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On the spectrum of the Dirichlet-to-Neumann operator acting on forms of a Euclidean domain*

S. Raulot and A. Savo

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Abstract

We compute the whole spectrum of the Dirichlet-to-Neumann operator acting on differential p -forms on the unit Euclidean ball. Then, we prove a new upper bound for its first eigenvalue on a domain Ω in Euclidean space in terms of the isoperimetric ratio $\text{Vol}(\partial\Omega)/\text{Vol}(\Omega)$.

1 Introduction

Let (Ω^{n+1}, g) be an $(n + 1)$ -dimensional compact and connected Riemannian manifold with smooth boundary Σ . The Dirichlet-to-Neumann operator on functions associates, to each function defined on the boundary, the normal derivative of its harmonic extension to Ω . More precisely, if $f \in C^\infty(\Sigma)$, its harmonic extension \widehat{f} is the unique smooth function on Ω satisfying

$$\begin{cases} \Delta \widehat{f} = 0 \text{ in } \Omega, \\ \widehat{f} = f \text{ on } \Sigma \end{cases}$$

and the Dirichlet-to-Neumann operator $T^{[0]}$ is defined by:

$$T^{[0]}f := -\frac{\partial \widehat{f}}{\partial N}$$

where N is the inner unit normal to Σ . It is a well known result (see [14] for example) that $T^{[0]}$ is a first order elliptic, non-negative and self-adjoint pseudo-differential operator with discrete spectrum

$$0 = \nu_{1,0}(\Omega) < \nu_{2,0}(\Omega) \leq \nu_{3,0}(\Omega) \leq \dots \nearrow \infty.$$

As Ω is connected, $\nu_{1,0}(\Omega) = 0$ is simple, and its eigenspace consists of the constant functions. The first positive eigenvalue has the following variational characterization:

$$\nu_{2,0}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |df|^2}{\int_{\Sigma} f^2} : f \in C^\infty(\Omega) \setminus \{0\}, \int_{\Sigma} f = 0 \right\}. \quad (1)$$

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The study of the spectrum of $T^{[0]}$ was initiated by Steklov in [13]. We note that the Dirichlet-to-Neumann map is closely related to the problem of determining a complete Riemannian manifold with boundary from the Cauchy data of harmonic functions. Indeed, a striking result of Lassas, Taylor and Uhlmann [7] states that if the manifold Ω is real analytic and has dimension at least 3, then the knowledge of $T^{[0]}$ determines Ω up to isometry.

It can be easily seen that the eigenvalues of the Dirichlet-to-Neumann map of the unit ball \mathbf{B}^{n+1} in \mathbf{R}^{n+1} are $\nu_{k,0} = k$, with $k = 0, 1, 2, \dots$ and the corresponding eigenspace is given by the vector space of homogeneous harmonic polynomials of degree k restricted to the sphere $\partial\mathbf{B}^{n+1}$.

1.1 The Dirichlet-to-Neumann operator on forms

In [9], we extend the definition of the Dirichlet-to-Neumann map $T^{[0]}$ acting on functions to an operator $T^{[p]}$ acting on $\Lambda^p(\Sigma)$, the vector bundle of differential p -forms of $\Sigma = \partial\Omega$ for $0 \leq p \leq n$. This is done as follows. Let ω be a form of degree p on Σ , with $p = 0, 1, \dots, n$. Then there exists a unique p -form $\widehat{\omega}$ on Ω such that:

$$\begin{cases} \Delta\widehat{\omega} = 0 \\ J^*\widehat{\omega} = \omega, \quad i_N\widehat{\omega} = 0. \end{cases}$$

Here $\Delta = d\delta + \delta d$ is the Hodge Laplacian acting on $\Lambda^p(\Omega)$ (the bundle of p -forms on Ω) $J^* : \Lambda^p(\Omega) \rightarrow \Lambda^p(\Sigma)$ is the restriction map and i_N is the interior product of $\widehat{\omega}$ with the inner unit normal vector field N . The existence and uniqueness of the form $\widehat{\omega}$ (called the *harmonic tangential extension of ω*) is proved, for example, in Schwarz [11]. We let:

$$T^{[p]}\omega = -i_N d\widehat{\omega}.$$

Then $T^{[p]} : \Lambda^p(\Sigma) \rightarrow \Lambda^p(\Sigma)$ defines a linear operator, *the (absolute) Dirichlet-to-Neumann operator*, which reduces to the classical Dirichlet-to-Neumann operator $T^{[0]}$ acting on functions when $p = 0$. We proved in [9] that $T^{[p]}$ is an elliptic self-adjoint and non-negative pseudo-differential operator, with discrete spectrum

$$0 \leq \nu_{1,p}(\Omega) \leq \nu_{2,p}(\Omega) \leq \dots$$

tending to infinity. Note that $\nu_{1,p}(\Omega)$ can in fact be zero: it is not difficult to prove that $\text{Ker}T^{[p]}$ is isomorphic to $H^p(\Omega)$, the p -th absolute de Rham cohomology space of Ω with real coefficients.

The operator $T^{[p]}$ belongs to a family of operators first considered by G. Carron in [2]. Other Dirichlet-to-Neumann operators acting on differential forms, but different from ours, were introduced by Joshi and Lionheart in [6], and Belishev and Sharafutdinov in [1]. In fact, our operator $T^{[p]}$ appears in a certain matrix decomposition of the Joshi and Lionheart operator (see [9] for complete references). However, one advantage of our operator is its self-adjointness, which permits to study its spectral and variational properties. In particular one has (see [9]):

$$\nu_{1,p}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |d\omega|^2 + |\delta\omega|^2}{\int_{\Sigma} |\omega|^2} : \omega \in \Lambda^p(\Omega) \setminus \{0\}, \quad i_N\omega = 0 \text{ on } \Sigma \right\}. \quad (2)$$

For $p = 0, \dots, n$, we also have a dual operator $T_D^{[p]} : \Lambda^p(\Omega) \rightarrow \Lambda^p(\Omega)$ with eigenvalues $\nu_{k,p}^D(\Omega) = \nu_{k,n-p}(\Omega)$ (for its definition, we refer to [9]). Here we just want to observe that:

$$\nu_{1,p}^D(\Omega) = \inf \left\{ \frac{\int_{\Omega} |d\omega|^2 + |\delta\omega|^2}{\int_{\Sigma} |\omega|^2} : \omega \in \Lambda^{p+1}(\Omega) \setminus \{0\}, J^*\omega = 0 \text{ on } \Sigma \right\}. \quad (3)$$

In [9], we obtained sharp upper and lower bounds of $\nu_{1,p}(\Omega)$ in terms of the extrinsic geometry of its boundary: let us briefly explain the main lower bound.

Fix $x \in \Sigma$ and consider the principal curvatures $\eta_1(x), \dots, \eta_n(x)$ of Σ at x ; if $p = 1, \dots, n$ and $1 \leq j_1 < \dots < j_p \leq n$ is a multi-index, we call the number $\eta_{j_1}(x) + \dots + \eta_{j_p}(x)$ a p -curvature of Σ . We set:

$$\begin{aligned} \sigma_p(x) &= \inf \{ \eta_{j_1}(x) + \dots + \eta_{j_p}(x) : 1 \leq j_1 < \dots < j_p \leq n \} \\ \sigma_p(\Sigma) &= \inf \{ \sigma_p(x) : x \in \Sigma \} \end{aligned}$$

and say that Σ is p -convex if $\sigma_p(\Sigma) \geq 0$ that is, if all p -curvatures of Σ are non-negative. For example Σ is 1-convex if and only if it is convex in the usual sense, and it is n -convex if and only if it is mean-convex (that is, it has non-negative mean curvature everywhere).

We then proved that for a compact domain Ω in \mathbf{R}^{n+1} with p -convex boundary, one has

$$\nu_{1,p}(\Omega) > \frac{n-p+2}{n-p+1} \sigma_p(\Sigma) \quad (4)$$

for $0 \leq p < \frac{n+1}{2}$ and

$$\nu_{1,p}(\Omega) \geq \frac{p+1}{p} \sigma_p(\Sigma) \quad (5)$$

for $(n+1)/2 \leq p \leq n$. The inequality (4) is never sharp, but (5) is sharp for Euclidean balls and actually equality characterizes the ball when $p > (n+1)/2$. For all this, and for similar inequalities in Riemannian manifolds we refer to [9].

In this paper we continue the study of the spectral properties of $T^{[p]}$. Namely:

- We compute the whole spectrum of $T^{[p]}$ and describe its eigenforms on the unit ball in \mathbf{R}^{n+1} .
- We give a sharp upper bound for the first eigenvalue of $T^{[p]}$ on Euclidean domains, in terms of the isoperimetric ratio $\frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}$.

It is perhaps worth noticing that in dimension 3 we have the following interpretation of (2) and (3) in terms of vector fields. If Ω is a bounded domain in \mathbf{R}^3 , then for all vector fields X on Ω which are tangent to the boundary Σ one has:

$$\int_{\Omega} (|\text{div}X|^2 + |\text{curl}X|^2) \geq \nu_{1,1}(\Omega) \int_{\Sigma} |X|^2, \quad (6)$$

with equality iff X is harmonic and its dual 1-form restricts to an eigenform of $T^{[1]}$ associated to $\nu_{1,1}(\Omega)$. Recall that, on a three dimensional Riemannian manifold, the curl of a vector field X is the vector field defined by

$$\text{curl}X = (\star dX^\sharp)^\sharp$$

where \sharp denotes the musical isomorphism between the tangent space and the cotangent space. Combining (6) with the estimate (4) gives, for all Euclidean domains with convex boundary:

$$\int_{\Omega} \left(|\operatorname{div} X|^2 + |\operatorname{curl} X|^2 \right) > \frac{3}{2} \sigma_1(\Sigma) \int_{\Sigma} |X|^2, \quad (7)$$

where X is any vector field tangent to Σ and $\sigma_1(\Omega)$ is a lower bound of the principal curvatures of Σ . As a by-product of the calculation in Section 1.2, we will see that (7) is almost sharp, because for all vector fields on the unit ball (tangent to the boundary) we have the sharp inequality

$$\int_{\mathbf{B}^3} \left(|\operatorname{div} X|^2 + |\operatorname{curl} X|^2 \right) \geq \frac{5}{3} \int_{\partial \mathbf{B}^3} |X|^2 \quad (8)$$

(for the description of the minimizing vector fields for the inequality (8) we refer to Section 2.2). Similarly, let X be a vector field on a Euclidean domain Ω which is normal to the boundary. Then:

$$\int_{\Omega} \left(|\operatorname{div} X|^2 + |\operatorname{curl} X|^2 \right) \geq \nu_{1,2}(\Omega) \int_{\Sigma} |X|^2, \quad (9)$$

with equality iff X is harmonic and the Hodge-star of its dual 1-form restricts to an eigenform of $T^{[2]}$ associated to $\nu_{1,2}(\Omega)$. Using (5), we see that, if Σ is mean-convex:

$$\int_{\Omega} \left(|\operatorname{div} X|^2 + |\operatorname{curl} X|^2 \right) \geq \frac{3}{2} \sigma_2(\Sigma) \int_{\Sigma} |X|^2. \quad (10)$$

Note that $\sigma_2(\Sigma) = 2H$, where H is a lower bound of the mean curvature of Σ . In this situation, our inequality is sharp and is an equality if and only if Ω is a ball in \mathbf{R}^3 , in which case the lower bound is 3, and X is a multiple of the radial vector field $r \frac{\partial}{\partial r}$ (see Section 2.2).

We end this discussion by remarking that inequalities (7) and (10) continue to be true for all bounded domains in three-dimensional Riemannian manifolds with non-negative Ricci curvature.

1.2 The spectrum of $T^{[p]}$ on the unit Euclidean ball in \mathbf{R}^{n+1}

In this section we compute the spectrum of $T^{[p]}$ on the unit ball \mathbf{B}^{n+1} in \mathbf{R}^{n+1} .

Let $\bar{\Delta}$ (resp. Δ) denote the Hodge Laplacian acting on p -forms of \mathbf{S}^n (resp. \mathbf{R}^{n+1}). It will turn out that $\bar{\Delta}$ and $T^{[p]}$ have the same eigenspaces: so we describe in details the eigenspaces of $\bar{\Delta}$.

We start from the case $p = 0$, which is classical. The operator $\bar{\Delta}$ is simply the Laplacian on functions, and it is a well-known fact that its eigenfunctions are restrictions to \mathbf{S}^n of homogeneous polynomial harmonic functions on \mathbf{R}^{n+1} . Precisely, let $P_{k,0}$ be the vector space of all polynomial functions on \mathbf{R}^{n+1} of homogeneous degree k , where $k = 0, 1, 2, \dots$, and set:

$$H_{k,0} = \{f \in P_{k,0} : \Delta f = 0\}.$$

Then the spectrum of $\bar{\Delta}$ acting on functions of \mathbf{S}^n is given by the eigenvalues

$$\mu_{k,0}'' = k(n+k-1), \quad k = 0, 1, 2, \dots,$$

with multiplicity $M_{k,0} = \dim(H_{k,0})$ and associated eigenspace $J^*(H_{k,0})$.

Now fix an eigenfunction $f \in J^*(H_{k,0})$ so that, by assumption, its harmonic extension \hat{f} is a harmonic polynomial of homogeneous degree k . It is very easy to see that $T^{[0]}f = kf$: so, f is also a Dirichlet-to-Neumann eigenfunction associated to the eigenvalue k . A standard density argument shows that these are all possible eigenvalues of $T^{[0]}$. Therefore we have the following well-known result:

Theorem 1. *The spectrum of the Dirichlet-to-Neumann operator $T^{[0]}$ on the unit ball in \mathbf{R}^{n+1} consists of the eigenvalues $\nu''_{k,0} = k$, where $k = 0, 1, 2, \dots$, each with multiplicity $M_{k,0} = \dim(H_{k,0})$ and associated eigenspace $J^*(H_{k,0})$.*

Now assume $p = 1, \dots, n$. The calculation of the spectrum of the Hodge Laplacian on the sphere was first done in [3]. We follow the exposition in [4].

As the Hodge Laplacian commutes with both the differential and the codifferential, it preserves closed (resp. co-closed) p -forms. Moreover, any exact eigenform is the differential of a co-exact eigenform associated to the same eigenvalue. In the following, we denote by $\{\mu'_{k,p}\}$ (resp. $\{\mu''_{k,p}\}$) the spectrum of the Hodge Laplacian restricted to closed (resp. co-closed) p -forms on the sphere \mathbf{S}^n .

We can write a p -form on \mathbf{R}^{n+1} as:

$$\omega = \sum_{i_1 < \dots < i_p} \omega_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

and we say that ω is polynomial if each component $\omega_{i_1 \dots i_p}$ is a polynomial function.

Now let $P_{k,p}$ be the vector space of polynomial p -forms of homogeneous degree $k \geq 0$ on \mathbf{R}^{n+1} and set:

$$\begin{cases} H_{k,p} = \{\omega \in P_{k,p} : \Delta\omega = 0, \delta\omega = 0\}, \\ H'_{k,p} = \{\omega \in H_{k,p} : d\omega = 0\} \\ H''_{k,p} = \{\omega \in H_{k,p} : i_Z\omega = 0\} \end{cases}$$

where Z is the radial vector field $Z = r \frac{\partial}{\partial r} = \sum_{j=1}^{n+1} x_j \frac{\partial}{\partial x_j}$. On the boundary, $Z = -N$. It turns out that $H_{k,p} = H'_{k,p} \oplus H''_{k,p}$ and $d : H''_{k,p} \rightarrow H'_{k-1,p+1}$ is a linear isomorphism for all $k \geq 1$. We set

$$M_{k,p} = \dim(H''_{k,p})$$

(this number can be computed by representation theory, see Theorem 6.8 in [4]).

By the Hodge-de Rham decomposition, any Hodge-Laplace eigenspace splits into the direct sum of its co-exact, exact and harmonic parts. But in the range $1 \leq p \leq n$ the only harmonic forms occur in degree $p = n$, and are multiples of the volume form of \mathbf{S}^n . Moreover, for $p = n$ the co-exact part is reduced to zero. Then, there is a spectral resolution of $L^2(\Lambda^p(\mathbf{S}^n))$ which consists of the following three types of Hodge-Laplace eigenforms: $1 \leq p \leq n - 1$ and the eigenform ξ is co-exact; $1 \leq p \leq n$ and the eigenform ξ is exact; $p = n$ and ξ is a multiple of the volume form of \mathbf{S}^n . Correspondingly, we have the following three families of eigenvalues (see [4]):

- If $1 \leq p \leq n - 1$ and ξ is co-exact the associated eigenvalues are given by the family

$$\mu''_{k,p} = (k+p)(n+k-p-1) \quad k = 1, 2, \dots$$

with associated eigenspace $J^*(H''_{k,p})$ and multiplicity $M_{k,p}$.

- If $1 \leq p \leq n$ and ξ is exact the associated eigenvalues are given by the family

$$\mu'_{k,p} = (k+p-1)(n+k-p) \quad k = 1, 2, \dots$$

with associated eigenspace $J^*(H'_{k-1,p})$ and multiplicity $M_{k,p-1}$.

- If $p = n$ and ξ is the volume form of \mathbf{S}^n we have the associated eigenvalue $\mu''_{1,n} = 0$.

In conclusion, eigenforms of the Hodge Laplacian are suitable restrictions to \mathbf{S}^n of harmonic, co-closed polynomial forms.

In Section 2.1 we will prove that any Hodge-Laplace eigenform of \mathbf{S}^n is also a Dirichlet-to-Neumann eigenform (associated to a different eigenvalue). This will imply the following calculations.

Theorem 2. *Let $1 \leq p \leq n-1$. The spectrum of the Dirichlet-to-Neumann operator $T^{[p]}$ on the unit ball of \mathbf{R}^{n+1} is given by the following two families of eigenvalues $\{\nu''_{k,p}\}, \{\nu'_{k,p}\}$ indexed by a positive integer $k = 1, 2, \dots$:*

$$\begin{cases} \nu''_{k,p} = k+p & \text{with multiplicity } M_{k,p}, \\ \nu'_{k,p} = (k+p-1)\frac{n+2k+1}{n+2k-1} & \text{with multiplicity } M_{k,p-1}. \end{cases}$$

The eigenspace associated to $\nu''_{k,p}$ is $J^*(H''_{k,p})$ and consists of co-exact forms. The eigenspace associated to $\nu'_{k,p}$ is $J^*(H'_{k-1,p})$ and consists of exact forms.

In degree $p = n$:

Theorem 3. *The spectrum of the Dirichlet-to-Neumann operator $T^{[n]}$ on the unit ball of \mathbf{R}^{n+1} consists of the eigenvalues:*

$$\nu''_{1,n} = n+1,$$

with multiplicity one (the associated eigenspace being spanned by the volume form of \mathbf{S}^n) and, for $k \geq 1$:

$$\nu'_{k,n} = (k+n-1)\frac{n+2k+1}{n+2k-1},$$

with multiplicity $M_{k,n-1}$ and associated eigenspace $J^*(H'_{k-1,n})$.

From this result we deduce that the lowest eigenvalue of $T^{[n]}$ is $\nu_{1,n} = n+1$ with multiplicity one for $n \geq 2$ and three for $n = 1$. Indeed, in this last situation, we have $\nu''_{1,1} = \nu'_{2,1} = 2$ and the corresponding eigenspace is spanned by the the volume form v of \mathbf{S}^1 , $\bar{d}x$ and $\bar{d}y$ where \bar{d} denotes the differential on \mathbf{S}^1 . From the above results we obtain:

Corollary 4. *If $\nu_{1,p}$ denotes the first eigenvalue of $T^{[p]}$ on the unit ball in \mathbf{R}^{n+1} , then:*

$$\nu_{1,p} = \begin{cases} \frac{n+3}{n+1}p & \text{if } 1 \leq p \leq \frac{n+1}{2} \\ p+1 & \text{if } \frac{n+1}{2} \leq p \leq n. \end{cases}$$

The proof of Theorems 2 and 3 splits into two parts. In a first step (see Lemma 6 and 8), we compute the expression of the Hodge Laplacian on p -forms for rotationally symmetric manifolds. Then in Section 2 we apply these computations to construct the tangential harmonic extension to the unit ball of any p -eigenform of the Hodge Laplacian on $\mathbf{S}^n = \partial\mathbf{B}^{n+1}$.

1.3 A sharp upper bound by the isoperimetric ratio

As shown in [9], the existence of a parallel p -form implies upper bounds of the Dirichlet-to-Neumann eigenvalues by the isoperimetric ratio $\frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}$. These bounds are never sharp, unless $p = n$. In that case one has, for any $(n + 1)$ -dimensional Riemannian domain Ω :

$$\nu_{1,n}(\Omega) \leq \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}, \quad (11)$$

which is sharp for Euclidean balls. The proof of (11) is easily obtained by applying the min-max principle (2) to the test n -form $\omega = \star dE$, where E is the mean-exit time function, solution of the problem

$$\begin{cases} \Delta E = 1 & \text{on } \Omega, \\ E = 0 & \text{on } \Sigma. \end{cases} \quad (12)$$

If Ω is a Euclidean domain and equality holds in (11), then a famous result of Serrin implies that Ω is a ball. The equality case for general Riemannian domains is an open (and interesting) problem.

In this paper we generalize inequality (11) to any degree $p = 0, \dots, n$ when Ω is a Euclidean domain; in the range $p \geq (n + 1)/2$ the estimate is sharp and it also turns out that the ball is the unique maximizer. Namely, we prove:

Theorem 5. *Let Ω be a domain in \mathbf{R}^{n+1} and $p = 1, \dots, n$. Then:*

$$\nu_{1,p}(\Omega) \leq \frac{p + 1}{n + 1} \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}.$$

Equality holds iff $p \geq (n + 1)/2$ and Ω is a Euclidean ball.

We note that the corresponding inequality for the first *positive* eigenvalue of the Dirichlet-to-Neumann operator on functions:

$$\nu_{2,0}(\Omega) \leq \frac{1}{n + 1} \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)},$$

has been recently proved by Ilias and Makhoul in [5], but their approach does not extend to higher degrees.

2 The Dirichlet-to-Neumann spectrum of the unit Euclidean ball

We first give an expression of the Hodge Laplacian on the unit ball \mathbf{B}^{n+1} in \mathbf{R}^{n+1} . Note that $\mathbf{B}^{n+1} = [0, 1] \times \mathbf{S}^n$ with the metric $dr^2 \oplus r^2 ds_n^2$, where ds_n^2 is the canonical metric on \mathbf{S}^n .

We consider p -forms on the ball \mathbf{B}^{n+1} of the following type:

$$\omega(r, x) = Q(r)\xi(x) + P(r)dr \wedge \eta(x),$$

where $\eta \in \Lambda^{p-1}(\mathbf{S}^n)$, $\xi \in \Lambda^p(\mathbf{S}^n)$, and P, Q are smooth functions on $(0, 1)$. We will write for short

$$\omega = Q\xi + Pdr \wedge \eta. \quad (13)$$

Then we have:

Lemma 6. *Let \bar{d} and $\bar{\delta}$ denote, respectively, the differential and the co-differential acting on \mathbf{S}^n . Let ω be a p -form as in (13), then:*

$$\Delta\omega = \omega_1 + dr \wedge \omega_2,$$

where:

$$\begin{aligned} \omega_1 &= \frac{Q}{r^2} \bar{\Delta} \xi - \left(Q'' + \frac{n-2p}{r} Q' \right) \xi - \frac{2P}{r} \bar{d}\eta, \\ \omega_2 &= \frac{P}{r^2} \bar{\Delta} \eta - \left(P' + \frac{n-2p+2}{r} P \right)' \eta - \frac{2Q}{r^3} \bar{\delta} \xi. \end{aligned}$$

For the proof, we refer to Lemma 8 in Section 2.3.

2.1 Proof of Theorems 2 and 3

It is enough to show that any Hodge-Laplace eigenform of \mathbf{S}^n is also a Dirichlet-to-Neumann eigenform, associated to one of the eigenvalues $\nu'_{k,p}$ or $\nu''_{k,p}$ of Theorems 2 and 3. In fact, as the direct sum of all the Hodge-Laplace eigenspaces is dense in $L^2(\Lambda^p(\mathbf{S}^n))$, the list of Dirichlet-to-Neumann eigenvalues we have just found is complete, and the theorems follow.

We now use the classification of the eigenspaces of the Hodge Laplacian done in Section 1.2. It is then enough to prove the following:

Proposition 7.

a) *Assume $1 \leq p \leq n-1$, and let ξ be a co-exact Hodge p -eigenform on \mathbf{S}^n associated to*

$$\mu''_{k,p} = (k+p)(n+k-p-1)$$

for some $k \geq 1$. Then ξ is also a Dirichlet-to-Neumann eigenform associated to the eigenvalue

$$\nu''_{k,p} = k+p.$$

b) Assume $p = n$ and let ξ be a multiple of the volume form of \mathbf{S}^n . Then ξ is also a Dirichlet-to-Neumann eigenform associated to the eigenvalue $\nu''_{1,n} = n + 1$.

c) Assume $1 \leq p \leq n$ and let ξ be an exact Hodge-Laplace p -eigenform associated to

$$\mu'_{k,p} = (k + p - 1)(n + k - p)$$

for some $k \geq 1$. Then ξ is also a Dirichlet-to-Neumann p -eigenform associated to

$$\nu'_{k,p} = (k + p - 1) \frac{n + 2k + 1}{n + 2k - 1}.$$

For the proof, we explicitly determine in all three cases the tangential harmonic extension of ξ using Lemma 6.

Proof of a) Let us compute the tangential harmonic extension $\widehat{\xi}$ of ξ . We let $\widehat{\xi} = Q\xi$ with $Q = Q(r)$ to be determined so that $\Delta\widehat{\xi} = 0$. Note that $i_N\widehat{\xi} = 0$; moreover $J^*\widehat{\xi} = \xi$ whenever $Q(1) = 1$. As $\eta = 0 = \bar{\delta}\xi$, Lemma 6 gives

$$\Delta(Q\xi) = \left(\mu''_{k,p} \frac{Q}{r^2} - Q'' - \frac{n - 2p}{r} Q' \right) \xi$$

hence $Q\xi$ is harmonic if $Q(r)$ satisfies:

$$r^2 Q'' + (n - 2p)rQ' - \mu''_{k,p} Q = 0. \quad (14)$$

It is straightforward to check that a solution (at least C^2) on $[0, 1]$ is given by $Q(r) = r^{k+p}$. The tangential harmonic extension of ξ to the ball is then:

$$\widehat{\xi} = r^{k+p}\xi.$$

Recall that, by definition, $T^{[p]}(\xi) = -i_N d\widehat{\xi}$. Now:

$$d\widehat{\xi} = r^{k+p} d\xi + (k + p)r^{k+p-1} dr \wedge \xi.$$

On the boundary we have $r = 1$ and $dr(N) = -1$, then:

$$i_N d\widehat{\xi} = -(k + p)\xi$$

so that

$$T^{[p]}(\xi) = (k + p)\xi$$

as asserted.

Proof of b) Note that ξ is co-closed and the associated Hodge-Laplace eigenvalue is $\mu''_{1,n} = 0$. Proceeding as in a), one finds that the tangential harmonic extension of ξ is $\widehat{\xi} = r^{n+1}\xi$ and therefore $T^{[n]}(\xi) = (n + 1)\xi$.

Proof of c) We first observe that a Hodge-Laplace exact p -eigenform ξ associated to $\mu'_{k,p}$ is the differential of a co-exact $(p - 1)$ -eigenform ϕ , associated to the same eigenvalue $\mu''_{k,p-1} = \mu'_{k,p}$:

$$\xi = \bar{d}\phi.$$

From Lemma 6, we see that the p -form $Q\bar{d}\phi + Pdr \wedge \phi$ is harmonic iff:

$$\begin{cases} \mu''_{k,p-1}(rP - 2Q) = r^3 \left(P' + \frac{n-2p+2}{r}P \right)' \\ \mu''_{k,p-1}Q - 2rP = r^2 \left(Q'' + \frac{n-2p}{r}Q' \right). \end{cases} \quad (15)$$

We observe that, by part a) of the proposition, the $(p-1)$ -form ϕ is an eigenform for $T^{[p-1]}$ associated to $\nu''_{k,p-1} = k+p-1$ (note that if $p=1$ this fact follows directly from Theorem 1). Now consider the p -form:

$$\hat{\eta} = \alpha_{k,p}r^{k+p+1}\bar{d}\phi - \nu''_{k,p-1}r^{k+p}dr \wedge \phi,$$

where:

$$\alpha_{k,p} = \frac{(\nu''_{k,p-1})^2}{\mu''_{k,p-1}} = \frac{k+p-1}{n+k-p}.$$

The p -form $\hat{\eta}$ satisfies:

$$\begin{cases} \Delta\hat{\eta} = 0 \\ J^*\hat{\eta} = \alpha_{k,p}\bar{d}\phi, \quad i_N\hat{\eta} = \nu''_{k,p-1}\phi, \end{cases} \quad (16)$$

where the harmonicity follows from (15) by taking:

$$P(r) = -\nu''_{k,p-1}r^{k+p}, \quad Q(r) = \alpha_{k,p}r^{k+p+1}.$$

Now let $\hat{\phi}$ be the harmonic tangential extension of ϕ , that is:

$$\begin{cases} \Delta\hat{\phi} = 0 \\ J^*\hat{\phi} = \phi, \quad i_N\hat{\phi} = 0. \end{cases} \quad (17)$$

As ϕ is an eigenform for $T^{[p-1]}$ associated to $\nu''_{k,p-1}$, one has:

$$\begin{cases} \Delta d\hat{\phi} = 0 \\ J^*d\hat{\phi} = \bar{d}\phi, \quad i_N d\hat{\phi} = -\nu''_{k,p-1}\phi. \end{cases} \quad (18)$$

From (16) and (18) one sees that the form $\hat{\omega} = d\hat{\phi} + \hat{\eta}$ satisfies:

$$\begin{cases} \Delta\hat{\omega} = 0 \\ J^*\hat{\omega} = (\alpha_{k,p} + 1)\bar{d}\phi, \quad i_N\hat{\omega} = 0. \end{cases}$$

From the definition of $T^{[p]}$, we have:

$$T^{[p]}((\alpha_{k,p} + 1)\bar{d}\phi) = -i_N d\hat{\omega}.$$

Now:

$$d\hat{\omega} = d\hat{\eta} = (\nu''_{k,p-1} + \alpha_{k,p}(k+p+1))r^{k+p}dr \wedge \bar{d}\phi$$

and, restricted to the boundary:

$$i_N d\widehat{\omega} = -(\nu''_{k,p-1} + \alpha_{k,p}(k+p+1))\bar{d}\phi.$$

This means that

$$T^{[p]}((\alpha_{k,p} + 1)\bar{d}\phi) = (\nu''_{k,p-1} + \alpha_{k,p}(k+p+1))\bar{d}\phi$$

and so $\xi = \bar{d}\phi$ is a Dirichlet-to-Neumann eigenform associated to the eigenvalue

$$\frac{\nu''_{k,p-1} + \alpha_{k,p}(k+p+1)}{\alpha_{k,p} + 1} = (k+p-1)\frac{n+2k+1}{n+2k-1}$$

as asserted.

This ends the proof of the proposition.

2.2 On a variational problem for vector fields

Recall the variational problems defined in (6) and (9). Let \mathbf{B}^3 be the unit ball in \mathbf{R}^3 . As a consequence of Corollary 4 we have that, for any vector field X on \mathbf{B}^3 tangent to $\partial\mathbf{B}^3 = \mathbf{S}^2$:

$$\int_{\mathbf{B}^3} (|\operatorname{div}X|^2 + |\operatorname{curl}X|^2) \geq \frac{5}{3} \int_{\mathbf{S}^2} |X|^2, \quad (19)$$

because $\nu_{1,1}(\mathbf{B}^3) = \nu'_{1,1}(\mathbf{B}^3) = \frac{5}{3}$. Let us describe the minimizing vector fields. The eigenspace associated to $\nu'_{1,1}$ is 3-dimensional, spanned by $\bar{d}\phi$, where ϕ is a linear function on \mathbf{R}^3 . Hence the vector field X attains equality in (19) iff it is dual to the harmonic tangential extension of $\bar{d}\phi$. Take for example $\phi = x_1$. As a consequence of the calculation done in the previous section, we see that the harmonic tangential extension of $\bar{d}x_1$ to the unit ball in \mathbf{R}^3 is, in rectangular coordinates:

$$\widehat{\xi} = (2 - 2x_1^2 + x_2^2 + x_3^2)dx_1 - 3x_1x_2dx_2 - 3x_1x_3dx_3.$$

Note that $\widehat{\xi}$ is a polynomial (not homogeneous) 1-form. The dual vector field

$$X = (2 - 2x_1^2 + x_2^2 + x_3^2)\frac{\partial}{\partial x_1} - 3x_1x_2\frac{\partial}{\partial x_2} - 3x_1x_3\frac{\partial}{\partial x_3}$$

is then a minimizer for the variational problem (19).

On the other hand, as $\nu_{1,2}(\mathbf{B}^3) = \nu''_{1,2}(\mathbf{B}^3) = 3$, one sees that, for any vector field normal to \mathbf{S}^2 we have:

$$\int_{\mathbf{B}^3} (|\operatorname{div}X|^2 + |\operatorname{curl}X|^2) \geq 3 \int_{\mathbf{S}^2} |X|^2. \quad (20)$$

Now $\nu''_{1,2}$ is simple and is spanned by the volume form v of \mathbf{S}^2 . The tangential harmonic extension of v to the unit ball is

$$\widehat{v} = x_1dx_2 \wedge dx_3 + x_2dx_3 \wedge dx_1 + x_3dx_1 \wedge dx_2.$$

The space of minimizing vector fields for (20) is then one-dimensional, spanned by the dual of $\star\widehat{v}$, that is, by the radial vector field

$$Y = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}.$$

2.3 Proof of Lemma 6

Assume that (M^{n+1}, g) is a rotationally symmetric manifold, that is

$$M^{n+1} = [0, \infty) \times \mathbf{S}^n,$$

endowed with the Riemannian metric

$$g = dr^2 \oplus \theta(r)^2 ds_n^2,$$

where ds_n^2 is the canonical metric on \mathbf{S}^n and θ is a smooth positive function on $(0, \infty)$. Of course, one gets the space form of curvature $-1, 0, 1$ when $\theta = \sinh r, r, \sin r$, respectively. To prove Lemma 6 we will take $\theta(r) = r$. In this setting, any p -form can be split into its tangential and normal components:

$$\omega = \omega_1 + dr \wedge \omega_2,$$

where ω_1 and ω_2 are forms of degrees p and $p-1$, respectively. We assume that we can separate the variables, that is:

$$\omega_1(r, x) = Q(r)\xi(x), \quad \omega_2(r, x) = P(r)\eta(x),$$

where $\xi \in \Lambda^p(\mathbf{S}^n)$, $\eta \in \Lambda^{p-1}(\mathbf{S}^n)$ and P, Q are radial functions so that

$$\omega = Q\xi + dr \wedge (P\eta).$$

2.3.1 A suitable orthonormal frame

Fix a point $(r, x) \in M$ with $r \in \mathbf{R}$ and $x \in \mathbf{S}^n$ and let $(\bar{E}_1, \dots, \bar{E}_n)$ be an orthonormal frame on the sphere which is geodesic at x for the canonical metric ds_n^2 . If we set $Z = \partial/\partial r$ and $E_j = \theta^{-1}\bar{E}_j$ for $1 \leq j \leq n$, it is obvious that the frame (Z, E_1, \dots, E_n) is g -orthonormal. From the Koszul formula, we compute:

$$\nabla_{E_j} E_k = -\delta_{jk} \frac{\theta'}{\theta} Z, \quad \nabla_{E_j} Z = \frac{\theta'}{\theta} E_j, \quad \nabla_Z E_j = 0, \quad \nabla_Z Z = 0 \quad (21)$$

where ∇ denotes the Levi-Civita connection on (M, g) . In particular, we have $\nabla_{E_j} E_k = 0$ if $j \neq k$ and

$$\nabla_{E_j} E_j = -\frac{\theta'}{\theta} Z, \quad \sum_j \nabla_{E_j} E_j = -n \frac{\theta'}{\theta} Z. \quad (22)$$

2.3.2 Computing the divergence and the differential

In this part, we first compute the divergence of a p -form using the orthonormal frame of the previous section. If ω is a p -form on M , then from the definition of the divergence and using the relations (21) and (22), we have:

$$\begin{aligned} \delta\omega(E_1, \dots, E_{p-1}) &= -\sum_{j=1}^n E_j \cdot \omega(E_j, E_1, \dots, E_{p-1}) - Z \cdot \omega(Z, E_1, \dots, E_{p-1}) \\ &\quad - (n-p+1) \frac{\theta'}{\theta} \omega(Z, E_1, \dots, E_{p-1}) \end{aligned}$$

and

$$\delta\omega(Z, E_1, \dots, E_{p-2}) = \sum_{j=1}^n E_j \cdot \omega(Z, E_j, E_1, \dots, E_{p-2}).$$

Let ϕ be a p -form on \mathbf{S}^n . As the frame $(\bar{E}_1, \dots, \bar{E}_n)$ of Σ is geodesic at the point $x \in \Sigma$ and the function $\phi(\bar{E}_{j_1}, \dots, \bar{E}_{j_p})$ is constant in the r -direction for any choice of j_1, \dots, j_p , we have:

$$\begin{cases} \sum_{j=1}^n E_j \cdot \phi(E_j, E_1, \dots, E_{p-1}) = -\frac{1}{\theta^2} \bar{\delta}\phi(E_1, \dots, E_{p-1}) \\ Z \cdot \phi(E_1, \dots, E_p) = -p \frac{\theta'}{\theta} \phi(E_1, \dots, E_p) \end{cases}$$

Since the vectors E_1, \dots, E_p can be replaced by any set of tangential vectors in the chosen frame, a straightforward calculation using the above equations shows that, for $\eta \in \Lambda^{p-1}(\mathbf{S}^n)$ and $\xi \in \Lambda^p(\mathbf{S}^n)$:

$$\begin{cases} \delta(dr \wedge (P\eta)) = dr \wedge \left(-\frac{P}{\theta^2} \bar{\delta}\eta\right) - \left[P' + (n - 2p + 2) \frac{\theta'}{\theta} P\right] \eta \\ \delta(Q\xi) = \frac{Q}{\theta^2} \bar{\delta}\xi. \end{cases} \quad (23)$$

For the differential, it is clear that we directly have:

$$\begin{cases} d(dr \wedge (P\eta)) = -dr \wedge (P\bar{d}\eta) \\ d(Q\xi) = dr \wedge (Q'\xi) + Q\bar{d}\xi. \end{cases} \quad (24)$$

At this point, using (23) and (24), one proves, after some standard work:

Lemma 8. *Let $\omega = Q\xi + Pdr \wedge \eta$ where $\eta \in \Lambda^{p-1}(\mathbf{S}^n)$ and $\xi \in \Lambda^p(\mathbf{S}^n)$. Then:*

$$\Delta\omega = \omega_1 + dr \wedge \omega_2,$$

where:

$$\begin{cases} \omega_1 = \frac{Q}{\theta^2} \bar{\Delta}\xi - (Q'' + (n - 2p) \frac{\theta'}{\theta} Q') \xi - 2 \frac{\theta'}{\theta} P \bar{d}\eta \\ \omega_2 = \frac{P}{\theta^2} \bar{\Delta}\eta - (P' + (n - 2p + 2) \frac{\theta'}{\theta} P)' \eta - 2Q \frac{\theta'}{\theta^3} \bar{\delta}\xi. \end{cases}$$

To prove Lemma 6, we now set $\theta(r) = r$.

Remark 9. Let B_R be the geodesic ball of radius R centered at the pole of a rotationally symmetric manifold, that is $B_R = \{(r, x) \in [0, \infty) \times \mathbf{S}^n : r \leq R\}$. One can construct eigenforms of the Dirichlet-to-Neumann operator on B_R by solving ordinary differential equations.

For example, one sees from the above expression of the Laplacian that the harmonic tangential extension of a co-closed eigenform $\xi \in \Lambda^p(\mathbf{S}^n)$ for the Hodge Laplacian on the sphere associated with $\mu''_{k,p}$ is given by $\hat{\xi} = Q\xi$, where $Q(r)$ is the smooth function satisfying:

$$\begin{cases} \frac{1}{\theta^{n-2p}} \left(\theta^{n-2p} Q'\right)' = \frac{\mu''_{k,p} Q}{\theta^2}, \\ Q(R) = 1. \end{cases}$$

It follows directly from the definition that then

$$\nu''_{k,p} = Q'(R)$$

is an eigenvalue of $T^{[p]}$.

Now take $p = n$. Then, the only possible value of $\mu''_{k,n}$ is zero, corresponding to the volume form of \mathbf{S}^n , which is parallel hence harmonic. In that case the previous equation reduces to

$$\left(\theta^{-n}Q'\right)' = 0,$$

hence, up to multiples, $Q'(r) = \theta^n(r)$. Then:

$$\nu''_{1,n}(B_R) = \frac{\theta^n(R)}{\int_0^R \theta^n(r)dr} = \frac{\text{Vol}(\partial B_R)}{\text{Vol}(B_R)} \quad (25)$$

is an eigenvalue for the Dirichlet-to-Neumann operator on n -forms.

In [9] we called a domain Ω *harmonic* if the mean-exit time function E of Ω , solution of the problem (12), has constant normal derivative on Σ : we then observed that for a harmonic domain $\text{Vol}(\Sigma)/\text{Vol}(\Omega)$ is always an eigenvalue of $T^{[n]}$. A geodesic ball B_R , being rotationally invariant, is a harmonic domain because the function E is also rotationally invariant. Therefore the above result (25) is not a surprise.

3 Proof of Theorem 5

Let Ω be a Riemannian domain with smooth boundary Σ and inner unit normal vector field N . Consider the shape operator of Σ , defined by $S(X) = -\nabla_X N$ for all $X \in T\Sigma$. Then S can be extended to a self-adjoint operator $S^{[p]}$ acting on p -forms of Σ by the rule:

$$S^{[p]}(\omega)(X_1, \dots, X_p) = \sum_{j=1}^p \omega(X_1, \dots, S(X_j), \dots, X_p),$$

for all $\omega \in \Lambda^p(\Sigma)$ and for all vectors $X_1, \dots, X_p \in T\Sigma$.

Let (e_1, \dots, e_n) be an orthonormal basis of principal directions, so that $S(e_j) = \eta_j e_j$ for all j , where η_1, \dots, η_n are the principal curvatures of Σ . Then, for any multi-index $1 \leq j_1 < \dots < j_p \leq n$ one has

$$S^{[p]}(\omega)(e_{j_1}, \dots, e_{j_p}) = (\eta_{j_1} + \dots + \eta_{j_p})\omega(e_{j_1}, \dots, e_{j_p}).$$

In particular, if ω is an n -form:

$$S^{[n]}(\omega) = nH\omega,$$

where H is the mean curvature of Σ .

For later use we observe that, if ξ is a p -form on Ω and $L_N \xi = di_N \xi + i_N d\xi$ is its Lie derivative along N , then it follows directly from the definitions that, on Σ , one has (see for example Lemma 18 in [8]):

$$J^*(L_N \xi) = J^*(\nabla_N \xi) - S^{[p]}(J^* \xi). \quad (26)$$

The proof of Theorem 5 is based on the following estimates.

Proposition 10. *Let ξ be an exact parallel p -form on Ω , with $p = 1, \dots, n$. If $p = 1$, then*

$$\nu_{2,0}(\Omega) \int_{\Omega} |\xi|^2 \leq \int_{\Sigma} |i_N \xi|^2. \quad (27)$$

and if $p = 2, \dots, n$:

$$\nu_{1,p-1}(\Omega) \int_{\Omega} |\xi|^2 \leq \int_{\Sigma} |i_N \xi|^2. \quad (28)$$

Moreover, if equality holds in (27) or (28), then:

$$S^{[p]}(J^* \xi) = \nu J^* \xi,$$

where $\nu = \nu_{2,0}(\Omega)$ when $p = 1$ and $\nu = \nu_{1,p-1}(\Omega)$ when $p \geq 2$.

Proof. Inequalities (27) and (28) were proved in [9] and hold, more generally, only assuming that ξ is exact and satisfies $d\xi = \delta\xi = 0$ on Ω . Let us recall the main argument. As ξ is exact, there exists a unique co-exact $(p-1)$ -form θ , called the *canonical primitive* of ξ , satisfying:

$$\begin{cases} d\theta = \xi & \text{on } \Omega, \\ i_N \theta = 0 & \text{on } \Sigma. \end{cases}$$

When $p = 1$, we take θ as the unique primitive of ξ such that $\int_{\Sigma} \theta = 0$.

We now take θ as test $(p-1)$ -form in the min-max principles (1) and (2) and the inequalities (27) and (28) follow after some easy work (see [9]). Now assume that equality holds: then θ is an eigenform associated to ν , so that $i_N d\theta = -\nu J^* \theta$. This means:

$$i_N \xi = -\nu J^* \theta$$

on Σ . Differentiating on Σ one gets $d^{\Sigma} i_N \xi = -\nu J^* \xi$, and in turn, as ξ is closed:

$$J^*(L_N \xi) = -\nu J^* \xi.$$

As ξ is parallel, (26) gives $J^*(L_N \xi) = -S^{[p]}(J^* \xi)$ and then the assertion follows. \square

We can now give the proof of Theorem 5. We can assume that $p \leq n-1$: in fact, the assertion for $p = n$ is a direct consequence of Theorem 5 and Corollary 6 in [9].

Let \mathcal{P} be the family of unit length parallel vector fields on \mathbf{R}^{n+1} : then \mathcal{P} is naturally identified with \mathbf{S}^n . For $V_1, \dots, V_p \in \mathcal{P}$ consider the parallel p -form

$$\xi = V_1^* \wedge \dots \wedge V_p^*$$

where V_j^* denotes the dual 1-form of V_j . Note that V_j^* is the differential of a linear function, so ξ is also exact. Then, denoting by ν the eigenvalue as in Proposition 10, we have from (27) and (28)

$$\nu \int_{\Omega} |\xi|^2 \leq \int_{\Sigma} |i_N \xi|^2. \quad (29)$$

We wish to integrate this inequality over all $(V_1, \dots, V_p) \in (\mathbf{S}^n)^p$. To that end, we introduce on $\mathcal{P} = \mathbf{S}^n$ the measure:

$$d\mu = \frac{n+1}{\text{Vol}(\mathbf{S}^n)} ds_n^2,$$

where ds_n^2 is the canonical measure on \mathbf{S}^n . The normalization is chosen so that, at each point $x \in \mathbf{R}^{n+1}$, and for all tangent vectors X, Y at x one has:

$$\int_{\mathbf{S}^n} \langle V, X \rangle \langle V, Y \rangle d\mu(V) = \langle X, Y \rangle. \quad (30)$$

A straightforward calculation using (30) (explicitly done in Lemma 2.1 of [10]) then gives, at each point of Ω (respectively, of Σ):

$$\begin{aligned} \int_{(\mathbf{S}^n)^p} |V_1^* \wedge \dots \wedge V_p^*|^2 d\mu(V_1) \dots d\mu(V_p) &= p! \binom{n+1}{p}, \\ \int_{(\mathbf{S}^n)^p} |i_N(V_1^* \wedge \dots \wedge V_p^*)|^2 d\mu(V_1) \dots d\mu(V_p) &= p! \binom{n}{p-1}. \end{aligned}$$

We now integrate both sides of (29) with respect to $(V_1, \dots, V_p) \in (\mathbf{S}^n)^p$. If $p = 1$ we get:

$$\nu_{2,0}(\Omega) \leq \frac{1}{n+1} \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)},$$

and if $p = 2, \dots, n$ we get:

$$\nu_{1,p-1}(\Omega) \leq \frac{p}{n+1} \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)},$$

which, after replacing p by $p+1$, are the inequalities stated in the theorem.

We now discuss the equality case. If equality holds then, from Proposition 10, we see that

$$S^{[p]}(J^* \xi) = \nu J^* \xi \quad (31)$$

for all such $\xi = V_1^* \wedge \dots \wedge V_p^*$. Fix a point $x \in \Sigma$ and an orthonormal frame (e_1, \dots, e_n) of principal directions at x . Fix a multi-index $j_1 < \dots < j_p$ and choose:

$$V_1 = e_{j_1}, \dots, V_p = e_{j_p}.$$

At x , one has $J^* \xi(e_{j_1}, \dots, e_{j_p}) = 1$ and then, by the definition of $S^{[p]}$:

$$S^{[p]}(J^* \xi)(e_{j_1}, \dots, e_{j_p}) = \eta_{j_1}(x) + \dots + \eta_{j_p}(x),$$

the corresponding p -curvature at x . From (31) we then get:

$$\nu = \eta_{j_1}(x) + \dots + \eta_{j_p}(x).$$

This holds for all multi-indices $j_1 < \dots < j_p$ and for all $x \in \Sigma$: hence, all p -curvatures are constant on Σ , and equal to ν . If $p < n$, this immediately implies that Σ is totally umbilical, hence it is a sphere. If $p = n$, we have that the mean curvature of Σ is constant; by the well-known Alexandrov theorem Σ is, again, a sphere.

Finally, from Theorem 2, we have that if Ω is a ball, then equality holds for $\nu_{2,0}$ and for $\nu_{1,p-1}$ provided that $p-1 \geq (n+1)/2$. The proof is complete.

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