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On some properties of arbitrarily partitionable balloons with five branches

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Abstract A connected graph $G = (V, E)$ is said to be *arbitrarily partitionable* (AP for short) if for any partition (τ_1, \dots, τ_k) of $|V|$ there exists a partitioning (V_1, \dots, V_k) of V such that each V_i induces a connected subgraph of G on τ_i vertices. Some stronger versions of this property were introduced, namely the ones of being *online arbitrarily partitionable* and *recursively arbitrarily partitionable* (OL-AP and R-AP for short, respectively), in which the subgraphs induced by a partitioning must not only be connected but also fulfil some additional conditions. A *balloon* is a 2-connected graph obtained by connecting two distinct vertices by means of several *branches*, *i.e.* by vertex-disjoint paths. Through many investigations, it appeared that AP balloons are interesting since some of their properties can be easily generalized to 2-connected AP graphs. In this paper, we first show that OL-AP balloons cannot have more than five branches, this bound meeting the equivalent known one for R-AP balloons. We then investigate the cases of OL-AP and R-AP balloons having this many branches. In particular, we prove that there exists an infinite number of them and give a first upper bound to the size of their smallest branch. We then use the latter result to show that a 2-connected OL-AP or R-AP graph must contain some small components.

Keywords online arbitrarily partitionable graphs · recursively arbitrarily partitionable graphs · 2-connected graphs · balloons · traceability

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1 Introduction

Let $G = (V, E)$ be a graph with order n . A sequence $\tau = (\tau_1, \dots, \tau_k)$ of positive integers is said to be *admissible for G* if it performs a partition of n , that is if $\sum_{i=1}^k \tau_i = n$. When, for such an admissible sequence for G , we can find a partition (V_1, \dots, V_k) of V such that each V_i induces a connected subgraph of G on τ_i vertices, then τ is called *realizable in G* . If any admissible sequence for G is also realizable in it, this graph is said to be *arbitrarily partitionable* (AP for short).

The notion of AP graphs was recently introduced in [1] to deal with the following computer science problem. Suppose that we want to share a network of n computing resources between k users, where the i^{th} one needs τ_i resources. Of course, for the sake of performance, we do not want the sharing to be performed arbitrarily, but in such a way that the following two conditions are met:

- a resource must be allocated to exactly one user,
- the subnetwork attributed to a user must be connected¹.

We can use the previously introduced notions to deduce an optimal sharing of our resources. Indeed, let G be the graph modelling our network; then we can satisfy our users with this specific resource demand if the sequence (τ_1, \dots, τ_k) is realizable in G . Moreover, observe that, regarding this allocation problem, the networks which are of most interest are those which can be shared between an arbitrary number of users no matter how many resources they need. Clearly, these networks are the ones having an AP graph topology.

It appears that the structure of AP graphs is not obvious in general, although we still do not know if the problem of determining whether a graph is AP is complete. The interested reader can refer to [1], [2] or [10] for a review of some of the most important results about AP graphs. An older interesting result proved independently by Györi and Lovász on the partitioning of a k -connected graph can be found in [7] and [9], illustrating the fact that partitioning graphs into connected subgraphs was studied long before the introduction of AP graphs.

Observe that the above definition of AP graphs is quite static and thus not representative of the difficulties we can encounter while actually partitioning a network; notably, one could point out the following two issues:

1. In our definition, a graph is fully partitioned at once; from the network sharing point of view, this constraint is like waiting for every single resource of our network to be needed before eventually satisfying our users. This is, of course, not satisfying since we would like to satisfy them as soon as possible (immediately, ideally).
2. When a sequence is realized in a graph, the induced subgraphs must only meet the connectivity constraint. But according to our network analogy, it would be more convenient to make sure that the allocated subnetworks

¹ In the sense that two resources of a subnetwork must be able to communicate within it.

themselves have the property of being shareable at will. This can be quite useful if, for some reasons, a user wants himself to share his subnetwork between several other users or if he wants to delimit it to accomplish many different tasks simultaneously.

Of course, more arguments could be pointed out to show that the definition of AP graph does not fit so well with the above problem; but these two are enough to justify the introduction of the two upcoming stronger versions of the definition.

Firstly, a graph G is said to be *online arbitrarily partitionable* (OL-AP for short) if it is either isomorphic to K_1 , or if for any integer $\lambda \in [1, n]$ we can find a subset $S_\lambda \subseteq V$ of λ vertices such that $G[S_\lambda]$ is connected and $G[V \setminus S_\lambda]$ is OL-AP. Secondly, G is said to be *recursively arbitrarily partitionable* (R-AP for short) if it is either an isolated vertex or if for any sequence $\tau = (\tau_1, \dots, \tau_k)$ admissible for G we can partition V into k parts (V_1, \dots, V_k) such that each V_i induces an R-AP subgraph of G on τ_i vertices. Observe that OL-AP graphs (resp. R-AP graphs) can be used to deal with our network sharing problem taking issue 1 (resp. issue 2) pointed out above into account.

It appears that the properties of being OL-AP and R-AP are quite similar, previous investigations having shown that every R-AP graph is also OL-AP and that, in the context of some fully characterized classes of graphs (like *trees* and so-called *suns*), there only exists a few OL-AP graphs being not R-AP [4].

Theorem 1 [4] *Every R-AP is also OL-AP, but the contrary does not necessarily holds.*

The similarity between OL-AP and R-AP graphs apart, we still do not know much about the structure of these graphs though. However, since some previous studies, we feel that the property of being R-AP is even more closely related to *traceability*² than is the one of being just AP; indeed, we think that a graph should be "nearly traceable" in order to be R-AP.

Question 1 Does there exist a small constant c such that if a graph is R-AP then it necessarily contains an elementary path of size at most $n - c$?

In [4] was also introduced and studied the class of *balloon* graphs. Let b_1, \dots, b_k be $k \geq 3$ positive integers; the *balloon with k branches* $B(b_1, \dots, b_k)$ (sometimes called *k -balloon* for short) is the 2-connected graph obtained by linking two distinct vertices r_1 and r_2 by means of k vertex-disjoint paths of orders b_1, \dots, b_k respectively. By "linking" we mean that for each of these paths we add an edge between one of its degree-one vertices and r_1 , and similarly for its other degree-one vertex and r_2 . Observe that the structure of 2-connected AP graphs is closely related to the one of AP balloons:

Observation 1 *Let G be a 2-connected graph, u and v be two vertices forming an articulation pair of G , and n_1, \dots, n_k be the orders of the $k \geq 2$ connected components of $G[V \setminus \{u, v\}]$. If G is AP, then $B(n_1, \dots, n_k)$ is AP.*

² A graph is *traceable* if it admits a spanning path.

A similar observation can be deduced about OL-AP or R-AP graphs, although it has some more important consequences in the latter context. Indeed, it was proved in [4] that R-AP balloons cannot have too many branches.

Theorem 2 [4] *An R-AP balloon cannot have more than five branches.*

This result, combined with an R-AP version of Observation 1, gives us a property of 2-connected R-AP graphs. Indeed, the graph resulting from the deletion of an articulation pair of a 2-connected R-AP graph should not be composed of too many components (at most five) since otherwise we could find a counterexample to Theorem 2. Thus, the study of R-AP balloons is a good way to deduce general properties of 2-connected R-AP graphs; of course, this observation also holds for OL-AP balloons and 2-connected OL-AP graphs.

This paper is organized as follows. We first begin by providing, in Section 2, some definitions and tools necessary to understand our results. Then, we show in Section 3 that the bound exhibited in Theorem 2 on the maximum degree of a R-AP balloon also holds for OL-AP graphs. We afterwards prove, in Section 4, that there exists an unbounded number of R-AP 5-balloons, and, thus, of OL-AP 5-balloons according to Theorem 1. Finally, we give a first upper bound on the size of the smallest branch of an OL-AP or R-AP 5-balloon in Section 5. With this result, we also exhibit a property of 2-connected OL-AP and 2-connected R-AP graphs.

2 Terminology and preliminary results

Let $x \geq 1$ be an integer. In what follows, the set $\{1, \dots, x\}$ is denoted by $[1, x]$, while x^+ denotes an arbitrary integer y such that $y \geq x$.

We deal with connected, non-oriented and simple graphs, using mainly the terminology of [6]. The vertex and edges sets of a graph G are respectively denoted by $V(G)$ and $E(G)$ (or simply by V and E when no ambiguity is possible). The order of G , commonly denoted by n , is its number of vertices, *i.e.* $|V|$. Given a subset of vertices $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by the vertices of S . If $F \subseteq E$ is a subset of edge of G , we denote by $G \setminus F$ the partial graph of G obtained by removing all the edges of F from G .

We denote by P_n the path of size n . Given k positive integers a_1, \dots, a_k , the k -pode $P(a_1, \dots, a_k)$ is the tree obtained by linking one central node to one extremity of each one of k disjoint paths of size a_1, \dots, a_k , respectively. Since previous investigations [4] [5], a 3-pode $P(1, a_2, a_3)$ is often referenced as a *caterpillar* and is also denoted by $Cat(a_2 + 1, a_3 + 1)$ for convenience³. The next two theorems give a complete characterization of OL-AP and R-AP trees [8] [4].

Theorem 3 [8] *A tree is OL-AP iff it is either isomorphic to a path, to a caterpillar $Cat(a, b)$ with a and b given in Table 1, or to the 3-pode $P(2, 4, 6)$.*

a	b
2, 4	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod{6}$
7	8, 9, 11, 13, 15
8	11, 19
9, 10	11
11	12

Table 1 Values a and b , $b \geq a$, such that $Cat(a, b)$ is OL-AP.

Theorem 4 [4] *A tree is R-AP iff it is either isomorphic to a path, to a caterpillar $Cat(a, b)$ with a and b given in Table 2, or to the 3-pode $P(2, 4, 6)$.*

a	b
2, 4	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 2 Values a and b , $b \geq a$, such that $Cat(a, b)$ is R-AP.

These results were proved using the following two observations, which provide two alternative methods to check whether a graph is OL-AP or R-AP (which will be widely used later on).

Observation 2 *A graph G is R-AP iff for any $\lambda \in [1, \lfloor \frac{n}{2} \rfloor]$ we can partition V into two parts S_λ and $S_{n-\lambda}$ such that $G[S_\lambda]$ and $G[S_{n-\lambda}]$ are R-AP on λ and $n - \lambda$ vertices respectively.*

Observation 3 *The property of being OL-AP (resp. R-AP) is closed under edge additions, i.e. a graph having an OL-AP spanning subgraph (resp. an R-AP spanning subgraph) is OL-AP (resp. R-AP).*

We now give some more notations associated with the notion of balloon graphs. Let $B = B(b_1, \dots, b_k)$ be a k -balloon. The vertices r_1 and r_2 of degree k in B are called the *roots of B* , while the path of size b_i connecting them is said to be the i^{th} *branch of B* . For any $i \in [1, k]$, the vertices of the i^{th} branch of B are denoted by $v_1^i, \dots, v_{b_i}^i$ where $v_1^i r_1 \in E$, $v_{b_i}^i r_2 \in E$, and for any $j \in [1, b_i - 1]$, $v_j^i v_{j+1}^i \in E$. Finally, for any $i \in [1, k]$, we denote by $b_i(B)$ the number of vertices composing the i^{th} branch of B .

We denote by $PB(b_1, \dots, b_x, \overline{b_{x+1}}, \dots, \overline{b_{x+y}}, \overline{b_{x+y+1}}, \dots, \overline{b_{x+y+z}})$ the graph $B(b_1, \dots, b_{x+y+z}) \setminus (\bigcup_{i=x+1}^{x+y} \{v_{b_i}^i r_2\}, \bigcup_{i=x+y+1}^{x+y+z} \{r_1 v_1^i\})$ obtained by removing $y +$

³ Observe that $Cat(a, b)$ has order $a + b$.

z edges to a $(x + y + z)$ -balloon. Such a graph is called a *partial balloon with $(x + y + z)$ branches*, or *partial $(x + y + z)$ -balloon* for short. In this paper, the notations introduced above for balloons are used in an analogous way for partial balloons. For convenience, the vertices of an *hanging branch* of size b_i of a partial balloon B (that is, a branch linked to only one root of B) are successively denoted by $v_1^i, \dots, v_{b_i}^i$ where v_1^i is the degree-one vertex of the branch and $v_{b_i}^i$ is the one adjacent to one root of B .

We now give the following two properties of AP balloons which were found during previous investigations.

Observation 4 [4] *Let B be an AP balloon. If n is odd, then B has at most three branches of odd orders. Otherwise, B has at most two branches of odd orders.*

Lemma 1 [3] *Let $B(b_1, \dots, b_k)$ be an AP balloon with $b_1 \leq \dots \leq b_k$. Then for any $i \leq k$ we have $2b_i \geq \sum_{j < i} b_j$.*

3 An OL-AP 5-balloon cannot have more than five branches

In this section, we prove that the bound of Theorem 2 also holds for OL-AP balloons, which is not surprising since the properties of being OL-AP and R-AP showed to be closely related in previous investigations.

Theorem 5 *An OL-AP balloon can not have more than five branches.*

Proof The proof is by contradiction. Let us suppose that there exist OL-AP k -balloons with $k \geq 6$, and let B denote one of those having the least order.

By definition, we know that B is OL-AP iff for any $\lambda \in [1, n]$ we can partition V into two parts S_λ and $S_{n-\lambda}$ with size λ and $n - \lambda$ respectively and such that $B[S_\lambda]$ is connected and $B[S_{n-\lambda}]$ is OL-AP. Observe that, by the minimality of B , $B[S_{n-\lambda}]$ cannot be a partial k -balloon since otherwise we could find an OL-AP k -balloon with order strictly less than B (Observation 3). Hence, for each λ , one of the following statement must be true (Theorem 3):

1. $B[S_{n-\lambda}]$ is an OL-AP $(k - 1)$ -balloon, *i.e.* S_λ contains all the vertices composing one branch of B , or
2. $B[S_{n-\lambda}]$ is a path, *i.e.* S_λ contains all the vertices of B but some consecutive ones from one branch, or
3. $B[S_{n-\lambda}]$ is an OL-AP caterpillar, *i.e.* S_λ contains one root r_1 of B and all the vertices of $k - 2$ branches of it but one linked to the root of B different from r_1 , or
4. $B[S_{n-\lambda}]$ is the 3-pode $P(2, 4, 6)$, *i.e.* S_λ contains one root r_1 of B and all the vertices of $k - 2$ branches of it but two adjacent vertices, one of them being adjacent to the root of B different from r_1 .

We claim that B has branches with orders 1, 2, 3, 4, 5 and 6; indeed, let $\lambda \in [1, 6]$ and suppose that B does not have a branch with order λ . For any value of λ , it is not possible to partition V into two subsets of size λ and $n - \lambda$ satisfying the above conditions:

- $\lambda \in [1, 3]$: for any choice of S_λ , $B[S_{n-\lambda}]$ is either a partial k -balloon having less vertices than B , or a tree with maximum degree at least 4 since S_λ can only contain all the vertices of at most two branches of B . In both cases, $B[S_{n-\lambda}]$ is not OL-AP.
- $\lambda = 4$: so far, notice that B must have branches of sizes 1, 2 and 3. Similarly as in the previous case, observe that for any choice of S_λ , $B[S_{n-\lambda}]$ is either a partial k -balloon having less vertices than B , which can not be OL-AP by the minimality of B , or a tree having maximum degree at least 3. For the latter case, the only correct possibility is to make $B[S_{n-\lambda}]$ being isomorphic to $Cat(3, 4)$, which is only possible when $k = 6$ and $B = B(1, 1, 1, 1, 2, 3)$. But then we know that B is not AP by Lemma 1, and thus not OL-AP.
- $\lambda = 5$: by the previous cases we know that B has branches composed by respectively 1, 2, 3 and 4 vertices. Notice that $B[S_{n-\lambda}]$ cannot be a path since five vertices cannot cover four branches plus a root of B . If $B[S_{n-\lambda}]$ is isomorphic to an OL-AP caterpillar, then necessarily $k = 6$ and $B[S_{n-\lambda}]$ is isomorphic to $Cat(4, 5)$. In this case, we would have $B = B(1, 1, 1, 2, 3, 4)$ which is not AP according to Observation 4 and thus is not OL-AP. Finally, observe that we cannot take S_λ in such a way that $B[S_{n-\lambda}]$ is $P(2, 4, 6)$. Indeed, since $k \geq 6$, then S_λ should contain one root of B , cover all of its branches of size 1, and then contain at least three vertices from its branches of size 3 and 2; thus, we need to cover at least six vertices, which is impossible since we want S_λ to contain only five vertices.
- $\lambda = 6$: by the previous cases, we know that B has branches of sizes 1, 2, 3, 4 and 5. Since $k \geq 6$, B has one additional branch of size b_i . Let us first suppose that $b_i \leq 5$; by Lemma 1, we must have $b_i = 1$. Thus $B = B(1, 1, 2, 3, 4, 5, \dots)$. But now the only potential choice for S_λ would be to cover the three smallest branches of B , one of its roots and one additional vertex from another branch; observe that this choice would not be a correct one since $B[S_{n-\lambda}]$ would be either isomorphic to a 3-pode different from an R-AP caterpillar or from $P(2, 4, 6)$, or to a tree with maximum degree at least 4. Thus, $b_i \geq 7$; but, again, we cannot find a correct subset S_λ for the same reasons as above or because it would contradict the minimality of B . Hence B is not OL-AP.

Finally, B must be $B(1, 2, 3, 4, 5, 6, \dots)$ which is not AP following Lemma 1; it thus cannot be OL-AP.

4 An infinite family of OL-AP and R-AP 5-balloons

The number of branches of OL-AP and R-AP balloons being upper bounded (Theorems 2 and 5), one could think that we could step forward a complete characterization of them (as it was done for trees). In what follows, we prove that, for any $k \geq 1$, the 5-balloon $B(1, 1, 2, 3, 2k)$ is R-AP, providing us an infinite family of R-AP 5-balloons (and of OL-AP 5-balloons according to Theorem 1). We use the following two lemmas for this purpose.

Lemma 2 *The partial balloon $PB(1, 1, 2, \overline{k})$ is R-AP for any $k \geq 1$.*

Proof Observe that this fact is true whenever $k = 1$, $k = 2$ or $k = 3$ since, in these cases, the corresponding partial balloons are spanned by $Cat(2, 5)$, $Cat(3, 5)$ and $Cat(4, 5)$ respectively.

Suppose now that this claim holds for any $i \leq k - 1$ and consider the partial balloon $B = PB(1, 1, 2, \overline{k})$. By Observation 2, we know that B is R-AP if it can be partitioned, for any $\lambda \in [1, \lfloor \frac{n}{2} \rfloor]$, into two R-AP subgraphs having size λ and $n - \lambda$ respectively. For $\lambda \leq 4$, B can be easily partitioned into two traceable subgraphs. When $\lambda = 5$, it can be partitioned into $Cat(2, 3)$ and P_{k+1} , while for $\lambda = 6$ we can obtain $B(1, 1, 2)$, which is traceable, and P_k . Finally, for $\lambda \geq 7$, we can partition B into $PB(1, 1, 2, \overline{\lambda - 6})$ and $P_{k-\lambda+6}$, the first one of these two being R-AP by induction. \square

Lemma 3 *The partial balloon $PB(1, 2, 3, \overline{k})$ is R-AP for any $k \geq 1$.*

Proof The proof is by induction on k . If we first suppose that $k = 1$, $k = 2$, or $k = 3$, then observe that the corresponding partial 4-balloons are R-AP since they are spanned by the R-AP caterpillars $Cat(2, 7)$, $Cat(3, 7)$ and $Cat(3, 8)$, respectively.

Let us secondly suppose that the lemma holds for any $i \leq k - 1$, and consider the partial balloon $B = PB(1, 2, 3, \overline{k})$; once again, by Observation 2, it is sufficient, to prove that B is R-AP, to show that we can partition it into two R-AP subgraphs on λ and $n - \lambda$ vertices respectively, and this for every $\lambda \in [1, \lfloor \frac{n}{2} \rfloor]$. For any such $\lambda \leq 6$, observe that we can easily partition B into two traceable graphs. When $\lambda = 7$, it can be partitioned into $Cat(3, 4)$ and P_{k+1} , while for $k = 8$ we can find a partition of its vertices inducing $B(1, 2, 3)$ and P_k . Finally, for any $\lambda \geq 9$, there exists a partition of B into $PB(1, 2, 3, \overline{\lambda - 8})$ and $P_{n-\lambda}$, the first one being R-AP by the induction hypothesis. \square

We are now ready to prove the main result of this section:

Proposition 1 *The partial balloon $PB(1, 1, 2, 3, \overline{2k})$ is R-AP for any $k \geq 1$.*

Proof Let $B = PB(1, 1, 2, 3, \overline{2k})$ be a partial balloon, where $k \geq 1$; by Observation 2, recall that B is R-AP iff we can partition it into two R-AP subgraphs $B[S_\lambda]$ and $B[S_{n-\lambda}]$ having size λ and $n - \lambda$ respectively, and this for any $\lambda \in [1, \lfloor \frac{n}{2} \rfloor]$. Here are some partitions we can consider for the first values of λ :

- $\lambda = 1$: P_1 and $B(1, 2, 3, \overline{2k})$ (R-AP by Lemma 3),
- $\lambda = 2$: P_2 and $B(1, 1, 3, \overline{2k})$ (spanned by $Cat(2, 5 + 2k)$),
- $\lambda = 3$: P_3 and $B(1, 1, 2, \overline{2k})$ (R-AP by Lemma 2),
- $\lambda = 4$: P_4 and $Cat(4, 2k + 1)$,
- $\lambda = 5$: $Cat(2, 3)$ and P_{2k+4} ,
- $\lambda = 6$: P_6 and $Cat(2, 2k + 1)$,
- $\lambda = 7$: $Cat(3, 4)$ and P_{2k+2} .

By now, it should be clear that the proposition holds for any partial balloon $PB(1, 1, 2, 3, \overline{2k})$ such that $n \leq 15$ (*i.e.* whenever $k \leq 3$). Let us suppose, as an induction hypothesis, that the claim is true for any $i \leq k - 1$; we now consider the partitioning of $B = PB(1, 1, 2, 3, \overline{2k})$ into two R-AP subgraphs for the remaining values of λ , that is for any $\lambda \in [8, \lfloor \frac{n}{2} \rfloor]$.

- When $\lambda \geq 9$ is odd, then B can be partitioned into $PB(1, 1, 2, 3, \overline{\lambda - 9})$, which is R-AP by induction since $\lambda - 9$ is even and positive, and $P_{n-\lambda}$.
- When $\lambda \geq 8$ is even, observe that $\lambda \leq 2k$ since $\lambda \leq \lfloor \frac{n}{2} \rfloor$ and we handled the cases where $k \leq 3$. We can thus partition B into P_λ , and either $B(1, 1, 2, 3)$ when $k = 4$ or $PB(1, 1, 2, 3, \overline{2k - \lambda})$ otherwise, which are respectively R-AP according to Lemma 3 and by induction since $2k - \lambda$ is even.

□

By Observation 3, we can deduce the following corollary ensuring that there exists arbitrary many R-AP 5-balloons; this result also holds for OL-AP 5-balloons since any R-AP graph is also OL-AP (Theorem 1).

Corollary 1 *The 5-balloon $B(1, 1, 2, 3, 2k)$ is OL-AP and R-AP for any $k \geq 1$.*

5 OL-AP and R-AP 5-balloons have their smallest branch of bounded size

In this section, we mainly deal with R-AP 5-balloons, but our result can be easily derived for OL-AP ones. Since, by Corollary 1, we know that there exists arbitrarily many R-AP 5-balloons, one can next wonder about the repartition of the vertices of a such one along its branches. Observe that any member of the exhibited family has its longest path of size $n - 2$, and thus does not compromise Question 1. Although, this family is not representative enough of the entirety of R-AP 5-balloons; hence, it would be interesting to show that 5-balloons in general cannot counter our intuition. In what follows, we make a first step towards this goal by giving a first upper bound on the size of the smallest branch of an R-AP 5-balloon.

Before giving the main result of this section, we introduce some lemmas showing that some particular graphs can never be R-AP.

Lemma 4 *The partial balloon $PB(8^+, 8^+, \overline{x}, y)$ cannot be R-AP for any $x, y \geq 1$.*

Proof We prove this claim by induction on $x + y$. Let us first suppose that $x = y = 1$; if $B = PB(8^+, 8^+, \overline{1}, \underline{1})$ were R-AP then, by Observation 2, for any $\lambda \in [1, \lfloor \frac{n}{2} \rfloor]$ we should be able to find a subset $S_\lambda \subset V$ with λ such that $B[S_\lambda]$ and $B[V \setminus S_\lambda]$ are both R-AP. Let us consider a part with size $\lambda = 2$; obviously, every possible choice for S_2 does not fulfil these conditions:

- $S_2 = \{v_1^3, r_1\}$: $B[V \setminus S_2]$ is isomorphic to $Cat(9^+, 9^+)$.

- $S_2 = \{v_1^1, v_2^1\}$: $B[V \setminus S_2]$ is isomorphic to $Cat(7^+, 11^+)$ which is R-AP only when the first branch of B has size 8 while its second one has size belonging in $\{8, 10, 12\}$. But observe that the corresponding graphs $PB(8, 8, \bar{1}, \underline{1})$, $PB(8, 10, \bar{1}, \underline{1})$ and $PB(8, 12, \bar{1}, \underline{1})$ are not R-AP since they cannot be partitioned into two parts inducing R-AP subgraphs with size 3 and $n - 3$.
- $S_2 = \{v_2^1, v_3^1\}$: $B[V \setminus S_2]$ is a tree having two degree-three vertices.
- For any other choice of S_2 , $B[V \setminus S_2]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

Let us now suppose that $x + y = 3$; without loss of generality, we can consider that $x = 2$ and $y = 1$. We claim that it is not possible to partition $B = PB(8^+, 8^+, \bar{2}, \underline{1})$ into two parts with size 3 and $n - 3$ inducing R-AP subgraphs: indeed, for any possible choice S_3 of three vertices, the remaining graph $B[V \setminus S_3]$ cannot be R-AP:

- $S_3 = \{r_1, v_2^3, v_1^3\}$: $B[V \setminus S_3]$ is isomorphic to $Cat(9^+, 9^+)$.
- $S_3 = \{v_1^4, r_2, v_{b_1(B)}^1\}$: $B[V \setminus S_3]$ is isomorphic to a non-caterpillar 3-pode different from $P(2, 4, 6)$.
- $S_3 = \{v_1^1, v_2^1, v_3^1\}$: $B[V \setminus S_3]$ is isomorphic to $Cat(6^+, 12^+)$ which is R-AP only when the first branch of B has size 9 while its second one has size 9 or 11. But observe that the corresponding graphs $PB(8, 9, \bar{2}, \underline{1})$ and $PB(8, 11, \bar{2}, \underline{1})$ are not R-AP since they cannot be partitioned into two parts with size 4 and $n - 4$ inducing R-AP subgraphs.
- $S_3 = \{v_2^1, v_3^1, v_4^1\}$: $B[V \setminus S_3]$ is a tree with two degree-three vertices.
- For any other choice of S_3 , $B[V \setminus S_3]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

Consider now that the claim holds for any graph $PB(8^+, 8^+, \bar{x}, \underline{y})$ such that $x + y \leq k$ with $k > 3$. We now prove that this is also the case when $x + y = k + 1$. There are two cases to consider:

- $x > 1$ and $y > 1$: $B = PB(8^+, 8^+, \bar{x}, \underline{y})$ is not R-AP since we cannot partition its vertices into two parts S_1 and S_{n-1} inducing R-AP subgraphs with size 1 and $n - 1$ respectively. Observe that, indeed, S_1 cannot contain one vertex different from the degree-one ones belonging to the two hanging branches of B : otherwise, $B[V \setminus S_1]$ would be either disconnected or isomorphic to a big caterpillar or a tree with two degree-three vertices, and thus would not be R-AP. We must thus consider $S_1 = \{v_1^3\}$ or $S_1 = \{v_1^4\}$ but in both cases the remaining graph $B[S_{n-1}]$ is isomorphic to $PB(8^+, 8^+, \bar{x}', \underline{y}')$, where $x' \leq x$, $y' \leq y$ and $x' + y' = x + y - 1 \leq k$, which is not R-AP by induction.
- $x = 1$ and $y > 2$: consider we want to partition B into two R-AP subgraphs $B[S_2]$ and $B[S_{n-2}]$ of size 2 and $n - 2$ respectively. For the same reasons as previously, we have to consider $S_2 = \{v_1^4, v_2^4\}$. But obviously this partitioning is not correct since $B[S_{n-2}]$ is isomorphic to $PB(8^+, 8^+, \bar{x}, \underline{y - 2})$ which is not R-AP by induction. These arguments hold analogously when $x > 2$ and $y = 1$. \square

Lemma 5 *The partial balloon $PB(8^+, 8^+, \bar{x}, \bar{y}, \underline{z})$ cannot be R-AP for any $x, y, z \geq 1$.*

Proof We prove this claim by induction on $x + y + z$ in an analogous way as what we did for Lemma 4. Let us first suppose that $x = y = z = 1$ and consider the associated graph $B = PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{1})$. Once again, B is not R-AP since its vertex set cannot be partitioned into two parts S_2 and S_{n-2} inducing R-AP subgraphs with size 2 and $n - 2$ respectively:

- $S_2 = \{v_1^5, r_2\}$: $B[V \setminus S_2]$ is a tree with maximum degree 4.
- $S_2 = \{v_1^1, v_2^1\}$: $B[V \setminus S_2]$ is a tree with two degree-three vertices.
- $S_2 = \{v_2^1, v_3^1\}$: $B[V \setminus S_2]$ is a tree having maximum degree 4.
- For any other choice of S_2 , $B[V \setminus S_2]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

Let us now suppose that $x + y + z = 4$; in this case, B is either isomorphic to $PB(8^+, 8^+, \bar{2}, \bar{1}, \underline{1})$ or $PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{2})$ and thus, straightforwardly, is not R-AP. Indeed, on the one hand, if $B = PB(8^+, 8^+, \bar{2}, \bar{1}, \underline{1})$ then it cannot be partitioned into two R-AP graphs $B[S_2]$ and $B[S_{n-2}]$ with size 2 and $n - 2$ respectively:

- $S_2 = \{v_1^3, v_2^3\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \underline{1})$, which is not R-AP according to Lemma 4.
- $S_2 = \{v_1^5, r_2\}$: $B[V \setminus S_2]$ is a tree with maximum degree 4.
- $S_2 = \{v_1^1, v_2^1\}$: $B[V \setminus S_2]$ is a tree with two degree-three vertices.
- $S_2 = \{v_2^1, v_3^1\}$: $B[V \setminus S_2]$ is a tree having maximum degree 4.
- For any other choice of S_2 , $B[V \setminus S_2]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

On the other hand, if $B = PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{2})$ then it cannot be divided into two R-AP subgraphs with size 1 and $n - 1$ respectively:

- $S_1 = \{v_1^3\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \underline{2})$, which is not R-AP according to Lemma 4.
- $S_1 = \{v_1^5\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{1})$, which is not R-AP by induction.
- $S_1 = \{v_1^1\}$: $B[V \setminus S_1]$ is a tree with two degree-three vertices.
- $S_1 = \{v_2^1\}$: $B[V \setminus S_1]$ is a tree having maximum degree 4.
- For any other choice of S_1 , $B[V \setminus S_1]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

Suppose now that this claim holds whenever $x + y + z \leq k$ for some $k > 4$, and consider the associated graph $B = PB(8^+, 8^+, \bar{x}, \bar{y}, \underline{z})$ where $x + y + z = k + 1$. Once again, we consider two main cases:

- $z > 1$: if B were R-AP, then it would be possible to partition its vertex set into two parts S_1 and S_{n-1} with size 1 and $n - 1$ respectively such that $B[S_1]$ and $B[S_{n-1}]$ are R-AP. Notice that we must consider $S_1 = \{v_1^3\}$ (resp. $S_1 = \{v_1^4\}$) or $S_1 = \{v_1^5\}$ since any other choice of S_1 would make

$B[S_{n-1}]$ being disconnected or isomorphic to a non R-AP tree (*i.e.* to a tree having several degree-three vertices, or one degree-four one). But, in both cases, $B[S_{n-1}]$ is either not R-AP according to Lemma 4 (notably when $x = 1$ (resp. $y = 1$)) or by induction. Hence, B cannot be R-AP when $z > 1$.

- $z = 1$: once again, B cannot be R-AP under this condition since it cannot be partitioned into two R-AP subgraphs $B[S_2]$ and $B[S_{n-2}]$ with size 2 and $n - 2$ respectively. For the same reasons as above, one must consider $S_2 = \{v_1^3, v_2^3\}$ (or $S_2 = \{v_1^4, v_2^4\}$); but the remaining graph $B[S_{n-2}]$ can never be R-AP, either by induction when $x > 2$ (resp. $y = 2$) or according to Lemma 4 otherwise. \square

Lemma 6 *The partial balloon $PB(8^+, 8^+, 8^+, \bar{x}, \underline{y})$ cannot be R-AP for any $x, y \geq 1$.*

Proof Once more, let us prove this claim by induction on $x + y$. Consider first that $x = y = 1$ and the associated partial balloon $B = PB(8^+, 8^+, 8^+, \bar{1}, \underline{1})$. Obviously, this graph cannot be R-AP: it is indeed not possible to partition it into two subgraphs $B[S_2]$ and $B[S_{n-2}]$ with size 2 and $n - 2$ respectively. Here are all the possibilities for S_2 :

- $S_2 = \{v_1^4, r_1\}$: $B[V \setminus S_2]$ is a tree with maximum degree 4.
- $S_2 = \{v_1^1, v_2^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \underline{6^+}, \underline{1})$, which is not R-AP according to Lemma 5.
- $S_2 = \{v_2^1, v_3^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{5^+}, \underline{1})$, which cannot be R-AP since otherwise the 6-balloon $B(8^+, 8^+, 5^+, 1, 1, 1)$ would be R-AP too.
- For any other choice of S_2 , $B[V \setminus S_2]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

Let us now consider that $x + y = 3$, that is $x = 2$ and $y = 1$ without loss of generality, and the partial balloon $B = PB(8^+, 8^+, 8^+, \bar{2}, \underline{1})$. Once again, B cannot be R-AP since its vertex set cannot be partitioned into two parts S_3 and S_{n-3} having size 3 and $n - 3$ respectively:

- $S_3 = \{v_2^4, v_1^4, r_1\}$: $B[V \setminus S_3]$ is a tree with maximum degree 4.
- $S_3 = \{v_1^5, r_2, v_{b_1(B)}^1\}$: $B[V \setminus S_3]$ is a tree having maximum degree 4.
- $S_3 = \{v_1^1, v_2^1, v_3^1\}$: $B[V \setminus S_3]$ isomorphic to $PB(8^+, 8^+, \bar{1}, \underline{5^+}, \underline{1})$, which is not R-AP according to Lemma 5.
- $S_3 = \{v_2^1, v_3^1, v_4^1\}$: $B[V \setminus S_3]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{4^+}, \underline{1})$, which cannot be R-AP since otherwise the 6-balloon $B(8^+, 8^+, 4^+, 1, 1, 1)$ would be R-AP too.
- For any other choice of S_3 , $B[V \setminus S_3]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

We thus suppose that this claim holds for any $x + y \leq k$, where $k > 3$, and consider it regarding the partial balloon $B = PB(8^+, 8^+, 8^+, \bar{x}, \underline{y})$ where $x + y = k + 1$. Let us take the following two cases in consideration:

- $x > 1$ and $y > 1$: it is straightforward to notice that, in this situation, B cannot be R-AP since it cannot be partitioned into two R-AP subgraphs $B[S_1]$ and $B[S_{n-1}]$ having size 1 and $n - 1$ respectively. Indeed, observe that, for some similar reasons as the ones we used for the previous claims, we have to consider $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$ (if there exists another correct choice of S_1 , then it means that either an R-AP 6-balloon exists or that Lemma 5 is wrong); but in both cases, $B[S_{n-1}]$ cannot be R-AP by induction since it is either isomorphic to $PB(8^+, 8^+, 8^+, \overline{x-1}, \underline{y})$ or to $PB(8^+, 8^+, 8^+, \overline{x}, \underline{y-1})$.
- $x = 1$ and $y > 2$: once again, the graph $B = PB(8^+, 8^+, 8^+, \overline{1}, \underline{y})$ cannot be R-AP: this time, we cannot find a partitioning of V into two parts S_2 and S_{n-2} such that $B[S_2]$ and $B[S_{n-2}]$ are R-AP and with size 2 and $n - 2$ respectively. Indeed, observe that we must consider $S_2 = \{v_1^5, v_2^5\}$ since otherwise there would exist an R-AP 6-balloon, an R-AP tree having maximum degree 4, or a graph contradicting Lemma 5. But for this choice of S_2 , we have $B[S_{n-2}] = PB(8^+, 8^+, 8^+, \overline{1}, \underline{y-2})$ which is not R-AP by induction. An analogous proof can be led when $x > 2$ and $y = 1$. \square

Lemma 7 *The partial balloon $PB(8^+, 8^+, \overline{x}, \overline{y}, \overline{z})$ cannot be R-AP for any $x, y, z \geq 1$.*

Proof We prove this claim by induction on $x + y + z$. First, let us suppose that $x = y = z = 1$ and consider the associated partial balloon $B = PB(8^+, 8^+, \overline{1}, \overline{1}, \overline{1})$. It is straightforward to notice that B is not R-AP since it cannot be partitioned into two R-AP subgraphs $B[S_2]$ and $B[S_{n-2}]$ with size 2 and $n - 2$ respectively: indeed, any choice of S_2 makes $B[V \setminus S_2]$ being either disconnected or isomorphic to a tree with maximum degree at least 4.

Observe that this argument also holds to prove that $PB(8^+, 8^+, \overline{x}, \overline{y}, \overline{z})$ is not R-AP when $x = 2$ and $y = z = 1$ by considering a partitioning of it into two R-AP parts with size 3 and $n - 3$ respectively. Such a reasoning is also correct when $x = y = 2$ and $z = 1$.

Suppose now that this claim holds by induction whenever $x + y + z \leq k$ for $k > 5$, and consider a partial balloon $B = PB(8^+, 8^+, \overline{x}, \overline{y}, \overline{z})$ where $x + y + z = k + 1$. We distinguish the following two main cases depending on the size of x , y and z :

- $x > 1$, $y > 1$ and $z > 1$: suppose we want to partition B into two parts S_1 and S_{n-1} inducing R-AP graphs on, respectively, 1 and $n - 1$ vertices; obviously, we must consider $S_1 = \{v_1^3\}$, $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$ since, for any other choice of S_1 , the remaining graph $B[S_{n-1}]$ is either disconnected or isomorphic to a tree having maximum degree at least 4. But in any of these cases, the subgraph $B[S_{n-1}]$ is not R-AP by induction; hence, B is not R-AP under these conditions.
- $z = 1$: in this situation, B cannot be R-AP for the same reason as above but for a partitioning of it into two parts S_α and $S_{n-\alpha}$ with size α and $n - \alpha$, where $\alpha = \min(\{2, 3, 4\} \setminus \{x, y\})$. Indeed, for any choice of S_α , the remaining graph $B[S_{n-\alpha}]$ is not R-AP either by induction or because it is

isomorphic to a non-connected graph or a tree with maximum degree at least 4. We can use a very similar argument when $x = 1$ or $y = 1$. \square

Lemma 8 *The partial balloon $PB(8^+, 8^+, 8^+, \bar{x}, \bar{y})$ cannot be R-AP for any $x, y \geq 1$.*

Proof Once again, we prove this claim by induction on the value of $x + y$. We firstly suppose that $x = y = 1$ and consider the associated partial balloon $B = PB(8^+, 8^+, 8^+, \bar{1}, \bar{1})$; similarly as in the proof of the previous cases, B cannot be R-AP. Indeed, it cannot be partitioned into two R-AP parts $B[S_2]$ and $B[S_{n-2}]$ having size 2 and $n - 2$ respectively. Below are described all the possibilities for choosing S_2 .

- $S_2 = \{v_1^1, v_2^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{6^+})$, which is not R-AP according to Lemma 5.
- $S_2 = \{v_{b_1(B)}^1, v_{b_1(B)-1}^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, \overline{6^+}, \bar{1}, \bar{1})$, which is not R-AP according to Lemma 7.
- $S_2 = \{v_2^1, v_3^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \bar{1}, \underline{5^+})$, which cannot be R-AP since otherwise the 6-balloon $B(8^+, 8^+, 5^+, 1, 1, 1)$ would be R-AP too.
- $S_2 = \{r_2, v_{b_1(B)}^1\}$: $B[V \setminus S_2]$ is a tree with maximum degree 5.
- For any other choice of S_2 , $B[V \setminus S_2]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

We now suppose that $x = 2$ and $y = 1$ and consider the associated partial balloon $B = PB(8^+, 8^+, 8^+, \bar{2}, \bar{1})$. Once again, B cannot be R-AP since its vertex set cannot be partitioned into two parts S_3 and S_{n-3} inducing R-AP subgraphs of B and containing 3 and $n - 3$ vertices respectively:

- $S_3 = \{v_1^1, v_2^1, v_3^1\}$: $B[V \setminus S_3]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \underline{5^+})$, which is not R-AP according to Lemma 5
- $S_3 = \{v_{b_1(B)}^1, v_{b_1(B)-1}^1, v_{b_1(B)-2}^1\}$: $B[V \setminus S_3]$ is isomorphic to $PB(8^+, 8^+, \overline{5^+}, \bar{1}, \bar{1})$, which is not R-AP according to Lemma 7.
- $S_3 = \{v_2^1, v_3^1, v_4^1\}$: $B[V \setminus S_3]$ is isomorphic to $PB(8^+, 8^+, \bar{1}, \bar{1}, \bar{1}, \underline{4^+})$, which cannot be R-AP since otherwise the 6-balloon $B(8^+, 8^+, 4^+, 1, 1, 1)$ would be R-AP too.
- $S_3 = \{r_2, v_{b_1(B)}^1, v_{b_1(B)-1}^1\}$: $B[V \setminus S_3]$ is a tree with maximum degree 5.
- For any other choice of S_3 , $B[V \setminus S_3]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

We finally suppose that the induction is correct whenever $x + y \leq k$ for some $k > 3$, and consider a partial balloon $B = PB(8^+, 8^+, 8^+, \bar{x}, \bar{y})$ such that $x + y = k + 1$. We distinguish two main cases, depending on the values of x and y , to prove that B is not R-AP:

- $x > 1$ and $y > 1$: in this case, B cannot be R-AP since it cannot be partitioned into two R-AP subgraphs $B[S_1]$ and $B[S_{n-1}]$ having size 1 and $n - 1$ respectively. Indeed, one must consider $S_1 = \{v_1^4\}$ or $S_1 = \{v_1^5\}$ since

any other choice for S_1 would make $B[S_{n-1}]$ being either disconnected, or isomorphic to a tree with maximum degree 5 or to a graph that can never be R-AP according to Lemmas 5 and 7. But observe that in both cases, the graph $B[S_{n-1}]$ is not R-AP by induction.

- $x > 2$ and $y = 1$: the above arguments hold when we consider two parts S_2 and S_{n-2} with size 2 and $n - 2$. They can also be used analogously to prove that B is not R-AP when $x = 1$ and $y > 2$. \square

Lemma 9 *The partial balloon $PB(8^+, 8^+, 8^+, 8^+, \bar{x})$ cannot be R-AP for any $x \geq 1$.*

Proof Let us prove this claim by induction on x . We first suppose that $x = 1$ and consider the partitioning of $B = PB(8^+, 8^+, 8^+, 8^+, \bar{1})$ into two R-AP parts $B[S_2]$ and $B[S_{n-2}]$ with size 2 and $n - 2$ respectively. Obviously, such a partitioning does not exist if we consider all the possibilities for S_2 :

- $S_2 = \{v_1^5, r_1\}$: $B[V \setminus S_2]$ is a tree with maximum degree 4.
- $S_2 = \{v_{b_1(B)}^1, r_2\}$: $B[V \setminus S_2]$ is a tree with maximum degree 5.
- $S_2 = \{v_1^1, v_2^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, 8^+, \bar{1}, \underline{6^+})$, which is not R-AP according to Lemma 6.
- $S_2 = \{v_{b_1(B)}^1, v_{b_1(B)-1}^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, 8^+, \bar{1}, \overline{6^+})$, which is not R-AP according to 8.
- $S_2 = \{v_2^1, v_3^1\}$: $B[V \setminus S_2]$ is isomorphic to $PB(8^+, 8^+, 8^+, \bar{1}, \bar{1}, \underline{5^+})$, which cannot be R-AP since otherwise the 6-balloon $B(8^+, 8^+, 8^+, 5^+, 1, 1)$ would be R-AP too.
- For any other choice of S_2 , $B[V \setminus S_2]$ is either not R-AP for a reason similar to one of those above or because it is not connected.

Suppose now that $PB(8^+, 8^+, 8^+, 8^+, \bar{x})$ is not R-AP for every $x \leq k$ where $k > 1$, and consider the graph $B = PB(8^+, 8^+, 8^+, 8^+, \overline{k+1})$. If it were R-AP, then it would be possible to partition it into two R-AP subgraphs $B[S_1]$ and $B[S_{n-1}]$ with size 1 and $n - 1$ respectively. But it is obviously not the case regarding all the possibilities for S_1 :

- $S_1 = \{v_1^5\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, 8^+, 8^+, \bar{k})$, which is not R-AP by induction.
- $S_1 = \{r_2\}$: $B[V \setminus S_1]$ is a tree with maximum degree 5.
- $S_1 = \{v_1^1\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, 8^+, \overline{k+1}, \underline{7^+})$, which is not R-AP according to Lemma 6.
- $S_1 = \{v_{b_1(B)}^1\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, 8^+, \overline{k+1}, \overline{7^+})$, which is not R-AP according to Lemma 8.
- $S_1 = \{v_2^1\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, 8^+, \overline{k+1}, \bar{1}, \underline{6^+})$, which cannot be R-AP since otherwise the 6-balloon $B(8^+, 8^+, 8^+, 6^+, k+1, 1)$ would be R-AP too.
- For any other choice of S_1 , $B[V \setminus S_1]$ is either not R-AP for a reason similar to one of those above or because it is not connected. \square

Thanks to Lemmas 4 to 9, we can now prove that the smallest branch of an R-AP 5-balloon has size at most 7.

Theorem 6 *Let $B = (b_1, b_2, b_3, b_4, b_5)$ be a 5-balloon with $b_1 \leq \dots \leq b_5$. If B is R-AP, then $b_1 \leq 7$.*

Proof Let $B = B(8^+, 8^+, 8^+, 8^+, 8^+)$ be a 5-balloon; we claim that it cannot be R-AP since it is not possible to partition V into two parts S_1 and S_{n-1} inducing R-AP graphs with size 1 and $n - 1$ respectively:

- $S_1 = \{r_1\}$: $B[V \setminus S_1]$ is a tree with maximum degree 4.
- $S_1 = \{v_1^1\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, 8^+, 8^+, \underline{7})$, which is not R-AP according to Lemma 9.
- $S_1 = \{v_2^1\}$: $B[V \setminus S_1]$ is isomorphic to $PB(8^+, 8^+, 8^+, 8^+, \underline{6})$, which is not R-AP since otherwise the 6-balloon $B(1, 6^+, 8^+, 8^+, 8^+, 8^+)$ would be R-AP too.
- For any other choice of S_1 , $B[V \setminus S_1]$ is not R-AP for a reason similar to one of those above.

Finally, a 5-balloon cannot be R-AP when its smallest branch has size at least 8, proving the claim. \square

The same arguments can be used to deduce analogously that OL-AP 5-balloons have their smallest branch of size at most 11. Notice though that this result might not be sharp since there is a gap between 7 and 2, the size of the smallest branch of the R-AP 5-balloon known to have the biggest one⁴. However, since the biggest elementary path of a 5-balloon $B(b_1, \dots, b_5)$ has size $n - (b_1 + b_2)$ when $b_1 \leq \dots \leq b_5$, the bound of Theorem 6 concurs with Question 1 despite it does not ensure that 5-balloons cannot counter it since we do not know if the second smallest branch of an R-AP 5-balloon can be arbitrarily big or not. In a more general context, this result allows us to deduce, according to an R-AP version of Observation 1, that 2-connected R-AP graphs should have some small components in their structure:

Corollary 2 *Let G be a 2-connected R-AP graph, and u and v be two vertices forming an articulation pair of it. If the deletion of $\{u, v\}$ from G disconnects it into exactly five components, then one is of order at most seven.*

Notice that, using the mentioned equivalent upper bound for OL-AP 5-balloons, we could derive a similar result for 2-connected OL-AP graphs.

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⁴ Which is $B(2, 2, 3, 4, 6)$.

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