones we used to deal with the base cases, we have to consider $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$. But in both cases, $B[S_{n-1}]$ cannot be OL-AP by the induction hypothesis.
- $x = 1$ and $y > 2$: once again, observe that B cannot be OL-AP-partitioned for 2. Indeed, observe that we must consider $S_2 = \{v_1^5, v_2^5\}$ since otherwise there would exist an OL-AP 6-balloon, an OL-AP tree having maximum degree 4, or a graph contradicting Lemma 38. But for this choice of S_2 , we have $B[S_{n-2}] = PB(12^+, 12^+, 12^+, \bar{1}, \underline{y-2})$ which is not OL-AP according to our induction hypothesis.

□

Lemma 28 (Section 5) *The graph $PB(12^+, 12^+, 12^+, \bar{x})$ is not OL-AP for every $x \geq 1$.*

Proof Once again, this claim is proved by induction on x . Let us first suppose that $x = 1$ and let B be the partial balloon $PB(12^+, 12^+, 12^+, \bar{1})$. This time, B is not OL-AP since it cannot be OL-AP-partitioned for 2: for every possible choice of S_2 , the remaining graph $B[S_{n-2}]$ is not OL-AP since it is disconnected, isomorphic to a tree with maximum degree 4, to a non-caterpillar 3-pode different from $P(2, 4, 6)$ or to a partial balloon which is not OL-AP by Lemma 26, 37 or 38.

Let us now suppose that this claim holds for every $x \leq k - 1$ and some $k > 3$. To complete the proof, observe that a graph $B = PB(12^+, 12^+, 12^+, \bar{k})$ is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every choice of S_1 , the subgraph $B[S_{n-1}]$ is not OL-AP according to the induction hypothesis, or because of one reason used to deal with the base case.

□

Lemma 39 *The graph $PB(12^+, 12^+, \bar{x}, \bar{y}, \bar{z})$ is not OL-AP for every $x, y, z \geq 1$.*

Proof We prove this claim by induction on $x+y+z$. First, let us suppose that $x = y = z = 1$ and consider the partial balloon $B = PB(12^+, 12^+, 12^+, \bar{1}, \bar{1}, \bar{1})$. Notice that B is not OL-AP since it cannot be OL-AP-partitioned for 2. Indeed, every choice of S_2 implies that $B[S_{n-2}]$ is either disconnected or isomorphic to a tree with maximum degree at least 4. Analogously, observe that neither $PB(12^+, 12^+, \bar{1}, \bar{1}, \bar{2})$ nor $PB(12^+, 12^+, \bar{1}, \bar{2}, \bar{2})$ are OL-AP since they cannot be OL-AP-partitioned for 3.

Suppose now that this claim holds by induction whenever $x + y + z \leq k - 1$ for a $k > 6$, and consider a partial balloon $B = PB(12^+, 12^+, \bar{x}, \bar{y}, \bar{z})$ where $x + y + z = k$. We distinguish the following two main cases depending on x , y and z :

- $x > 1$, $y > 1$ and $z > 1$: suppose we want to OL-AP-partition B for 1. Then, we must consider $S_1 = \{v_1^3\}$, $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$ since, for every other choice of S_1 , the remaining graph $B[S_{n-1}]$ is either disconnected or isomorphic to a tree having maximum degree at least 4. But in any of these three choices for S_1 , the subgraph $B[S_{n-1}]$ is not OL-AP by the induction hypothesis. Thus, B is not OL-AP.
- $x = 1$: let $\alpha = \min(\{2, 3, 4\} - \{y, z\})$. In this situation, B cannot be OL-AP for the same reason as above but for an OL-AP-partition of B for α . Indeed, for every coherent choice of S_α , the remaining graph $B[S_{n-\alpha}]$ is not OL-AP either according to the induction hypothesis, or because it is isomorphic to a non-connected graph or a tree with maximum degree at least 4. □

Lemma 40 *The graph $PB(12^+, 12^+, 12^+, \bar{x}, \bar{y})$ is not OL-AP for every $x, y \geq 1$.*

Proof Once again, we prove this claim by induction on $x + y$. We first suppose that $x = y = 1$ and let $B = PB(12^+, 12^+, 12^+, \bar{1}, \bar{1})$. Similarly as in the proofs of the previous lemmas, B is not OL-AP for it cannot be OL-AP-partitioned for 2. Indeed, for every possible choice as S_2 , the remaining graph $B[S_{n-2}]$ is not connected, a tree with maximum degree 5, a partial balloon not OL-AP according to Lemma 38 or 39, or a partial 6-balloon. For the latter case, recall that a partial 6-balloon cannot be OL-AP since otherwise there would exist a 6-balloon contradicting Theorem 20. Similarly, observe that $PB(12^+, 12^+, 12^+, \bar{1}, \bar{2})$ is not OL-AP since it cannot be OL-AP-partitioned for 3.

We finally suppose that the induction hypothesis is true whenever $x + y \leq k - 1$ for some $k > 4$, and consider a partial balloon $B = PB(12^+, 12^+, 12^+, \bar{x}, \bar{y})$ where $x + y = k$. We distinguish two main cases, depending on the values of x and y , to prove that B is not OL-AP.

- $x > 1$ and $y > 1$: in this situation, B is not OL-AP since it cannot be OL-AP-partitioned for 1. Indeed, for every choice of S_1 , the remaining graph $B[S_{n-1}]$ is not OL-AP either for one of the reasons used to deal with the base cases or according to the induction hypothesis.
- $x = 1$ and $y > 2$: the above arguments hold to prove that B cannot be OL-AP-partitioned for 2. Thus, B is not OL-AP. □

Lemma 29 (Section 5) *The graph $PB(12^+, 12^+, 12^+, 12^+, \bar{x})$ is not OL-AP for every $x \geq 1$.*

Proof Let us prove this claim by induction on x . We first suppose that $x = 1$ and consider the OL-AP-partition of $B = PB(12^+, 12^+, 12^+, 12^+, \bar{1})$ for 2. Such a partition does not exist since for every choice of S_2 , the remaining graph $B[S_{n-2}]$ cannot be OL-AP: indeed, this subgraph is either not connected, a tree with maximum degree at least 4, a partial balloon which cannot be OL-AP according to Lemma 27 or 40, or a partial 6-balloon. In the latter case, such a graph cannot be OL-AP since otherwise there would exist a graph contradicting Theorem 20.

Suppose now that $PB(12^+, 12^+, 12^+, 12^+, \bar{x})$ is not OL-AP for every $x \leq k - 1$ and some $k > 3$, and consider a graph $B = PB(12^+, 12^+, 12^+, 12^+, \bar{k})$. Once again, B cannot be OL-AP since it cannot OL-AP-partitioned for 1. Indeed, for every possible choice for S_1 , the graph $B[S_{n-1}]$ cannot be OL-AP either according to the induction hypothesis or because of one of the reasons used to deal with the base case. □