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# A multilayer anisotropic plate model with warping functions for the study of vibrations reformulated from Woodcock's work

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## Abstract

In this paper, a plate model suitable for static and dynamic analysis of inhomogeneous anisotropic multilayered plates is described. This model takes transverse shear variation through the thickness of the plate into account by means of warping functions. Warping functions are determined by enforcing kinematic and static assumptions at the interfaces of the layers. This model leads to a  $10 \times 10$  behavior matrix in which membrane strains, bending and torsion curvatures, and transverse shear  $x$  and  $y$ -derivatives are coupled to each other, and to a classical  $2 \times 2$  shear behavior matrix. This model has been proven to be very efficient, especially when high ratios –up to  $10^5$ – between the stiffnesses of layers are present.

This work is related to Woodcock's model, so it can be seen as a reformulation of his work. However, it propose several enhancements: the displacement field is made explicit; it is reformulated with commonly used plate notations; laminate equations of motion are fully detailed; the place of this model relatively to other plate models is now easy to see and is discussed; the link between this formulation and the original one is completely written with all necessary proofs; misses and errors have been found in the energy coefficients of the original work, and then have been corrected; it is now easy to improve or to adapt the model for specific applications with the choice of refined or specific warping functions.

*Keywords:* Multilayered, plate, model, laminate, vibration, warping functions

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## 1. Introduction

After the early works of Kirchhoff, Rayleigh, Love, done at the end of the 19th century, followed by Reissner, Mindlin, Uflyand in the middle of the 20th century, dealing respectively with thin and moderately thick homogeneous plates, the behavior of anisotropic multilayered plates began to be studied by Lekhnitskii and Ambartsumyan.

Since these works, static and dynamic behavior of anisotropic multilayered plates have been extensively studied. In composite applications, the high ratio between shear and longitudinal stiffnesses and the strong inhomogeneities between layers make standard models inefficient. It has incited searchers to develop more refined theories, able to handle the behavior of such structures.

As done by the previously cited early works, these refined theories attempt to describe the displacement field in the thickness direction by means of kinematic and/or static hypothesis. The difference is that in the recent works, this description is more refined. In particular, the transverse shear evolution through the thickness has received many attention. The more recent and pertinent theories have been built using *warping functions* that describe this evolution. In these models, first spatial derivatives of the transverse shear are coupled with membrane and bending terms, leading to a more complex behavior matrix of size  $10 \times 10$  compared to the classical  $6 \times 6$  behavior matrix of the previous models.

The model presented in this paper belongs to this last category as it will be shown later. It has been first formulated in an early work of Sun & Whitney [1], it is the second model described in this reference. Even if the model is presented for the more general case, authors have written equations of the model only for symmetric isotropic laminate. This model was improved by other researchers [2, 3] to study the dynamic behavior of more

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general multilayered plates and also to simulate the sound transmission through these structures. This model has been proved to be very efficient for the study of viscoelastic behavior of strongly inhomogeneous structures like plates damped with viscoelastic PCLD patches [4, 5]. All these works were limited to on-axis orthotropic plies. Woodcock has extended his first work to laminates including off-axes orthotropic plies [6].

The present work reformulates last Woodcock's model [6]. This reformulation has been motivated by the following disadvantages of the original work:

- the displacement field is not made explicit. It is only known implicitly and involves constraints on both displacement and stress fields;
- the notations used by the author are not the standard notations for plate models, which makes the work difficult to understand;
- some helpful formulas are not given and missing terms and errors have been found on the given energy coefficients;
- due to the complexity of the model and its formulas, implementation is difficult;
- links between this model and other plate models are not easy to foresee.

The reformulation of this work leads to the following enhancements:

- the displacement field is now fully explicit. It involves four warping functions  $\varphi_{\alpha\beta}(z)$ ;
- the global plate behavior is given in terms of classical (membrane / bending / shearing) generalized displacements, forces, and stiffnesses, using the most common plate theories notations;
- the link between the present model and Woodcock's model is fully detailed;
- helpful formulas for computing Woodcock's coefficients  $\alpha_x^\ell, \alpha_{xy}^\ell, \beta_x^\ell \dots$  are given;
- the corrected list of all the 69 Woodcock's strain energy coefficients  $\lambda_i$  is given. It is shown that the list reduces to 55 independent coefficients in the most general case;
- a corrected and extended list (18 instead of 15) of Woodcock's coefficients  $\delta_i$  for the kinetic energy is given;
- the place of this model relatively to other plate theories is now easy to see, and it is discussed;
- as the model only involves integrals through the thickness, it is easy to implement;
- the present reformulation makes improvements easy to implement: warping functions may be changed or enhanced without complicating the model.

## 2. Model

### 2.1. Laminate definition

The laminate is composed of  $n$  layers. In this study, all the quantities will be related to those of the first layer, which then plays a central role. Figure 1 helps to visualize the following definitions:

- $z^\ell$  is the elevation/offsetting of the middle plane of the layer  $\ell$
- the  $\ell$ th layer is located between elevations  $\zeta^{\ell-1}$  and  $\zeta^\ell$

With these definitions:

- there is  $n$  parameters  $z^\ell$  with, by definition,  $z^1 = 0$ ,
- there is  $n + 1$  parameters  $\zeta^i$  with  $i$  taking values from 0 to  $n$ ,
- the thickness of the layer  $\ell$  is  $h^\ell = \zeta^\ell - \zeta^{\ell-1}$

Plate models take classically as reference plane the laminate middle plane, then the laminate is located between  $h/2$  and  $-h/2$  where  $h$  is the total height of the laminate. This model has been developed in a slightly different way: the middle plane of the first layer is the reference plane. This is particularly useful, although not essential, to study plates damped with patches [4, 5]. In this last case, the number of layers can vary from a point to another and the definition of a middle plane is difficult to precise. Of course one can change the reference plane in this formulation without changing the behavior of the model, but all formulas have to be changed in consequence.

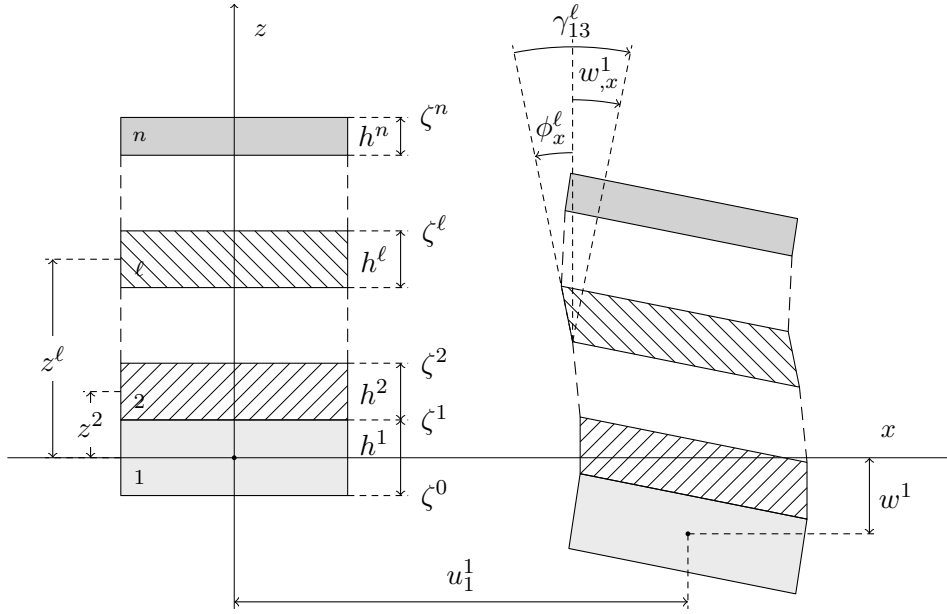


Figure 1: Geometrical parameters of the undeformed laminate on the left side, and its deformed shape with corresponding quantities on the right side.

## 2.2. Kinematic and static assumptions of the model

In the following, Greek subscripts takes values 1 or 2 and Latin subscripts takes values 1, 2 or 3. The Einstein's summation convention is used for subscripts only. The comma used as a subscript index means the partial derivative with respect to the following(s) index(ices).

### 2.2.1. Displacement field

First, kinematic assumptions of Woodcock's model are reported below after a change of notations in order to agree with most of plate models notations. They are:

$$\begin{cases} u_\alpha^\ell(x, y, z) = u_\alpha^\ell(x, y, z^\ell) + (z^\ell - z)(w_{,\alpha}^1(x, y) - \gamma_{\alpha 3}^\ell(x, y)) \\ u_3^\ell(x, y, z) = w^1(x, y) \end{cases} \quad (1)$$

where  $\ell \in [1..n]$  is the number of the layer,  $z^\ell$ ,  $u_\alpha^\ell(x, y, z)$ ,  $\gamma_{\alpha 3}^\ell(x, y)$  and  $u_3^\ell(x, y, z)$  are respectively the  $z$ -coordinate of the mid plane, the in-plane displacements, the engineering transverse shear strains<sup>1</sup>, and the transverse displacement within the layer  $\ell$ ,  $w^1(x, y)$  is the transverse displacement at the mid-plane of the first layer. We can see that the transverse displacement is equal for all the layers and does not depends on  $z$ . However, the superscript 1 is kept to show that the reference is, in this model, the middle plane of the first layer. Note also that this model needs at this time  $4n + 1$  unknowns functions of the  $(x, y)$  coordinates.

The kinematic assumptions of the present formulation are:

$$\begin{cases} u_\alpha(x, y, z) = u_\alpha^1(x, y) - zw_{,\alpha}^1(x, y) + \varphi_{\alpha\beta}(z)\gamma_{\beta 3}^1(x, y) \\ u_3(x, y, z) = w^1(x, y) \end{cases} \quad (2)$$

The main difference is the lack of the  $\ell$  superscript and the consideration of four warping functions  $\varphi_{\alpha\beta}(z)$ . They will be determined with the use of equations (4a) and (4b).

<sup>1</sup>Note that in formula (2) of Woodcock's work, there is a plus sign in the term  $(z^\ell - z)(w_{,\alpha}^1(x, y) + \varphi_{\alpha 3}^\ell(x, y))$  which is not coherent with the definition "the shear" given for the corresponding quantities (denoted  $\varphi_x^n$  and  $\varphi_y^n$ ). These quantities are obviously "the rotations due to shear", this explains some sign differences between the formulations which appear in the following.

### 2.2.2. Additional assumptions

In addition, Woodcock's model (as the second model of reference [1]) requires the continuity of in-plane displacements and transverse shear stresses at each of the  $n - 1$  interfaces:

$$\begin{cases} u_\alpha^\ell(x, y, \zeta^\ell) = u_\alpha^{\ell+1}(x, y, \zeta^\ell) \\ \sigma_{\alpha 3}^\ell(x, y, \zeta^\ell) = \sigma_{\alpha 3}^{\ell+1}(x, y, \zeta^\ell) \end{cases} \quad (3a)$$

$$(3b)$$

where  $\sigma_{\alpha 3}^\ell$  are the transverse shear stresses in layer  $\ell$ . These  $4 \times (n - 1)$  equations permit to eliminate all the unknowns function of the upper layers as it is shown in references [2, 4, 6]. This is done by means of transfer matrices and leads to the definition and the computation of coefficients  $\alpha_x^\ell, \alpha_{xy}^\ell, \beta_x^\ell \dots$ . As the overall process is quite complicated, it is not reproduced here but it is presented in details in Appendix A. The model reduces finally to  $4n + 1 - (4n - 4) = 5$  unknown functions of the first layer: two in-plane displacements  $u_\alpha^1(x, y, z^1)$ , a transverse displacement  $w^1(x, y)$ , and two transverse shear strains  $\gamma_{\alpha 3}^1(x, y)$ . These generalized displacements differentiate from the Mindlin-Reissner plate model in which rotations  $\psi_\alpha^1(x, y) = w_{,\alpha}^1(x, y) - \gamma_{\alpha 3}^1(x, y)$  are considered. There is another difference: these quantities are those of the first layer, whereas in plate models, they are generally defined at the mid-plane of the laminate. However it is possible to pass from one description to another. The former one is particularly suitable to patch problems because the first layer is common to the entire structure.

For the present formulation, which is already written in function of the above five unknown functions, the same physical assumptions are done. They are written in a slightly different manner:

$$\begin{cases} \lim_{z \rightarrow \zeta^\ell -} u_\alpha(x, y, z) = \lim_{z \rightarrow \zeta^\ell +} u_\alpha(x, y, z) \\ \lim_{z \rightarrow \zeta^\ell -} \sigma_{\alpha 3}(x, y, z) = \lim_{z \rightarrow \zeta^\ell +} \sigma_{\alpha 3}(x, y, z) \end{cases} \quad (4a)$$

$$(4b)$$

The advantage of this formulation is that, as shown below, the four warping functions  $\varphi_{\alpha\beta}(z)$  are easily obtained by means of the above conditions, making explicit the displacement field.

### 2.2.3. Final form

As told before, the elimination of the displacements of the upper layers in Woodcock's formulation is a quite complicated process and does not lead to an explicit form for the displacement field. The displacement field is then written independently for each layer with the help of coefficients which must be computed for each layer.

For the present model, the use of equations (4a) and (4b) to build the warping functions is presented below. First, let us compute the strain field from equations (2):

$$\begin{cases} \epsilon_{\alpha\beta}(x, y, z) = \epsilon_{\alpha\beta}^1(x, y) - zw_{,\alpha\beta}^1(x, y) + \frac{1}{2}(\varphi_{\alpha\gamma}(z)\gamma_{\gamma 3,\beta}^1(x, y) + \varphi_{\beta\gamma}(z)\gamma_{\gamma 3,\alpha}^1(x, y)) \\ \epsilon_{\alpha 3}(x, y, z) = \frac{1}{2}\varphi'_{\alpha\beta}(z)\gamma_{\beta 3}^1(x, y) \\ \epsilon_{33}(x, y, z) = 0 \end{cases} \quad (5a)$$

$$(5b)$$

$$(5c)$$

Note that  $\epsilon_{\alpha\beta}^1, w_{,\alpha\beta}^1, \gamma_{\alpha 3,\beta}^1$  and  $\gamma_{\alpha 3}^1$  form a set of 12 independent generalized strains to consider in this model.

With the help of equations (5a) and Hooke's law, the transverse shear stresses can be formulated:

$$\sigma_{\alpha 3}(x, y, z) = 2C_{\alpha 3\beta 3}\epsilon_{\beta 3} = C_{\alpha 3\beta 3}(z)\varphi'_{\beta\gamma}(z)\gamma_{\gamma 3}^1(x, y) \quad (6)$$

On the above formula, the 2 factor comes from the symmetry of the stiffness and strain tensors:

$$\sigma_{\alpha 3} = C_{\alpha 3\beta 3}\epsilon_{\beta 3} + C_{\alpha 3\beta 3}\epsilon_{\beta 3} = 2C_{\alpha 3\beta 3}\epsilon_{\beta 3} \quad (7)$$

The additional conditions (4a) and (4b) can now be written in terms of conditions on the warping functions:

$$\begin{cases} \lim_{z \rightarrow \zeta^\ell -} \varphi_{\alpha\beta}(z) = \lim_{z \rightarrow \zeta^\ell +} \varphi_{\alpha\beta}(z) \\ \lim_{z \rightarrow \zeta^\ell -} C_{\alpha 3\beta 3}(z)\varphi'_{\beta\gamma}(z) = \lim_{z \rightarrow \zeta^\ell +} C_{\alpha 3\beta 3}(z)\varphi'_{\beta\gamma}(z) \end{cases} \quad (8a)$$

$$(8b)$$

Equations (8a) represent the continuity of the warping functions at the  $n - 1$  interfaces whereas equations (8b) represent “jump” conditions on the derivatives. For a laminate, the simplest way to choose the warping function is to take their derivatives as step functions, then, the warping functions are continuous piecewise linear functions of  $z$ . For the first layer *i. e.* for  $z \in [\zeta^0, \zeta^1]$ , the choice of  $\varphi'_{\alpha\beta}(z) = \delta_{\alpha\beta}^K$  is done, where  $\delta_{\alpha\beta}^K$  is the Kronecker symbol. This choice corresponds to Woodcock’s model.

This leads to the following expressions for the derivative of the warping functions:

$$\varphi'_{\alpha\beta}(z) = 4S_{\alpha 3 \gamma 3}(z)C_{\gamma 3 \delta 3}(z^1)\delta_{\delta\beta} = 4S_{\alpha 3 \gamma 3}(z)C_{\gamma 3 \beta 3}(z^1) \quad (9)$$

The warping functions are obtained integrating (9). Integration constants are chosen to give  $\varphi_{\alpha\beta}(0) = 0$ . That means that the transverse shear strains have no effect on the displacement of the middle plane of the first layer, which is normal because it is precisely the reference plane for the model. Hence the warping functions could be written:

$$\varphi_{\alpha\beta}(z) = 4C_{\gamma 3 \beta 3}(z^1) \int_0^z S_{\alpha 3 \gamma 3}(\zeta) d\zeta \quad (10)$$

### 2.3. Stresses

The strains obtained in formula (5) permit to compute the stresses:

$$\begin{cases} \sigma_{\alpha\beta}(x, y, z) = Q_{\alpha\beta\gamma\delta}(z) (\epsilon_{\gamma\delta}^1(x, y) - zw_{,\gamma\delta}^1(x, y) + \varphi_{\gamma\mu}(z)\gamma_{\mu 3, \delta}^1(x, y)) & (11a) \\ \sigma_{\alpha 3}(x, y, z) = C_{\alpha 3 \beta 3}\varphi'_{\beta\mu}(z)\gamma_{\mu 3}^1(x, y) & (11b) \\ \sigma_{33}(x, y, z) = 0 & (11c) \end{cases}$$

where  $Q_{\alpha\beta\gamma\delta}$  are the reduced generalized plane strain stiffnesses. The vanishing of the term  $\frac{1}{2}$  of equation (5a) into equation (11a) is not evident. Let us demonstrate it, omitting the  $x$ ,  $y$ , and  $z$  coordinates:

$$\begin{aligned} \frac{1}{2}Q_{\alpha\beta\gamma\delta} (\varphi_{\gamma\mu}\gamma_{\mu 3, \delta}^1 + \varphi_{\delta\mu}\gamma_{\mu 3, \gamma}^1) &= \frac{1}{2}Q_{\alpha\beta\gamma\delta}\varphi_{\gamma\mu}\gamma_{\mu 3, \delta}^1 + \frac{1}{2}Q_{\alpha\beta\gamma\delta}\varphi_{\delta\mu}\gamma_{\mu 3, \gamma}^1 \\ &= \frac{1}{2}Q_{\alpha\beta\gamma\delta}\varphi_{\gamma\mu}\gamma_{\mu 3, \delta}^1 + \frac{1}{2}Q_{\alpha\beta\delta\gamma}\varphi_{\gamma\mu}\gamma_{\mu 3, \delta}^1 \\ &= Q_{\alpha\beta\gamma\delta}\varphi_{\gamma\mu}\gamma_{\mu 3, \delta}^1 \end{aligned} \quad (12)$$

### 2.4. Strain energy

It is possible to compute the strain energy density  $\delta J = 1/2\epsilon_{ij}\sigma_{ij}$  from formulas (5) and (11) and integrate it over the thickness to obtain a strain energy surface density  $J(x, y)$ . This is the method followed by Woodcock and other previous authors. Woodcock has found an expression which is the sum of 69 terms. In fact there is only 55 independent terms as we shall see below.

In this paper, this method was also reproduced to find Woodcock’s coefficients given in Appendix B but in the following lines, a different approach is presented. Let us start with the strain energy surface density:

$$\begin{aligned} J &= \frac{1}{2} \int_{\zeta_0}^{\zeta_n} \epsilon_{ij}\sigma_{ij} dz = \frac{1}{2} \int_{\zeta_0}^{\zeta_n} (\epsilon_{\alpha\beta}\sigma_{\alpha\beta} + 2\epsilon_{\alpha 3}\sigma_{\alpha 3} + \epsilon_{33}\sigma_{33}) dz \\ &= \frac{1}{2} \int_{\zeta_0}^{\zeta_n} \left[ \left( \epsilon_{\alpha\beta}^1 - zw_{,\alpha\beta}^1 + \frac{1}{2} (\varphi_{\alpha\gamma}(z)\gamma_{\gamma 3, \beta}^1 + \varphi_{\beta\gamma}(z)\gamma_{\gamma 3, \alpha}^1) \right) \sigma_{\alpha\beta} + 2\frac{1}{2}\varphi'_{\alpha\beta}(z)\gamma_{\beta 3}^1\sigma_{\alpha 3} \right] dz \\ &= \frac{1}{2} \int_{\zeta_0}^{\zeta_n} \left[ (\epsilon_{\alpha\beta}^1 - zw_{,\alpha\beta}^1 + \varphi_{\alpha\gamma}(z)\gamma_{\gamma 3, \beta}^1) \sigma_{\alpha\beta} + \varphi'_{\alpha\beta}(z)\gamma_{\beta 3}^1\sigma_{\alpha 3} \right] dz \end{aligned} \quad (13)$$

It can also be written

$$J = \frac{1}{2} [\epsilon_{\alpha\beta}^1 N_{\alpha\beta} - w_{,\alpha\beta}^1 M_{\alpha\beta} + \gamma_{\gamma 3, \beta}^1 P_{\gamma\beta} + \gamma_{\beta 3}^1 Q_{\beta}] \quad (14)$$

naturally introducing the following quantities which are the generalized forces,

$$\left\{ \begin{array}{l} \{N_{\alpha\beta}, M_{\alpha\beta}, P_{\gamma\beta}\} = \int_{\zeta^0}^{\zeta^n} \{1, z, \varphi_{\alpha\gamma}(z)\} \sigma_{\alpha\beta}(z) dz \\ Q_{\beta} = \int_{\zeta^0}^{\zeta^n} \varphi'_{\alpha\beta}(z) \sigma_{\alpha 3}(z) dz \end{array} \right. \quad (15a)$$

$$\left\{ \begin{array}{l} Q_{\beta} = \int_{\zeta^0}^{\zeta^n} \varphi'_{\alpha\beta}(z) \sigma_{\alpha 3}(z) dz \end{array} \right. \quad (15b)$$

each associated with a corresponding generalized displacement in the strain energy formula(14).

*Interpretation.:*  $N_{\alpha\beta}$  and  $M_{\alpha\beta}$  are respectively the classical plate membrane forces and bending moments, and  $P_{\alpha\beta}$  and  $Q_{\alpha}$  are special moments associated with the warping functions, *i. e.* associated with the transverse shear behavior. Note that  $P_{\alpha\beta} \neq P_{\beta\alpha}$  in the general case, leading to a set of 12 generalized forces.

Let us follow the computation of these generalized forces with the help of equations (11) and (15). They are (some explanations are given below):

$$\left\{ \begin{array}{l} N_{\alpha\beta} = A_{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta}^1 + B_{\alpha\beta\gamma\delta} (-w_{,\gamma\delta}^1) + E_{\alpha\beta\mu\delta} \gamma_{\mu 3,\delta}^1 \\ M_{\alpha\beta} = B_{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta}^1 + D_{\alpha\beta\gamma\delta} (-w_{,\gamma\delta}^1) + F_{\alpha\beta\mu\delta} \gamma_{\mu 3,\delta}^1 \\ P_{\alpha\beta} = E_{\gamma\delta\alpha\beta} \epsilon_{\gamma\delta}^1 + F_{\gamma\delta\alpha\beta} (-w_{,\gamma\delta}^1) + G_{\alpha\beta\mu\delta} \gamma_{\mu 3,\delta}^1 \\ Q_{\alpha} = H_{\alpha 3\beta 3} \gamma_{\beta 3}^1 \end{array} \right. \quad (16a)$$

$$\left\{ \begin{array}{l} M_{\alpha\beta} = B_{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta}^1 + D_{\alpha\beta\gamma\delta} (-w_{,\gamma\delta}^1) + F_{\alpha\beta\mu\delta} \gamma_{\mu 3,\delta}^1 \end{array} \right. \quad (16b)$$

$$\left\{ \begin{array}{l} P_{\alpha\beta} = E_{\gamma\delta\alpha\beta} \epsilon_{\gamma\delta}^1 + F_{\gamma\delta\alpha\beta} (-w_{,\gamma\delta}^1) + G_{\alpha\beta\mu\delta} \gamma_{\mu 3,\delta}^1 \end{array} \right. \quad (16c)$$

$$\left\{ \begin{array}{l} Q_{\alpha} = H_{\alpha 3\beta 3} \gamma_{\beta 3}^1 \end{array} \right. \quad (16d)$$

where the following generalized stiffnesses have been introduced:

$$\left\{ \begin{array}{l} \{A_{\alpha\beta\gamma\delta}, B_{\alpha\beta\gamma\delta}, D_{\alpha\beta\gamma\delta}, E_{\alpha\beta\mu\delta}, F_{\alpha\beta\mu\delta}, G_{\nu\beta\mu\delta}\} = \int_{\zeta^0}^{\zeta^n} Q_{\alpha\beta\gamma\delta} \{1, z, z^2, \varphi_{\gamma\mu}(z), z\varphi_{\gamma\mu}(z), \varphi_{\alpha\nu}(z)\varphi_{\gamma\mu}(z)\} dz \\ H_{\alpha 3\beta 3} = \int_{\zeta^0}^{\zeta^n} \varphi'_{\gamma\alpha}(z) C_{\gamma 3\delta 3} \varphi'_{\delta\beta}(z) dz \end{array} \right. \quad (17a)$$

$$\left\{ \begin{array}{l} H_{\alpha 3\beta 3} = \int_{\zeta^0}^{\zeta^n} \varphi'_{\gamma\alpha}(z) C_{\gamma 3\delta 3} \varphi'_{\delta\beta}(z) dz \end{array} \right. \quad (17b)$$

The  $N_{\alpha\beta}$  and  $M_{\alpha\beta}$  computation is straightforward but special attention must be paid to the computation of the  $P_{\alpha\beta}$ , revealing some uncommon symmetries:

$$\begin{aligned} P_{\alpha\beta} &= \int_{\zeta^0}^{\zeta^n} \varphi_{\mu\alpha}(z) \sigma_{\mu\beta}(z) dz \\ &= \int_{\zeta^0}^{\zeta^n} \varphi_{\mu\alpha}(z) Q_{\mu\beta\gamma\delta}(z) (\epsilon_{\gamma\delta}^1 - z w_{,\gamma\delta}^1 + \varphi_{\gamma\nu}(z) \gamma_{\nu 3,\delta}^1) dz \\ &= E_{\gamma\delta\alpha\beta} \epsilon_{\gamma\delta}^1 + F_{\gamma\delta\alpha\beta} (-w_{,\gamma\delta}^1) + G_{\alpha\beta\mu\delta} \gamma_{\mu 3,\delta}^1 \end{aligned} \quad (18)$$

In this last expression, the  $E_{\gamma\delta\alpha\beta}$  and  $F_{\gamma\delta\alpha\beta}$  are identified with the help of the major symmetry of the  $Q_{\mu\beta\gamma\delta}(z)$  tensor.

The **A**, **B** and **D** tensors inherits the symmetries of Hooke's tensor, a symmetry for each pair of indices, called the *minor symmetries* and the *major symmetry* which permits the swap of the two pairs of indices, this last one being related to the existence of a strain energy. The **E** and **F** tensors loose the symmetry on the last pair of indices, forcing the major symmetry to disappear. The **G** tensor looses the symmetry on the two pairs of indices, but keep the major symmetry:

$$\left\{ \begin{array}{l} \text{for the } \mathbf{A}, \mathbf{B}, \mathbf{D} \text{ tensors: } A_{\beta\alpha\gamma\delta} = A_{\alpha\beta\gamma\delta} = A_{\gamma\delta\alpha\beta} = A_{\gamma\delta\beta\alpha} \\ \text{for the } \mathbf{E}, \mathbf{F} \text{ tensors: } E_{\beta\alpha\gamma\delta} = E_{\alpha\beta\gamma\delta} \neq E_{\gamma\delta\alpha\beta} \neq E_{\gamma\delta\beta\alpha} \\ \text{for the } \mathbf{G} \text{ tensor: } G_{\beta\alpha\gamma\delta} \neq G_{\alpha\beta\gamma\delta} = G_{\gamma\delta\alpha\beta} \neq G_{\delta\gamma\beta\alpha} \end{array} \right. \quad (19)$$

Hence there is 6 independent components for **A**, **B** and **D**, 12 for **E** and **F**, 10 for **G**, and 3 for **H**. So this plate model has a total of 55 independent stiffness coefficients in the most general case.

### 2.5. Static laminate behavior

The generalized forces are set, by type, into vectors:

$$\mathbf{N} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} \quad \mathbf{M} = \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} \quad \mathbf{P} = \begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \\ P_{21} \end{Bmatrix} \quad \mathbf{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \quad (20)$$

and the same is done for the corresponding generalized strains:

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_{11}^1 \\ \epsilon_{22}^1 \\ \epsilon_{12}^1 \end{Bmatrix} \quad \boldsymbol{\kappa} = \begin{Bmatrix} -w_{,11}^1 \\ -w_{,22}^1 \\ -2w_{,12}^1 \end{Bmatrix} \quad \boldsymbol{\Gamma} = \begin{Bmatrix} \gamma_{13,1}^1 \\ \gamma_{23,2}^1 \\ \gamma_{13,2}^1 \\ \gamma_{23,1}^1 \end{Bmatrix} \quad \boldsymbol{\gamma} = \begin{Bmatrix} \gamma_{13}^1 \\ \gamma_{23}^1 \end{Bmatrix} \quad (21)$$

Generalized forces are linked with the generalized strains by the  $10 \times 10$  and  $2 \times 2$  following behavior matrices:

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{P} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} \\ \mathbf{B} & \mathbf{D} & \mathbf{F} \\ \mathbf{E}^T & \mathbf{F}^T & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\Gamma} \end{Bmatrix} \quad \{\mathbf{Q}\} = [\mathbf{H}] \{\boldsymbol{\gamma}\} \quad (22)$$

### 2.6. Kinetic energy

The kinetic energy surface density  $E_c(x, y)$  of the structure is:

$$\begin{aligned} E_c(x, y) &= \frac{1}{2} \int_{\zeta^0}^{\zeta^n} \rho(x, y, z) \dot{u}_i(x, y, z) \dot{u}_i(x, y, z) dz \\ &= \frac{1}{2} \int_{\zeta^0}^{\zeta^n} \rho(z) \left[ (\dot{u}_\alpha^1 - z \dot{w}_{,\alpha}^1 + \varphi_{\alpha\beta}(z) \dot{\gamma}_{\beta 3}^1) (\dot{u}_\alpha^1 - z \dot{w}_{,\alpha}^1 + \varphi_{\alpha\beta}(z) \dot{\gamma}_{\beta 3}^1) + (\dot{w}^1)^2 \right] dz \\ &= \frac{1}{2} \int_{\zeta^0}^{\zeta^n} \rho(z) \left[ \dot{u}_\alpha^1 \dot{u}_\alpha^1 - 2z \dot{u}_\alpha^1 \dot{w}_{,\alpha}^1 + 2\dot{u}_\alpha^1 \varphi_{\alpha\beta}(z) \dot{\gamma}_{\beta 3}^1 + z^2 \dot{w}_{,\alpha}^1 \dot{w}_{,\alpha}^1 \right. \\ &\quad \left. - 2z \dot{w}_{,\alpha}^1 \varphi_{\alpha\beta}(z) \dot{\gamma}_{\beta 3}^1 + \varphi_{\alpha\beta}(z) \dot{\gamma}_{\beta 3}^1 \varphi_{\alpha\mu}(z) \dot{\gamma}_{\mu 3}^1 + (\dot{w}^1)^2 \right] dz \end{aligned} \quad (23)$$

for concision  $x$  and  $y$  have been omitted for the last two lines of this formula.

Let us now introduce the following generalized mass:

$$\{R, S, T, U_{\alpha\beta}, V_{\alpha\beta}, W_{\alpha\beta}\} = \int_{\zeta^0}^{\zeta^n} \rho(z) \{1, z, z^2, \varphi_{\alpha\beta}(z), \varphi_{\alpha\beta}(z)z, \varphi_{\mu\alpha}(z)\varphi_{\mu\beta}(z)\} dz \quad (24)$$

Note that the  $U_{\alpha\beta}$  and  $V_{\alpha\beta}$  are antisymmetric tensors but  $W_{\alpha\beta}$  is symmetrical. Then, there is 14 independent mass coefficients to consider. The kinetic energy surface density can now be written:

$$\begin{aligned} E_c(x, y) &= \frac{1}{2} \left( R \dot{u}_\alpha^1 \dot{u}_\alpha^1 - 2S \dot{u}_\alpha^1 \dot{w}_{,\alpha}^1 + 2U_{\alpha\beta} \dot{u}_\alpha^1 \dot{\gamma}_{\beta 3}^1 + T \dot{w}_{,\alpha}^1 \dot{w}_{,\alpha}^1 \right. \\ &\quad \left. - 2V_{\alpha\beta} \dot{w}_{,\alpha}^1 \dot{\gamma}_{\beta 3}^1 + W_{\alpha\beta} \dot{\gamma}_{\alpha 3}^1 \dot{\gamma}_{\beta 3}^1 + R(\dot{w}^1)^2 \right) dz \end{aligned} \quad (25)$$

### 2.7. Laminate equations of motion

Let us recall the equilibrium conditions within a solid. Without loss of generality, body forces are neglected here, and the previous convention on indices is kept:

$$\begin{cases} \sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} = \rho \ddot{u}_\alpha & (26a) \\ \sigma_{\alpha 3,\alpha} + \sigma_{33,3} = \rho \ddot{u}_3 & (26b) \end{cases}$$



Integrating the equations of equilibrium (26) over the thickness, with the help of formulas (2), (15) and (24) leads to:

$$\begin{cases} N_{\alpha\beta,\beta} + [\sigma_{\alpha 3}(z)]_{\zeta_0^c}^{\zeta_0^n} = R\ddot{u}_\alpha^1 - S\ddot{w}_{,\alpha}^1 + U_{\alpha\beta}\ddot{\gamma}_{\beta 3}^1 & (27a) \\ Q_{\alpha,\alpha}^c + [\sigma_{33}(z)]_{\zeta_0^c}^{\zeta_0^n} = R\ddot{w}^1 & (27b) \end{cases}$$

where the  $Q_\alpha^c$  are the classical shear forces. In order to get more equations, weighted integrals over the thickness of equation (26a) are computed. Weight functions are  $z$  and  $\varphi_{\alpha\gamma}(z)$ . It gives four more equations:

$$\begin{cases} M_{\alpha\beta,\beta} + [\sigma_{\alpha 3}(z)z]_{\zeta_0^c}^{\zeta_0^n} - Q_\alpha^c = S\ddot{u}_\alpha^1 - T\ddot{w}_{,\alpha}^1 + V_{\alpha\beta}\ddot{\gamma}_{\beta 3}^1 & (28a) \\ P_{\gamma\beta,\beta} + [\varphi_{\alpha\gamma}(z)\sigma_{\alpha 3}(z)]_{\zeta_0^c}^{\zeta_0^n} - Q_\gamma = U_{\alpha\gamma}\ddot{u}_\alpha^1 - V_{\alpha\gamma}\ddot{w}_{,\alpha}^1 + W_{\gamma\beta}\ddot{\gamma}_{\beta 3}^1 & (28b) \end{cases}$$

Let denote  $q = [\sigma_{33}(z)]_{\zeta_0^c}^{\zeta_0^n}$  the value of the transverse loading and suppose there is no tangential forces on the top and bottom of the plate, so  $\sigma_{\alpha 3}(-h/2) = \sigma_{\alpha 3}(h/2) = 0$ . We shall note that there is no generalized strains corresponding to the classical shear forces  $Q_\alpha^c$ . They must be eliminated. It is done setting values of  $Q_\alpha^c$  obtained from formula (28a) into equation (27b). This leads to the plate equilibrium system of equations:

$$\begin{cases} N_{\alpha\beta,\beta} = R\ddot{u}_\alpha^1 - S\ddot{w}_{,\alpha}^1 + U_{\alpha\beta}\ddot{\gamma}_{\beta 3}^1 & (29a) \\ M_{\alpha\beta,\beta\alpha} + q = R\ddot{w}^1 + S\ddot{u}_{\alpha,\alpha}^1 - T\ddot{w}_{,\alpha\alpha}^1 + V_{\alpha\beta}\ddot{\gamma}_{\beta 3,\alpha}^1 & (29b) \\ P_{\alpha\beta,\beta} - Q_\alpha = U_{\beta\alpha}\ddot{u}_\beta^1 - V_{\beta\alpha}\ddot{w}_{,\beta}^1 + W_{\alpha\beta}\ddot{\gamma}_{\beta 3}^1 & (29c) \end{cases}$$

## 2.8. Link with Woodcock's model

### 2.8.1. Strain energy

The behavior matrices of equation (22) can be formulated using Woodcock's 69  $\lambda_i$  energy coefficients. Computation of these coefficients is detailed in Appendix A and Appendix B and the proof of the link with stiffnesses of formulas 17 is given in Appendix C. The two matrices of equation (22) can be written:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} \\ \mathbf{B}^T & \mathbf{D} & \mathbf{F} \\ \mathbf{E}^T & \mathbf{F}^T & \mathbf{G} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4\lambda_3 & 2\lambda_{21} & 2\lambda_{47} & -2\lambda_5 & -2\lambda_{15} & -\lambda_{45} & -2\lambda_6 & -2\lambda_{20} & -2\lambda_{46} & -2\lambda_{52} \\ 2\lambda_{21} & 4\lambda_9 & 2\lambda_{62} & -2\lambda_{15} & -2\lambda_{11} & -\lambda_{60} & -2\lambda_{18} & -2\lambda_{12} & -2\lambda_{61} & -2\lambda_{67} \\ 2\lambda_{47} & 2\lambda_{62} & 4\lambda_{25} & -2\lambda_{41} & -2\lambda_{56} & -\lambda_{29} & -2\lambda_{44} & -2\lambda_{59} & -2\lambda_{32} & -2\lambda_{34} \\ -2\lambda_5 & -2\lambda_{15} & -2\lambda_{41} & 4\lambda_1 & 2\lambda_{13} & \lambda_{39} & 2\lambda_4 & 2\lambda_{14} & 2\lambda_{40} & 2\lambda_{48} \\ -2\lambda_{15} & -2\lambda_{11} & -2\lambda_{56} & 2\lambda_{13} & 4\lambda_7 & \lambda_{54} & 2\lambda_{16} & 2\lambda_{10} & 2\lambda_{55} & 2\lambda_{63} \\ -\lambda_{45} & -\lambda_{60} & -\lambda_{29} & \lambda_{39} & \lambda_{54} & \lambda_{22} & \lambda_{42} & \lambda_{57} & \lambda_{27} & \lambda_{28} \\ -2\lambda_6 & -2\lambda_{18} & -2\lambda_{44} & 2\lambda_4 & 2\lambda_{16} & \lambda_{42} & 4\lambda_2 & 2\lambda_{17} & 2\lambda_{43} & 2\lambda_{50} \\ -2\lambda_{20} & -2\lambda_{12} & -2\lambda_{59} & 2\lambda_{14} & 2\lambda_{10} & \lambda_{57} & 2\lambda_{17} & 4\lambda_8 & 2\lambda_{58} & 2\lambda_{65} \\ -2\lambda_{46} & -2\lambda_{61} & -2\lambda_{32} & 2\lambda_{40} & 2\lambda_{55} & \lambda_{27} & 2\lambda_{43} & 2\lambda_{58} & 4\lambda_{23} & 2\lambda_{31} \\ -2\lambda_{52} & -2\lambda_{67} & -2\lambda_{34} & 2\lambda_{48} & 2\lambda_{63} & \lambda_{28} & 2\lambda_{50} & 2\lambda_{65} & 2\lambda_{31} & 4\lambda_{24} \end{bmatrix} \quad (30)$$

and:

$$[\mathbf{H}] = \frac{1}{4} \begin{bmatrix} 4\lambda_{37} & 2\lambda_{69} \\ 2\lambda_{69} & 4\lambda_{38} \end{bmatrix} \quad (31)$$

with:

$$\begin{aligned} \lambda_{15} &= \lambda_{19} \\ \lambda_{25} &= \lambda_{26} = \lambda_{36}/2 \\ \lambda_{29} &= \lambda_{30} \\ \lambda_{32} &= \lambda_{33} \\ \lambda_{34} &= \lambda_{35} \\ \lambda_{41} &= \lambda_{49} = \lambda_{45}/2 \\ \lambda_{44} &= \lambda_{51} \\ \lambda_{47} &= \lambda_{53} \end{aligned} \quad (32)$$

$$\begin{aligned}
\lambda_{56} &= \lambda_{64} = \lambda_{60}/2 \\
\lambda_{59} &= \lambda_{66} \\
\lambda_{62} &= \lambda_{68}
\end{aligned}$$

These above 14 equalities show that only  $69 - 14 = 55$  strain energy coefficients are independent, which is the number previously announced after the examination of symmetries of tensors done in section 2.4 starting from formula (19).

The presence of minus sign in the blocs corresponding to **B** and **E** blocs is not easy to understand because it is due to two sign differences between the two formulations:  $-w_{\alpha\beta}$  are used here instead of  $w_{\alpha\beta}$  in Woodcock's work, and  $\gamma_{\alpha 3}$  are used here instead of  $\varphi_{\alpha} = -\gamma_{\alpha 3}$  (hence  $\varphi_{\alpha,\beta} = -\gamma_{\alpha 3,\beta}$ ) in Woodcock's work. The combination of these two sign changes gives explanation to this point.

### 2.8.2. Kinetic energy

Let us put the generalized speeds into a vector:

$$\boldsymbol{\Omega}^T = \{ \dot{u}_1^1 \ \dot{u}_2^1 \ \dot{w}^1 \ -\dot{w}_{,1}^1 \ -\dot{w}_{,2}^1 \ \dot{\gamma}_{13}^1 \ \dot{\gamma}_{23}^1 \} \quad (33)$$

which permits to define a generalized mass matrix  $\Xi$ :

$$E_c(x, y) = \frac{1}{2} \boldsymbol{\Omega}^T \Xi \boldsymbol{\Omega} \quad (34)$$

Identifying equation (25) with formula (18) of Woodcock's work<sup>2</sup> shows that the expression of the generalized mass matrix can be put in the two alternative forms:

$$\Xi = \begin{bmatrix} R & 0 & 0 & S & 0 & U_{11} & U_{12} \\ 0 & R & 0 & 0 & S & U_{21} & U_{22} \\ 0 & 0 & R & 0 & 0 & 0 & 0 \\ \hline S & 0 & 0 & T & 0 & V_{11} & V_{12} \\ 0 & S & 0 & 0 & T & V_{21} & V_{22} \\ \hline U_{11} & U_{21} & 0 & V_{11} & V_{21} & W_{11} & W_{12} \\ U_{12} & U_{22} & 0 & V_{12} & V_{22} & W_{12} & W_{22} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\delta_3 & 0 & 0 & -\delta_5 & 0 & -\delta_6 & -\delta_{18} \\ 0 & 2\delta_9 & 0 & 0 & -\delta_{11} & -\delta_{17} & -\delta_{12} \\ 0 & 0 & 2\delta_{13} & 0 & 0 & 0 & 0 \\ \hline -\delta_5 & 0 & 0 & 2\delta_1 & 0 & \delta_4 & \delta_{15} \\ 0 & -\delta_{11} & 0 & 0 & 2\delta_7 & \delta_{16} & \delta_{10} \\ \hline -\delta_6 & -\delta_{17} & 0 & \delta_4 & \delta_{16} & 2\delta_2 & \delta_{14} \\ -\delta_{18} & -\delta_{12} & 0 & \delta_{15} & \delta_{10} & \delta_{14} & 2\delta_8 \end{bmatrix} \quad (35)$$

Note that:

$$\begin{aligned}
\delta_3 &= \delta_9 = \delta_{13} \\
\delta_5 &= \delta_{11} \\
\delta_1 &= \delta_7
\end{aligned} \quad (36)$$

These above 4 equalities show that only  $13 - 4 = 9$  coefficients were independent in Woodcock's work (that can be verified in Appendix A. of reference [6]), but with the 5 missing coefficients, this lead to the announced 14 independent generalized mass coefficients. Complete proof of this relationship, including the missing coefficients, is given in Appendix C.

### 3. Place of this model relatively to other plate models

The present model is more general than the Love-Kirchoff, Mindlin-Reissner and Reddy's models because it integrates additional information on the laminate behavior. Setting  $\varphi_{\alpha\beta}(z) = 0$  in the present model gives the Love-Kirchoff model. Matrices **E**, **F**, **G** and **H** are null, so the system (22) can be reduced to the classical

<sup>2</sup>The terms  $\delta_i$  for  $14 \leq i \leq 18$  are missing in Woodcock's formula (18) and also in the list of  $\delta_i$  he produced. They are introduced here, but the corrected formula (18) is not given: one can easily obtain it with the present mass matrix or following Woodcock's procedure.

$6 \times 6$  four-block membrane-bending behavior matrix. The shear matrix  $\mathbf{H}$  is null. Mass terms  $U_{\alpha\beta}$ ,  $V_{\alpha\beta}$ ,  $W_{\alpha\beta}$  are also null, so the motion equations reduces to three equations:

$$\begin{cases} N_{\alpha\beta,\beta} = R\ddot{u}_\alpha^1 - S\ddot{w}_{,\alpha}^1 & (37a) \\ M_{\alpha\beta,\beta\alpha} + q = R\ddot{w}^1 + S\ddot{u}_{\alpha,\alpha}^1 - T\ddot{w}_{,\alpha\alpha}^1 & (37b) \end{cases}$$

Setting  $\varphi_{\alpha\beta}(z) = z\delta_{\alpha\beta}^K$  (where  $\delta_{\alpha\beta}^K$  is the Kronecker symbol) in the present model gives the Mindlin-Reissner model.  $P_{\alpha\beta}$  becomes symmetric and equals  $M_{\alpha\beta}$  and  $Q_\alpha$  are now the classical shear forces (previously denoted  $Q_\alpha^c$  in this document). If  $\gamma_{13,2}$  and  $\gamma_{12,3}$  are summed, matrices  $\mathbf{E}$ ,  $\mathbf{F}$  and  $\mathbf{G}$  can be reduced to  $3 \times 3$  matrices, and doing this,  $\mathbf{E} = \mathbf{A}$ ,  $\mathbf{F} = \mathbf{B}$  and  $\mathbf{G} = \mathbf{D}$  so that the  $-w_{,\alpha\beta}$  and the  $\gamma_{\alpha\beta,3}$  can be combined to give the classical symmetrical part  $\psi_{\alpha\beta} = 1/2(\phi_{\alpha,\beta} + \phi_{\beta,\alpha})$  of the gradient of rotations  $\phi_\alpha = -w_{,\alpha} + \gamma_{\alpha 3}$  of the Mindlin-Reissner model and the classical  $6 \times 6$  four-block membrane-bending behavior matrix. The shear matrix  $\mathbf{H}$  is not null, and it can be enhanced by shear correction factors. The rotations time second derivative  $\ddot{\phi}_\alpha = -\ddot{w}_{,\alpha} + \ddot{\gamma}_{\alpha 3}$  will also appear. Note also that  $U_{\alpha\beta} = S\delta_{\alpha\beta}^K$ ,  $V_{\alpha\beta} = T\delta_{\alpha\beta}^K$  and  $W_{\alpha\beta} = T\delta_{\alpha\beta}^K$ . The system (22) can then be reduced, in a slightly different manner than for the previous case, with the help of (27b), leading to 5 equations:

$$\begin{cases} N_{\alpha\beta,\beta} = R\ddot{u}_\alpha^1 + S\ddot{\phi}_\alpha & (38a) \\ M_{\alpha\beta,\beta} - Q_\alpha = S\ddot{u}_\alpha^1 + T\ddot{\phi}_\alpha & (38b) \\ Q_{\alpha,\alpha} + q = R\ddot{w}^1 & (38c) \end{cases}$$

Reddy's original third-order theory [7] is very efficient for a single layer plate because the kinematic assumption is very close to the three-dimensional elasticity solution. It can be simulated setting  $\varphi_{\alpha\beta}(z) = (z - 4/3z^3)\delta_{\alpha\beta}^K$  in the present model. This will work perfectly with the present formulation, but the link with Woodcock's work is not valid anymore, due to the presence of cubic terms in  $z$ . But this model is not adapted to high modulus ratio between adjacent layers because the warping functions are those of an homogeneous plate.

More sophisticated models with refined warping functions have been proposed. We shall pay attention only on those for which the number of unknowns does not depend on the number of layers. Among these models, we can cite the references [8, 9]. These authors obtain warping functions for the general case, starting from undetermined cubic functions in each layer. Then they develop a (different) method to combine the layer polynomial functions into 4 laminate warping functions. It is also evident at this point that these warping functions can be put into our model, but it was not the purpose of this paper.

#### 4. Conclusion

In this work, a multilayered plate model has been reformulated from Woodcock's work [6]. First, the pertinence of this model, especially when there is high modulus ratios between the layers, has been proved by comparing to the literature and to three-dimensional finite element studies [4, 5]. The reformulation has been motivated by several reasons. First, in the original work, there are missing terms in formulas for the energy coefficients, leading to an erroneous list of those coefficients. It is problematic because these energy coefficients are precisely the core of the method. Second, this work is not presented with the standard notations of plate models, that makes it very difficult to understand and to situate relatively to other known models. For example, the displacement field, which is not explicit in the original work, has been made explicit in the present paper by means of warping functions.

A mechanical behavior law involving a  $10 \times 10$  matrix for the membrane / bending / shear warping behavior and their respective coupling, and a  $2 \times 2$  matrix for the shear behavior, is given. This is suitable for static analysis. Complete equations of motion with generalized mass are also given. It has been proven that 55 generalized stiffnesses components (or strain energy coefficients) and 15 generalized mass (or kinetic energy coefficients) are independent.

A complete link has been established between the original model and its new formulation, with all necessary proofs. In addition, the original work of Woodcock has been completed by additional formulas and detailed procedures to implement it in numerical simulations. It is ready to use in analytical works or in numerical simulations with Rayleigh-Ritz or finite element discretization for example.

The place of the model amongst all multilayered plate models is now easy to see. That permits to understand why the model is pertinent in the cases discussed above, and also to understand its limits, especially the one

due to the hypothesis of constant transverse shear stresses, which is a poor hypothesis for thick plates without shear correction coefficients. This limit is not so problematic, because the present model can be, without effort, enhanced by a choice of more refined warping functions. Results of interesting works like [8] can be easily implemented for dynamic studies with the help of this work, which was definitely not the case from initial formulation.

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## Appendix A. Computation of Woodcock’s layer coefficients $\alpha_x^\ell, \alpha_{xy}^\ell, \beta_x^\ell \dots$

In this section, the computation of the transfer matrix given in Woodcock’s work at formula (14) is made. The notations used are those of section 2.2.1.

### Appendix A.1. Link between the layer $\ell + 1$ and the layer $\ell$

We start with the kinematic and “static” assumptions of formulas (3a) and (3b). Let us consider that  $x$  and  $y$  are fixed, hence they can be omitted in the terms. Enforcing condition (3a) into (1) leads to:

$$u_\alpha^{\ell+1}(z^{\ell+1}) + (z^{\ell+1} - \zeta^\ell)(w_{,\alpha}^1 - \gamma_{\alpha 3}^{\ell+1}) = u_\alpha^\ell(z^\ell) + (z^\ell - \zeta^\ell)(w_{,\alpha}^1 - \gamma_{\alpha 3}^\ell) \quad (\text{A.1})$$

Let  $[\mathbf{C}_s^\ell]$  be the transverse shear stiffness matrix of layer  $\ell$  defined as:

$$[\mathbf{C}_s^\ell] = \begin{bmatrix} C_{1313}^\ell & C_{1323}^\ell \\ C_{1323}^\ell & C_{2323}^\ell \end{bmatrix} \quad (\text{A.2})$$

Hence, with the “static” assumption (3b):

$$\begin{Bmatrix} \gamma_{13}^{\ell+1} \\ \gamma_{23}^{\ell+1} \end{Bmatrix} = [\mathbf{C}_s^{\ell+1}]^{-1} \begin{Bmatrix} \sigma_{13}^{\ell+1} \\ \sigma_{23}^{\ell+1} \end{Bmatrix} = [\mathbf{C}_s^{\ell+1}]^{-1} \begin{Bmatrix} \sigma_{13}^\ell \\ \sigma_{23}^\ell \end{Bmatrix} = [\mathbf{C}_s^{\ell+1}]^{-1} [\mathbf{C}_s^\ell] \begin{Bmatrix} \gamma_{13}^\ell \\ \gamma_{23}^\ell \end{Bmatrix} \quad (\text{A.3})$$

Let us define:

$$[\mathcal{A}^{\ell+1}] = [\mathbf{C}_s^{\ell+1}]^{-1} [\mathbf{C}_s^\ell] \quad (\text{A.4})$$

hence:

$$\begin{Bmatrix} \gamma_{13}^{\ell+1} \\ \gamma_{23}^{\ell+1} \end{Bmatrix} = [\mathcal{A}^{\ell+1}] \begin{Bmatrix} \gamma_{13}^\ell \\ \gamma_{23}^\ell \end{Bmatrix} \quad (\text{A.5})$$

For clarity we denote  $\bar{u}_\alpha^\ell$  the mid-layer membrane displacements  $u_\alpha^\ell(z^\ell)$ . Equation (A.1) can then be written:

$$\begin{Bmatrix} \bar{u}_1^{\ell+1} \\ \bar{u}_2^{\ell+1} \end{Bmatrix} = \begin{Bmatrix} \bar{u}_1^\ell \\ \bar{u}_2^\ell \end{Bmatrix} + (z^\ell - z^{\ell+1}) \begin{Bmatrix} w_{,1}^1 \\ w_{,2}^1 \end{Bmatrix} - (z^\ell - \zeta^\ell) \begin{Bmatrix} \gamma_{13}^\ell \\ \gamma_{23}^\ell \end{Bmatrix} + (z^{\ell+1} - \zeta^\ell) [\mathcal{A}^{\ell+1}] \begin{Bmatrix} \gamma_{13}^\ell \\ \gamma_{23}^\ell \end{Bmatrix} \quad (\text{A.6})$$

And finally (minus signs are left to fit Woodcock's formulas):

$$\begin{Bmatrix} \bar{u}_1^{\ell+1} \\ \bar{u}_2^{\ell+1} \end{Bmatrix} = [\mathbf{I}] \begin{Bmatrix} \bar{u}_1^\ell \\ \bar{u}_2^\ell \end{Bmatrix} - [\mathcal{D}^{\ell+1}] \begin{Bmatrix} -\gamma_{13}^\ell \\ -\gamma_{23}^\ell \end{Bmatrix} + [\mathcal{B}^{\ell+1}] \begin{Bmatrix} w_1^1 \\ w_2^1 \end{Bmatrix} \quad (\text{A.7})$$

with:

$$[\mathcal{B}^{\ell+1}] = (z^\ell - z^{\ell+1})[\mathbf{I}] \quad \text{and} \quad [\mathcal{D}^{\ell+1}] = \frac{h^\ell}{2}[\mathbf{I}] + \frac{h^{\ell+1}}{2}[\mathcal{A}^{\ell+1}] \quad (\text{A.8})$$

Equations (A.4) and (A.7) can be written :

$$\begin{Bmatrix} \bar{u}_1^{\ell+1} \\ \bar{u}_2^{\ell+1} \\ -\gamma_{13}^{\ell+1} \\ -\gamma_{23}^{\ell+1} \\ w_1^{\ell+1} \\ w_2^{\ell+1} \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathcal{D}^{\ell+1} & \mathcal{B}^{\ell+1} \\ 0 & \mathcal{A}^{\ell+1} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \bar{u}_1^\ell \\ \bar{u}_2^\ell \\ -\gamma_{13}^\ell \\ -\gamma_{23}^\ell \\ w_1^\ell \\ w_2^\ell \end{Bmatrix} \quad (\text{A.9})$$

Let  $\Upsilon^\ell$  denote the vector involved in the right side of equation (A.9) and  $\Omega^{\ell+1}$  the corresponding block matrix:

$$\Upsilon^{\ell+1} = \Omega^{\ell+1} \Upsilon^\ell \quad (\text{A.10})$$

*Appendix A.2. Link between the layer  $\ell$  and the layer 1 (for  $\ell > 1$ )*

Let us try to compute a matrix which links the layer  $\ell$  and the layer 1:

$$\Upsilon^\ell = \Omega^\ell \Omega^{\ell-1} \dots \Omega^2 \Upsilon^1 = \hat{\Omega}^\ell \Upsilon^1 \quad (\text{A.11})$$

It is shown in the following lines that the matrix  $\hat{\Omega}^\ell$  can be written:

$$\hat{\Omega}^\ell = \begin{bmatrix} \mathbf{I} & -\mathcal{D}^\ell & \mathcal{B}^\ell \\ 0 & \mathfrak{A}^\ell & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \quad (\text{A.12})$$

where:

$$\mathfrak{A}^\ell = [\mathbf{C}_s^\ell]^{-1} [\mathbf{C}_s^1] \quad (\text{A.13a})$$

$$\mathcal{D}^\ell = \frac{h^1}{2} \mathfrak{A}^1 + \sum_{m=2}^{\ell-1} h^m \mathfrak{A}^m + \frac{h^\ell}{2} \mathfrak{A}^\ell \quad (\text{A.13b})$$

$$\mathcal{B}^\ell = (z^1 - z^\ell) \mathbf{I} = -z^\ell \mathbf{I} \quad (\text{A.13c})$$

Let us make the proof by recurrence. Setting  $\ell = 1$  into equations (A.4) and (A.8) leads to:

$$\mathfrak{A}^2 = [\mathcal{A}^2] = [\mathbf{C}_s^2]^{-1} [\mathbf{C}_s^1] \quad (\text{A.14a})$$

$$\mathcal{D}^2 = [\mathcal{D}^2] = \frac{h^1}{2} [\mathbf{I}] + \frac{h^2}{2} [\mathcal{A}^2] \quad (\text{A.14b})$$

$$\mathcal{B}^2 = [\mathcal{B}^2] = (z^1 - z^2) [\mathbf{I}] = -z^2 [\mathbf{I}] \quad (\text{A.14c})$$

Computing  $\hat{\Omega}^{\ell+1} = \Omega^{\ell+1} \hat{\Omega}^\ell$  leads to:

$$\hat{\Omega}^{\ell+1} = \begin{bmatrix} \mathbf{I} & -\mathcal{D}^\ell - \mathcal{D}^{\ell+1} \mathfrak{A}^\ell & \mathcal{B}^\ell + \mathcal{B}^{\ell+1} \\ 0 & \mathcal{A}^{\ell+1} \mathfrak{A}^\ell & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \quad (\text{A.15})$$

Computation of the three non trivial blocks shows that the formulas (A.13) work also for the layer  $\ell + 1$ ,

$$\begin{aligned}\mathfrak{A}^{\ell+1} &= \mathcal{A}^{\ell+1}\mathfrak{A}^\ell \\ &= [\mathbf{C}_s^{\ell+1}]^{-1}[\mathbf{C}_s^\ell][\mathbf{C}_s^\ell]^{-1}[\mathbf{C}_s^1] \\ &= [\mathbf{C}_s^{\ell+1}]^{-1}[\mathbf{C}_s^1]\end{aligned}\tag{A.16a}$$

$$\begin{aligned}\mathfrak{D}^{\ell+1} &= \mathfrak{D}^\ell + \mathcal{D}^{\ell+1}\mathfrak{A}^\ell \\ &= \frac{h^1}{2}\mathfrak{A}^1 + \sum_{m=2}^{\ell-1} h^m \mathfrak{A}^m + \frac{h^\ell}{2}\mathfrak{A}^\ell + \left(\frac{h^\ell}{2}\mathbf{I} + \frac{h^{\ell+1}}{2}\mathcal{A}^{\ell+1}\right)\mathfrak{A}^\ell \\ &= \frac{h^1}{2}\mathfrak{A}^1 + \sum_{m=2}^{\ell} h^m \mathfrak{A}^m + \frac{h^{\ell+1}}{2}\mathfrak{A}^{\ell+1}\end{aligned}\tag{A.16b}$$

$$\begin{aligned}\mathfrak{B}^{\ell+1} &= \mathfrak{B}^\ell + \mathcal{B}^{\ell+1} \\ &= (z^1 - z^\ell)\mathbf{I} + (z^\ell - z^{\ell+1})\mathbf{I} \\ &= (z^1 - z^{\ell+1})\mathbf{I} = -z^{\ell+1}\mathbf{I}\end{aligned}\tag{A.16c}$$

that ends the proof.

The  $\hat{\Omega}^\ell$  matrix contains terms which are here denoted in a slightly different manner than in Woodcock's work:

$$\hat{\Omega}^\ell = \begin{bmatrix} \mathbf{I} & -\mathfrak{D}^\ell & \mathfrak{B}^\ell \\ 0 & \mathfrak{A}^\ell & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \delta_{xx}^\ell & \delta_{xy}^\ell & \beta^\ell & 0 \\ 0 & 1 & \delta_{yx}^\ell & \delta_{yy}^\ell & 0 & \beta^\ell \\ 0 & 0 & \alpha_{xx}^\ell & \alpha_{xy}^\ell & 0 & 0 \\ 0 & 0 & \alpha_{yx}^\ell & \alpha_{yy}^\ell & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\tag{A.17}$$

This notation permits to write generic formulas like (C.6) for example. The correspondence with Woodcock's notation and the matrix in formula (14) of Woodcock's article [6] can be obtained following the rules:

- replace superscripts  $\ell$  by  $n - 1$  ( $n$  would have been a more judicious choice),
- replace  $\delta_{xx}^{n-1}$  by  $\gamma_x^{n-1}$  and  $\delta_{yy}^{n-1}$  by  $\gamma_y^{n-1}$ ,
- replace  $\alpha_{xx}^{n-1}$  by  $\alpha_x^{n-1}$  and  $\alpha_{yy}^{n-1}$  by  $\alpha_y^{n-1}$ ,
- makes appear a  $\beta_x^{n-1}$  and a  $\beta_y^{n-1}$  instead of the two (identical)  $\beta^{n-1}$ ,
- swap rows and columns of matrix following  $(1, 2, 3, 4, 5, 6) \rightarrow (5, 3, 1, 6, 4, 2)$

## Appendix B. Corrected lists of Woodcock's energies coefficients

### Appendix B.1. Important note

Below are listed the corrected 69 Woodcock's strain energy coefficients and the corrected 18 kinetic energy coefficients as they were defined in the article [6].

Computation has been done with the help of a symbolic computation software. The coefficients have been computed following Woodcock's procedure: once established that the transfer matrix of equation (A.17) could link each layer to the first one, strain and kinetic energies are expressed in function of the first layer displacement field. Integration of these strain and kinetic energies over the thickness of the plate leads to functionals given in reference [6] at formulas (24) and (18), and in Appendix B. Woodcock found that these functional has respectively 69 and 13 coefficients, some of them being equal. Reproducing Woodcock's procedure, the computation shows that mistakes occur in the given lists. A rapid examen of these lists shows that none of the coefficients exhibit a  $\delta_{xy}$  or a  $\alpha_{xy}$  term which are present in the transfer matrix of formula (14) of reference [6].

Another proof is given in this work, mainly in Appendix C, in which these same coefficients are computed from a different method involving the warping functions of formula (10).

### Appendix B.2. Preliminary remarks

Some preliminary remarks must be done at this time:

1. Energy coefficients has been given in Woodcock's work using the two index Voigt's notation for the fourth order stiffness tensor, so we choose to keep here this index convention. We shall recall here that the change is made according to the well known rule  $(11, 22, 33, 23, 13, 12) \leftrightarrow (1, 2, 3, 4, 5, 6)$ .
2. Note that in this document a  $\ell$  superscript has been chosen for the layer index instead of the  $n$  superscript in Woodcock's formulas. Note also that an awkward choice of numbering in Woodcock's work has lead to a mix of  $n$  and  $n - 1$  superscript which hides the true nature of inner terms of coefficients, which are simply "weights" for each layer into generalized stiffnesses or mass.
3. The density  $\mu^n$  of the  $n$ th layer in Woodcock's work has been changed with  $\rho^\ell$  in the following formulas.

Some equalities between coefficients are listed in equations (32) and (36). It can be verified that these relations obtained by symmetry considerations on the energies are confirmed in the lists below. Note that there remain only 55 and 15 independent coefficients for respectively strain and kinetic energies, in the most general case.

### Appendix B.3. Definition of coefficients $\lambda_i$

$$\begin{aligned}
\lambda_1 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (h^\ell)^3 + h^\ell (\beta^\ell)^2 \right) Q_{11}^\ell \right] \\
\lambda_2 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (\alpha_x^\ell)^2 (h^\ell)^3 + (\gamma_x^\ell)^2 h^\ell \right) Q_{11}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{yx}^\ell h^\ell \right) Q_{16}^\ell \right. \\
&\quad \left. + \left( \frac{1}{12} (\alpha_{yx}^\ell)^2 (h^\ell)^3 + (\delta_{yx}^\ell)^2 h^\ell \right) Q_{66}^\ell \right] \\
\lambda_3 &= \sum_{\ell=1}^n \left[ h^\ell Q_{11}^\ell \right] \\
\lambda_4 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell (h^\ell)^3 + 2\gamma_x^\ell h^\ell \beta^\ell \right) Q_{11}^\ell + \left( \frac{1}{6} \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{yx}^\ell h^\ell \beta^\ell \right) Q_{16}^\ell \right] \\
\lambda_5 &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{11}^\ell \right] \\
\lambda_6 &= \sum_{\ell=1}^n \left[ 2\gamma_x^\ell h^\ell Q_{11}^\ell + 2\delta_{yx}^\ell h^\ell Q_{16}^\ell \right] \\
\lambda_7 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (h^\ell)^3 + h^\ell (\beta^\ell)^2 \right) Q_{22}^\ell \right] \\
\lambda_8 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (\alpha_y^\ell)^2 (h^\ell)^3 + (\gamma_y^\ell)^2 h^\ell \right) Q_{22}^\ell + \left( \frac{1}{6} \alpha_y^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{xy}^\ell h^\ell \right) Q_{26}^\ell \right. \\
&\quad \left. + \left( \frac{1}{12} (\alpha_{xy}^\ell)^2 (h^\ell)^3 + (\delta_{xy}^\ell)^2 h^\ell \right) Q_{66}^\ell \right] \\
\lambda_9 &= \sum_{\ell=1}^n \left[ h^\ell Q_{22}^\ell \right] \\
\lambda_{10} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_y^\ell (h^\ell)^3 + 2\gamma_y^\ell h^\ell \beta^\ell \right) Q_{22}^\ell + \left( \frac{1}{6} \alpha_{xy}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell h^\ell \beta^\ell \right) Q_{26}^\ell \right]
\end{aligned}$$

$$\begin{aligned}
\lambda_{11} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{22}^\ell \right] \\
\lambda_{12} &= \sum_{\ell=1}^n \left[ 2\gamma_y^\ell h^\ell Q_{22}^\ell + 2\delta_{xy}^\ell h^\ell Q_{26}^\ell \right] \\
\lambda_{13} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} (h^\ell)^3 + 2h^\ell (\beta^\ell)^2 \right) Q_{12}^\ell \right] \\
\lambda_{14} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_y^\ell (h^\ell)^3 + 2\gamma_y^\ell h^\ell \beta^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} \alpha_{xy}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell h^\ell \beta^\ell \right) Q_{16}^\ell \right] \\
\lambda_{15} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{12}^\ell \right] \\
\lambda_{16} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell (h^\ell)^3 + 2\gamma_x^\ell h^\ell \beta^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{yx}^\ell h^\ell \beta^\ell \right) Q_{26}^\ell \right] \\
\lambda_{17} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell \alpha_y^\ell (h^\ell)^3 + 2\gamma_x^\ell \gamma_y^\ell h^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{xy}^\ell h^\ell \right) Q_{16}^\ell \right. \\
&\quad \left. + \left( \frac{1}{6} \alpha_y^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{yx}^\ell h^\ell \right) Q_{26}^\ell + \left( \frac{1}{6} \alpha_{xy}^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell \delta_{yx}^\ell h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{18} &= \sum_{\ell=1}^n \left[ 2\gamma_x^\ell h^\ell Q_{12}^\ell + 2\delta_{yx}^\ell h^\ell Q_{26}^\ell \right] \\
\lambda_{19} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{12}^\ell \right] \\
\lambda_{20} &= \sum_{\ell=1}^n \left[ 2\gamma_y^\ell h^\ell Q_{12}^\ell + 2\delta_{xy}^\ell h^\ell Q_{16}^\ell \right] \\
\lambda_{21} &= \sum_{\ell=1}^n \left[ 2h^\ell Q_{12}^\ell \right] \\
\lambda_{22} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} (h^\ell)^3 + 4h^\ell (\beta^\ell)^2 \right) Q_{66}^\ell \right] \\
\lambda_{23} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (\alpha_{yx}^\ell)^2 (h^\ell)^3 + (\delta_{yx}^\ell)^2 h^\ell \right) Q_{22}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{yx}^\ell h^\ell \right) Q_{26}^\ell \right. \\
&\quad \left. + \left( \frac{1}{12} (\alpha_x^\ell)^2 (h^\ell)^3 + (\gamma_x^\ell)^2 h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{24} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (\alpha_{xy}^\ell)^2 (h^\ell)^3 + (\delta_{xy}^\ell)^2 h^\ell \right) Q_{11}^\ell + \left( \frac{1}{6} \alpha_y^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{xy}^\ell h^\ell \right) Q_{16}^\ell \right. \\
&\quad \left. + \left( \frac{1}{12} (\alpha_y^\ell)^2 (h^\ell)^3 + (\gamma_y^\ell)^2 h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{25} &= \sum_{\ell=1}^n \left[ h^\ell Q_{66}^\ell \right] \\
\lambda_{26} &= \sum_{\ell=1}^n \left[ h^\ell Q_{66}^\ell \right] \\
\lambda_{27} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} \alpha_{yx}^\ell (h^\ell)^3 + 4\delta_{yx}^\ell h^\ell \beta^\ell \right) Q_{26}^\ell + \left( \frac{1}{3} \alpha_x^\ell (h^\ell)^3 + 4\gamma_x^\ell h^\ell \beta^\ell \right) Q_{66}^\ell \right]
\end{aligned}$$



$$\begin{aligned}
\lambda_{28} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} \alpha_{xy}^\ell (h^\ell)^3 + 4\delta_{xy}^\ell h^\ell \beta^\ell \right) Q_{16}^\ell + \left( \frac{1}{3} \alpha_y^\ell (h^\ell)^3 + 4\gamma_y^\ell h^\ell \beta^\ell \right) Q_{66}^\ell \right] \\
\lambda_{29} &= \sum_{\ell=1}^n \left[ 4h^\ell \beta^\ell Q_{66}^\ell \right] \\
\lambda_{30} &= \sum_{\ell=1}^n \left[ 4h^\ell \beta^\ell Q_{66}^\ell \right] \\
\lambda_{31} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{xy}^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell \delta_{yx}^\ell h^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{xy}^\ell h^\ell \right) Q_{16}^\ell \right. \\
&\quad \left. + \left( \frac{1}{6} \alpha_y^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{yx}^\ell h^\ell \right) Q_{26}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_y^\ell (h^\ell)^3 + 2\gamma_x^\ell \gamma_y^\ell h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{32} &= \sum_{\ell=1}^n \left[ 2\delta_{yx}^\ell h^\ell Q_{26}^\ell + 2\gamma_x^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{33} &= \sum_{\ell=1}^n \left[ 2\delta_{yx}^\ell h^\ell Q_{26}^\ell + 2\gamma_x^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{34} &= \sum_{\ell=1}^n \left[ 2\delta_{xy}^\ell h^\ell Q_{16}^\ell + 2\gamma_y^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{35} &= \sum_{\ell=1}^n \left[ 2\delta_{xy}^\ell h^\ell Q_{16}^\ell + 2\gamma_y^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{36} &= \sum_{\ell=1}^n \left[ 2h^\ell Q_{66}^\ell \right] \\
\lambda_{37} &= \sum_{\ell=1}^n \left[ (\alpha_{yx}^\ell)^2 h^\ell C_{44}^\ell + 2\alpha_x^\ell \alpha_{yx}^\ell h^\ell C_{45}^\ell + (\alpha_x^\ell)^2 h^\ell C_{55}^\ell \right] \\
\lambda_{38} &= \sum_{\ell=1}^n \left[ (\alpha_y^\ell)^2 h^\ell C_{44}^\ell + 2\alpha_y^\ell \alpha_{xy}^\ell h^\ell C_{45}^\ell + (\alpha_{xy}^\ell)^2 h^\ell C_{55}^\ell \right] \\
\lambda_{39} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} (h^\ell)^3 + 4h^\ell (\beta^\ell)^2 \right) Q_{16}^\ell \right] \\
\lambda_{40} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{yx}^\ell h^\ell \beta^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} \alpha_x^\ell (h^\ell)^3 + 2\gamma_x^\ell h^\ell \beta^\ell \right) Q_{16}^\ell \right] \\
\lambda_{41} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{16}^\ell \right] \\
\lambda_{42} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} \alpha_x^\ell (h^\ell)^3 + 4\gamma_x^\ell h^\ell \beta^\ell \right) Q_{16}^\ell + \left( \frac{1}{3} \alpha_{yx}^\ell (h^\ell)^3 + 4\delta_{yx}^\ell h^\ell \beta^\ell \right) Q_{66}^\ell \right] \\
\lambda_{43} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{yx}^\ell h^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} (\alpha_x^\ell)^2 (h^\ell)^3 + 2(\gamma_x^\ell)^2 h^\ell \right) Q_{16}^\ell \right. \\
&\quad \left. + \left( \frac{1}{6} (\alpha_{yx}^\ell)^2 (h^\ell)^3 + 2(\delta_{yx}^\ell)^2 h^\ell \right) Q_{26}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{yx}^\ell h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{44} &= \sum_{\ell=1}^n \left[ 2\gamma_x^\ell h^\ell Q_{16}^\ell + 2\delta_{yx}^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{45} &= \sum_{\ell=1}^n \left[ 4h^\ell \beta^\ell Q_{16}^\ell \right]
\end{aligned}$$

$$\begin{aligned}
\lambda_{46} &= \sum_{\ell=1}^n \left[ 2\delta_{yx}^\ell h^\ell Q_{12}^\ell + 2\gamma_x^\ell h^\ell Q_{16}^\ell \right] \\
\lambda_{47} &= \sum_{\ell=1}^n \left[ 2h^\ell Q_{16}^\ell \right] \\
\lambda_{48} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{xy}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell h^\ell \beta^\ell \right) Q_{11}^\ell + \left( \frac{1}{6} \alpha_y^\ell (h^\ell)^3 + 2\gamma_y^\ell h^\ell \beta^\ell \right) Q_{16}^\ell \right] \\
\lambda_{49} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{16}^\ell \right] \\
\lambda_{50} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{xy}^\ell h^\ell \right) Q_{11}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_y^\ell (h^\ell)^3 + \frac{1}{6} \alpha_{xy}^\ell \alpha_{yx}^\ell (h^\ell)^3 \right. \right. \\
&\quad \left. \left. + 2\gamma_x^\ell \gamma_y^\ell h^\ell + 2\delta_{xy}^\ell \delta_{yx}^\ell h^\ell \right) Q_{16}^\ell + \left( \frac{1}{6} \alpha_y^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{yx}^\ell h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{51} &= \sum_{\ell=1}^n \left[ 2\gamma_x^\ell h^\ell Q_{16}^\ell + 2\delta_{yx}^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{52} &= \sum_{\ell=1}^n \left[ 2\delta_{xy}^\ell h^\ell Q_{11}^\ell + 2\gamma_y^\ell h^\ell Q_{16}^\ell \right] \\
\lambda_{53} &= \sum_{\ell=1}^n \left[ 2h^\ell Q_{16}^\ell \right] \\
\lambda_{54} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} (h^\ell)^3 + 4h^\ell (\beta^\ell)^2 \right) Q_{26}^\ell \right] \\
\lambda_{55} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{yx}^\ell h^\ell \beta^\ell \right) Q_{22}^\ell + \left( \frac{1}{6} \alpha_x^\ell (h^\ell)^3 + 2\gamma_x^\ell h^\ell \beta^\ell \right) Q_{26}^\ell \right] \\
\lambda_{56} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{26}^\ell \right] \\
\lambda_{57} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{3} \alpha_y^\ell (h^\ell)^3 + 4\gamma_y^\ell h^\ell \beta^\ell \right) Q_{26}^\ell + \left( \frac{1}{3} \alpha_{xy}^\ell (h^\ell)^3 + 4\delta_{xy}^\ell h^\ell \beta^\ell \right) Q_{66}^\ell \right] \\
\lambda_{58} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_y^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{yx}^\ell h^\ell \right) Q_{22}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_y^\ell (h^\ell)^3 + \frac{1}{6} \alpha_{xy}^\ell \alpha_{yx}^\ell (h^\ell)^3 \right. \right. \\
&\quad \left. \left. + 2\gamma_x^\ell \gamma_y^\ell h^\ell + 2\delta_{xy}^\ell \delta_{yx}^\ell h^\ell \right) Q_{26}^\ell + \left( \frac{1}{6} \alpha_x^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{xy}^\ell h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{59} &= \sum_{\ell=1}^n \left[ 2\gamma_y^\ell h^\ell Q_{26}^\ell + 2\delta_{xy}^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{60} &= \sum_{\ell=1}^n \left[ 4h^\ell \beta^\ell Q_{26}^\ell \right] \\
\lambda_{61} &= \sum_{\ell=1}^n \left[ 2\delta_{yx}^\ell h^\ell Q_{22}^\ell + 2\gamma_x^\ell h^\ell Q_{26}^\ell \right] \\
\lambda_{62} &= \sum_{\ell=1}^n \left[ 2h^\ell Q_{26}^\ell \right] \\
\lambda_{63} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{xy}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell h^\ell \beta^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} \alpha_y^\ell (h^\ell)^3 + 2\gamma_y^\ell h^\ell \beta^\ell \right) Q_{26}^\ell \right]
\end{aligned}$$

$$\begin{aligned}
\lambda_{64} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell Q_{26}^\ell \right] \\
\lambda_{65} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_y^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{xy}^\ell h^\ell \right) Q_{12}^\ell + \left( \frac{1}{6} (\alpha_{xy}^\ell)^2 (h^\ell)^3 + 2(\delta_{xy}^\ell)^2 h^\ell \right) Q_{16}^\ell \right. \\
&\quad \left. + \left( \frac{1}{6} (\alpha_y^\ell)^2 (h^\ell)^3 + 2(\gamma_y^\ell)^2 h^\ell \right) Q_{26}^\ell + \left( \frac{1}{6} \alpha_y^\ell \alpha_{xy}^\ell (h^\ell)^3 + 2\gamma_y^\ell \delta_{xy}^\ell h^\ell \right) Q_{66}^\ell \right] \\
\lambda_{66} &= \sum_{\ell=1}^n \left[ 2\gamma_y^\ell h^\ell Q_{26}^\ell + 2\delta_{xy}^\ell h^\ell Q_{66}^\ell \right] \\
\lambda_{67} &= \sum_{\ell=1}^n \left[ 2\delta_{xy}^\ell h^\ell Q_{12}^\ell + 2\gamma_y^\ell h^\ell Q_{26}^\ell \right] \\
\lambda_{68} &= \sum_{\ell=1}^n \left[ 2h^\ell Q_{26}^\ell \right] \\
\lambda_{69} &= \sum_{\ell=1}^n \left[ 2\alpha_y^\ell \alpha_{yx}^\ell h^\ell C_{44}^\ell + (2\alpha_x^\ell \alpha_y^\ell h^\ell + 2\alpha_{xy}^\ell \alpha_{yx}^\ell h^\ell) C_{45}^\ell + 2\alpha_x^\ell \alpha_{xy}^\ell h^\ell C_{55}^\ell \right]
\end{aligned} \tag{B.1}$$

Appendix B.4. Definition of coefficients  $\delta_i$

$$\begin{aligned}
\delta_1 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (h^\ell)^3 + h^\ell (\beta^\ell)^2 \right) \rho^\ell \right] \\
\delta_2 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (\alpha_x^\ell)^2 (h^\ell)^3 + \frac{1}{12} (\alpha_{yx}^\ell)^2 (h^\ell)^3 + (\gamma_x^\ell)^2 h^\ell + (\delta_{yx}^\ell)^2 h^\ell \right) \rho^\ell \right] \\
\delta_3 &= \sum_{\ell=1}^n \left[ h^\ell \rho^\ell \right] \\
\delta_4 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell (h^\ell)^3 + 2\gamma_x^\ell h^\ell \beta^\ell \right) \rho^\ell \right] \\
\delta_5 &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell \rho^\ell \right] \\
\delta_6 &= \sum_{\ell=1}^n \left[ 2\gamma_x^\ell h^\ell \rho^\ell \right] \\
\delta_7 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (h^\ell)^3 + h^\ell (\beta^\ell)^2 \right) \rho^\ell \right] \\
\delta_8 &= \sum_{\ell=1}^n \left[ \left( \frac{1}{12} (\alpha_y^\ell)^2 (h^\ell)^3 + \frac{1}{12} (\alpha_{xy}^\ell)^2 (h^\ell)^3 + (\gamma_y^\ell)^2 h^\ell + (\delta_{xy}^\ell)^2 h^\ell \right) \rho^\ell \right] \\
\delta_9 &= \sum_{\ell=1}^n \left[ h^\ell \rho^\ell \right] \\
\delta_{10} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_y^\ell (h^\ell)^3 + 2\gamma_y^\ell h^\ell \beta^\ell \right) \rho^\ell \right] \\
\delta_{11} &= \sum_{\ell=1}^n \left[ 2h^\ell \beta^\ell \rho^\ell \right]
\end{aligned}$$

$$\begin{aligned}
\delta_{12} &= \sum_{\ell=1}^n \left[ 2\gamma_y^\ell h^\ell \rho^\ell \right] \\
\delta_{13} &= \sum_{\ell=1}^n \left[ h^\ell \rho^\ell \right] \\
\delta_{14} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_x^\ell \alpha_{xy}^\ell (h^\ell)^3 + \frac{1}{6} \alpha_y^\ell \alpha_{yx}^\ell (h^\ell)^3 + 2\gamma_x^\ell \delta_{xy}^\ell h^\ell + 2\gamma_y^\ell \delta_{yx}^\ell h^\ell \right) \rho^\ell \right] \\
\delta_{15} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{xy}^\ell (h^\ell)^3 + 2\delta_{xy}^\ell h^\ell \beta^\ell \right) \rho^\ell \right] \\
\delta_{16} &= \sum_{\ell=1}^n \left[ \left( \frac{1}{6} \alpha_{yx}^\ell (h^\ell)^3 + 2\delta_{yx}^\ell h^\ell \beta^\ell \right) \rho^\ell \right] \\
\delta_{17} &= \sum_{\ell=1}^n \left[ 2\delta_{yx}^\ell h^\ell \rho^\ell \right] \\
\delta_{18} &= \sum_{\ell=1}^n \left[ 2\delta_{xy}^\ell h^\ell \rho^\ell \right]
\end{aligned} \tag{B.2}$$

## Appendix C. Proof of the link between behavior and mass matrices and Woodcock's coefficients

### Appendix C.1. Preliminary results

We recall here that the height of each layer is written  $h^\ell = \zeta^\ell - \zeta^{\ell-1}$  and that with the help of formulas (A.16c) and (A.17) it can be seen that  $(\zeta^\ell + \zeta^{\ell-1})/2 = z^\ell = -\beta^\ell$  is the layer mid-plane elevation. Let us first compute:

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} dz = h^\ell \tag{C.1a}$$

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} z dz = \frac{1}{2} ((\zeta^\ell)^2 - (\zeta^{\ell-1})^2) = h^\ell z^\ell = h^\ell (-\beta^\ell) \tag{C.1b}$$

$$\begin{aligned}
\int_{\zeta^{\ell-1}}^{\zeta^\ell} z^2 dz &= \frac{1}{3} ((\zeta^\ell)^3 - (\zeta^{\ell-1})^3) = \frac{1}{12} ((\zeta^\ell)^3 - (\zeta^{\ell-1})^3) + \frac{1}{4} ((\zeta^\ell)^3 - (\zeta^{\ell-1})^3) \\
&= \frac{1}{12} (\zeta^\ell - \zeta^{\ell-1})^3 + \frac{1}{4} ((\zeta^\ell)^3 - (\zeta^{\ell-1})^3 + (\zeta^\ell)^2 \zeta^{\ell-1} - \zeta^\ell (\zeta^{\ell-1})^2) \\
&= \frac{1}{12} (h^\ell)^3 + h^\ell (z^\ell)^2 = \frac{1}{12} (h^\ell)^3 + h^\ell (\beta^\ell)^2
\end{aligned} \tag{C.1c}$$

These above three formulas help to compute:

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} (z - z^\ell) dz = h^\ell (z^\ell - z^\ell) = 0 \tag{C.2a}$$

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} (z - z^\ell) z dz = \frac{1}{12} (h^\ell)^3 + h^\ell (z^\ell)^2 - h^\ell (z^\ell)^2 = \frac{1}{12} (h^\ell)^3 \tag{C.2b}$$

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} (z - z^\ell)^2 dz = \frac{1}{12} (h^\ell)^3 + h^\ell (z^\ell)^2 - 2h^\ell (z^\ell)^2 + (z^\ell)^2 h^\ell = \frac{1}{12} (h^\ell)^3 \tag{C.2c}$$

Another important preliminary result consists in the transformation of the following integral on a sum over the layers, written in a particular manner which leads to easier calculus on the following steps:

$$\int_0^z S_{\alpha_3\gamma_3}(\zeta) d\zeta = S_{\alpha_3\gamma_3}^1(\zeta^1 - 0) + \sum_{m=2}^{\ell-1} S_{\alpha_3\gamma_3}^m(\zeta^m - \zeta^{m-1}) + S_{\alpha_3\gamma_3}^\ell(z - \zeta^{\ell-1})$$

$$= S_{\alpha_3\gamma_3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\alpha_3\gamma_3}^m h^m + S_{\alpha_3\gamma_3}^\ell \frac{h^\ell}{2} + S_{\alpha_3\gamma_3}^\ell (z - z^\ell) \quad (\text{C.3})$$

*Appendix C.2. Warping function integrals*

In this section, for clarity, the notations of equation (A.17) for Woodcock's coefficients are used. The link with Woodcock's notations is explained just after equation (A.17).

*Computation of the  $E_{\alpha\beta\gamma\delta}$ .* It requires the calculus of:

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) dz = 4C_{\gamma_3\beta_3}(z^1) \int_{\zeta^{\ell-1}}^{\zeta^\ell} \left( \int_0^z S_{\alpha_3\gamma_3}(\zeta) d\zeta \right) dz \quad (\text{C.4})$$

according to preliminary results of formulas (C.1a), (C.2a) and (C.3):

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) dz = 4C_{\gamma_3\beta_3}(z^1) \left( S_{\alpha_3\gamma_3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\alpha_3\gamma_3}^m h^m + S_{\alpha_3\gamma_3}^\ell \frac{h^\ell}{2} \right) h^\ell \quad (\text{C.5})$$

One can see now with the help of Appendix A that the integrals of the warping functions over the layer  $\ell$  are linked with Woodcock's coefficients  $\delta_{\alpha\beta}^\ell$ :

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) dz = -\delta_{\alpha\beta}^\ell h^\ell \quad (\text{C.6})$$

*Computation of the  $F_{\alpha\beta\gamma\delta}$ .* It requires the calculus of:

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) z dz = 4C_{\gamma_3\beta_3}(z^1) \int_{\zeta^{\ell-1}}^{\zeta^\ell} \left( \int_0^z S_{\alpha_3\gamma_3}(\zeta) d\zeta \right) z dz \quad (\text{C.7})$$

according to preliminary results of formulas (C.1b), (C.2b) and (C.3):

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) z dz = 4C_{\gamma_3\beta_3}(z^1) \left[ \left( S_{\alpha_3\gamma_3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\alpha_3\gamma_3}^m h^m + S_{\alpha_3\gamma_3}^\ell \frac{h^\ell}{2} \right) (-h^\ell \beta^\ell) + S_{\alpha_3\gamma_3}^\ell \frac{(h^\ell)^3}{12} \right] \quad (\text{C.8})$$

One can see now with the help of Appendix A that the first moments of the warping functions over the layer  $\ell$  are linked with Woodcock's coefficients  $\delta_{\alpha\beta}^\ell$ ,  $\beta^\ell$  and  $\alpha_{\alpha\beta}^\ell$ :

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) z dz = \delta_{\alpha\beta}^\ell \beta^\ell h^\ell + \alpha_{\alpha\beta}^\ell \frac{(h^\ell)^3}{12} \quad (\text{C.9})$$

*Computation of the  $G_{\alpha\beta\gamma\delta}$ .* It requires the calculus of:

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) \varphi_{\gamma\delta}(z) dz = 16C_{\lambda_3\beta_3}(z^1) C_{\mu_3\delta_3}(z^1) \int_{\zeta^{\ell-1}}^{\zeta^\ell} \left( \int_0^z S_{\alpha_3\lambda_3}(\zeta) d\zeta \int_0^z S_{\gamma_3\mu_3}(\zeta) d\zeta \right) dz \quad (\text{C.10})$$

according to preliminary results of formulas (C.1a), (C.2a), (C.2c) and (C.3):

$$\begin{aligned} \int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) \varphi_{\gamma\delta}(z) dz = & 16C_{\lambda_3\beta_3}(z^1) C_{\mu_3\delta_3}(z^1) \int_{\zeta^{\ell-1}}^{\zeta^\ell} \left[ \right. \\ & \left. \left( S_{\alpha_3\lambda_3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\alpha_3\lambda_3}^m h^m + S_{\alpha_3\lambda_3}^\ell \frac{h^\ell}{2} \right) \left( S_{\gamma_3\mu_3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\gamma_3\mu_3}^m h^m + S_{\gamma_3\mu_3}^\ell \frac{h^\ell}{2} \right) \right. \\ & \left. + \left( S_{\alpha_3\lambda_3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\alpha_3\lambda_3}^m h^m + S_{\alpha_3\lambda_3}^\ell \frac{h^\ell}{2} \right) S_{\gamma_3\mu_3}^\ell (z - z^\ell) \right] \end{aligned}$$

$$\begin{aligned}
& + \left( S_{\gamma 3 \mu 3}^1 \frac{h^1}{2} + \sum_{m=2}^{\ell-1} S_{\gamma 3 \mu 3}^m h^m + S_{\gamma 3 \mu 3}^\ell \frac{h^\ell}{2} \right) S_{\alpha 3 \lambda 3}^\ell (z - z^\ell) \\
& + S_{\gamma 3 \mu 3}^\ell S_{\alpha 3 \lambda 3}^\ell (z - z^\ell)^2 \Big] dz
\end{aligned} \tag{C.11}$$

One can see now with the help of [Appendix A](#) that this integral over the layer  $\ell$  is linked with Woodcock's coefficients  $\delta_{\alpha\beta}^\ell$  and  $\alpha_{\alpha\beta}^\ell$ :

$$\int_{\zeta^{\ell-1}}^{\zeta^\ell} \varphi_{\alpha\beta}(z) \varphi_{\gamma\delta}(z) dz = \delta_{\alpha\beta} \delta_{\gamma\delta} h^\ell + \alpha_{\alpha\beta} \alpha_{\gamma\delta} \frac{(h^\ell)^3}{12} \tag{C.12}$$

*Appendix C.3. Computation of matrices [A], [B], [D], [E], [F], [G], [H] and [Ξ]*

*Matrices [A], [B] and [D].* The computation of components of these matrices is classical. Formula (17) shows that results can be written directly in a matrix form:

$$\{[\mathbf{A}], [\mathbf{B}], [\mathbf{D}]\} = \int_{\zeta^0}^{\zeta^n} [\mathbf{Q}]\{1, z, z^2\} dz = \sum_{\ell=1}^n [\mathbf{Q}]^\ell \left\{ h^\ell, -h^\ell \beta^\ell, \frac{(h^\ell)^3}{12} + h^\ell (\beta^\ell)^2 \right\} \tag{C.13}$$

One can see that these formulas corresponds to Woodcock's coefficients  $\lambda_i$  with  $i \in (3, 9, 21, 25, 47, 62)$  for  $[\mathbf{A}]$ , to  $-\lambda_i$  with  $i \in (5, 11, 15, 29, 45, 60)$  for  $[\mathbf{B}]$  and to  $\lambda_i$  with  $i \in (1, 7, 13, 22, 39, 54)$  for  $[\mathbf{D}]$ . Note that some Woodcock's coefficients must be divided by 2 or by 4 according to the correspondence shown in formula (30).

*Matrix [E].* Formulas (17) and (C.6) gives:

$$E_{\alpha\beta\mu\delta} = \int_{\zeta^0}^{\zeta^n} Q_{\alpha\beta\gamma\delta} \varphi_{\gamma\mu}(z) dz = - \sum_{\ell=1}^n Q_{\alpha\beta\gamma\delta}^\ell \delta_{\gamma\mu}^\ell h^\ell \tag{C.14}$$

This formula gives Woodcock's coefficients  $-\lambda_i$  with  $i \in (6, 12, 18, 20, 32, 34, 44, 46, 52, 59, 61, 67)$ . Let us finish the demonstration for some of them, changing the notations on the fly to fit Woodcock's coefficients:

$$E_{1111} = - \sum_{\ell=1}^n (Q_{1111}^\ell \delta_{11}^\ell + Q_{1121}^\ell \delta_{21}^\ell) h^\ell = - \sum_{\ell=1}^n (Q_{11}^\ell \gamma_x^\ell + Q_{16}^\ell \delta_{yx}^\ell) h^\ell = - \frac{\lambda_6}{2} \tag{C.15a}$$

$$E_{2211} = - \sum_{\ell=1}^n (Q_{2211}^\ell \delta_{11}^\ell + Q_{2221}^\ell \delta_{21}^\ell) h^\ell = - \sum_{\ell=1}^n (Q_{12}^\ell \gamma_x^\ell + Q_{26}^\ell \delta_{yx}^\ell) h^\ell = - \frac{\lambda_{18}}{2} \tag{C.15b}$$

$$E_{1122} = - \sum_{\ell=1}^n (Q_{1112}^\ell \delta_{12}^\ell + Q_{1122}^\ell \delta_{22}^\ell) h^\ell = - \sum_{\ell=1}^n (Q_{16}^\ell \delta_{xy}^\ell + Q_{12}^\ell \gamma_y^\ell) h^\ell = - \frac{\lambda_{20}}{2} \tag{C.15c}$$

$$E_{1221} = - \sum_{\ell=1}^n (Q_{1211}^\ell \delta_{12}^\ell + Q_{1221}^\ell \delta_{22}^\ell) h^\ell = - \sum_{\ell=1}^n (Q_{16}^\ell \delta_{xy}^\ell + Q_{66}^\ell \gamma_y^\ell) h^\ell = - \frac{\lambda_{34}}{2} \tag{C.15d}$$

*Matrix [F].* Formulas (17) and (C.9) gives:

$$F_{\alpha\beta\mu\delta} = \int_{\zeta^0}^{\zeta^n} Q_{\alpha\beta\gamma\delta} \varphi_{\gamma\mu}(z) z dz = \sum_{\ell=1}^n Q_{\alpha\beta\gamma\delta}^\ell \left( \delta_{\gamma\mu}^\ell \beta^\ell h^\ell + \alpha_{\gamma\mu}^\ell \frac{(h^\ell)^3}{12} \right) \tag{C.16}$$

This formula gives Woodcock's coefficients  $\lambda_i$  with  $i \in (4, 10, 14, 16, 27, 28, 40, 42, 48, 55, 57, 63)$ . Let us finish the demonstration for some of them, changing the notations on the fly to fit Woodcock's coefficients:

$$\begin{aligned}
F_{2222} &= \sum_{\ell=1}^n \left[ Q_{2212}^\ell \left( \delta_{12}^\ell \beta^\ell h^\ell + \alpha_{12}^\ell \frac{(h^\ell)^3}{12} \right) + Q_{2222}^\ell \left( \delta_{22}^\ell \beta^\ell h^\ell + \alpha_{22}^\ell \frac{(h^\ell)^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{26}^\ell \left( \delta_{xy}^\ell \beta^\ell h^\ell + \alpha_{xy}^\ell \frac{(h^\ell)^3}{12} \right) + Q_{22}^\ell \left( \gamma_y^\ell \beta^\ell h^\ell + \alpha_y^\ell \frac{(h^\ell)^3}{12} \right) \right] = \frac{\lambda_{10}}{2}
\end{aligned} \tag{C.17a}$$

$$\begin{aligned}
F_{1122} &= \sum_{\ell=1}^n \left[ Q_{1112}^{\ell} \left( \delta_{12}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{12}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{1122}^{\ell} \left( \delta_{22}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{22}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{16}^{\ell} \left( \delta_{xy}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{xy}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{12}^{\ell} \left( \gamma_y^{\ell} \beta^{\ell} h^{\ell} + \alpha_y^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] = \frac{\lambda_{14}}{2}
\end{aligned} \tag{C.17b}$$

$$\begin{aligned}
F_{1121} &= \sum_{\ell=1}^n \left[ Q_{1111}^{\ell} \left( \delta_{12}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{12}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{2221}^{\ell} \left( \delta_{22}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{22}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{11}^{\ell} \left( \delta_{xy}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{xy}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{16}^{\ell} \left( \gamma_y^{\ell} \beta^{\ell} h^{\ell} + \alpha_y^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] = \frac{\lambda_{48}}{2}
\end{aligned} \tag{C.17c}$$

$$\begin{aligned}
F_{2212} &= \sum_{\ell=1}^n \left[ Q_{2212}^{\ell} \left( \delta_{11}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{11}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{2222}^{\ell} \left( \delta_{21}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{21}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{26}^{\ell} \left( \gamma_x^{\ell} \beta^{\ell} h^{\ell} + \alpha_x^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{22}^{\ell} \left( \delta_{yx}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{yx}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] = \frac{\lambda_{55}}{2}
\end{aligned} \tag{C.17d}$$

Matrix  $[\mathbf{G}]$ . Formulas (17) and (C.12) gives:

$$G_{\nu\beta\mu\delta} = \int_{\zeta^0}^{\zeta^n} Q_{\alpha\beta\gamma\delta} \varphi_{\alpha\nu}(z) \varphi_{\gamma\mu}(z) dz = \sum_{\ell=1}^n Q_{\alpha\beta\gamma\delta} \left( \delta_{\alpha\nu} \delta_{\gamma\mu} h^{\ell} + \alpha_{\alpha\nu} \alpha_{\gamma\mu} \frac{(h^{\ell})^3}{12} \right) \tag{C.18}$$

This formula gives Woodcock's coefficients  $\lambda_i$  with  $i \in (2, 8, 17, 23, 24, 31, 43, 50, 58, 65)$ . Let us finish the demonstration for some of them, changing the notations on the fly to fit Woodcock's coefficients:

$$\begin{aligned}
G_{1111} &= \sum_{\ell=1}^n \left[ Q_{1111}^{\ell} \left( \delta_{11}^{\ell} \delta_{11}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{11}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{1121}^{\ell} \left( \delta_{11}^{\ell} \delta_{21}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{21}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{2111}^{\ell} \left( \delta_{21}^{\ell} \delta_{11}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{11}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{2121}^{\ell} \left( \delta_{21}^{\ell} \delta_{21}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{21}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{11}^{\ell} \left( (\gamma_x^{\ell})^2 h^{\ell} + (\alpha_x^{\ell})^2 \frac{(h^{\ell})^3}{12} \right) + 2Q_{16}^{\ell} \left( \gamma_x^{\ell} \delta_{yx}^{\ell} h^{\ell} + \alpha_x^{\ell} \alpha_{yx}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{66}^{\ell} \left( (\delta_{yx}^{\ell})^2 h^{\ell} + (\alpha_{yx}^{\ell})^2 \frac{(h^{\ell})^3}{12} \right) \right] = \lambda_2
\end{aligned} \tag{C.19a}$$

$$\begin{aligned}
G_{1212} &= \sum_{\ell=1}^n \left[ Q_{1212}^{\ell} \left( \delta_{11}^{\ell} \delta_{11}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{11}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{1222}^{\ell} \left( \delta_{11}^{\ell} \delta_{21}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{21}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{2212}^{\ell} \left( \delta_{21}^{\ell} \delta_{11}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{11}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{2222}^{\ell} \left( \delta_{21}^{\ell} \delta_{21}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{21}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{66}^{\ell} \left( (\gamma_x^{\ell})^2 h^{\ell} + (\alpha_x^{\ell})^2 \frac{(h^{\ell})^3}{12} \right) + 2Q_{26}^{\ell} \left( \gamma_x^{\ell} \delta_{yx}^{\ell} h^{\ell} + \alpha_x^{\ell} \alpha_{yx}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{22}^{\ell} \left( (\delta_{yx}^{\ell})^2 h^{\ell} + (\alpha_{yx}^{\ell})^2 \frac{(h^{\ell})^3}{12} \right) \right] = \lambda_{23}
\end{aligned} \tag{C.19b}$$

$$\begin{aligned}
G_{1221} &= \sum_{\ell=1}^n \left[ Q_{1211}^{\ell} \left( \delta_{11}^{\ell} \delta_{12}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{12}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{1221}^{\ell} \left( \delta_{11}^{\ell} \delta_{22}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{22}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{2211}^{\ell} \left( \delta_{21}^{\ell} \delta_{12}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{12}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{2221}^{\ell} \left( \delta_{21}^{\ell} \delta_{22}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{22}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{16}^{\ell} \left( \gamma_x^{\ell} \delta_{xy}^{\ell} h^{\ell} + \alpha_x^{\ell} \alpha_{xy}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{66}^{\ell} \left( \gamma_x^{\ell} \gamma_y^{\ell} h^{\ell} + \alpha_x^{\ell} \alpha_y^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{12}^{\ell} \left( \delta_{yx}^{\ell} \delta_{xy}^{\ell} h^{\ell} + \alpha_{yx}^{\ell} \alpha_{xy}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{26}^{\ell} \left( \delta_{yx}^{\ell} \gamma_y^{\ell} h^{\ell} + \alpha_{yx}^{\ell} \alpha_y^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] = \frac{\lambda_{31}}{2}
\end{aligned} \tag{C.19c}$$

$$\begin{aligned}
G_{1121} &= \sum_{\ell=1}^n \left[ Q_{1111}^{\ell} \left( \delta_{11}^{\ell} \delta_{12}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{12}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{1121}^{\ell} \left( \delta_{11}^{\ell} \delta_{22}^{\ell} h^{\ell} + \alpha_{11}^{\ell} \alpha_{22}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + Q_{2111}^{\ell} \left( \delta_{21}^{\ell} \delta_{12}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{12}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{2121}^{\ell} \left( \delta_{21}^{\ell} \delta_{22}^{\ell} h^{\ell} + \alpha_{21}^{\ell} \alpha_{22}^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] \\
&= \sum_{\ell=1}^n \left[ Q_{11}^{\ell} \left( \gamma_x^{\ell} \delta_{xy}^{\ell} h^{\ell} + \alpha_x^{\ell} \alpha_{xy}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{16}^{\ell} \left( \gamma_x^{\ell} \gamma_y^{\ell} h^{\ell} + \alpha_x^{\ell} \alpha_y^{\ell} \frac{(h^{\ell})^3}{12} \right) \right. \\
&\quad \left. + \delta_{yx}^{\ell} \delta_{xy}^{\ell} h^{\ell} + \alpha_{yx}^{\ell} \alpha_{xy}^{\ell} \frac{(h^{\ell})^3}{12} \right) + Q_{66}^{\ell} \left( \delta_{yx}^{\ell} \gamma_y^{\ell} h^{\ell} + \alpha_{yx}^{\ell} \alpha_y^{\ell} \frac{(h^{\ell})^3}{12} \right) \right] = \frac{\lambda_{50}}{2} \tag{C.19d}
\end{aligned}$$

Note the specific structure of these coefficients, some of them involving 3 different stiffness tensor components (with also different coefficients structures) like  $\lambda_{23}$  and  $\lambda_{50}$ , and some of them involving 4 different stiffness tensor components, like  $\lambda_{31}$ .

*Matrix [H].* Equations (17), (9) and (A.16a) gives:

$$H_{\alpha 3\beta 3} = \int_{\zeta^0}^{\zeta^n} \varphi'_{\gamma\alpha}(z) C_{\gamma 3\delta 3} \varphi'_{\delta\beta}(z) dz = \sum_{\ell=1}^n C_{\gamma 3\delta 3}^{\ell} \alpha_{\gamma\alpha}^{\ell} \alpha_{\delta\beta}^{\ell} h^{\ell} \tag{C.20}$$

This formula gives Woodcock's coefficients  $\lambda_i$  with  $i \in \{37, 38, 69\}$ . Let us finish the demonstration for the  $\lambda_{69}$ :

$$\begin{aligned}
H_{1323} &= \sum_{\ell=1}^n \left[ C_{1313}^{\ell} (\alpha_{11}^{\ell} \alpha_{12}^{\ell} h^{\ell}) + C_{2313}^{\ell} (\alpha_{21}^{\ell} \alpha_{12}^{\ell} h^{\ell}) + C_{1323}^{\ell} (\alpha_{11}^{\ell} \alpha_{22}^{\ell} h^{\ell}) + C_{2323}^{\ell} (\alpha_{21}^{\ell} \alpha_{22}^{\ell} h^{\ell}) \right] \\
&= \sum_{\ell=1}^n \left[ C_{55}^{\ell} \alpha_x^{\ell} \alpha_{xy}^{\ell} h^{\ell} + C_{45}^{\ell} (\alpha_{yx}^{\ell} \alpha_{xy}^{\ell} + \alpha_x^{\ell} \alpha_y^{\ell}) h^{\ell} + C_{44}^{\ell} \alpha_{yx}^{\ell} \alpha_y^{\ell} h^{\ell} \right] = \frac{\lambda_{69}}{2} \tag{C.21}
\end{aligned}$$

*Matrix [Ξ].* Equations (24), (C.1a), (C.1b), (C.1c), (C.6), (C.9) and (C.12) permits to write:

$$R = \int_{\zeta^0}^{\zeta^n} \rho(z) dz = \sum_{\ell=1}^n \rho^{\ell} h^{\ell} = \delta_3 = \delta_9 = \delta_{13} \tag{C.22}$$

$$S = \int_{\zeta^0}^{\zeta^n} \rho(z) z dz = - \sum_{\ell=1}^n \rho^{\ell} \beta^{\ell} h^{\ell} = -\delta_5 = -\delta_{11} \tag{C.23}$$

$$T = \int_{\zeta^0}^{\zeta^n} \rho(z) z^2 dz = \sum_{\ell=1}^n \rho^{\ell} \left( \frac{1}{12} (h^{\ell})^3 + h^{\ell} (\beta^{\ell})^2 \right) = \delta_1 = \delta_7 \tag{C.24}$$

$$U_{\alpha\beta} = \int_{\zeta^0}^{\zeta^n} \rho(z) \varphi_{\alpha\beta} dz = - \sum_{\ell=1}^n \rho^{\ell} \delta_{\alpha\beta}^{\ell} h^{\ell} \quad \text{are the } -\frac{\delta_i}{2} \text{ for } i \in \{6, 12, 17, 18\} \tag{C.25}$$

$$V_{\alpha\beta} = \int_{\zeta^0}^{\zeta^n} \rho(z) \varphi_{\alpha\beta} z dz = \sum_{\ell=1}^n \rho^{\ell} \left( \delta_{\alpha\beta}^{\ell} \beta^{\ell} h^{\ell} + \alpha_{\alpha\beta}^{\ell} \frac{(h^{\ell})^3}{12} \right) \quad \text{are the } \frac{\delta_i}{2} \text{ for } i \in \{4, 10, 15, 16\} \tag{C.26}$$

$$W_{\alpha\beta} = \int_{\zeta^0}^{\zeta^n} \rho(z) \varphi_{\mu\alpha} \varphi_{\mu\beta} dz = \sum_{\ell=1}^n \rho^{\ell} \left( \delta_{\mu\alpha} \delta_{\mu\beta} h^{\ell} + \alpha_{\mu\alpha} \alpha_{\mu\beta} \frac{(h^{\ell})^3}{12} \right) \quad \text{are the } \delta_i \text{ for } i \in \{2, 8\} \text{ and } \frac{\lambda_{14}}{2} \tag{C.27}$$