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## Quasi-recognizable vs MSO definable languages of one-dimensional overlapping tiles

February 19, 2012

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# Quasi-recognizable vs MSO definable languages of one-dimensional overlapping tiles

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## Abstract

It has been shown [6] that, within McAlister inverse monoid [8], whose element can be seen as overlapping one-dimensional tiles, the class of languages recognizable by finite monoids is fairly smaller than the class of languages definable in Monadic Second Order Logic (MSO). In this paper, we aim at capturing the expressive power of MSO definability of languages of tiles by means of a weakening of the notion of algebraic recognizability we call quasi-recognizability.

For that purpose, since the collapse of algebraic recognizability is intrinsically linked with the notion of monoid homomorphisms itself, we consider instead prehomomorphisms. Introduced by McAlister and Reilly [10] these are monotonic mappings over ordered monoids that are only required to be sub-multiplicative with respect to the monoid product, i.e. mapping  $\varphi$  such that for all  $x$  and  $y$ ,  $\varphi(xy) \leq \varphi(x)\varphi(y)$ .

Doing so, we indeed obtain, with some extra but rather natural closure conditions, the expected quasi-algebraic characterization of MSO definable languages of positive tiles. This result is achieved via a non trivial embedding of any (finite) monoid  $S$  into a (sufficiently well-behaved) ordered monoid  $\mathcal{Q}(S)$ .

## 1 Introduction

One-dimensional overlapping tiles, defined years ago in inverse semigroup theory as elements of McAlister inverse monoid  $M_A$  [8], seem to convey a considerable but yet quite unexploited modeling potential in computer application fields (see e.g. [5] for application in music modeling).

Indeed, despite a rich algebraic structure - overlapping tiles are equipped with a product that allow the modeling of unbounded contextual compatibility constraints - it has been shown [6] that the class of Monadic Second Order

(MSO) definable languages of such tiles is both *robust* - it is closed under product, iterated products, residual and inverse operations - and *simple* - it essentially consists of finite sums of cartesian products of regular languages of finite words.

How strongly languages of tiles and languages of words can be related? The class of regular languages of words is known to be especially robust. It has many equivalent characterizations from finite state (or finite monoid) recognizability up to logical definability to mention just a few.

In the case of languages of tiles however, it has been shown [6] that recognizability by finite monoid collapses. Indeed, the rich structure of McAlister monoid, transferred via morphisms to finite monoids, induces far too strong constraints for preserving a good expressive power.

More precisely, inverse tiles, available in McAlister monoids  $M_A$ , somehow force the (underlying) finite state automaton to be invertible in some sense. This is not a surprise since this phenomenon has already been observed in similar research contexts such as the study of word languages recognizable by finite inverse monoids [9] or the study of free inverse monoid languages [15].

However, even restricted to the submonoid of positive tiles  $T_A$ , henceforth without inverse tiles, algebraic recognizability still collapses [6]. It occurs that this collapse is intrinsically linked with the notion of monoid morphism itself and the fact that some product of (even positive) tiles may equal zero when compatibility constraints are violated.

Relaxing the notion of recognizability by finite monoids, we aim at recovering for languages of tiles, as for languages of words, the expressive power of MSO definability. In some sense, our study can also be understood as an attempt to adapt the standard notion of algebraic recognizability to the more general case of Rees quotients of monoids, i.e. monoid with zero.

## 1.1 Main result

In this paper, we consider prehomomorphisms instead of homomorphisms. More precisely, following McAlister and Reilly [10], prehomomorphisms are defined as monotonic mappings between *ordered monoids* that are only required to be submultiplicative w.r.t. the monoid product, i.e. for all  $x$  and  $y$  we only require that  $\varphi(xy) \leq \varphi(x)\varphi(y)$ .

Doing so, we obtained the desired quasi-algebraic characterization of (a natural restriction of) the class of MSO definable languages of positive tiles by means of (well-behaved) prehomomorphisms and (well-behaved) finite ordered monoids.

This result is obtained via a non trivial embedding of any (finite) monoid  $S$  into a sufficiently well-behaved (finite) ordered monoid  $\mathcal{Q}(S)$ . In some sense, standard algebraic recognizability in  $A^*$  is shifted up to quasi-algebraic recognizability in  $\mathcal{Q}(A^*)$  with the monoid  $T_A$  turning out to be a submonoid of  $\mathcal{Q}(A^*)$ .

## 1.2 Related studies and further developments

In mathematics, the class of (well-behaved) ordered monoids we consider seems to be strongly related with a two-sided variant of Ehresmann semigroups [3]. In this paper, our definition are oriented towards ordered monoids. It must be mentioned that in another study [7], we give an essentially equivalent axiomatic (bi-unary) definition of the class of well-behaved ordered monoids that makes the relationship with restriction or Ehresmann semigroups [3] explicit. However, the status of our proposal in this especially rich research field (see [4]) still need to be clarified.

As far as we know, the presented embedding of arbitrary monoid  $S$  into a well-behaved ordered monoid  $\mathcal{Q}(S)$  induces, though not in the standard category of homomorphisms on monoids but, rather, in the category of prehomomorphisms on ordered monoids, a yet unknown expansion in the sense of Birget and Rhodes [14]. As such, it may have many more interested consequences in that mathematical field. But that claim need to be further evaluated.

It computer science, it also seems that the notion of well-behaved ordered monoids (or quasi-inverse monoids as they are called in [7]) can be further developed.

For instance, one may consider these monoids for a theoretical characterization of two-way automata over words. Following the work of Pécuchet [12], it is already known that the domain of a partial run of a two-way automaton can just be seen as a tile in McAlister monoid. It follows that inverse monoids provide a natural algebraic framework for mimicking two-way runs with back and forth readings of subwords. However, as already mentioned, this approach is probably bound to fail since recognizability by finite inverse monoid does collapse [9]. Alternatively, it may be the case however that any run of a two-way automata can be defined by means of some combination of *positive tiles* only. If so, our approach can then lead to a positive answer to the question raised by Birget [1] for an algebraic characterization of two-way automata.

In the field of finite tree automata theory, it is also known that there is a tight connection between regular (or MSO definable) languages of trees and tree-walking automata (see e.g.[2]). Again, back and forth walks translated in algebraic terms seems to require inverse monoids. However, tree walking automata may not necessarily *need* such backward readings. More precisely, it may be enough to allow these automata only to jump back to their last branching moves. If so, the notion of quasi-inverse monoids may just be enough for characterizing these automata, backward jumps being modeled by context operator applications.

Of course, these are long standing challenging topics and they are, for now, still left open for further studies.

## 1.3 Some notations

In the remainder of the text, given a monoid  $S$  with neutral 1 and zero 0, for all  $x$  and  $y \in S$ , we generally write  $xy$  in place for the product  $x.y$  of  $x$  and  $y$

in  $S$ .

Given  $X \subseteq S$ , and  $x \in S$ , we also write  $xX$  (resp.  $Xx$ ) for the set  $xX = \{xy \in S : y \in X\}$  (resp.  $Xx = \{yx \in S : y \in X\}$ ).

The prefix preorder  $\leq_p$  (resp. the suffix preorder  $\leq_s$ ) is defined, for all  $x$  and  $y \in S$  by  $x \leq_p y$  when  $xz = y$  for some  $z \in S$  (resp.  $x \leq_s y$  when  $zx = y$  for some  $z \in S$ ). Under both prefix and suffix preorder, 1 is the least element of  $S$  and, when there is such, 0 is the greatest.

We also write  $x^{-1}(y) = \{z \in S : xz = y\}$  and  $(y)x^{-1} = \{z \in S : zx = y\}$ . This notation extend to sets as follows. For all  $x \in S$  and  $Y \subseteq S$  we write  $x^{-1}(Y) = \{z \in S : xz \in Y\}$  and  $(Y)x^{-1} = \{z \in S : zx \in Y\}$ .

## 2 The monoids of positive tiles

We review here the definition of the monoid of positive tiles and we recall few of its properties. We review then the characterization of MSO definable languages we have already obtained. All proofs and many more properties of this monoid and the associated languages can be found in [6].

In this section, for every two words  $u$  and  $v \in A^* + 0$  we shall write  $u^{-1}(v) \in A^*$  (resp.  $(v)u^{-1} \in A^*$ ) the unique word, if it exists, such that  $v = uu^{-1}(v)$  (resp.  $v = (v)u^{-1}u$ ) or 0 otherwise. We also write  $u \vee_p v$  (resp.  $u \vee_s v$ ) for the smallest word  $w \in A^* + 0$  such that  $u \leq_p w$  and  $v \leq_p w$  (resp.  $u \leq_s w$  and  $v \leq_s w$ ).

A positive tile over the alphabet  $A$  is defined to be a triple of words  $u = (u_1, u_2, u_3) \in A^* \times A^* \times A^*$ . An extra tile, written 0 and called the *undefined tile*, is also assumed to be.

For any two positive tiles  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$ , the *sequential product* of  $u$  and  $v$  is defined to be

$$u.v = ((u_1.u_2 \vee_s v_1)u_2^{-1}, u_2v_2, v_2^{-1}(u_3 \vee_p v_2v_3))$$

when the *left matching constraint*  $u_1.u_2 \vee_s v_1 \neq 0$ , i.e.  $u_1u_2$  and  $v_1$  are suffix one of the other, and the *right matching constraint*  $u_3 \vee_p v_2v_3 \neq 0$ , i.e.  $u_3$  and  $v_2v_3$  are prefix one of the other, are both satisfied. This partial product is completed by  $u.v = 0$  when the matching constraint are not satisfied and by  $u.0 = 0.u = 0$ .

The monoid of positive tiles  $T_A$  is defined as the set  $A^* \times A^* \times A^* + 0$  equipped with the above product. The neutral tile  $(1, 1, 1)$  is written 1.

Monoid  $T_A$  is conveniently seen as an ordered monoid with order relation defined for all  $u \in T_A$  by  $0 \leq u$  and for all  $u$  and  $v \in T_A$  with  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$ , by  $u \leq v$  when  $v_1 \leq_s u_1$ ,  $v_2 = u_2$  and  $v_3 \leq_p u_3$ .

Defining, for all non zero tile  $u = (u_1, u_2, u_3) \in T_A$ , the left (resp. right) context of tile  $u$  by  $u_L = (u_1u_2, 1, u_3)$  (resp.  $u_R = (u_1, 1, u_2, u_3)$ ) and taking  $0_L = 0_R = 0$ , we can prove that, for all  $u$  and  $v \in T_A$ ,  $u \leq v$  if and only if  $u = u_Rvu_L$ .

It is also shown that  $T_A$  is (isomorphic to) the submonoid of McAlister monoid  $M_A$  [8] induced by the (canonical) images of  $A^*$  in  $M_A$  and closed

under product, and left and right context operators with, for every positive tile  $x \in M_A$ ,  $x_L = x^{-1}x$  and  $x_R = xx^{-1}$ .

Given a language  $L \subseteq T_A - 0$ , we define then the *word congruence*  $\simeq_L$  associated to  $L$  as the greatest word congruence such that, for all  $u$  and  $v \in A^*$ ,  $u \simeq_L v$  when for all  $w_1, w_2$  and  $w_3 \in A^*$ , the following equivalences hold:  $(w_1uw_2, w_3, w_4) \in L \Leftrightarrow (w_1vw_2, w_3, w_4) \in L$ ,  $(w_1, w_2uw_3, w_4) \in L \Leftrightarrow (w_1, w_2vw_3, w_4) \in L$ , and  $(w_1, w_2, w_3uw_4) \in L \Leftrightarrow (w_1, w_2, w_3vw_4) \in L$ .

Although the class of MSO definable languages of positive tiles is closed under product, iterated product, and even left and right residuals, these languages remains simple. It is shown [6] that:

**Theorem 1** *For all language  $L \subseteq T_A - 0$ , language  $L$  is MSO-definable if and only if the associated word congruence  $\simeq_L$  over  $A^*$  is of finite index or, equivalently,  $L = \Sigma_{(u,v,w) \in L} [u]_L \times [v]_L \times [w]_L$  and this sum is finite.*

### 3 Well-behaved ordered monoids

Our aim is to provide an algebraic characterization of MSO definable languages of positive tiles. It has been shown [6] that with monoid morphism this cannot be achieved. As an alternative, well-behaved ordered monoids are presented in this section. Well-behaved prehomomorphisms are then presented in the next section.

#### 3.1 Ordered monoids

The following definition is adapted from [13]. For technical reason, we also assume that zero is always there and that it is the least element.

**Definition 2** *An ordered monoid is a monoid  $S$  with zero denoted by  $0$  equipped with a partial order relation  $\leq$  such that  $0$  is the least element and for all  $x, y$  and  $z \in S$ , if  $x \leq y$  then  $xz \leq yz$  and  $zx \leq zy$ , i.e. the order relation is stable by product.*

Any monoid  $S$  is trivially embedded in an ordered monoid by taking  $S^0 = S + 0$  (with a new zero element) ordered by the order relation defined, for all  $x$  and  $y \in S^0$ , by  $x \leq y$  when  $x = 0$  or  $x = y$ . In the remainder of the text,  $S^0$  is called the *trivial ordered extension* of  $S$ .

**Definition 3** *Let  $S$  be an ordered monoid and let  $U(S) = \{x \in S : x \leq 1\}$ . Elements of  $U(S)$  are called the subunits of  $S$ .*

**Lemma 4** *If  $S$  is a ordered monoid then, by stability assumption,  $U(S)$  is a submonoid of  $S$ . Moreover, if  $U(S) \subseteq E(S)$  (the idempotents of  $S$ ) then  $U(S)$  is a meet semi-lattice with, for all  $x$  and  $y \in U(S)$ ,  $x \wedge y = xy = yx$ .*



*Proof.* (1) Let  $x$  and  $y \in U(S)$ . By definition  $x \leq 1$  and  $y \leq 1$  hence, by stability again,  $xy \leq 1$ .

(2) Assume that all elements of  $U(S)$  are idempotent and let  $x$  and  $y \in U(S)$ . Since both  $x \leq 1$  and  $y \leq 1$  we have by stability  $xy \leq x$  and  $xy \leq y$ .

Let then  $z \in U(S)$  such that  $z \leq x$  and  $z \leq y$ . By stability, we have  $zz \leq xy$  hence by idempotence  $z \leq xy$ .

This proves that  $x \wedge y = xy$  and thus, by symmetrical argument,  $x \wedge y = yx$ .

□

Observe that even with  $U(S) \subseteq E(S)$  it still may happen that  $xy \neq yx$  for some  $x$  and  $y \in E(S) - U(S)$ . In the theory developed here, the fact that  $U(S)$  is distinct from  $E(S)$  plays a central role, i.e. idempotents do not necessarily commute.

### 3.2 Well behaved ordered monoid

An ordered monoid  $S$  with order  $\leq$  is a *well-behaved ordered monoid* when the following properties are satisfied:

- (A1)  $U(S) \subseteq E(S)$ , i.e. all subunits are idempotents,
- (A2) for all  $x$  and  $y \in S$ , if  $x \leq y$  then there is  $e$  and  $f \in U(S)$  such that  $x = e y f$ , i.e. the order relation is a natural order,
- (A3) for all non empty  $X \subseteq U(S)$ ,  $\bigwedge X \in S$  exists, i.e.  $U(S)$  is  $\wedge$ -complete,
- (A4) for all non empty  $X \subseteq U(S)$ , for all  $x \in S$ ,  $\bigwedge(xX) = x(\bigwedge X)$  and  $\bigwedge(Xx) = (\bigwedge X)x$ , i.e. both mappings  $e \mapsto ex$  and  $f \mapsto xf$  from  $U(S)$  to  $S$  are  $\wedge$ -continuous.

Observe that axiom (A2) really enforces an alternative (natural) definition of the order relation. In fact, by stability hypothesis, whenever  $x = e y f$  for some  $e$  and  $f \in U(S)$ , one has  $x \leq y$ , i.e. this axiom makes the order a two-sided variation of Nambooripad's natural order [11].

Observe also that for all monoid, the trivial extension  $S^0$  of  $S$  is a well-behaved ordered monoid. Nevertheless, we will prove below that there are plenty of non trivial well-behaved ordered monoids.

### 3.3 Left and right element contexts

It occurs that in a well-behaved ordered monoid  $S$  every element  $x \in S$  can be mapped to a left context element  $x_L$  and a right context element  $x_R$  that somehow capture all elements above  $x$ .

**Definition 5** Let  $S$  be a well-behaved monoid. For all  $x \in S$ , let  $x_L$  (resp.  $x_R$ ), called the left context (resp. right context) element associated to  $x$ , defined by

$$x_L = \bigwedge \{e \in U(S) : xe = x\} \text{ (resp. } x_R = \bigwedge \{f \in U(S) : fx = x\}.$$

**Lemma 6** Let  $S$  be a well-behaved monoid. For all  $x \in S$ , one has  $x_L \in U(S)$  and  $xx_L = x$  (resp.  $x_R \in U(S)$  and  $x_Rx = x$ ).

*Proof.* Let  $x \in S$  and let  $L_x = \{e \in U(S) : xe = x\}$  with  $x_L = \bigwedge L_x$ . Observe first  $L_x$  is non empty since  $1 \in L_x$  hence, by axiom (A3),  $x_L$  is well-defined with  $x_L \leq 1$  hence  $x_L \in U(S)$ . We conclude then by applying axiom (A4) that ensure  $xx_L = x$  since  $xL_x = \{x\}$ .

Similar arguments hold for the  $x_R$  case.  $\square$

**Remark.** The class of well-behaved monoids equipped with left and right context operators have many more properties that are described in [7]. In particular, this class has many similarities with the class of (two sided version of) Ehresmann monoids [3].

However, the resulting order in Ehresmann monoids is not in general well-behaved. In our alternative bi-unary axiomatic presentation of well-behaved monoids [7] we essentially add a single extra axiom to the axiomatization of Ehresmann monoid that ensure stability of the order relation w.r.t. the product.

**Corollary 7** *Let  $S$  be a well-behaved ordered monoid. For all  $x$  and  $y \in S$ ,  $x \leq y$  if and only  $x = x_Ryx_L$ .*

*Proof.* Let  $x$  and  $y \in S$ . Assume  $x = x_Ryx_L$ . By Lemma 6 we have  $x_L$  and  $x_R \in U(S)$  hence, by stability of the order under product,  $x = x_Ryx_L \leq y$ .

Conversely, assume  $x \leq y$  hence, by axiom (A2),  $x = eyf$  for some  $e$  and  $f \in U(S)$ . By idempotence of both  $e$  and  $f$ , we have  $ex = x$  and  $xf = x$  hence, by definition of  $x_L$  and  $x_R$ ,  $x_R \leq e$  and  $x_L \leq f$  hence, by Lemma 4,  $x_Re = x_R$  and  $x_Lf = x_L$ . Now, by Lemma 6, we have  $x = x_Rxx_L$  hence  $x = x_Reyfx_L$  and thus  $x = x_Ryx_L$ .  $\square$

Last, we observe that left and right context operators are tightly related with Green's relations  $\mathcal{L}$  and  $\mathcal{R}$ , i.e. two elements  $x$  and  $y \in S$  are  $\mathcal{L}$ -equivalent (resp.  $\mathcal{R}$ -equivalent) when both  $x \leq_s y$  and  $y \leq_s x$  (resp. both  $x \leq_p y$  and  $y \leq_p x$ ).

**Lemma 8** *Let  $S$  be a well-behaved ordered monoid. For all  $x$  and  $y \in S$ , if  $x$  and  $y$  are  $\mathcal{L}$ -equivalent (resp.  $\mathcal{R}$ -equivalent) then  $x_L = y_L$  (resp.  $x_R = y_R$ ).*

*Proof.* This immediately follows Lemma 6 and the fact that if  $x \leq_p y$  (resp.  $x \leq_s y$ ) then, for all  $e \in U(S)$ , if  $x = ex$  (resp.  $x = xe$ ) then  $y = ey$  (resp.  $y = ye$ ).  $\square$

### 3.4 $F^*$ -property

**Definition 9** *A well-behaved ordered monoid  $S$  is said to be (stably)  $F^*$  when it moreover satisfies the following properties:*

- (A5) *any non zero element  $x \in S$  lies beneath a unique maximal element  $\widehat{x} \in S$ ,*
- (A6) *for all pair of non zero elements  $x$  and  $y \in S$ ,  $\widehat{x\widehat{y}} \neq 0$  and  $\widehat{x\widehat{y}} = \widehat{x\widehat{y}}$ .*

Observe that when  $S$  satisfies the  $F^*$ -property, for all  $x \in S$ , we have  $x = x_R \hat{x} x_L$  and  $\hat{S}$  is a submonoid of  $S$ .

**Lemma 10** *A  $F^*$ -well-behaved ordered monoid is meet complete. For all non empty  $X \subseteq S$ , either  $\bigwedge X = 0$  or there is  $\hat{x} \in S$  maximal such that  $\bigwedge X = (\bigwedge_{x \in X} x_R) \hat{x} (\bigwedge_{x \in X} x_L)$ .*

*Proof.* Let  $X \subseteq S$ . Assume there is some non zero  $z \in S$  such that  $z \leq x$  for all  $x \in X$ . Otherwise,  $\bigwedge X = 0$  and nothing has to be done.

By  $F^*$ -property, this means there exists some  $\hat{z} = \hat{x}$  maximal in  $S$  such that for all  $x \in X$ ,  $x \leq \hat{x}$  hence  $x = x_R \hat{x} x_L$ .

Let then  $x_0 = (\bigwedge_{x \in X} x_R) \hat{x} (\bigwedge_{x \in X} x_L)$ . By stability of the order relation, we have  $x_0 \leq x$  for all  $x \in X$ . Moreover, since  $z_R \leq x_R$  and  $z_L \leq x_L$  for all  $x \in X$ , we also have  $z \leq x_0$ .

As this is true for arbitrary lower bound  $z$  of  $X$  this conclude the proof.  $\square$

The formulation of the previous Lemma, distinguishing the cases  $\bigwedge X = 0$  and  $\bigwedge X \neq 0$  may seem a little strange.

But we must put the emphasis on the fact that in an  $F^*$ -well-behaved monoid  $S$ , say with  $X = \{x, y\} \subseteq S$ , even when both  $x_R y x_L$  and  $y_R x y_L$  are non zero, nothing ensure they are equal and thus it can still be the case that  $x \wedge y = 0$ .

In fact, the extension presented below provide plenty of such examples.

## 4 Prehomomorphisms and quasi-recognizability

We define in this section the notion of well-behaved prehomomorphism and the notion quasi-recognizability it induces. Then we prove that quasi-recognizable languages of positive tiles are definable in MSO.

### 4.1 Prehomomorphisms

The following definition is adapted from McAlister and Reilly [10].

**Definition 11** *Let  $S$  and  $T$  be two ordered monoids. A mapping  $\varphi : S \rightarrow T$  is a prehomomorphism when  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , for all  $x$  and  $y \in S$ , if  $x \leq y$  then  $\varphi(x) \leq \varphi(y)$  and, for all  $x$  and  $y \in S$ ,  $\varphi(xy) \leq \varphi(x)\varphi(y)$*

*A prehomomorphism  $\varphi$  such that  $\varphi(xy) < \varphi(x)\varphi(y)$  if and only if  $xy = 0$  is called a trivial prehomomorphism.*

Observe that given a prehomomorphism  $\varphi : S \rightarrow T$ ,  $(\varphi(S))^*$  is a submonoid of  $T$  while, in general,  $\varphi(S)$  may not be closed under product.

**Lemma 12** *For every prehomomorphism  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow U$ , the mapping  $\varphi\psi : S \rightarrow U$  defined for all  $x \in S$  by  $\varphi\psi(x) = \psi(\varphi(x))$  is a prehomomorphism.*

In other words, ordered monoids and prehomomorphisms forms a category. The following lemma shows how prehomomorphisms behave with subunits and idempotents.

**Lemma 13** *Let  $\varphi : S \rightarrow T$  be a prehomomorphism with  $S$  and  $T$  two ordered monoids. We have  $\varphi(U(S)) \subseteq U(T)$  and  $\varphi(U(S) \cap E(S)) \subseteq U(T) \cap E(T)$ .*

*Proof.* Let  $x \in U(S)$ . (1) By monotonicity argument,  $\varphi(x) \in U(T)$ . (2) Assume  $x \in E(S)$ . By idempotence and submultiplicativity,  $\varphi(x) \leq \varphi(x)\varphi(x)$  hence, by stability since  $\varphi(x) \leq 1$ ,  $\varphi(x)\varphi(x) \leq \varphi(x)$  hence  $\varphi(x) = \varphi(x)\varphi(x)$ .  $\square$

This lemma says in particular that if  $U(S) \subseteq E(S)$  - as in well-behaved ordered monoids - then, by Lemma 4,  $\varphi(U(S))$  is a meet semi-lattice with, for all  $x$  and  $y \in U(S)$ ,  $\varphi(xy) = \varphi(x) \wedge \varphi(y)$  within  $\varphi(U(S))$ . However, even if  $U(T) \subseteq E(T)$ , this *does not imply* that  $\varphi(xy) = \varphi(x)\varphi(y)$  since we can have strict inclusion  $\varphi(U(S)) \subset U(T)$ . Again, this shows that, although  $U(S)$  is a submonoid of  $S$ , its image  $\varphi(U(S))$  is *not necessarily* a submonoid of  $T$ .

Observe also that under prehomomorphism  $\varphi$ , the image  $\varphi(e)$  of an idempotent  $e \in E(S) - U(S)$  that is not a subunit need not to be idempotent itself. We only have  $\varphi(e) \leq \varphi(e)\varphi(e)$ .

More generally, the weakening of the morphism assumption  $\varphi(uv) = \varphi(u)\varphi(v)$  into the weaker submultiplicativity assumption  $\varphi(uv) \leq \varphi(u)\varphi(v)$  just breaks many standard and useful properties of morphisms. In particular, the MSO definability of  $\varphi(uv)$  by means of some MSO definable combination of the MSO definability of  $\varphi(u)$  and  $\varphi(v)$  - the monoid version of Shelah decomposition theorem -, essential in standard algebraic recognizability, is just lost with prehomomorphism.

## 4.2 Well-behaved prehomomorphisms

**Definition 14** *Let  $S$  and  $T$  be two  $F^*$ -well-behaved ordered monoid. A prehomomorphism  $\varphi : S \rightarrow T$  is a well-behaved prehomomorphism when the following condition are satisfied:*

$$(P1) \text{ for all } x \text{ and } y \in \widehat{S}, \varphi(xy) \in \widehat{T},$$

$$(P2) \text{ for all } x, y \text{ and } z \in \widehat{S}, \varphi(x_L y z_R) = (\varphi(x))_L \varphi(y) (\varphi(z))_R,$$

where  $\widehat{S}$  (resp.  $\widehat{T}$ ) denotes the set of maximal elements of  $S$  (resp.  $T$ ).

In particular, as an immediate consequence of (P1) and (P2), for all non zero element  $x \in S$ , if  $\varphi(x) \neq 0$ , we have  $\widehat{\varphi(x)} = \varphi(\widehat{x})$  and, if  $x \in \widehat{S}$ ,  $\varphi(x_L) = (\varphi(x))_L$  and  $\varphi(x_R) = (\varphi(x))_R$ .

## 4.3 Quasi-recognizability

**Definition 15** *A set  $L \subseteq S$  is quasi-recognizable when there exists a finite  $T$  and a well-behaved prehomomorphism  $\varphi : S \rightarrow T$  such that  $L = \varphi^{-1}(\varphi(L))$ .*

Since any monoid  $S$  is embedded into its trivial ordered extension  $S^0$  with  $S$  isomorphic with  $\widehat{S^0}$ , this definition generalizes the definition of algebraic recognizability.

**Theorem 16** *Let  $\varphi : T_A \rightarrow S$  be a well-behaved prehomomorphism with  $S$  finite. Let  $X \subseteq S$  be some (finite) subset of  $S$ . Language  $L = \varphi^{-1}(X)$  is definable in monoid second order logic.*

*Proof.* Let  $\varphi : T_A \rightarrow S$  be a well-behaved prehomomorphism with  $S$  finite. Since  $S$  is finite, it suffices to show that, for all  $x \in S$ ,  $\varphi^{-1}(x)$  is MSO-definable. Moreover, we can restrict to non zero elements since we have  $\varphi^{-1}(0) = T_A - \bigcup_{x \neq 0} \varphi^{-1}(x)$ .

Let then  $\widehat{T}_A = \{(1, u, 1) \in T_A : u \in A^*\}$  the maximal elements of  $T_A$ . Since  $\varphi$  is a well-behaved prehomomorphism, we have  $\varphi(\widehat{T}_A) \subseteq \widehat{S}$  and the restriction  $\widehat{\varphi} : \widehat{T}_A \rightarrow \widehat{S}$  of  $\varphi$  to maximal elements of  $T_A$  is just a morphism. It follows that for all  $x \in S$ , language  $\varphi^{-1}(x) \cap \widehat{T}_A$  is MSO definable.

Now we conclude by observing that for all  $(u, v, w) \in T_A$ , since  $(u, v, w) = u_L v_C w_R$  with  $u_L = (u, 1, 1)$ ,  $v_C = (1, v, 1)$  and  $w_R = (1, 1, w)$ , we have

$$\varphi((u, v, w)) = (\varphi(u_C))_L \varphi(v_C) (\varphi(w_C))_R$$

hence checking that  $\varphi((u, v, w)) = x$  for some non zero  $x \in S$  amounts to check that there exists  $x_1, x_2$  and  $x_3 \in \widehat{S}$  such that  $x = (x_1)_L x_2 (x_3)_R$  with  $\widehat{\varphi}(u_C) = x_1$ ,  $\widehat{\varphi}(v_C) = x_2$  and  $\widehat{\varphi}(w_C) = x_3$  which are, altogether, definable in MSO.  $\square$

#### 4.4 Context coherence

Does the converse of Theorem 16 hold ? The answer is no as it turns out, by Lemma 8, that quasi-recognizability induces a normalization of languages of tiles. More precisely, given a well-behaved prehomomorphism  $\varphi : T_A \rightarrow S$ , for all  $u$  and  $v \in A^*$ , if  $\varphi(u_C)$  and  $\varphi(v_C)$  are  $\mathcal{L}$ -equivalent (resp.  $\mathcal{R}$ -equivalent) then  $\varphi(u_L) = \varphi(v_L)$  (resp.  $\varphi(u_R) = \varphi(v_R)$ ).

From an optimistic point of view, given a quasi-recognizable languages, images of context elements act as they are expected to : they just induce product constraints. It follows that, in  $T_A$  whenever two context elements induces equivalent product constraints under a prehomomorphism defining  $L$  they are just mapped to the same elements.

This observation leads us to the following definition.

**Definition 17** *Let  $L \subseteq T_A$  be a language of positive tile and, for every word  $u \in A^*$ , let  $[u]$  be the equivalence class of  $u$  under the word congruence  $\simeq_L$  induced by  $L$ . Language  $L$  is context-coherent when, for every non zero tile  $(u, v, w) \in L$ , for every word  $u'$  and  $w' \in A^*$ , if  $A^*[u] = A^*[u']$  and  $[w]A^* = [w']A^*$  then  $(u', v, w') \in L$ .*

Our aim is now to prove, under the context coherence hypothesis, that MSO definable and context-coherent languages of tiles are quasi-recognizable. This is done in the next section via the non trivial ordered extension  $\mathcal{Q}(S)$  of arbitrary monoid  $S$ .

## 5 A well-behaved extension

In this section, we define from arbitrary monoid  $S$ , sort of a monoid of positive tiles  $\mathcal{Q}(S)$  much in the same way the monoid  $T_A$  of positive tiles [6] over the alphabet  $A$  is built upon  $A^*$  with a canonical injection from  $S$  to  $\mathcal{Q}(S)$ . It is then shown that any homomorphism  $\psi : A^* \rightarrow S$  can be lifted up to a prehomomorphism  $\varphi : T_A \rightarrow \mathcal{Q}(S)$ . The expected converse of Theorem 16 can then be proved.

### 5.1 Prefix and suffix upper sets

Let  $S$  be a monoid. Let  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) defined to be the set of non empty upward closed subsets of  $S^0$  preordered by  $\leq_p$  (resp.  $\leq_s$ ) the prefix (resp. suffix) preorder.

More precisely, as  $S$  is a monoid hence with  $1 \in S$ ,  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) is the set of non empty  $U \subseteq S$  such that  $US^0 = U$  (resp.  $S^0U = U$ ).

For both  $x = p$  or  $x = s$ , elements of  $\mathcal{U}_x(S)$  are from now on called  $x$ -upper set. The set  $\mathcal{U}_x(S)$  is turned into a monoid by taking  $\cap$  as product.

Indeed, the intersection of two  $x$ -upper sets is a  $x$ -upper set and the neutral (or maximal) element is  $S^0$  itself, and  $\{0\}$  is the absorbant (or minimal) element. For convenience, we also write 0 for that zero.

**Remark.** In semigroup theory elements of  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) are often called right ideals (resp. left ideals) of  $S^0$ . As ideals in order theory must satisfy some extra condition we prefer to stick to the notion of  $p$ -upper or  $s$ -upper sets.

**Lemma 18** *Let  $S$  be some monoid and let  $x \in S$ . Set  $xS^0$  is a  $p$ -upper set (resp.  $S^0x$  a  $s$ -upper set) and, for every  $p$ -upper set (resp.  $s$ -upper set)  $U \subseteq S^0$ :*

- (1) *if  $x \in U$  then  $x^{-1}(U) = S^0$  (resp.  $(U)x^{-1} = S^0$ ),*
- (2)  *$xU$  is a  $p$ -upper set (resp.  $Ux$  is a  $s$ -upper set),*
- (3)  *$x^{-1}(U)$  is a  $p$ -upper set (resp.  $(U)x^{-1}$  is a  $s$ -upper sets),*
- (4)  *$xx^{-1}(U) \subseteq U \subseteq x^{-1}(xU)$  (resp.  $(U)x^{-1}x \subseteq U \subseteq (Ux)x^{-1}$ ),*

*Proof.* Straightforward. □

### 5.2 The extension ordered monoid

**Definition 19** *Let  $S$  be a monoid. The quasi-inverse extension of  $S$  is defined to be  $\mathcal{Q}(S) = (\mathcal{U}_s(S) - 0) \times S \times (\mathcal{U}_p(S) - 0) + 0$  with, for all non zero element  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$  the product  $u_1u_2$  defined by*

$$u_1u_2 = (L_1 \cap (L_2)x_1^{-1}, x_1x_2, R_2 \cap x_2^{-1}(R_1))$$

*when both compatibility constraints  $L_1 \cap (L_2)x_1^{-1} \neq 0$  and  $R_2 \cap x_2^{-1}(R_1) \neq 0$  are satisfied, and by  $u_1u_2 = 0$  otherwise.*

For all two non zero elements  $u = (L, x, R)$  and  $v = (M, y, N) \in \mathcal{Q}(S)$ , we say that  $u \leq v$  when  $L \subseteq M$ ,  $x = y$  and  $R \subseteq N$ . This relation is extended to zero by taking  $0 \leq u$  for all  $u \in \mathcal{Q}(S)$ .

**Lemma 20** *The extension monoid  $\mathcal{Q}(S)$  ordered by the extension order is a (stable) ordered monoid.*

*Proof.* The fact that relation  $\leq$  is a partial order is immediate from the definition.

Let then  $u = (L_1, x_1, R_1)$ ,  $v = (L_2, x_2, R_2)$  and  $w = (M, y, N)$ . Assume  $u \leq v$ . By definition, we have  $x_1 = x_2$  from now on denoted by  $x$  and both  $L_1 \subseteq L_2$  and  $R_1 \subseteq R_2$ .

By definition of the product we have  $uw = (L_1 \cap (M)x^{-1}, xy, N \cap y^{-1}(R_1))$  and  $vw = (L_2 \cap (M_2)x^{-1}, xy, N \cap y^{-1}(R_2))$ . Hence  $uw \leq vw$  by definition of the extension order and stability of the inclusion order by intersection and residual.

Symmetrical arguments show that  $wu \leq wv$ .  $\square$

**Lemma 21** *The mapping  $i : S^0 \rightarrow \mathcal{Q}(S)$  that maps zero to zero and any non zero element  $x \in S$  to  $i(x) = (S^0, x, S^0)$  is a one-to-one monoid morphism. The image  $i(S)$  of  $S \subseteq S^0$  is a submonoid of  $\mathcal{Q}(S)$  that contains exactly all maximal elements of  $\mathcal{Q}(S)$ .*

Moreover, the mapping  $\pi : \mathcal{Q}(S) \rightarrow S^0$  that maps 0 to 0 and any non zero element  $(L, x, R)$  to  $\pi(L, x, R) = x$  is an onto well-behaved prehomomorphism with  $\pi \circ i = 1_{S^0}$ .

*Proof.* The fact that  $i$  is a monoid morphism is immediate and the product on elements of  $i(S)$ , all of the form  $(S^0, x, S^0)$ , just mimics the product in  $S$  since, for all  $x \in S$ ,  $x^{-1}(S^0) = (S^0)x^{-1} = S^0$ .

Mapping  $\pi$  is obviously onto since  $\pi \circ i = 1_{S^0}$ . Then we check that it is a (trivial) prehomomorphism, i.e. for all  $u_1$  and  $u_2 \in \mathcal{Q}(S)$  either  $u_1 u_2 \neq 0$  and then  $\pi(u_1 u_2) = \pi_S(u_1) \pi_S(u_2)$  or  $u_1 u_2 = 0$  and thus  $\pi(u_1 u_2) = 0$ .  $\square$

**Theorem 22** *For all monoid  $S$ , the monoid  $\mathcal{Q}(S)$  ordered by the extension order  $\leq$  is a  $F^*$ -well-behaved ordered monoid.*

*Proof.* This essentially follows from Lemma 20 that tells  $\mathcal{Q}(S)$  is indeed a (stable) ordered monoid, from Lemma 23 below that ensures subunits are idempotents and from Lemma 24 below that show the extension order is natural.

Other axioms and  $F^*$ -property immediately follows from definitions and from the fact that over  $p$ -upper sets (resp.  $s$ -upper sets) intersection distribute with left (resp. right) product and residual by an elements  $x \in S$ .  $\square$

The set of subunits of  $\mathcal{Q}(S)$  is, by definition of the extension order, the set  $U(\mathcal{Q}(S)) = 0 + \{(L, x, R) \in \mathcal{Q}(S) : x = 1\}$ .

The next lemma shows that these indeed are idempotents elements.

**Lemma 23** *A non zero triple  $(L, x, R) \in \mathcal{Q}(S)$  is idempotent if and only if  $x \in S$  is idempotent,  $L \subseteq (L)x^{-1}$  and  $R \subseteq x^{-1}(R)$ .*

*Proof.* Let  $(L, x, R)$  be an idempotent of  $\mathcal{Q}(S)$ . By definition of the product in  $\mathcal{Q}(S)$  this means that  $xx = x$  hence  $x$  is idempotent in  $S$ ,  $L \cap (L)x^{-1} = L$  hence  $L \subseteq (L)x^{-1}$  and  $R \cap x^{-1}(R) = R$  hence  $R \subseteq x^{-1}R$ . The converse is immediate.  $\square$

**Lemma 24** *For all non zero  $u = (L, x, R)$  and  $v = (M, y, N) \in \mathcal{Q}(S)$ , if  $u \leq v$  then  $(L, x, R) = (L, 1, xR) \cdot (M, y, N) \cdot (Lx, 1, R)$ .*

*Proof.* Assume  $u \leq v$ . By definition of the extension order this implies that  $x = y$  and  $L \subseteq M$  and  $R \subseteq N$ . Taking  $e = (L, 1, xR)$  and  $f = (Lx, 1, R)$  on has

$$evf = (L \cap M \cap (Lx)y^{-1}, y, y^{-1}(xR) \cap N \cap R)$$

But, by assumption,  $L \subseteq M$  and  $R \subseteq N$  and, since  $x = y$ , we also have by Lemma 18  $L \subseteq (Lx)y^{-1}$  and  $R \subseteq y^{-1}(xR)$  hence  $evf = u$ .  $\square$

Last, the next Lemma shows that given the elements above are indeed the left and right contexts elements associated with  $u$ .

**Lemma 25** *For all non zero  $u = (L, x, R)$ , we have  $u_L = (Lx, 1, R)$  and  $u_R = (L, 1, xR)$ , i.e.  $u_R u = u = u u_L$  and these are the least subunits that satisfies this properties.*

*Proof.* Let  $u_L$  and  $u_R$  be defined as in 3.3. Let  $u'_L = (Lx, 1, R)$ . Observe first that we do have  $u u'_L = u$ . In fact  $u u'_L = (L \cap (Lx)x^{-1}, x, R)$  with, by Lemma 18,  $L \subseteq (Lx)x^{-1}$  hence  $u u'_L = u$ . By definition of  $u_L$  this means  $u_L \leq u'_L$ .

Let then  $e = (L', 1, R')$  be a subunit of  $\mathcal{Q}(S)$  such that  $ue = u$ . This means that  $(L \cap (L')x^{-1}, x, R' \cap R) = (L, x, R)$  hence  $L \subseteq (L')x^{-1}$  henceforth  $Lx \subseteq L'$  and  $R \subseteq R'$ . It follows that  $u'_L \leq e$ . As this is true for all  $e$  as above, this means that  $u'_L \leq u_L$ .

Symmetrical arguments prove the claim for  $u_R$ .  $\square$

The following theorem says that our construction above essentially extend to arbitrary monoid the construction of the monoid of positive tiles [6] from the free monoid  $A^*$ .

**Theorem 26** *There is a one to one morphism  $i : T_A \rightarrow \mathcal{Q}(A^*)$  such that, moreover, for all  $u \in T_A$ ,  $i(u_L) = (i(u))_L$  and  $i(u_R) = (i(u))_R$ .*

*Proof.* Observe that  $A^*$  is totally ordered by  $\leq_s$  and  $\leq_p$ . It follows that for  $x = p$  and  $x = s$ , the mapping  $\varphi_x : A^* + 0 \rightarrow \mathcal{U}_x(A^* + 0)$  defined, for every  $u \in A^* + 0$ , by  $\varphi_x(u) = \{v \in A^* + 0 : u \leq_x v\}$  is one-to-one.

Is is then an easy task to check that  $i : T_A \rightarrow \mathcal{Q}(A^* + 0)$  defined by  $i(0) = 0$  and, for every tile  $(u, v, w) \in T_A$ ,  $i((u, v, w)) = (\varphi_s(u), v, \varphi_p(w))$  is a one-to-one morphism.

The last property is immediate.  $\square$



### 5.3 More on maximal elements in the extension

We observe now that the extension monoid is a non trivial well-ordered monoid.

**Lemma 27** *For all  $u$  and  $v \in \mathcal{Q}(S)$ , if  $v$  is maximal then  $v_R = u_R v_R$  (resp.  $v_L = v_L u_L$ ) if and only if  $u \leq_p v$  (resp.  $u \leq_s v$ ).*

*In particular, with  $v = 1$  hence  $v_R = v_L = 1$ , we have  $u_R = 1$  (resp.  $u_L = 1$ ) if and only if  $uw = 1$  (resp.  $wu = 1$ ) for some  $w \in \mathcal{Q}(S)$ , i.e.  $u$  admits a right (resp. left) group inverse.*

*Proof.* Let then  $u = (L, x, R)$  hence  $u_R = (L, 1, xR)$  and let  $v = (S^0, y, S^0)$  some maximal element of  $\mathcal{Q}(S)$ .

If  $u \leq_p v$  then we immediately have, by definition of the right context operator  $u_R v_R = v_R$ .

Assume that  $v_R = u_R v_R$  henceforth, by commutation and right product by  $v$ ,  $u_R v = v$ . By definition of the product, we have  $u_R v = (L, y, y^{-1}(xR))$ . It follows that (a)  $S^0 = L$  and that  $S^0 = y^{-1}(xR)$  with  $y^{-1}(xR) = \{z \in S^0 : yz \in xR\}$ . Since  $1 \in S^0$ , this means that  $y \in xR$  and thus (b)  $y = xz$  for some  $z \in R$ . Taking then  $w = (S^0, z, S^0)$  we have, by definition of the product,

$$uw = (L \cap (S^0)x^{-1}, xz, S^0 \cap z^{-1}(R))$$

Now by (a)  $L = S^0$  and since  $x \in S$ , by Lemma 18,  $(S^0)x^{-1} = S^0$ . By (b)  $xz = y$  and, since  $z \in R$ , again by Lemma 18,  $z^{-1}(R) = S^0$ . It follows that  $uw = v$  henceforth  $u \leq_p v$ .

The case of left context and suffix preorder is obtained by symmetrical arguments.  $\square$

### 5.4 Morphism extensions

We prove here that any morphism from  $A^*$  to a monoid  $S$  can be lifted up to a prehomomorphism from  $T_A$  to  $\mathcal{Q}(S)$ . Aside of its interest in terms of structural description of well-behaved ordered monoids, this result is essential in the proof presented right after, that, under extra conditions, MSO definability implies quasi-recognizability.

**Theorem 28** *Let  $\psi : A^* \rightarrow S$  be a monoid morphism. The extension mapping  $\varphi : T_A \rightarrow \mathcal{Q}(S)$  defined for every non zero tile  $(u, v, w) \in T_A$  by  $\varphi((u, v, w)) = (S^0\psi(u), \psi(v), \psi(w)S^0)$ , is a well-behaved prehomomorphism.*

*Proof.* In order to simplify notation, for all  $u \in A^*$ , we write  $[u] \in S$  for the image of  $u$  by  $\psi$  in  $S$ . This notation is coherent with the fact that  $S$  can just be restricted to be the quotient of  $A^*$  by the congruence induced by  $\psi$ .

By construction,  $\varphi(1) = (S^0, 1, S^0)$  which is indeed the neutral element of  $\mathcal{Q}(S)$ .

For every  $u \in A^*$ , let us write,  $\varphi_s(u) = S^0[u]$  (resp.  $\varphi_p(u) = [u]S^0$ ). Mappings  $\varphi_s : A^* \rightarrow \mathcal{U}_s(S)$  and  $\varphi_p : A^* \rightarrow \mathcal{U}_p(S)$  are the left and right projection of

$\varphi$ . In particular, both for  $x = p$  and  $x = s$ , for every  $u$  and  $v \in A^*$ , if  $u \leq_x v$  then  $\varphi_x(u) \supseteq \varphi_x(v)$  and, if  $u \vee_x v \neq 0$  then  $\varphi_x(u \vee_x v) = \varphi_x(u) \cap \varphi_x(v)$ .

Let then  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$  be two non zero positive tiles in  $T_A$ . By definition,  $\varphi(u) = (\varphi_s(u_1), [u_2], \varphi_p(u_3))$ .

Assume for a while that  $u \leq v$ . This means that  $v_1 \leq_s u_1$ ,  $v_2 = u_2$  and  $v_3 \leq_p u_3$ . This means that  $[v_1] \leq_s [u_1]$  hence  $\varphi_s(u_1) \subseteq \varphi_s(v_1)$ ,  $[u_2] = [v_2]$  and  $[v_3] \leq_p [u_3]$  hence  $\varphi_p(u_3) \subseteq \varphi_p(v_3)$ . It follows, by definition of the extension order, that  $\varphi(u) \leq \varphi(v)$ . This means that  $\varphi$  is monotonic.

Assume again that  $u$  and  $v$  are arbitrary. We need to prove that  $\varphi(uv) \leq \varphi(u)\varphi(v)$ . If  $u.v = 0$  nothing has to be done since  $\varphi(0)$  is the smallest in  $\mathcal{Q}(S)$  hence the pre-homomorphism property is satisfied. Assume thus  $uv$  is non zero. we have

$$uv = ((u_1 u_2 \vee_s v_1) u_2^{-1}, u_2 v_2, v_2^{-1} (u_1 \vee_p v_2 v_3))$$

since  $[u_2 v_2] = [u_2][v_2]$ , by definition of the product of  $\varphi(u)\varphi(v)$  in  $\mathcal{Q}(S)$ , by definition of the extension order, we have to prove that

$$\varphi_s((u_1 u_2 \vee_s v_1) u_2^{-1}) \subseteq \varphi_s(u_1) \cap (\varphi_s(v_1)) [u_2]^{-1}$$

and

$$\varphi_p(v_2^{-1} (u_3 \vee_p v_2 v_3)) \subseteq [v_2]^{-1} (\varphi_p(u_3)) \cap \varphi_p(v_3)$$

By symmetry it suffices to prove it for the prefix case.

Observe that since the product  $uv$  is non zero, we have  $u_3 \vee_p v_2 v_3 \neq 0$ .

The first case is when  $u_3 \leq_p v_2 v_3$ . This means that  $v_2^{-1} (u_3 \vee_p v_2 v_3) = v_3$  hence  $\varphi_p(v_2^{-1} (u_3 \vee_p v_2 v_3)) = \varphi_p(v_3)$ . But this also means that  $[v_2] \in \varphi_p(u_3) = [u_3] S^0$  since  $[u_3] \leq_p [v_2]$  hence, by Lemma 18,  $[v_2]^{-1} (\varphi_p(u_3)) = S^0$ . Observe that in this case, we do have equality.

The second case is when  $v_2 v_3 \leq_p u_3$ . This means that  $v_2^{-1} (u_3 \vee_p v_2 v_3) = v_2^{-1} (u_3) = w$  for some  $w \in A^*$  such that  $u_3 = v_2 w$ . It follows that:

$$\varphi_p(v_2^{-1} (u_3 \vee_p v_2 v_3)) = \varphi_p(v_2^{-1} (u_3)) = [w] S^0$$

and

$$[v_2]^{-1} (\varphi_p(u_3)) = [v_2]^{-1} [v_2 w] S^0 = [v_2]^{-1} ([v_2][w] S^0)$$

hence, by Lemma 18, the result since  $[w] S^0 \subseteq [v_2]^{-1} ([v_2][w] S^0)$ .

Observe that in that case, hence in general, we do not have equality. This means that prehomomorphism  $\varphi$  from  $T_A$  to  $\mathcal{Q}(S)$  is not, in general, a trivial prehomomorphism, i.e. it may happen that, for some  $u$  and  $v \in T_A$ ,  $\varphi(uv) < \varphi(u)\varphi(v)$  with  $\varphi(uv) \neq 0$ .  $\square$

We may ask if such an extension morphism could not be extended a little more to become a prehomomorphism from  $\mathcal{Q}(A^*)$  to  $\mathcal{Q}(S)$ . If so, then our extension construction is a expansion in the sense of Birget and Rhodes [14] in the category of well-behaved ordered monoids with (well-behaved) prehomomorphisms.

## 5.5 From MSO-definability to quasi-recognizability

We conclude here by showing that, under context coherence hypothesis, any MSO definable languages of tiles is quasi-recognizable.

**Theorem 29** *If  $L \subseteq T_A$  is MSO definable and context coherent then  $L$  is quasi-recognizable.*

*Proof.* Let  $L \subseteq T_A$  be an MSO definable language of positive tiles and let  $S = A^*/\simeq_L$  be the finite monoid defined by the quotient of  $A^*$  under the (finite index) word congruence induced by  $L$ . For every word  $u \in A^*$ , let us write  $[u] \subseteq A^*$  the class of words of  $A^*$  equivalent under  $\simeq_L$  to  $u$ .

Let define  $\varphi : T_A \rightarrow \mathcal{Q}(S)$  by taking, for every non zero positive tile  $u = (u_1, u_2, u_3) \in T_A$ ,  $\varphi(u) = (S^0[u_1], [u_2], [u_3]S^0)$ . By Theorem 28 above, we know that  $\varphi$  is a well-behaved prehomomorphism. Moreover, by construction,  $L \subseteq \varphi^{-1}(\varphi(L))$ . The converse inclusion is then just ensured by the fact that  $L$  is context coherent.  $\square$

An attentive reader can observe that this last statement is slightly frustrating. More precisely, there are languages of tiles that are quasi-recognizable and not context coherent. In other words, context coherence is a little too strong restriction.

In some sense, we fail so far to identify an adequate notion of syntactic quasi-congruence that would play for quasi-recognizability the same role syntactic congruence plays for recognizability. Indeed, as already mentioned, the notion of quasi-recognizability is still a theory to be developed. . .

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