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# Quasi-recognizable vs MSO definable languages of one-dimensional overlapping tiles

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Quasi-recognizable vs MSO definable languages of  
one-dimensional overlapping tiles

April 23, 2012

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## Abstract

It has been shown [6] that, within the McAlister inverse monoid [9], whose elements can be seen as overlapping one-dimensional tiles, the class of languages recognizable by finite monoids collapses compared with the class of languages definable in Monadic Second Order Logic (MSO).

This paper aims at capturing the expressive power of the MSO definability of languages of tiles by means of a weakening of the notion of algebraic recognizability which we shall refer to as quasi-recognizability. For that purpose, since the collapse of algebraic recognizability is intrinsically linked with the notion of monoid morphism itself, we propose instead to use premorphisms, monotonic mappings on ordered monoids that are only required to be sub-multiplicative with respect to the monoid product, i.e. mapping  $\varphi$  so that for all  $x$  and  $y$ ,  $\varphi(xy) \leq \varphi(x)\varphi(y)$ .

In doing so, we indeed obtain, with additional but relatively natural closure conditions, the expected quasi-algebraic characterization of MSO definable languages of positive tiles. This result is achieved via the axiomatic definition of an original class of well-behaved ordered monoid so that quasi-recognizability implies MSO definability. An original embedding of any (finite) monoid  $S$  into a (finite) well-behaved ordered monoid  $\mathcal{Q}(S)$  is then used to prove the converse.

## 1 Introduction

This paper is rooted in formal language theory where, classically, the object of study are sets (languages) of words (or strings) over a finite alphabet  $A$ , that is, subsets of the free monoid  $A^*$ . A main topic of research is the characterization of various classes of languages and, for this, there are a number of approaches: recognition by automata of various kinds, recognition by monoids, definability through various logical systems. The vast theoretical corpus that has been developed has inspired the study of languages (subsets) in monoids other than free monoids. A notable example is the theory of trace languages [2] where traces model concurrent behaviors. Another example is the study of subsets of free inverse monoids [16].

In this paper, the monoid that replaces the free monoid is the monoid  $T_A$  of positive overlapping tiles. This monoid extends the free monoid  $A^*$  by adjoining left and right compatibility constraints to words which define whether the concatenation product is legal or not. The resulting objects are triples of words extended with 0 to complete the product. Indeed, the product of incompatible tiles is set to 0.

Our interest in languages of tiles originally stems from computational music theory [5] where the monoid of positive tiles is defined as an abstraction of sequential combinations of musical sequences with overlapping anacrusis or conclusions.

It is also possible to generalize these modeling perspectives. Indeed, overlapping tiles can be seen as extended models of *guarded processes*. The left compatibility constraint of a tile can be seen as a sequence of actions that must occur *before* a process is executed. Symmetrically, the right compatibility constraint of a tile can be seen as a sequence of actions that must occur *after* a process is executed. The behavior of a complex sequential process, with both past and future guards for each executions, can thus be modeled as the set of its possible guarded executions. This is a *language* of non zero tiles.

The study of languages of tiles have already been the subject of research on our part [6]. The class of languages of tiles definable in Monadic Second Order Logic (MSO) is shown to be *simple*, i.e. these languages are finite sums of cartesian products of regular languages of words. Since non zero tiles are just triples of words this result is not a surprise. This class is also shown to be *robust*. It is not only closed under boolean operators and projections, but also under (tiles) product, iterated product and left and right residuals. Since the product of tiles involves arbitrarily long pattern matching constraints, this good property was less expected.

Aiming at identifying efficient mechanism to manipulate these languages of tiles, we also have studied recognition by monoids. However, despite interesting mathematical properties related with covers of bi-infinite periodic words [6], the class of languages of tiles recognizable by finite monoids is shown to collapse. It thus provides no interesting notion of automata. Of course, we may try to define, by brute force, finite state machines for MSO definable languages of tiles. However, in order to be truly useful, this notion of machine needs to be compositional with respect to the product of (languages of) tiles. Since compositionality is guaranteed by algebraic approaches, we rather seek to identify ways of remedying the collapse of recognition by monoids.

Within the context of tiles, morphisms, in that they preserve products, convey far too much structure to induce enough expressive power. We thereby seek to identify a *relaxation* of the notion of morphism itself.

In this paper, we introduce the notion of quasi-recognizability: recognizability by means of *premorphisms* instead of morphisms. Defined on *ordered monoids*, premorphisms are mappings that are only required to be sub-multiplicative w.r.t. the monoid product [11], i.e.  $\varphi(xy) \leq \varphi(x)\varphi(y)$  instead of  $\varphi(xy) = \varphi(x)\varphi(y)$ .

But this proposal comes with a price: in general, quasi-recognizability does not imply MSO definability. This means that our proposal is necessarily two-fold. On

the one hand, we need to find an adequate *restriction* of the category of ordered monoids and premorphisms that preserves MSO definability. On the other, we need the induced subcategory to be as *large* as possible in order to recover our expressiveness yardstick : MSO definability.

Is such a delicate balance achievable? The answer to this question is as yet unknown and may well be negative but we provide here a solution that *essentially* fulfills these goals.

## 1.1 Main results

We define the subclass of strongly well-behaved ordered monoids and the subclass of well-behaved premorphisms. In the associated category, we prove that all quasi-recognizable languages of tiles are definable, i.e.  $QREC \subseteq MSO$ , i.e. Theorem 9. Our first goal is achieved. Conversely, provided languages of tiles satisfy a context-coherence closure property ( $CC$ ), we prove that MSO definable languages of tiles are quasi-recognizable, i.e.  $MSO \cap CC \subseteq QREC$ , i.e. Theorem 20. Our second goal is *essentially* achieved

As far as we can see, the context-coherence closure property seems inherently linked with the notion of overlapping tiles itself. Moreover, the class  $QREC$  is much larger than the class  $REC$  of (classically) recognizable languages of tiles. Indeed, the class  $QREC$  contains all (embeddings of) regular word languages. It thus truly generalizes the class of recognizable languages of words. In contrast, as far as the embedding of languages of words is concerned, it can be shown [6] that the classical recognizability over tiles leads to finite boolean combination of languages of words of the form  $u(vu)^k(vu)^*$  with  $u \in A^*$ ,  $v \in A^+$  and  $k \in \mathbb{N}$ . This class is even smaller than the related case of languages of words defined by finite inverse monoids [10].

Of course, our proposal may seem technical. These are new ideas and methods. Their presentation has not as yet benefited from the fine-tuning that tried and tested practice brings. Setting up an adequate subcategory also requires the definition of a non trivial subclass of ordered monoids. Further still, our proof that context-coherent MSO-definable languages are quasi-recognizable relies on a *original* embedding of any monoid  $S$  into a well-behaved ordered monoid  $\mathcal{Q}(S)$ . The mathematical elegance of this construction justifies, a posteriori, the technicalities that led to its definition.

## 1.2 Related works

Must we necessarily therefore conclude that our proposal is totally disconnected from former studies?

In mathematics, it results that the (well-behaved) ordered monoids, considered in this paper, generalize the two-sided version of Restriction or Ehresmann monoids [3]. Further still, it results that well-behaved ordered monoids coincide with a stable order restriction of Lawson's  $U$ -semiadequate monoids [8].

More precisely, the definitions advanced in this paper are based on ordered monoids. In another paper [7], we set forward an equivalent axiomatic (bi-unary) definition of the class of well-behaved ordered monoids. In doing so, the relationship with other known classes of monoids or semigroups is made clear in this particularly rich field of research of semigroups with local units (see [4]) at the frontier with the even richer inverse semigroup theory. Furthermore the main construction, the embedding of any monoid  $S$  into a well-behaved ordered monoid  $\mathcal{Q}(S)$ , proves to be an *expansion* in the sense of Birget and Rhodes [15] within the category of premorphisms on ordered monoids [7]. As such, it may have many more interesting consequences in that specific mathematical field.

In computer science itself, though more implicitly, the notion of tiles and the related notion of concatenation products has been a subject of research for many years. In his study of two-way automata, Pécuchet [13] implicitly defines partial runs as (positive or negative) overlapping tiles and combining two partial runs amounts to making the product of the underlying tiles. Later, Birget proposes an algebraic reduction of two-way automata into finite monoids [1] where these ideas are further developed. However, despite many attempts since then, the question raised by Birget: *does a true algebraic characterization of two-way automata in fact exist*, remains unanswered so far.

Altogether, overlapping tiles are partial runs of two-way automata. We do provide an algebraic setting for these languages. And even the context coherence requirement no longer stands when two special letters mark (as for two-way automata) the beginning and the end of the word upon which given tiles are considered. Thereby, our proposal may constitute a first step towards a positive answer to Birget's long standing question.

## 1.3 Some notations

For every monoid  $S$ , for every  $x$  and  $y \in S$ , we often write  $xy$  for the product  $x.y$  of  $x$  and  $y$  in  $S$ . By extension, given  $X \subseteq S$ , and  $x \in S$ , let  $xX$  (resp.  $Xx$ ) be the

set  $xX = \{xy \in S : y \in X\}$  (resp.  $XX = \{yx \in S : y \in X\}$ ).

The prefix preorder  $\leq_p$  (resp. the suffix preorder  $\leq_s$ ) is defined, for all  $x$  and  $y \in S$  by  $x \leq_p y$  when  $xz = y$  for some  $z \in S$  (resp.  $x \leq_s y$  when  $zx = y$  for some  $z \in S$ ). Under both prefix and suffix preorder, the neutral element 1 is the least element of  $S$ , and the absorbant element 0 (if there is such) is the greatest. We also write  $x^{-1}(y) = \{z \in S : xz = y\}$  and  $(y)x^{-1} = \{z \in S : zx = y\}$ . This notation extends to sets as follows: for all  $x \in S$  and  $Y \subseteq S$  we write  $x^{-1}(Y) = \{z \in S : xz \in Y\}$  and  $(Y)x^{-1} = \{z \in S : zx \in Y\}$ .

The free monoid  $A^*$  on the alphabet  $A$  is extended with 0 with, for all  $u \in A^*$ ,  $u0 = 0u = 0$ . Then, for every two words  $u$  and  $v$ ,  $u^{-1}(v)$  (resp.  $(v)u^{-1}$ ) is defined as the unique word, if it exists, such that  $v = uu^{-1}(v)$  (resp.  $v = (v)u^{-1}u$ ) or 0 otherwise. Last, we write  $u \vee_p v$  (resp.  $u \vee_s v$ ) for the smallest word  $w \in A^* + 0$  such that both  $u \leq_p w$  and  $v \leq_p w$  (resp.  $u \leq_s w$  and  $v \leq_s w$ ). In other words,  $u \vee_s v$  (resp.  $u \vee_p v$ ) equals the greatest word among  $u$  or  $v$  when they are suffix-comparable (resp. prefix-comparable) or it equals 0 otherwise.

## 2 Monoids and languages of positive tiles

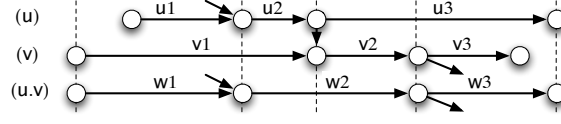
We review here the definition of the monoid of positive tiles. We also review how it can be seen as an ordered monoid with many additional properties. Our former characterization of MSO definable languages of tiles is then presented [6]. Here, this characterization is used as an alternative and simpler definition of this class of languages of tiles.

### 2.1 Positive tiles

The set  $T_A$  of positive tile on the alphabet  $A$  is defined as the set of triples of words  $u = (u_1, u_2, u_3) \in A^* \times A^* \times A^*$  extended with an extra tile 0 called the *undefined tile*.

For all non zero tiles  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$ , the *sequential product of  $u$  and  $v$*  is defined to be  $u.v = ((u_1.u_2 \vee_s v_1)u_2^{-1}, u_2v_2, v_2^{-1}(u_3 \vee_p v_2v_3))$  when both the *left matching constraint*  $u_1.u_2 \vee_s v_1 \neq 0$  ( $u_1u_2$  and  $v_1$  are suffix-comparable) and the *right matching constraint*  $u_3 \vee_p v_2v_3 \neq 0$  ( $u_3$  and  $v_2v_3$  are prefix-comparable) are satisfied. This definition is illustrated by the following figure where matching constraints are modeled, on the vertical dimension, by letter to letter equality.





The product is completed by  $u.v = 0$  when the matching constraint is not satisfied. For instance, with  $a, b$  and  $c$  three distinct letters, we have  $(a, b, c).(b, c, a) = (a, bc, a)$ ,  $(a, b, c).(a, b, c) = 0$  and  $(b, 1, ac).(ab, 1, a) = (ab, 1, ac)$ .

Set  $T_A$  equipped product of tiles is a monoid from now on called the monoid of positive tiles. The neutral element  $(1, 1, 1)$  is denoted by  $1$ .

## 2.2 Natural order

Monoid  $T_A$  is conveniently seen as an *ordered monoid* with the natural order relation defined for all  $u \in T_A$  by  $0 \leq u$  and for all  $u$  and  $v \in T_A$  with  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$ , by  $u \leq v$  when  $v_1 \leq_s u_1$ ,  $v_2 = u_2$  and  $v_3 \leq_p u_3$ .

The following lemma, proved in [6], summarizes some properties of the natural order proved. For all non zero tile  $u = (u_1, u_2, u_3)$ , let  $u_L = (u_1 u_2, 1, u_3)$  and let  $u_R = (u_1, 1, u_2 u_3)$ . This notation is extended to  $0$  by setting  $0_L = 0_R = 0$ .

**Lemma 1** *The set  $U(T_A) = \{t \in T_A : t \leq 1\}$  of subunits of  $T_A$ , is a commutative monoid of idempotent elements and, ordered by the natural order, it is a lattice with product as meet. Moreover, for all  $u \in T_A$ ,  $u_L = \bigwedge \{x \in U(T_A) : ux = u\}$  and  $u_R = \bigwedge \{x \in U(T_A) : xu = u\}$ , and, for all  $v \in T_A$ ,  $u \leq v$  if and only if  $u = u_R v u_L$ .*

The natural order also induces a structure theorem.

**Theorem 2** *Monoid  $T_A$  is completely determined by the submonoid  $\widehat{T}_A$  of its maximal elements (isomorphic to  $A^*$ ) and the lattice  $U(T_A)$  of its subunits.*

*Proof.* For all  $u \in A^*$ , let  $u_C = (1, u, 1)$ . The mapping  $u \mapsto u_C$  from  $A^*$  to  $T_A$  is a one-to-one morphism. These *canonical images* of words into tiles form the submonoid  $\widehat{T}_A$  of maximal elements w.r.t. the natural order of  $T_A$ . Staying coherent with notations above, for all  $u \in A^*$ , let also  $u_L = (u, 1, 1) = (u_C)_L$  and  $u_R = (1, 1, u) = (u_C)_R$ . Both  $u_L$  and  $u_R \in U(T_A)$ . Moreover, for all non zero tile  $(u, v, w) \in T_A$  we have  $(u, v, w) = u_L v_C w_R$ .  $\square$

## 2.3 MSO definable languages of tiles

Given a language  $L \subseteq T_A - 0$ , the *word congruence*  $\simeq_L$  associated to  $L$  is defined to be the greatest word congruence such that, for all  $u$  and  $v \in A^*$ ,  $u \simeq_L v$  when

for all  $w_1, w_2$  and  $w_3 \in A^*$ , the following equivalences hold:  $(w_1uw_2, w_3, w_4) \in L \Leftrightarrow (w_1vw_2, w_3, w_4) \in L$ ,  $(w_1, w_2uw_3, w_4) \in L \Leftrightarrow (w_1, w_2vw_3, w_4) \in L$ , and  $(w_1, w_2, w_3uw_4) \in L \Leftrightarrow (w_1, w_2, w_3vw_4) \in L$ . In [6], this word congruence is shown to capture, in the following sense, MSO definable languages.

**Theorem 3** *For all language  $L \subseteq T_A - 0$ ,  $L = \Sigma_{(u,v,w) \in L} [u]_L \times [v]_L \times [w]_L$  with, for all  $u \in A^*$ ,  $[u]_L = \{v \in A^* : u \simeq_L v\}$ . Moreover,  $L$  is MSO-definable if and only if the associated word congruence  $\simeq_L$  over  $A^*$  is of finite index and thus  $L$  equals a finite sum of Cartesian products of regular languages.*

### 3 Well-behaved ordered monoids and premorphisms

We provide in this section, the foundation of quasi-recognizability. It is based on the notion of well-behaved ordered monoids - an abstract generalization of the monoid of positive tiles, defined from the class of (stable) ordered monoid with zero [14] - and the notion of premorphisms. However, the adequate definition of quasi-recognizability itself is postponed to the next section.

#### 3.1 Well-behaved ordered monoids

Let  $S$  be an ordered monoid and let  $U(S) = \{x \in S : x \leq 1\}$ . Elements of  $U(S)$  are called the *subunits* of  $S$ . Monoid  $S$  is a *well-behaved ordered monoid* when it satisfies the following axioms:

- (A0)  $0 \in U(S)$ , i.e.  $0$  is a subunit,
- (A1) for all  $x, y$  and  $z \in S$ , if  $x \leq y$  then  $xz \leq yz$  and  $zx \leq zy$ , i.e. the order relation is *stable* by product,
- (A2) for all  $x \in U(S)$ ,  $x.x = x$ , i.e. *subunits are idempotents*,
- (A3) for all  $x$  and  $y \in S$ , if  $x \leq y$  then there is  $e$  and  $f \in U(S)$  such that  $x = eyf$ , i.e. the monoid order is the two-sided variant of Nambooripad's natural order [12].
- (A4) for all  $x \in S$ , both sets  $L_x = \{e \in U(S) : xe = x\}$  and  $R_x = \{e \in U(S) : ex = x\}$  have least element  $x_L = \bigwedge L_x$  and  $x_R = \bigwedge R_x$  with  $x_L \in L_x$  and  $x_R \in R_x$ .

Mappings  $x \mapsto x_L$  and  $x \mapsto x_R$  from  $S$  to  $U(S)$  are respectively called the *left* and the *right context operators* on  $S$ . One can check that these mappings are projective onto

mappings, i.e. for all  $x \in U(S)$ ,  $x_L = x_R = x$ . As they capture the order relation (see Lemma 4 below), this leads us, in [7], to propose an equivalent axiomatization of well-behaved monoids based on these left and right context operators.

But does there exist any well-behaved monoid? Actually yes, many! In particular, every monoid  $S$  can be completed into a well-behaved ordered monoid  $S^0$ . Indeed, let  $S^0$ , called the *trivial ordered extension* of  $S$ , be defined by  $S^0 = S + 0$  (with a new zero element) and the order relation defined, for all  $x$  and  $y \in S^0$ , by  $x \leq y$  if and only if  $x = 0$  or  $x = y$ . The fact that  $S^0$  is well-behaved is straightforward.

One can also check that the monoid of positive tiles is a well-behaved ordered monoid. The next two Lemmas illustrate how well-behaved monoids generalize the monoid of positive tiles.

**Lemma 4** *Let  $S$  be a well-behaved ordered monoid. Then  $0$  is the least element of  $S$  and  $U(S)$  is a submonoid of  $S$ ,  $U(S)$  ordered by natural order is a meet semi-lattice with product as meet, and, in particular,  $U(S)$  is a commutative submonoid.*

*Proof.* The fact that  $0$  is the least element easily follows from axioms (A0) and (A1). Properties of  $U(S)$  follows from axioms (A1), (A2) and reflexivity of the order relation.  $\square$

We observe that, in well-behaved ordered monoids, idempotents do not necessarily commute. More precisely, given  $E(S)$  the set of idempotents of  $S$ , axiom (A2) tells that  $U(S) \subseteq E(S)$  but still, it may happen that  $xy \neq yx$  for some  $x$  and  $y \in E(S) - U(S)$ . In the theory developed here, the distinction made between subunits and idempotents is essential.

**Lemma 5** *Let  $S$  be a well-behaved ordered monoid. Then for all  $x$  and  $y \in S$ ,  $x \leq y$  if and only if there exists  $e \in U(S)$  and  $f \in U(S)$  such that  $x = eyf$  if and only if  $x = x_R y x_L$ .*

*Proof.* Let  $x$  and  $y \in S$ . Assume  $x \leq y$ . By axiom (A3) there are  $e$  and  $f \in U(S)$  such that  $x = eyf$ . Now, by subunits idempotence (axiom A2), we have  $ex = x$  hence, by (A4)  $x_R \leq e$  and thus, by Lemma 4,  $x_R e = x_R$ . By symmetrical argument we also have  $f x_L = x_L$  and thus  $x = x_R e y f x_L$  hence  $x = x_R y x_L$ . The converse immediately follows from definition (both  $x_R$  and  $x_L \in U(S)$ ) and stability axiom (A4).  $\square$

### 3.2 Prehomomorphisms

The following definition is adapted from McAlister and Reilly [11]. Let  $S$  and  $T$  be two ordered monoids. A mapping  $\varphi : S \rightarrow T$  is a *prehomomorphism* when  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , for all  $x$  and  $y \in S$ , if  $x \leq y$  then  $\varphi(x) \leq \varphi(y)$  and, for all  $x$  and  $y \in S$ ,  $\varphi(xy) \leq \varphi(x)\varphi(y)$ .

Well-behaved ordered monoids and prehomomorphisms forms a category. Indeed, for every prehomomorphism  $\varphi : S \rightarrow T$  and  $\psi : T \rightarrow U$ , the mapping  $\varphi\psi : S \rightarrow U$  defined for all  $x \in S$  by  $\varphi\psi(x) = \psi(\varphi(x))$  is a prehomomorphism.

As a particular case, a prehomomorphism  $\varphi$  such that  $\varphi(xy) < \varphi(x)\varphi(y)$  if and only if  $xy = 0$  is called a *trivial prehomomorphism*. These prehomomorphisms are already worth being studied as illustrated by the following lemma.

**Lemma 6** *Let  $A$  be a finite alphabet and let  $L \subseteq A^*$  be a regular language. Let  $B = A + \#_1 + \#_2$  with  $\#_1$  and  $\#_2$  two distincts new letters. Let  $M = \{(1, \#_1 u \#_2, 1) \in T_B : u \in L\}$ . Then there is a finite monoid  $S$  and a trivial prehomomorphism  $\psi : T_B \rightarrow S^0$  such that  $M = \psi^{-1}(\psi(M))$ .*

*Proof.* Since  $L \subseteq A^*$  is recognizable so if  $L' = \#_1 L \#_2 \subseteq B^*$ . It follows that there is a finite monoid  $S$  and a morphism  $\varphi : B^* \rightarrow S$  such that  $L' = \varphi^{-1}(\varphi(L'))$ . Let then  $\psi : T_B \rightarrow S^0$  defined, for all  $(u, v, w) \in T_B$  by  $\psi((u, v, w)) = \varphi(v)$  when  $uvw \in \#_1 A^* \#_2$  and  $\psi((u, v, w)) = 0$  otherwise. Then one can easily check that  $M = \psi^{-1}(\varphi(L'))$  hence the claim.  $\square$

This Lemma tells us that with pre-images of pre-morphism from monoids of positives tiles to finite well-behaved ordered monoid, words being models as maximal tiles, we indeed generalize classical recognizability. However, we cannot take this as a definition of quasi-recognizability!

Indeed, the weakening of the morphism axiom  $\varphi(uv) = \varphi(u)\varphi(v)$  into the prehomomorphism axiom  $\varphi(uv) \leq \varphi(u)\varphi(v)$  just breaks many standard and useful properties of morphisms. For instance,  $\varphi(S)$  is not in general a submonoid of  $S$  since nothing ensures it is closed under product, i.e.  $(\varphi(S))^*$  is the submonoid induced by  $\varphi(S)$ . Even worse, over tiles, MSO definability just fails without extra hypothesis. These extra axioms are proposed in the next section.

## 4 From Quasi-recognizability to MSO definability

In this section, within the category of well-behaved monoid and prehomomorphisms, we want to settle our definition of quasi-recognizability in such a way that quasi-

recognizability implies MSO definability. This is achieved by paying a special attention to maximal elements and the way premorphisms behave on them.

## 4.1 Strongly well-behaved ordered monoids

A well-behaved ordered monoid  $S$  is said to be *strongly* well-behaved when it satisfies the following additional axioms:

- (A5) for all non zero  $x \in S$  there is a unique maximal element  $\hat{x} \in S$  such that  $x \leq \hat{x}$ , i.e.  $S$  is closed in some order theoretical sense,
- (A6) for all pair of non zero elements  $x$  and  $y \in S$ ,  $\hat{x}\hat{y} \neq 0$  and  $\widehat{\hat{x}\hat{y}} = \hat{x}\hat{y}$ , i.e. maximal elements form a submonoid.

**Theorem 7** *Every strongly well-behaved ordered monoid  $S$  is completely determined by the submonoid  $\hat{S}$  of its maximal elements and the semi-lattice  $U(S)$  of its subunits.*

*Proof.* By axiom (A5) and Lemma 5, for all  $x \in S$ ,  $x = x_R\hat{x}x_L$  with both  $x_R$  and  $x_L \in U(S)$ .  $\square$

The monoid  $T_A$  of positive tiles is strongly well-behaved and Theorem 7 generalizes to strongly well-behaved ordered monoids what Theorem 2 says about monoid  $T_A$ .

We review here more properties of these monoids.

**Lemma 8** *Every strongly well-behaved ordered monoid  $S$  is a meet semi-lattice. For all  $x$  and  $y \in S$ , either  $x \wedge y = 0$  or  $0 < x \wedge y = x_Ryx_L = y_Rxy_L$ .*

*Proof.* Let  $x$  and  $y \in S$  and let  $z \in S$  such that  $z \leq x$  and  $z \leq y$ . If this holds only for  $z = 0$  then  $x \wedge y = 0$  and we are done. Otherwise, assume  $z$  is non zero. By axiom (A5), we have  $\hat{z} = \hat{x} = \hat{y}$  hence  $y = y_R\hat{z}y_L$  and  $x = x_R\hat{z}x_L$ .

Let then  $t = x_Ryx_L$ . By commutation of subunits, we also have  $t = y_Rxy_L$ . We claim that  $t \leq y$ . Indeed,  $x_R \leq_p t$  hence  $t_Rx_R = t_R$ . Likewise,  $x_L \leq_s t$  hence  $t_Lx_L = t_L$ . But then, because  $t = x_Ryx_L$ , we also have  $t = t_Rtt_L = t_Rx_Ryx_Lt_L = t_Ryt_L$  hence  $t \leq y$ . A symmetrical argument show that  $t \leq x$ .

It remains to prove that  $z \leq t$ . Since  $z \leq x$  we have  $z = z_Rxz_L$ , hence  $x_Rzx_L = z$ . But since  $z \leq y$  we also have  $z = z_Ryz_L$  hence  $z = x_Rz_Ryz_Lx_R$  hence  $z = z_Rt_Lz_L$ .  $\square$

The formulation of the previous Lemma, distinguishing between the cases  $x \wedge y$  is zero or not, may seem a little strange. But we must put the emphasis on the fact that in an strongly well-behaved monoid  $S$ , even when both  $x_Ryx_L$  and  $y_Rxy_L$  are non zero for some  $x$  and  $y \in S$ , nothing ensures they are equal. Henceforth it

may still be the case that  $x \wedge y = 0$ . The well-behaved extension  $\mathcal{Q}(S)$  of arbitrary monoid  $S$  defined in the last section provides plenty of such examples.

The behavior of premorphisms on maximal elements is then restricted as follows. A premorphism  $\varphi : S \rightarrow T$  is a *well-behaved premorphism* when both  $S$  and  $T$  are strongly well-behaved monoids and the following condition are satisfied:

(P1) for all  $x$  and  $y \in \widehat{S}$ ,  $\varphi(xy) \in \widehat{T}$ ,

(P2) for all  $x, y$  and  $z \in \widehat{S}$ ,  $\varphi(x_L y z_R) = (\varphi(x))_L \varphi(y) (\varphi(z))_R$ ,

where  $\widehat{S}$  (resp.  $\widehat{T}$ ) denotes the set of maximal elements of  $S$  (resp.  $T$ ).

## 4.2 Quasi-recognizable languages

We say that  $L \subseteq S$  is *quasi-recognizable (QREC)* when there is a well-behaved premorphism  $\varphi : S \rightarrow M$  with finite  $M$  such that  $L = \varphi^{-1}(\varphi(L))$ .

**Theorem 9** *Quasi-recognizable subsets of  $T_A$  are definable in MSO, i.e. QREC  $\subseteq$  MSO.*

*Proof.* (sketch of) Let  $\varphi : T_A \rightarrow S$  be a well-behaved premorphism with  $S$  finite. It suffices to show that, for all  $x \in S$ ,  $\varphi^{-1}(x)$  is MSO-definable. Moreover, we can restrict to non zero elements since we have  $\varphi^{-1}(0) = T_A - \bigcup_{x \neq 0} \varphi^{-1}(x)$  and MSO definable languages are closed under finite boolean combination.

Let  $(u, v, w) \in T_A$ . By Theorem 2,  $(u, v, w) = u_L v_C w_R$  with  $u_L = (u, 1, 1)$ ,  $v_C = (1, v, 1)$  and  $w_R = (1, 1, w)$ . It follows, by applying axiom (P2), that  $\varphi((u, v, w)) = (\varphi(u_C))_L \varphi(v_C) (\varphi(w_C))_R$ . In other words, given  $\varphi_C : A^* \rightarrow S$  defined, for all  $u \in A^*$ , by  $\varphi_C(u) = \varphi(u_C)$ , for all non zero  $x \in S$ ,

$$\varphi^{-1}(x) = \bigcup \{ \varphi_C^{-1}(y) \times \varphi_C^{-1}(y') \times \varphi_C^{-1}(y'') \subseteq T_A : (y, y', y'') \in \widehat{S}, x = (y)_L y' (y'')_R \}$$

Since  $S$  is finite, this union is finite. Moreover, axiom (P1) ensures that  $\varphi_C$  is a morphism (since  $\widehat{T_A} = \{u_C \in T_A : u \in A^*\}$ ) hence, for all  $y \in \widehat{S}$ ,  $\varphi_C^{-1}(y) \subseteq A^*$  is a regular language. We conclude by applying Theorem 3.  $\square$

Does the converse of Theorem 9 hold? In general no. But this comes from a rather welcome property: left and right constraints in tiles are...just product constraints. It follows that quasi-recognizable languages satisfy a closure property on left and right constraints that is studied below.

### 4.3 Context coherence closure property

The following lemma tells that quasi-recognizable languages are closed w.r.t. to equivalent left and right constraints.

**Lemma 10** *Let  $\varphi : T_A \rightarrow S$  be a well-behaved premorphism with finite  $S$ . For all  $x \in S$ , for all  $(u, v, w) \in \varphi^{-1}(x)$ , for all  $u'$  and  $w' \in A^*$ , if  $\varphi(u)$  is  $\mathcal{L}$ -equivalent with  $\varphi(u')$  and  $\varphi(w)$  is  $\mathcal{R}$ -equivalent with  $\varphi(w')$ , then  $(u', v, w') \in \varphi^{-1}(x)$ .*

*Proof.* Let us first recall that two elements  $x$  and  $y \in S$  are  $\mathcal{L}$ -equivalent (resp.  $\mathcal{R}$ -equivalent) when both  $x \leq_s y$  and  $y \leq_s x$  (resp. both  $x \leq_p y$  and  $y \leq_p x$ ).

Since  $\varphi((u, v, w)) = (\varphi(u_C))_L \varphi(v_C) (\varphi(w_C))_R$ , the statement then follows from the easy observation that, for all  $x$  and  $y \in S$ , since  $S$  is well-behaved, if  $x$  and  $y$  are  $\mathcal{L}$ -equivalent (resp.  $\mathcal{R}$ -equivalent) then  $x_L = y_L$  (resp.  $x_R = y_R$ ).  $\square$

Observe that the underlying closure property is rather subtle. Indeed, we still lack of explicit canonical minimal structures (as syntactical monoids) that characterize quasi-recognizable languages of tiles. This means we still do not have a way to define that closure property in a minimal way. However, the word congruence  $\simeq_L$  associated to every language of tiles still gives us a canonical definition instead.

A language of tiles  $L \subseteq T_A$  is *context-coherent* when, given the induced word congruence  $\simeq_L$  associated to  $L$ , for all tiles  $(u, v, w) \in L$ , for all  $u'$  and  $v' \in A^*$ , if  $u$  and  $u'$  are  $\mathcal{L}$ -equivalent with respect to  $\simeq_L$  and  $w$  and  $w'$  are  $\mathcal{R}$ -equivalent with respect to  $\simeq_L$ , then  $(u', v, w') \in L$ .

Is this closure property a real loss in expressive power? We have seen in Lemma 6 that *plugs* can be used on words so that  $\mathcal{R}$ -equivalence on right constraints and  $\mathcal{L}$ -equivalence on left constraints trivialize in some sense. It follows that, from a modeling perspective, the context-coherence constraint is just a matter of modeling choice! In particular, as already mentioned in the introduction, this is probably enough in order to model the behavior of two-way automata on words where left and right plugs are classically used to mark words' extremities.

## 5 From MSO-definability to quasi-recognizability

Given a MSO definable language  $L \subseteq T_A$ , assumed to be context-coherent, we need now to provide a finite strongly well-behaved monoid that quasi-recognizes  $L$ .

By Theorem 3, since  $L$  is MSO definable, the word congruence  $\simeq_L$  associated to  $L$  is finite. Our point is then to built from  $S = A^* / \simeq_L$  (the finite monoid

induced by that congruence) a strongly well-behaved ordered monoid  $\mathcal{Q}(S)$  that quasi-recognizes  $L$  itself. This well-behaved extension  $\mathcal{Q}(S)$  of  $S$ , is presented in this section.

We will show that this extension is made in such a way that the monoid of positive tiles  $T_A$  itself becomes a submonoid of the well-behaved extension  $\mathcal{Q}(A^*)$  of the free monoid  $A^*$ . In other words, any morphism  $\varphi : A^* \rightarrow S$  can be lifted to a well-behaved premorphism from  $\mathcal{Q}(\varphi) : \mathcal{Q}(A^*) \rightarrow \mathcal{Q}(S)$ .

## 5.1 Prefix and suffix upper sets

Let  $S$  be a monoid. Let  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) be defined as the set of upward closed subsets of  $S$  preordered by  $\leq_p$  (resp.  $\leq_s$ ) the prefix (resp. suffix) preorder. More precisely, as  $S$  is a monoid hence with  $1 \in S$ ,  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) is the set of subsets  $U \subseteq S$  such that  $US = U$  (resp.  $SU = U$ ).

For both  $x = p$  or  $x = s$ , elements of  $\mathcal{U}_x(S)$  are from now on called *x-upper set*. The set  $\mathcal{U}_x(S)$  is turned into a monoid by taking  $\cap$  as product. Indeed, the intersection of two  $x$ -upper sets is a  $x$ -upper set and the neutral (or maximal) element is  $S$  itself, and  $\emptyset$  is the absorbant (or minimal) element.

In semigroup theory, non empty elements of  $\mathcal{U}_p(S)$  (resp.  $\mathcal{U}_s(S)$ ) are often called right ideals (resp. left ideals) of  $S$ . As ideals in order theory must satisfy some extra condition we prefer to stick to the notion of  $p$ -upper or  $s$ -upper sets.

**Lemma 11** *Let  $S$  be some monoid and let  $x \in S$ . Set  $xS$  is a  $p$ -upper set (resp.  $Sx$  a  $s$ -upper set) and, for every  $p$ -upper set (resp.  $s$ -upper set)  $U \subseteq S$ :*

- (1) *if  $x \in U$  then  $x^{-1}(U) = S$  (resp.  $(U)x^{-1} = S$ ),*
- (2)  *$xU$  is a  $p$ -upper set (resp.  $Ux$  is a  $s$ -upper set),*
- (3)  *$x^{-1}(U)$  is a  $p$ -upper set (resp.  $(U)x^{-1}$  is a  $s$ -upper sets),*
- (4)  *$xx^{-1}(U) \subseteq U \subseteq x^{-1}(xU)$  (resp.  $(U)x^{-1}x \subseteq U \subseteq (Ux)x^{-1}$ ),*

## 5.2 The strongly well-behaved extension

Let  $S$  be a monoid. The extension  $\mathcal{Q}(S)$  of  $S$  is defined to be  $\mathcal{Q}(S) = (\mathcal{U}_s(S) - \emptyset) \times S \times (\mathcal{U}_p(S) - \emptyset) + 0$  with, for all non zero element  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2)$  the product  $u_1 u_2$  defined by  $u_1 . u_2 = (L_1 \cap (L_2)x_1^{-1}, x_1 x_2, R_2 \cap x_2^{-1}(R_1))$  when both compatibility constraints  $L_1 \cap (L_2)x_1^{-1} \neq \emptyset$  and  $R_2 \cap x_2^{-1}(R_1) \neq \emptyset$  are satisfied, and by  $u_1 . u_2 = 0$  otherwise.



The expected natural order is defined as follows. For all pairs of non zero elements  $u_1 = (L_1, x_1, R_1)$  and  $u_2 = (L_2, x_2, R_2) \in \mathcal{Q}(S)$ , we say that  $u_1 \leq u_2$  when  $L_1 \subseteq L_2$ ,  $x_1 = x_2$  and  $R_1 \subseteq R_2$ . This relation is extended to zero by taking  $0 \leq u$  for all  $u \in \mathcal{Q}(S)$ .

As an ordered monoid, we already have:

**Lemma 12** *The mapping  $i : S^0 \rightarrow \mathcal{Q}(S)$  that maps zero to zero and any non zero element  $x \in S$  to  $i(x) = (S, x, S)$  is a one-to-one monoid morphism. The set  $i(S)$ , isomorphic to  $S$ , is the submonoid of  $\mathcal{Q}(S)$  that contains exactly all maximal elements of  $\mathcal{Q}(S)$ . The mapping  $\pi : \mathcal{Q}(S) \rightarrow S^0$  that maps 0 to 0 and any non zero element  $(L, x, R)$  to  $\pi(L, x, R) = x$  is an onto trivial premorphism with  $\pi \circ i = I_{S^0}$ , the identity mapping in  $S^0$ .*

*Proof.* The fact that  $i$  is a monoid morphism is immediate and the product on elements of  $i(S)$ , all of the form  $(S, x, S)$ , just mimics the product in  $S$  since, for all  $x \in S$ ,  $x^{-1}(S) = (S)x^{-1} = S$ .

Mapping  $\pi$  is obviously onto since  $\pi \circ i = I_{S^0}$ . Then we check that it is a (trivial) premorphism, i.e. for all  $u_1$  and  $u_2 \in \mathcal{Q}(S)$  either  $u_1 u_2 \neq 0$  and then  $\pi(u_1 u_2) = \pi_S(u_1) \pi_S(u_2)$  or  $u_1 u_2 = 0$  and thus  $\pi(u_1 u_2) = 0$ .  $\square$

Moreover, as intended:

**Theorem 13** *For all monoid  $S$ , the monoid  $\mathcal{Q}(S)$  ordered by the extension order  $\leq$  is a strongly well-behaved ordered monoid.*

*Proof.* This essentially follows from Lemma 14 that tells  $\mathcal{Q}(S)$  is indeed a (stable) ordered monoid, from Lemma 15 that ensures subunits are idempotents and from Lemma 16 that show the extension order is natural.

Other axioms immediately follows from definitions and from the fact that over  $p$ -upper sets (resp.  $s$ -upper sets) intersection distribute with left (resp. right) product and residual by an elements  $x \in S$ .  $\square$

**Lemma 14** *The extension monoid  $\mathcal{Q}(S)$  ordered by the extension order is a (stable) ordered monoid.*

*Proof.* The fact that relation  $\leq$  is a partial order is immediate from the definition. Let then  $u = (L_1, x_1, R_1)$ ,  $v = (L_2, x_2, R_2)$  and  $w = (M, y, N)$ . Assume  $u \leq v$ . By definition, we have  $x_1 = x_2$  from now on denoted by  $x$  and both  $L_1 \subseteq L_2$  and  $R_1 \subseteq R_2$ . By definition of the product we have  $uw = (L_1 \cap (M)x^{-1}, xy, N \cap y^{-1}(R_1))$  and  $vw = (L_2 \cap (M_2)x^{-1}, xy, N \cap y^{-1}(R_2))$ . Hence  $uw \leq vw$  by definition of the

extension order and stability of the inclusion order by intersection and residual. Symmetrical arguments show that  $wu \leq wv$ .  $\square$

The set of subunits of  $\mathcal{Q}(S)$  is, by definition of the extension order, the set  $U(\mathcal{Q}(S)) = 0 + \{(L, x, R) \in \mathcal{Q}(S) : x = 1\}$ . The next lemma shows that these indeed are idempotent elements.

**Lemma 15** *A non zero triple  $(L, x, R) \in \mathcal{Q}(S)$  is idempotent if and only if  $x \in S$  is idempotent,  $L \subseteq (L)x^{-1}$  and  $R \subseteq x^{-1}(R)$ .*

*Proof.* Let  $(L, x, R)$  be an idempotent of  $\mathcal{Q}(S)$ . By definition of the product in  $\mathcal{Q}(S)$  this means that  $xx = x$  hence  $x$  is idempotent in  $S$ ,  $L \cap (L)x^{-1} = L$  hence  $L \subseteq (L)x^{-1}$  and  $R \cap x^{-1}(R) = R$  hence  $R \subseteq x^{-1}R$ . The converse is immediate.  $\square$

**Lemma 16** *For all non zero  $u = (L, x, R)$  and  $v = (M, y, N) \in \mathcal{Q}(S)$ , if  $u \leq v$  then  $(L, x, R) = (L, 1, xR).(M, y, N).(Lx, 1, R)$ .*

*Proof.* Assume  $u \leq v$ . By definition of the extension order this implies that  $x = y$  and  $L \subseteq M$  and  $R \subseteq N$ . Taking  $e = (L, 1, xR)$  and  $f = (Lx, 1, R)$  one has

$$evf = (L \cap M \cap (Lx)y^{-1}, y, y^{-1}(xR) \cap N \cap R)$$

But, by assumption,  $L \subseteq M$  and  $R \subseteq N$  and, since  $x = y$ , we also have by Lemma 11  $L \subseteq (Lx)y^{-1}$  and  $R \subseteq y^{-1}(xR)$  hence  $evf = u$ .  $\square$

Last, the next Lemma shows that the elements used above are indeed the left and right contexts elements associated with  $u$ .

**Lemma 17** *For all non zero  $u = (L, x, R)$ , we have  $u_L = (Lx, 1, R)$  and  $u_R = (L, 1, xR)$ , i.e.  $u_Ru = u = uu_L$  and these are the least subunits that satisfies this properties.*

*Proof.* Let  $u_L$  and  $u_R$  be defined as in axiom (A3). Let  $u'_L = (Lx, 1, R)$ . Observe first that we do have  $uu'_L = u$ . In fact  $uu'_L = (L \cap (Lx)x^{-1}, x, R)$  with, by Lemma 11,  $L \subseteq (Lx)x^{-1}$  hence  $uu'_L = u$ . By definition of  $u_L$  this means  $u_L \leq u'_L$ .

Let then  $e = (L', 1, R')$  be a subunit of  $\mathcal{Q}(S)$  such that  $ue = u$ . This means that  $(L \cap (L')x^{-1}, x, R' \cap R) = (L, x, R)$  hence  $L \subseteq (L')x^{-1}$  henceforth  $Lx \subseteq L'$  and  $R \subseteq R'$ . It follows that  $u'_L \leq e$ . As this is true for all  $e$  as above, this means that  $u'_L \leq u_L$ .

Symmetrical arguments prove the claim for  $u_R$ .  $\square$

Last, the following theorem says that our construction above essentially extends to arbitrary monoids the way the monoid of positive tiles is built from the free monoid  $A^*$ .

**Theorem 18** *There is a one to one morphism  $i : T_A \rightarrow \mathcal{Q}(A^*)$  such that, for all  $u \in T_A$ ,  $i(u_L) = (i(u))_L$  and  $i(u_R) = (i(u))_R$ .*

*Proof.* Observe that  $A^*$  is totally ordered by  $\leq_s$  and  $\leq_p$ . It follows that for  $x = p$  and  $x = s$ , the mapping  $\varphi_x : A^* + 0 \rightarrow \mathcal{U}_x(A^*)$  defined, for every  $u \in A^*$ , by  $\varphi_x(u) = \{v \in A^* : u \leq_x v\}$  is one-to-one. It is then an easy task to check that  $i : T_A \rightarrow \mathcal{Q}(A^*)$  defined by  $i(0) = 0$  and, for every tile  $(u, v, w) \in A_T$ ,  $i((u, v, w)) = (\varphi_s(u), v, \varphi_p(w))$  is a one-to-one morphism. The last property is immediate.  $\square$

### 5.3 More on maximal elements in the extension

We observe now that the extension monoid is a non trivial well-ordered monoid.

**Lemma 19** *For all  $u$  and  $v \in \mathcal{Q}(S)$ , if  $v$  is maximal then  $v_R = u_R v_R$  (resp.  $v_L = v_L u_L$ ) if and only if  $u \leq_p v$  (resp.  $u \leq_s v$ ).*

*In particular, with  $v = 1$  hence  $v_R = v_L = 1$ , we have  $u_R = 1$  (resp.  $u_L = 1$ ) if and only  $uw = 1$  (resp.  $wu = 1$ ) for some  $w \in \mathcal{Q}(S)$ , i.e.  $u$  admits a right (resp. left) group inverse.*

*Proof.* Let then  $u = (L, x, R)$  hence  $u_R = (L, 1, xR)$  and let  $v = (S, y, S)$  some maximal element of  $\mathcal{Q}(S)$ .

If  $u \leq_p v$  then we immediately have, by definition of the right context operator  $u_R v_R = v_R$ .

Assume that  $v_R = u_R v_R$  henceforth, by commutation and right product by  $v$ ,  $u_R v = v$ . By definition of the product, we have  $u_R v = (L, y, y^{-1}(xR))$ . It follows that (a)  $S = L$  and that  $S = y^{-1}(xR)$  with  $y^{-1}(xR) = \{z \in S : yz \in xR\}$ . Since  $1 \in S$ , this means that  $y \in xR$  and thus (b)  $y = xz$  for some  $z \in R$ . Taking then  $w = (S, z, S)$  we have, by definition of the product,

$$uw = (L \cap (S)x^{-1}, xz, S \cap z^{-1}(R))$$

Now by (a)  $L = S$  and since  $x \in S$ , by Lemma 11,  $(S)x^{-1} = S$ . By (b)  $xz = y$  and, since  $z \in R$ , again by Lemma 11,  $z^{-1}(R) = S$ . It follows that  $uw = v$  henceforth  $u \leq_p v$ .

The case of left context and suffix preorder is obtained by symmetrical arguments.  $\square$

## 5.4 From MSO-definability to quasi-recognizability

We are now ready to make the picture complete. More precisely:

**Theorem 20** *If  $L \subseteq T_A$  is MSO definable and context coherent then  $L$  is quasi-recognizable.*

*Proof.* Let  $L \subseteq T_A$  be an MSO definable language of positive tiles and let  $S = A^* / \simeq_L$  be the finite monoid defined by the quotient of  $A^*$  under the (finite index) word congruence induced by  $L$ . For every word  $u \in A^*$ , let us write  $[u] \in S$  the class of words of  $A^*$  equivalent under  $\simeq_L$  to  $u$ .

Let define  $\varphi : T_A \rightarrow \mathcal{Q}(S)$  by taking, for every non zero positive tile  $u = (u_1, u_2, u_3) \in T_A$ ,  $\varphi(u) = (S[u_1], [u_2], [u_3]S)$ . By Lemma 21 below, we know that  $\varphi$  is a well-behaved premorphism. Moreover, by construction,  $L \subseteq \varphi^{-1}(\varphi(L))$ . The converse inclusion follows from the fact that  $L$  is context coherent.  $\square$

**Lemma 21** *Let  $\psi : A^* \rightarrow S$  be a monoid morphism. The extension mapping  $\varphi : T_A \rightarrow \mathcal{Q}(S)$  defined for every non zero tile  $(u, v, w) \in T_A$  by  $\varphi((u, v, w)) = (S\psi(u), \psi(v), \psi(w)S)$ , is a well-behaved premorphism.*

*Proof.* Let  $\psi : A^* \rightarrow S$  and  $\varphi : T_A \rightarrow \mathcal{Q}(S)$  as above.

In order to simplify notation, for all  $u \in A^*$ , we write  $[u] \in S$  for the image of  $u$  by  $\psi$  in  $S$ . This notation is coherent with the fact that  $S$  can just be restricted to be the quotient of  $A^*$  by the congruence induced by  $\psi$ .

By construction,  $\varphi(1) = (S, 1, S)$  which is indeed the neutral element of  $\mathcal{Q}(S)$ .

For every  $u \in A^*$ , let us write,  $\varphi_s(u) = S[u]$  (resp.  $\varphi_p(u) = [u]S$ ). Mappings  $\varphi_s : A^* \rightarrow \mathcal{U}_s(S)$  and  $\varphi_p : A^* \rightarrow \mathcal{U}_p(S)$  are the left and right projection of  $\varphi$ . In particular, both for  $x = p$  and  $x = s$ , for every  $u$  and  $v \in A^*$ , if  $u \leq_x v$  then  $\varphi_x(u) \supseteq \varphi_x(v)$  and, if  $u \vee_x v \neq 0$  then  $\varphi_x(u \vee_x v) = \varphi_x(u) \cap \varphi_x(v)$ .

Let then  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$  be two non zero positive tiles in  $T_A$ . By definition,  $\varphi(u) = (\varphi_s(u_1), [u_2], \varphi_p(u_3))$ .

We now prove monotonicity. If  $u \leq v$  we have  $v_1 \leq_s u_1$ ,  $v_2 = u_2$  and  $v_3 \leq_p u_3$ . This means that  $[v_1] \leq_s [u_1]$  hence  $\varphi_s(u_1) \subseteq \varphi_s(v_1)$ ,  $[u_2] = [v_2]$  and  $[v_3] \leq_p [u_3]$  hence  $\varphi_p(u_3) \subseteq \varphi_p(v_3)$ . It follows, by definition of the extension order, that  $\varphi(u) \leq \varphi(v)$ . Since  $u$  and  $v$  are arbitrary, this means  $\varphi$  is monotonic.

We now prove that  $\varphi(uv) \leq \varphi(u)\varphi(v)$ . If  $uv = 0$  nothing has to be done since  $\varphi(0)$  is the smallest in  $\mathcal{Q}(S)$  hence the premorphism property is satisfied. Assume

thus  $uv$  is non zero. we have

$$uv = ((u_1u_2 \vee_s v_1)u_2^{-1}, u_2v_2, v_2^{-1}(u_1 \vee_p v_2v_3))$$

since  $[u_2v_2] = [u_2][v_2]$ , by definition of the product of  $\varphi(u)\varphi(v)$  in  $\mathcal{Q}(S)$ , by definition of the extension order, we have to prove that

$$\varphi_s((u_1u_2 \vee_s v_1)u_2^{-1}) \subseteq \varphi_s(u_1) \cap (\varphi_s(v_1))[u_2]^{-1}$$

and

$$\varphi_p(v_2^{-1}(u_3 \vee_p v_2v_3)) \subseteq [v_2]^{-1}(\varphi_p(u_3)) \cap \varphi_p(v_3)$$

By symmetry it suffices to prove it for the prefix case.

Observe that since the product  $uv$  is non zero, we have  $u_3 \vee_p v_2v_3 \neq 0$ .

The first case is when  $u_3 \leq_p v_2v_3$ . This means that  $v_2^{-1}(u_3 \vee_p v_2v_3) = v_3$  hence  $\varphi_p(v_2^{-1}(u_3 \vee_p v_2v_3)) = \varphi_p(v_3)$ . But this also means that  $[v_2] \in \varphi_p(u_3) = [u_3]S$  since  $[u_3] \leq_p [v_2]$  hence, by Lemma 11,  $[v_2]^{-1}(\varphi_p(u_3)) = S$ . Observe that in this case, we do have equality.

The second case is when  $v_2v_3 \leq_p u_3$ . This means that  $v_2^{-1}(u_3 \vee_p v_2v_3) = v_2^{-1}(u_3) = w$  for some  $w \in A^*$  such that  $u_3 = v_2w$ . It follows that:

$$\varphi_p(v_2^{-1}(u_3 \vee_p v_2v_3)) = \varphi_p(v_2^{-1}(u_3)) = [w]S$$

and

$$[v_2]^{-1}(\varphi_p(u_3)) = [v_2]^{-1}([v_2w]S) = [v_2]^{-1}([v_2][w]S)$$

hence, by Lemma 11, the result since  $[w]S \subseteq [v_2]^{-1}([v_2][w]S)$ .

Observe that in that case, hence in general, we do not have equality. This means that premorphism  $\varphi$  from  $T_A$  to  $\mathcal{Q}(S)$  is not, in general, a trivial premorphism, i.e. it may happen that, for some  $u$  and  $v \in T_A$ ,  $\varphi(uv) < \varphi(u)\varphi(v)$  with  $\varphi(uv) \neq 0$ .  $\square$

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