On a question of Serre

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On a question of Serre

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Dedicated to the memory of Fritz Grunewald.

Abstract.

Consider the Borel–Serre compactification [6] of the quotient of hyperbolic 3–space $\mathcal{H}$ by a finite index subgroup Γ in a Bianchi group, and in particular the following question which Serre poses on page 514 of the quoted article. Consider the map α induced on homology when attaching the boundary into the Borel–Serre compactification.
How can one determine the kernel of α (in degree 1) ?
Serre uses a global topological argument and obtains the rank of the kernel of α. But in the quoted article, he kept asking what submodule precisely this kernel is. With a local topological study, we can decompose the kernel of α into its parts associated to each cusp.

Sur une question de Serre

Résumé. Considérons la compactification de Borel–Serre [6] du quotient de l'espace hyperbolique $\mathcal{H}$ à trois dimensions par un sous-groupe Γ d'indice fini dans un groupe de Bianchi, et en particulier la question suivante que Serre pose sur la page 514 de l'article cité. Considérons l'application α induite en homologie quand le bord est attaché dans la compactification de Borel–Serre. *Comment peut-on déterminer le noyau de α (en degré 1) ?* Serre se sert d'un argument topologique global et obtient le rang du noyau de α. Mais dans l'article cité, il continuait de demander de quel sous-module précisément il s'agit pour ce noyau. A travers d'une étude topologique locale, nous pouvons décomposer le noyau de α dans ses parties associées à chacune des pointes.

Consider an imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, and its ring of integers $\mathcal{O}$. The *Bianchi groups* are the groups $\mathrm{SL}_2(\mathcal{O})$. We will exclusively consider the Bianchi groups *with only units* $\{\pm 1\}$ in $\mathcal{O}$, by which we mean that we exclude the two Bianchi groups where $\mathcal{O}$ are the Gaussian or Eisenstein integers. Those two special cases can easily be treated separately. A wealth of information on the Bianchi groups can be found in the monographs [2], [3], [4]. For a subgroup Γ of finite index in $\mathrm{SL}_2(\mathcal{O})$, consider the Borel–Serre compactification ${}_\Gamma\backslash\widehat{\mathcal{H}}$ of the orbit space ${}_\Gamma\backslash\mathcal{H}$, constructed in the appendix of [6]. As a fundamental domain in hyperbolic space, we make use of the polyhedron with missing vertices at the cusps, described by Bianchi [1], and which we will call the *Bianchi fundamental polyhedron*. Our main result is the following, for which we consider the cellular structure on ${}_\Gamma\backslash\widehat{\mathcal{H}}$ induced by the Bianchi fundamental polyhedron.

Theorem 1. *The boundary $\partial({}_\Gamma\backslash\widehat{\mathcal{H}})$ is included as a sub-cellular chain complex into the Borel–Serre compactification ${}_\Gamma\backslash\widehat{\mathcal{H}}$ in the following way.*

(0) *All vertices of ${}_\Gamma\backslash\widehat{\mathcal{H}}$ are equivalent modulo the image of the 1–cells (they define the same class in degree 0 homology).*
(1) *For each orbit of cusps, exactly one of the two attached 1–cells is the boundary of a 2–chain.*
(2) *The boundary of the Bianchi fundamental polyhedron is the union over the attached 2–cells.*

Proof.

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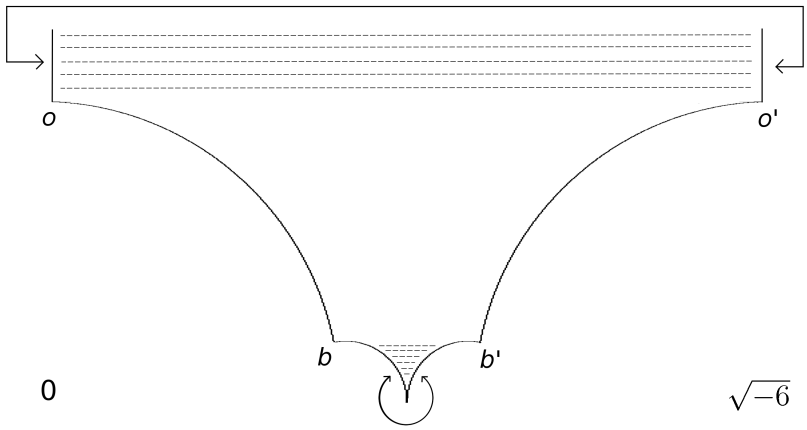

FIGURE 1. Intersection of the fundamental polyhedron for $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$ with the imaginary half-plane

(0) is obvious since the orbit space $_\Gamma\backslash\widehat{\mathcal{H}}$ is path-wise connected.
(1) will be proved cusp-wise in lemma 2.
(2) follows from Poincaré's theorem on fundamental polyhedra, which tells us that the 2–dimensional facets of the Bianchi fundamental polyhedron inside hyperbolic space appear in pairs modulo the action of Γ, with opposite signs. The 2–cells which we attach at the cusps are by construction unique modulo the action of Γ.

□

Throughout this article, we use the upper-half space model for hyperbolic space, as it is the one used by Bianchi. Recall [6] that the Borel–Serre compactification joins a 2-torus $\mathbf{T}$ to $_\Gamma\backslash\mathcal{H}$ at every cusp. We decompose $\mathbf{T}$ in the classical way into a 2–cell, two 1–cells and a vertex (see figure 2).

Lemma 2. *The inclusion of* $\mathbf{T}$ *into the Borel–Serre compactification of* $_\Gamma\backslash\mathcal{H}$ *makes exactly one of the* 1*–cells of* $\mathbf{T}$ *become the boundary of a* 2*–chain.*

Proof. Consider the fundamental rectangle $\mathcal{F}$ for the action of the cusp stabiliser on the plane joined to $\mathcal{H}$ at our cusp. There is a sequence of rectangles in $\mathcal{H}$ obtained as translates of $\mathcal{F}$ orthogonal to all the geodesic arcs emanating from the cusp. This way, the portion of the fundamental polyhedron which is nearest to the cusp, is trivially foliated by (locally homeomorphic to the Cartesian product of a geodesic arc with) translates of $\mathcal{F}$ (see figure 1 for the case $\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$, where the fundamental polyhedron admits one cusp at ∞ and one cusp at $\frac{\sqrt{-6}}{2}$). The boundaries of these translates are subject to the same identifications by Γ as the boundary of $\mathcal{F}$. So in the quotient space by the action of Γ, the image of $\mathbf{T}$ is wrapped into a sequence of layers of tori. And therefore in turn, the 3-dimensional interior of the Bianchi fundamental polyhedron is wrapped around the image of $\mathbf{T}$ along the entire surface of the latter. Hence, there is a neighbourhood of the image of $\mathbf{T}$ that is homeomorphic to Euclidean 3–space with the interior of a full torus removed. Now, considering the cell structure of the torus (see figure 2), we see that precisely one of the 1–cells can be contracted in the interior of the image of the Bianchi fundamental polyhedron. The other 1–cell is linked with the removed full torus and thus remains uncontractible in the Borel–Serre compactification of $_\Gamma\backslash\mathcal{H}$. □

Corollary 3. *The map* α *induced on integral homology by the inclusion of the boundary* $\partial(_\Gamma\backslash\widehat{\mathcal{H}})$ *into the Borel–Serre compactification* $_\Gamma\backslash\widehat{\mathcal{H}}$ *is determined as follows.*

- *In degree* 0*, it is the augmentation map.*
- *In degree* 1*, it is a surjection onto the non-cuspidal part of the homology, for each orbit of cusps vanishing on exactly one of the two attached* 1*–cells, and injective from the set of remaining* 1*–cells.*
- *In degree* 2*, it is the map the kernel of which admits as generator the union of all torus* 2*–cells.*

Proof. Consider the cellular decomposition of $_\Gamma\backslash\mathcal{H}$ with only 3–cell the Bianchi fundamental polyhedron. Consider the chain map induced by including the boundary $\partial\left({}_\Gamma\backslash\widehat{\mathcal{H}}\right)$ as a sub-cellular chain complex into the Borel–Serre compactification $_\Gamma\backslash\widehat{\mathcal{H}}$. As a sub-chain inclusion, this chain map is injective and preserves cycles. So we only have to divide out the boundaries in $_\Gamma\backslash\widehat{\mathcal{H}}$. In degree n, we carry this out applying theorem 1 (n). □

Proposition 4. *For any Bianchi group* $\mathrm{SL}_2(\mathcal{O})$ *with only units* $\{\pm 1\}$*, the boundary of the vertical* 2*–cell of constant imaginary part zero (above the real axis) in the boundary of the Bianchi fundamental polyhedron passes to the quotient by* $\mathrm{SL}_2(\mathcal{O})$ *as follows. All of its edges in* $\mathcal{H}$ *appear in pairs with opposite orientation signs, so only the top edge of this* 2*–cell lying in the attached infinitely far torus contributes non-trivially, and it becomes a loop because its boundary vertices are identified.*

Proof. By the natural inclusion of $\mathrm{SL}_2(\mathbb{Z})$ into $\mathrm{SL}_2(\mathcal{O})$, we obtain a natural inclusion of the modular tree for $\mathrm{SL}_2(\mathbb{Z})$ into the cell complex of the Bianchi group, when including the upper half-plane into three-dimensional upper half-space at constant imaginary part zero (above the real axis). The boundary of the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ in the upper half-plane vanishes when passing to the quotient of the action. The considered inclusion maps this fundamental domain to the 2–cell in question, so the assertion follows. □

It follows that the cycle obtained from this top edge by the matrix $M := \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ identifying its endpoints, generates the kernel of the map induced on first degree homology by sending the 2–torus into the compactified quotient space, replacing the cusp at infinity. Hence the matrix M is an element of the commutator subgroup of any Bianchi group. This can alternatively be seen from the decomposition of this matrix as a product of finite order matrices in $\mathrm{SL}_2(\mathbb{Z})$, which is included naturally into every Bianchi group. Together with theorem 1, we obtain the following corollary, which has already been obtained by Serre [6, page 519].

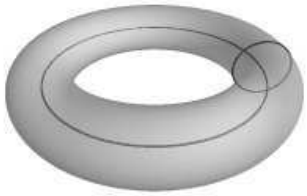

FIGURE 2. Cell structure of the 2–torus

Corollary 5. *The matrix* $\left(\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right)$*, with* ω *an element giving the ring of integers the form* $\mathbb{Z}[\omega]$*, represents a non-trivial element of the commutator factor group of the Bianchi group* $\mathrm{SL}_2(\mathbb{Z}[\omega])$*.*

The long exact sequence. Let Γ be a subgroup of finite index in a Bianchi group with only units $\{\pm 1\}$. Let $\mathbf{T}_i$ be the torus attached at the cusp i of Γ, and let x_i and y_i denote the cycles generating $\mathrm{H}_1(\mathbf{T}_i)$. Let $c(x_i)$ be a chain of hyperbolic 2-cells with boundary in $_\Gamma\backslash\widehat{\mathcal{H}}$ given by the cycle x_i. This is well-defined because of theorem 1(1). Let P be the Bianchi fundamental polyhedron of Γ, admitting as boundary $\cup_i\mathbf{T}_i$ by theorem 1(2).

Corollary 6. *Under the above assumptions and with the above notations, the long exact sequence induced on integral homology by the map* α *concentrates in*

$$\mathrm{H}_3({}_\Gamma\backslash\widehat{\mathcal{H}}) = 0 \to \langle P\rangle \longrightarrow \bigoplus_i\langle\mathbf{T}_i\rangle \xrightarrow{\alpha_2} \bigoplus_i\langle\mathbf{T}_i\rangle/_{\langle\cup_i\mathbf{T}_i\rangle} \oplus \mathrm{H}_2^{cusp}$$

$$\mathrm{H}_2^{cusp}\bigoplus_i\langle c(x_i)\rangle \longrightarrow \bigoplus_i\langle x_i, y_i\rangle \xrightarrow{\alpha_1} \bigoplus_i\langle y_i\rangle \oplus \mathrm{H}_1^{cusp}$$

$$\mathrm{H}_1^{cusp} \oplus \left\{\begin{smallmatrix}\text{vanishing}\\ \text{ideal}\end{smallmatrix}\right\} \longrightarrow \bigoplus_i\mathbb{Z} \xrightarrow[\text{augmentation}]{\alpha_0} \mathbb{Z} \to 0$$

where the maps without labels are the obvious restriction maps making the sequence exact; and where H_1^{cusp} *and* H_2^{cusp} *are generated by cycles from the interior of* $_\Gamma\backslash\mathcal{H}$*.*

Proof. We take into account that the virtual cohomological dimension of the Bianchi groups is 2, and apply corollary 3 to the long exact sequence in homology associated to the short exact sequence of chain complexes $\partial Y \xrightarrow{\alpha} Y \longrightarrow Y/_{\partial Y}$, where we write Y for the cellular chain complex of $_\Gamma\backslash\widehat{\mathcal{H}}$. □

***Remarks* 7.** (1) Note that the cuspidal homology satisfies equality of ranks between H_2^{cusp} and H_1^{cusp}, because the naive Euler-Poincaré characteristic vanishes.

(2) We obtain an analogous long exact sequence in cohomology with a coefficient module M; and $\mathrm{H}^1({}_\Gamma\backslash\widehat{\mathcal{H}};\, M) \cong \mathrm{Hom}(\bigoplus_i \langle y_i\rangle,\, M) \oplus \mathrm{H}^1_{cusp}$, with H^1_{cusp} being generated by cocycles from the interior of ${}_\Gamma\backslash\mathcal{H}$.

Example. The case $\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$. There are fifteen orbits of edges, labelled (s,s'), (s',s''), (r,∞), (∞,∞'), (∞',∞''), (b,a), (a,s'), (a,u), (u,v), (a'',v), (b,o), (o',b'), (u,r'), (r'',v) and (o,r'). The remaining edges of the Bianchi fundamental polyhedron, which we assemble from figures 1 and 3, are subject to identifications with those edges by $\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$.

The boundary of the quotient of the Bianchi fundamental polyhedron is $\mathbf{T}_\infty + \mathbf{T}_s$. The kernel of ∂_2 is the direct sum of $\langle \mathbf{T}_\infty\rangle$ and $\langle \mathbf{T}_s\rangle$. Therefore, $\mathrm{H}_2(\Gamma;\mathbb{Q}) \cong \mathbb{Q}$.

The kernel of ∂_1 is

$\langle (s,s')\rangle$
$\oplus\langle (s',s'')\rangle$
$\oplus\langle (a,u)+(u,v)+(v,a'')\rangle$
$\oplus\langle (\infty,\infty')\rangle$
$\oplus\langle (\infty',\infty'')\rangle$
$\oplus\langle (r',u)+(u,v)+(v,r'')\rangle$
$\oplus\langle (b,o)+(o',b')\rangle$
$\oplus\langle (o,r')+(r',u)+(u,a)+(a,b)+(b,o)\rangle$.

And the image of ∂_2 is

$\langle (s,s')\rangle$
$\oplus\langle (a,u)+(u,v)+(v,a'')-(s',s'')\rangle$
$\oplus\langle (\infty,\infty')\rangle$
$\oplus\langle (r',u)+(u,v)+(v,r'')-(\infty',\infty'')\rangle$
$\oplus\langle (o,r')+(r',u)+(u,a)+(a,b)+(b,o)\rangle$
$\oplus\langle -(o,r')+(a'',v)+(v,r'')-(a,b)+(o',b')\rangle$,

whence we se that $\mathrm{H}_1(\Gamma;\mathbb{Q}) \cong \mathbb{Q}^2$. In this case, this has been already known by Swan [7], and later been reobtained with different methods [8], [5].

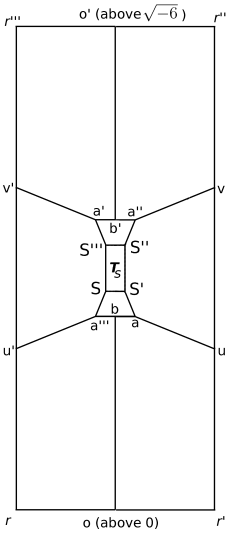

FIGURE 3. Bottom facets for $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$

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