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# CONVERGENCE OF LAGRANGE INTERPOLATION SERIES IN THE FOCK SPACES

ANDRÉ DUMONT AND KARIM KELLAY

ABSTRACT. We study the uniqueness sets, the weak interpolation sets, and convergence of the Lagrange interpolation series in radial weighted Fock spaces.

## 1. INTRODUCTION AND MAIN RESULTS.

In this paper we study the weighted Fock spaces  $\mathcal{F}_\varphi^2(\mathbb{C})$

$$\mathcal{F}_\varphi^2(\mathbb{C}) = \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_\varphi^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(|z|)} dm(z) < \infty \right\};$$

here  $dm$  is area measure and  $\varphi$  is an increasing function defined on  $[0, +\infty)$ ,  $\lim_{x \rightarrow \infty} \varphi(x) = \infty$ . We assume that  $\varphi(z) = \varphi(|z|)$  is  $C^2$  smooth and subharmonic on  $\mathbb{C}$ , and set

$$\rho(z) = (\Delta\varphi(z))^{-1/2}.$$

One more condition on  $\varphi$  is that for every fixed  $C$ ,

$$\rho(x + C\rho(x)) \asymp \rho(x), \quad 0 < x < \infty.$$

(In particular, this holds if  $\rho'(x) = o(1)$ ,  $x \rightarrow \infty$ .)

Typical  $\varphi$  are power functions

$$\varphi(r) = r^a, \quad a > 0.$$

Then

$$\rho(x) \asymp x^{1-a/2}, \quad x > 1.$$

Furthermore, if

$$\varphi(r) = (\log r)^2,$$

then

$$\rho(x) \asymp x, \quad x > 1.$$

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Given  $z, w \in \mathbb{C}$ , we define

$$d_\rho(z, w) = \frac{|z - w|}{\min(\rho(z), \rho(w))}.$$

We say that a subset  $\Lambda$  of  $\mathbb{C}$  is  $d_\rho$ -separated if

$$\inf_{\lambda \neq \lambda^*} \{d_\rho(\lambda, \lambda^*), \lambda, \lambda^* \in \Lambda\} > 0.$$

**Definition 1.** Given  $\gamma \in \mathbb{R}$ , we say that an entire function  $S$  belongs to the class  $\mathcal{S}_\gamma$  if

- (1) the zero set  $\Lambda$  of  $S$  is  $d_\rho$ -separated, and
- (2)

$$|S(z)| \asymp e^{\varphi(z)} \frac{d(z, \Lambda)}{\rho(z)} \frac{1}{(1 + |z|)^\gamma}, \quad z \in \mathbb{C}.$$

For constructions of such functions in radial weighted Fock spaces see, for example, [1] and [2].

In the standard Fock spaces ( $\varphi(r) = r^2$ ) the classes  $\mathcal{S}_\gamma$  were introduced by Lyubarskii in [4]. They are analogs of the sine type functions for the Paley–Wiener space, and their zero sets include rectangular lattices and their perturbations.

**Definition 2.** A set  $\Lambda \subset \mathbb{C}$  is called a weak interpolation set for  $\mathcal{F}_\varphi^2(\mathbb{C})$  if for every  $\lambda \in \Lambda$  there exists  $f_\lambda \in \mathcal{F}_\varphi^2(\mathbb{C})$  such that  $f_\lambda(\lambda) = 1$  and  $f_\lambda|_{\Lambda \setminus \{\lambda\}} = 0$ .

A set  $\Lambda \subset \mathbb{C}$  is called a uniqueness set for  $\mathcal{F}_\varphi^2(\mathbb{C})$  if  $f \in \mathcal{F}_\varphi^2(\mathbb{C})$  and  $f|_\Lambda = 0$  imply together that  $f = 0$ .

**Theorem 3.** Let  $S \in \mathcal{S}_\gamma$ , and denote by  $\Lambda$  the zero set of  $S$ . Then

- (a)  $\Lambda$  is a uniqueness set if and only if  $\gamma \leq 1$ ,
- (b)  $\Lambda$  is a weak interpolation set if and only if  $\gamma > 0$ .

Denote by  $\mathbb{k}_z$  the reproducing kernel in the space  $\mathcal{F}_\varphi^2(\mathbb{C})$ :

$$\langle f, \mathbb{k}_z, \rangle_{\mathcal{F}_\varphi^2(\mathbb{C})} = f(z), \quad f \in \mathcal{F}_\varphi^2(\mathbb{C}), \quad z \in \mathbb{C}.$$

It is known [6, 3, 2] that the space  $\mathcal{F}_\varphi^2(\mathbb{C})$  does not admit Riesz bases of the (normalized) reproducing kernels for regular  $\varphi$ ,  $\varphi(x) \gg (\log x)^2$ . On the other hand, by Theorem 3, for  $0 < \gamma \leq 1$ , the family  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  is a complete minimal family in  $\mathcal{F}_\varphi^2(\mathbb{C})$ . Then the family  $\{S/[S'(\lambda)(\cdot - \lambda)]\}_{\lambda \in \Lambda}$  is the biorthogonal system and we associate to any  $f \in \mathcal{F}_\varphi^2(\mathbb{C})$  the formal (Lagrange interpolation) series

$$f \sim \sum_{\lambda \in \Lambda} f(\lambda) \frac{S}{S'(\lambda)(\cdot - \lambda)}.$$

It is natural to ask whether this formal series converges if we modify the norm of the space.

Denote by  $\Lambda = \{\lambda_k\}$  the zero sequence of  $S$  ordered in such a way that  $|\lambda_k| \leq |\lambda_{k+1}|$ ,  $k \geq 1$ . Following Lyubarskii [4] and Lyubarskii–Seip [5] we obtain the following result:

**Theorem 4.** *Let  $0 \leq \beta \leq 1$ ,  $\gamma + \beta \in (1/2, 1)$ , and let  $S \in \mathcal{S}_\gamma$ . Suppose that*

$$r^{1-2\beta} = O(\rho(r)), \quad r \rightarrow +\infty. \quad (1)$$

*Then for every  $f \in \mathcal{F}_\varphi^2(\mathbb{C})$  we have*

$$\lim_{N \rightarrow \infty} \left\| f - S \sum_{k=1}^N \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)} \right\|_{\varphi_\beta} = 0,$$

where  $\varphi_\beta(r) = \varphi(r) + \beta \log(1+r)$ .

The result corresponding to  $\beta = 1/2$ ,  $\varphi(r) = r^2$ ,  $\rho(r) \asymp 1$  is contained in [5, Theorem 10]. On the other hand, in the case  $\varphi(r) = (\log r)^a$ ,  $1 < a \leq 2$ ,  $r > 2$ ,  $\rho(r) \asymp r$ , the space  $\mathcal{F}_\varphi^2(\mathbb{C})$  contains Riesz bases of (normalized) reproducing kernels [2]. Our theorem shows that in the case  $\rho(r) \asymp r$ , when  $S \in \mathcal{S}_\gamma$ ,  $\gamma \in (1/2, 1)$ , the interpolation series converges already in  $\mathcal{F}_\varphi^2(\mathbb{C})$ . Now, it is interesting to find out how sharp is condition (1) in Theorem 4.

**Theorem 5.** *Let  $0 < a \leq 2$ ,  $\varphi(r) = r^a$ ,  $r > 1$ ,  $\rho(x) \asymp x^{1-a/2}$ ,  $x > 1$ . If  $0 \leq \beta < a/4$ ,  $\gamma \in \mathbb{R}$ , and  $S \in \mathcal{S}_\gamma$ , then there exists  $f \in \mathcal{F}_\varphi^2(\mathbb{C})$  such that*

$$\left\| f - S \sum_{k=1}^N \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)} \right\|_{\varphi_\beta} \not\rightarrow 0, \quad N \rightarrow \infty.$$

Thus, for the power weights  $\varphi(r) = r^a$ ,  $0 < a \leq 2$ , we need to modify the norm to get the convergence, and the critical value of  $\beta$  is  $a/4$ .

The notation  $A \lesssim B$  means that there is a constant  $C$  independent of the relevant variables such that  $A \leq CB$ . We write  $A \asymp B$  if both  $A \lesssim B$  and  $B \lesssim A$ .

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## 2. PROOFS.

**2.1. Proof of Theorem 3.** (a) If  $\gamma > 1$ , then  $\mathcal{S}_\gamma \in \mathcal{F}_\varphi^2(\mathbb{C})$  and  $S|_\Lambda = 0$ . Hence,  $\Lambda$  is not a uniqueness set.

If  $\gamma \leq 1$ , then  $\mathcal{S}_\gamma \cap \mathcal{F}_\varphi^2(\mathbb{C}) = \emptyset$ . Suppose that there exists  $g \in \mathcal{F}_\varphi^2(\mathbb{C})$  such that  $g|_\Lambda = 0$ . Then  $g = FS$  for an entire function  $F$ , and

$$\int_{\mathbb{C}} |F(w)|^2 |S(w)|^2 e^{-2\varphi(w)} dm(w) < \infty. \quad (2)$$

Given  $\Omega \subset \mathbb{C}$ , denote

$$\mathcal{I}[\Omega] = \int_{\Omega} |F(w)|^2 \frac{d^2(w, \Lambda)}{(1 + |w|)^{2\gamma} \rho^2(w)} dm(w).$$

By (2),

$$\mathcal{I}[\mathbb{C}] < \infty.$$

Denote by  $D(z, r)$  the disc of radius  $r$  centered at  $z$ . Let

$$\Omega_\varepsilon = \bigcup_{\lambda \in \Lambda} D(\lambda, \varepsilon \rho(\lambda)),$$

where  $\varepsilon$  is such that the discs  $D(\lambda, 2\varepsilon \rho(\lambda))$  are pairwise disjoint. We have

$$\mathcal{I}[\mathbb{C}] = \mathcal{I}[\mathbb{C} \setminus \Omega_{2\varepsilon}] + \sum_{\lambda \in \Lambda} \mathcal{I}[D(\lambda, 2\varepsilon \rho(\lambda)) \setminus D(\lambda, \varepsilon \rho(\lambda))] + \mathcal{I}[D(\lambda, \varepsilon \rho(\lambda))].$$

It is clear that

$$\mathcal{I}[\mathbb{C} \setminus \Omega_{2\varepsilon}] \geq c_1 \int_{\mathbb{C} \setminus \Omega_{2\varepsilon}} \frac{|F(w)|^2}{(1 + |w|)^{2\gamma}} dm(w).$$

On the other hand,

$$\int_{D(\lambda, \varepsilon \rho(\lambda))} |F(w)|^2 dm(w) \leq c_2 \int_{D(\lambda, 2\varepsilon \rho(\lambda)) \setminus D(\lambda, \varepsilon \rho(\lambda))} |F(w)|^2 dm(w),$$

and, hence,

$$\mathcal{I}[D(\lambda, 2\varepsilon \rho(\lambda)) \setminus D(\lambda, \varepsilon \rho(\lambda))] \geq c_3 \mathcal{I}[D(\lambda, \varepsilon \rho(\lambda))].$$

Therefore

$$\int_{\mathbb{C}} \frac{|F(w)|^2}{(1 + |w|)^{2\gamma}} dm(w) < \infty,$$

the function  $F$  is constant, and  $g = cS$ . Since  $\mathcal{S}_\gamma \cap \mathcal{F}_\varphi^2(\mathbb{C}) = \emptyset$ , we get a contradiction. Statement (a) is proved.

(b) Let  $\gamma > 0$ . Set

$$f_\lambda(z) = \frac{S(z)}{S'(\lambda)(z - \lambda)}, \quad \lambda \in \Lambda.$$

It is obvious that  $f_\lambda \in \mathcal{F}_\varphi^2(\mathbb{C})$ ,  $f_\lambda|_{\Lambda \setminus \{\lambda\}} = 0$  and  $f_\lambda(\lambda) = 1$ . Hence  $\Lambda$  is a weak interpolation set. If  $\gamma \leq 0$ ,  $\lambda \in \Lambda$ , then by (a),  $\Lambda \setminus \{\lambda\}$  is a uniqueness set for  $\mathcal{F}_\varphi^2(\mathbb{C})$ . Therefore,  $\Lambda$  is not a weak interpolation set for  $\mathcal{F}_\varphi^2(\mathbb{C})$ .  $\square$

**2.2. Proof of Theorem 4.** We follow the scheme of proof proposed in [4, 5] and concentrate mainly on the places where the proofs differ. We need some auxiliary notions and lemmas. The proof of the first lemma is the same as in [1, Lemma 4.1].

**Lemma 6.** *For every  $\delta > 0$ , there exists  $C > 0$  such that for functions  $f$  holomorphic in  $D(z, \delta\rho(z))$  we have*

$$|f(z)|^2 e^{-2\varphi(z)} \leq \frac{C}{\rho(z)^2} \int_{D(z, \delta\rho(z))} |f(w)|^2 e^{-2\varphi(w)} dm(w).$$

**Definition 7.** *A simple closed curve  $\gamma = \{r(\theta)e^{-i\theta}, \theta \in [0, 2\pi]\}$  is called  $K$ -bounded if  $r$  is  $C^1$ -smooth and  $2\pi$ -periodic on the real line and  $|r'(\theta)| \leq K, \theta \in \mathbb{R}$ .*

Let  $\gamma \in \mathbb{R}$ , let  $S \in \mathcal{S}_\gamma$ , and let  $\Lambda = \{\lambda_k\}$  be the zero set of  $S$  ordered in such a way that  $|\lambda_k| \leq |\lambda_{k+1}|, k \geq 1$ . We can construct a sequence of numbers  $R_N \rightarrow \infty$  and a sequence of contours  $\Gamma_N$  such that

- (1)  $\Gamma_N = R_N \gamma_N$ , where  $\gamma_N$  are  $K$ -bounded, with  $K > 0$  independent of  $N$ .
- (2)  $d_\rho(\Lambda, \Gamma_N) \geq \varepsilon$  for some  $\varepsilon > 0$  independent of  $N$ .
- (3)  $\{\lambda_k\}_1^N$  lie inside  $\Gamma_N$  and  $\{\lambda_k\}_{N+1}^\infty$  lie outside  $\Gamma_N$ .
- (4)  $\Gamma_N \subset \{z : R_N - \rho(R_N) < |z| < R_N + \rho(R_N)\}$ .

Indeed, for some  $0 < \varepsilon < 1$  the discs  $D_k = D(\lambda_k, \varepsilon\rho(\lambda_k))$  are disjoint. For some  $\delta = \delta(\varepsilon) > 0$  we have

$$\varepsilon\rho(\lambda_k) > 4\delta\rho(\lambda_N), \quad ||\lambda_k| - |\lambda_N|| < \delta\rho(\lambda_N).$$

Fix  $\psi \in C_0^\infty[-1, 1], 0 < \psi < 1$ , such that  $\psi > 1/2$  on  $[-1/2, 1/2]$ .

Put  $\Xi = \{k : ||\lambda_k| - |\lambda_N|| < \frac{1}{4}\delta\rho(\lambda_N)\}$ , denote  $\lambda_k = r_k e^{i\theta_k}, k \in \Xi$ , and set

$$r(\theta) = 1 + \sum_{k \in \Xi} s_k \frac{\delta\rho(\lambda_N)}{|\lambda_N|} \psi\left(\frac{|\lambda_N|}{\delta\rho(\lambda_N)}(\theta - \theta_k)\right),$$

where  $s_k = 1, k \leq N, s_k = -1, k > N$ .

Finally, set

$$\gamma_N = \{r(\theta)e^{i\theta}, \theta \in [0, 2\pi]\}.$$

**Lemma 8.**

$$R_N \rho(R_N) \int_{\gamma_N} |f(R_N \zeta)|^2 e^{-2\varphi(R_N \zeta)} |d\zeta| \rightarrow 0, \quad N \rightarrow \infty.$$

*Proof.* Set  $C_N = \cup_{\zeta \in \gamma_N} D(R_N \zeta, \rho(R_N \zeta))$ . Since  $\rho(R_N \zeta) \asymp \rho(R_N)$ ,  $\zeta \in \gamma_N$ , by Lemma 6 we have

$$\begin{aligned} & R_N \rho(R_N) \int_{\gamma_N} |f(R_N \zeta)|^2 e^{-2\varphi(R_N \zeta)} |d\zeta| \\ & \lesssim R_N \rho(R_N) \int_{\gamma_N} \left[ \frac{1}{\rho(R_N \zeta)^2} \int_{D(R_N \zeta, \rho(R_N \zeta))} |f(w)|^2 e^{-2\varphi(w)} dm(w) \right] |d\zeta| \\ & \asymp \frac{R_N}{\rho(R_N)} \int_{C_N} |f(w)|^2 e^{-2\varphi(w)} \left( \int_{\gamma_N} \chi_{D(R_N \zeta, \rho(R_N \zeta))}(w) |d\zeta| \right) dm(w) \\ & \lesssim \int_{C_N} |f(w)|^2 e^{-2\varphi(w)} dm(w) \rightarrow 0, \quad N \rightarrow \infty. \end{aligned}$$

□

**Proof of Theorem 4.** Let  $\chi_N(z) = 1$ , if  $z$  lies inside  $\Gamma_N$  and 0 otherwise. Put

$$\Sigma_N(z, f) = S(z) \sum_{k=1}^N \frac{f(\lambda_k)}{S'(\lambda_k)(z - \lambda_k)},$$

and set

$$I_N(z, f) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)(z - \zeta)} d\zeta$$

The Cauchy formula gives us that

$$I_N(z, f) = \sum_{k=1}^N \frac{f(\lambda_k)}{S'(\lambda_k)(z - \lambda_k)} - \chi_N(z) \frac{f(z)}{S(z)}, \quad z \notin \Gamma_N.$$

Hence,

$$\Sigma_N(z, f) - f(z) = S(z) I_N(z, f) + (\chi_N(z) - 1) f(z),$$

and it remains only to prove that

$$\|S I_N(\cdot, f)\|_{\varphi_\beta} \rightarrow 0, \quad N \rightarrow \infty.$$

Let  $\omega$  be a Lebesgue measurable function such that

$$\int_0^\infty \int_0^{2\pi} |\omega(re^{it})|^2 e^{-2\varphi(r)} (1+r)^{-2\beta} r dr dt \leq 1, \quad (3)$$

and let

$$J_N(f, \omega) = \int_0^\infty \int_0^{2\pi} \omega(re^{it}) S(re^{it}) I_N(re^{it}, f) e^{-2\varphi(r)} (1+r)^{-2\beta} r dr dt.$$

It remains to show that

$$\sup |J_N(f, \omega)| \rightarrow 0, \quad N \rightarrow \infty,$$

where the supremum is taken over all  $\omega$  satisfying (3).

We have

$$\begin{aligned} 2\pi i J_N(f, \omega) &= \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)} \int_{\mathbb{C}} \frac{\omega(z)S(z)}{z - \zeta} e^{-2\varphi(z)} (1 + |z|)^{-2\beta} dm(z) d\zeta \\ &= \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)} \int_{\mathbb{C}} \frac{\phi(z)}{z - \zeta} (1 + |z|)^{-\beta-\gamma} dm(z) d\zeta, \end{aligned}$$

where

$$\phi(z) = [\omega(z)e^{-\varphi(z)}(1 + |z|)^{-\beta}] [S(z)e^{-\varphi(z)}(1 + |z|)^\gamma].$$

Note that

$$\int_{\mathbb{C}} |\phi(z)|^2 dm(z) \leq C.$$

Set  $\psi(z) = R_N \phi(R_N z)$ . We have

$$\int_{\mathbb{C}} |\psi(z)|^2 dm(z) \leq C.$$

Changing the variables  $z = R_N w$  and  $\zeta = R_N \eta$ , we get

$$2\pi i J_N(f, \omega) = R_N \int_{\gamma_N} \frac{f(R_N \eta)}{S(R_N \eta)} \int_{\mathbb{C}} \frac{\psi(w)}{w - \eta} (1 + R_N |w|)^{-\beta-\gamma} dm(w) d\eta.$$

Consider the operators

$$T_N(\psi)(\eta) = \int_{\mathbb{C}} \frac{\psi(w)}{w - \eta} |w|^{-\beta-\gamma} dm(w), \quad \psi \in L^2(\mathbb{C}, dm(w)).$$

Since  $\gamma + \beta \in (1/2, 1)$ , by [5, Lemma 13], the operators  $T_N$  are bounded from  $L^2(\mathbb{C}, dm(w))$  into  $L^2(\gamma_N)$  and

$$\sup_N \|T_N\| < \infty.$$

Hence, by Lemma 8 and by the property  $r^{1-2\beta} = O(\rho(r))$ ,  $r \rightarrow \infty$ , we get

$$\begin{aligned} J_N(f, \omega) &\lesssim R_N^{1-\beta-\gamma} \left| \int_{\gamma_N} \frac{f(R_N \eta)}{S(R_N \eta)} T_N(\psi)(\eta) d\eta \right| \\ &\lesssim R_N^{1-\beta} \int_{\gamma_N} |f(R_N \eta)| e^{-\varphi(R_N \eta)} |T_N(\psi)(\eta)| |d\eta| \\ &\lesssim \left( R_N \rho(R_N) \int_{\gamma_N} |f(R_N \eta)|^2 e^{-2\varphi(R_N \eta)} |d\eta| \right)^{1/2} \\ &\quad \times \|T_N(\psi)\|_{L^2(\mathbb{C}, dm(w))} \rightarrow 0, \quad N \rightarrow \infty. \end{aligned}$$

This completes the proof.  $\square$



**2.3. Proof of Theorem 5.** It suffices to find  $f \in \mathcal{F}_\varphi^2(\mathbb{C})$  and a sequence  $N_k$  such that (in the notations of the proof of Theorem 4)

$$A_k = \left\| S \chi_{N_k} \int_{\Gamma_{N_k}} \frac{f(\zeta)}{S(\zeta)(\cdot - \zeta)} d\zeta \right\|_{\varphi_\beta} \not\rightarrow 0, \quad k \rightarrow \infty. \quad (4)$$

We follow the method of the proof of [5, Theorem 11]. Let us write down the Taylor series of  $S$ :

$$S(z) = \sum_{n \geq 0} s_n z^n.$$

Since  $S \in \mathcal{S}_\gamma$ , we have

$$|s_n| \leq c \exp\left(-\frac{n}{a} \ln \frac{n}{ae} - \frac{\gamma}{a} \ln n\right), \quad n > 0.$$

Choose  $0 < \varepsilon < \frac{a}{2} - 2\beta$ . Given  $R > 0$  consider

$$S_R = \sum_{|n - aR^a| < R^{\frac{a}{2} + \varepsilon}} s_n z^n.$$

Then for every  $n$  we have

$$|S(z) - S_R(z)| e^{-|z|^\alpha} = O(|z|^{-n}), \quad ||z| - R| < \rho(R), \quad R \rightarrow \infty.$$

Next we use that for some  $c > 0$  independent of  $n$ ,

$$\int_0^\infty r^{2n+1} e^{-2r^a} dr \leq c \int_{|r - (\frac{n}{a})^{1/a}| < n^{(1/a) - (1/2)}} r^{2n+1} e^{-2r^a} dr.$$

Therefore,

$$\begin{aligned} \|S_R\|_\varphi^2 &= \sum_{|n - aR^a| < R^{\frac{a}{2} + \varepsilon}} \pi |s_n|^2 \int_0^\infty r^{2n+1} e^{-2r^a} dr \\ &\leq \sum_{|n - aR^a| < R^{\frac{a}{2} + \varepsilon}} c |s_n|^2 \int_{|r - (\frac{n}{a})^{1/a}| < n^{(1/a) - (1/2)}} r^{2n+1} e^{-2r^a} dr \\ &\leq \sum_{|n - aR^a| < R^{\frac{a}{2} + \varepsilon}} c |s_n|^2 \int_{|r - R| < c_1 R^{1 - \frac{a}{2} + \varepsilon}} r^{2n+1} e^{-2r^a} dr \\ &\leq \sum_{n \geq 0} c |s_n|^2 \int_{|r - R| < c_1 R^{1 - \frac{a}{2} + \varepsilon}} r^{2n+1} e^{-2r^a} dr \\ &= c \int_{|r - R| < c_1 R^{1 - \frac{a}{2} + \varepsilon}} \sum_{n \geq 0} |s_n|^2 r^{2n+1} e^{-2r^a} dr \\ &= c \int_{||z| - R| < c_1 R^{1 - \frac{a}{2} + \varepsilon}} |S(z)|^2 e^{-2|z|^\alpha} dm(z) \leq c_2 R^{2 - \frac{a}{2} + \varepsilon - 2\gamma}. \end{aligned}$$

Fix  $\varkappa$  such that

$$1 - \frac{a}{4} + \frac{\varepsilon}{2} - \gamma < \varkappa < 1 - \beta - \gamma.$$

Choose a sequence  $N_k$ ,  $k \geq 1$ , such that for  $R_k = |\lambda_{N_k}|$  we have  $R_{k+1} > 2R_k$ ,  $k \geq 1$ , and

$$\left| e^{-|z|^a} \sum_{m \neq k} S_{R_m}(z) R_m^{-\varkappa} \right| \leq \frac{1}{|z|^{\gamma+1}}, \quad ||z| - R_k| < \rho(R_k), \quad k \geq 1.$$

Set

$$f = \sum_{k \geq 1} S_{R_k} R_k^{-\varkappa}.$$

Then  $f \in \mathcal{F}_\varphi^2(\mathbb{C})$ , and

$$\frac{f}{S} = R_k^{-\varkappa} + O(R_k^{-1-\varkappa}) \quad \text{on } \Gamma_{N_k}, \quad k \rightarrow \infty.$$

Hence,

$$\left| S(z) \int_{\Gamma_{N_k}} \frac{f(\zeta)}{S(\zeta)(z - \zeta)} d\zeta \right| \geq c R_k^{-\varkappa} \frac{e^{|z|^a}}{(1 + |z|)^\gamma}, \quad |z| < \frac{R_k}{2},$$

and finally

$$A_k \geq c R_k^{-\varkappa} \left( \int_0^{R_k/2} \frac{r^{1-2\beta} dr}{(1+r)^{2\gamma}} \right)^{1/2} \rightarrow \infty, \quad k \rightarrow \infty.$$

This proves (4) and thus completes the proof of the theorem.  $\square$

## REFERENCES

- [1] A. Borichev, R. Dhuez, and K. Kellay, *Sampling and interpolation in large Bergman and Fock spaces*, J. Funct. Anal. **242** (2007), no. 2, 563–606.
- [2] A. Borichev, Yu. Lyubarskii, *Riesz bases of reproducing kernels in Fock type spaces*, Journal of the Institute of Mathematics of Jussieu **9** (2010) 449–461.
- [3] K. Isaev, R. Yulmukhametov, *On the nonexistence of unconditional bases of exponentials in Bergman spaces on non-polygonal domains*, Izv. Ross. Akad. Nauk Ser. Mat. **71** (2007), no. 6, 69–90 (Russian); English translation in Izv. Math. **71** (2007), no. 6, 1145–1166.
- [4] Yu. Lyubarskii, *Frames in the Bargmann space of entire functions*, in *Entire and Subharmonic Functions*, Adv. Soviet Math. 11 (1992) 167–180, American Mathematical Society, Providence, RI.
- [5] Yu. Lyubarskii, K. Seip, *Convergence and summability of Gabor expansions at the Nyquist density*, J. Fourier Anal. Appl. **5** (1999) 127–157.
- [6] K. Seip, *Interpolating and sampling in spaces of analytic functions*, American Mathematical Society, Providence, 2004.

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