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## ON A QUESTION OF SERRE

ALEXANDER D. RAHM

Consider the Borel–Serre compactification [1] of the quotient of hyperbolic 3–space  $\mathcal{H}$  by a finite index subgroup  $\Gamma$  in a Bianchi group, and in particular the following question which Serre poses on page 514 of the quoted article. Consider the map  $\alpha$  induced on homology when attaching the boundary  $\partial(\Gamma \backslash \widehat{\mathcal{H}})$  into the Borel–Serre compactification  $\Gamma \backslash \widehat{\mathcal{H}}$  of the orbit space  $\Gamma \backslash \mathcal{H}$ .

**Question 1.** *How can one determine the kernel of  $\alpha$  (in degree 1) ?*

Serre uses a global topological argument and obtains the rank of the kernel of  $\alpha$  in  $H_1(\partial(\Gamma \backslash \widehat{\mathcal{H}}))$ . But he keeps asking what submodule precisely this kernel is.

With a local topological study, we can decompose the kernel of  $\alpha$  into its parts associated to each cusp, both in first and second degree homology.

Our main result is the following, for which we consider the cellular structure on  $\Gamma \backslash \widehat{\mathcal{H}}$  induced by Bianchi’s fundamental polyhedron.

**Theorem 2.** *The boundary  $\partial(\Gamma \backslash \widehat{\mathcal{H}})$  is included as a sub-cellular chain complex into the Borel–Serre compactification  $\Gamma \backslash \widehat{\mathcal{H}}$  in the following way.*

- (0) *All vertices of  $\Gamma \backslash \widehat{\mathcal{H}}$  are equivalent modulo the image of the 1–cells (they define the same class in degree 0 homology).*
- (1) *For each orbit of cusps, exactly one of the two attached 1–cells is the boundary of a 2–chain.*
- (2) *The boundary of the Bianchi fundamental polyhedron is the union over the attached 2–cells.*

*Proof.*

- (0) is obvious since the orbit space  $\Gamma \backslash \widehat{\mathcal{H}}$  is path-wise connected.
- (1) will be proved cusp-wise in lemma 3.
- (2) follows from Poincaré’s theorem on fundamental polyhedra, which tells us that the 2–dimensional facets of the Bianchi fundamental polyhedron inside hyperbolic space appear in pairs modulo the action of  $\Gamma$ , with opposite signs. The 2–cells which we attach at the cusps are by construction unique modulo the action of  $\Gamma$ .

□

Let  $\mathbf{T}$  be the 2–torus joined to  $\Gamma \backslash \mathcal{H}$  at a cusp. We decompose  $\mathbf{T}$  in the classical way into a 2–cell, two 1–cells and a vertex (see figure 2).

**Lemma 3.** *The inclusion of  $\mathbf{T}$  into the Borel–Serre compactification of  $\Gamma \backslash \mathcal{H}$  makes exactly one of the 1–cells of  $\mathbf{T}$  become the boundary of a 2–chain.*

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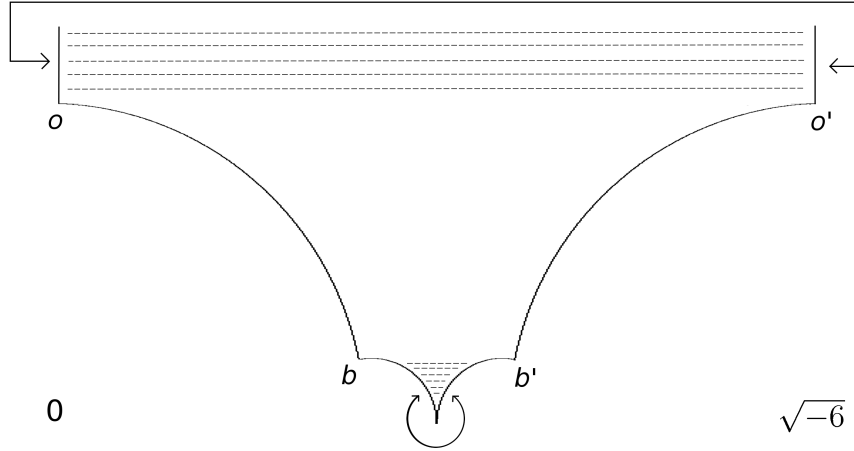


FIGURE 1. Intersection of the fundamental polyhedron for  $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$  with the imaginary half-plane

*Proof.* Consider the fundamental rectangle  $\mathcal{F}$  for the action of the cusp stabiliser on the plane joined to  $\mathcal{H}$  at our cusp. There is a sequence of rectangles in  $\mathcal{H}$  obtained as translates of  $\mathcal{F}$  orthogonal to all the geodesic arcs emanating from the cusp. This way, the portion of the fundamental polyhedron which is nearest to the cusp, is trivially foliated by (locally homeomorphic to the Cartesian product of a geodesic arc with) translates of  $\mathcal{F}$  (see figure 1 for the case  $\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$ , where the fundamental polyhedron admits one cusp at  $\infty$  and one cusp at  $\frac{\sqrt{-6}}{2}$ ). The boundaries of these translates are subject to the same identifications by  $\Gamma$  as the boundary of  $\mathcal{F}$ . So in the quotient space by the action of  $\Gamma$ , the image of  $\mathbf{T}$  is wrapped into a sequence of layers of tori. And therefore in turn, the 3-dimensional interior of the Bianchi fundamental polyhedron is wrapped around the image of  $\mathbf{T}$  along the entire surface of the latter. Hence, there is a neighbourhood of the image of  $\mathbf{T}$  that is homeomorphic to Euclidean 3-space with the interior of a full torus removed. Now, considering the cell structure of the torus (see figure 2), we see that precisely one of the 1-cells can be contracted in the interior of the image of the Bianchi fundamental polyhedron. The other 1-cell is linked with the removed full torus and thus remains uncontractible in the Borel-Serre compactification of  $\Gamma \backslash \mathcal{H}$ .  $\square$

**Corollary 4.** *The map  $\alpha$  induced on integral homology by the inclusion of the boundary  $\partial(\Gamma \backslash \widehat{\mathcal{H}})$  into the Borel-Serre compactification  $\Gamma \backslash \widehat{\mathcal{H}}$  is determined as follows.*

- In degree 0, it is the augmentation map.
- In degree 1, it is a surjection onto the non-cuspidal part of the homology, for each orbit of cusps vanishing on exactly one of the two attached 1-cells, and injective from the set of remaining 1-cells.
- In degree 2, it is the map the kernel of which admits as generator the union of all torus 2-cells.



FIGURE 2. Cell structure of the 2-torus

*Proof.* Consider the cellular decomposition of  $\Gamma \backslash \mathcal{H}$  with only 3-cell the Bianchi fundamental polyhedron. Consider the chain map induced by including the boundary  $\partial(\Gamma \backslash \widehat{\mathcal{H}})$  as a sub-cellular chain complex into the Borel–Serre compactification  $\Gamma \backslash \widehat{\mathcal{H}}$ . As a sub-chain inclusion, this chain map is injective and preserves cycles. So we only have to divide out the boundaries in  $\Gamma \backslash \widehat{\mathcal{H}}$ . In degree  $n$ , we carry this out applying theorem 2 ( $n$ ).  $\square$

Recall the identification [1, page 512] between coproduct of cusp stabilisers and first degree homology of the boundary of the quotient of hyperbolic 3-space by  $\Gamma$ ,

$$\left( \coprod \Gamma_{cusp} \cong \right) \coprod \Gamma_{cusp}^{ab} \cong \coprod H_1(\Gamma_{cusp}; \mathbb{Z}) \cong \coprod H_1(\mathbf{T}; \mathbb{Z}) \cong H_1(\partial(\Gamma \backslash \mathcal{H}); \mathbb{Z}),$$

where the coproduct runs over the orbits of cusps modulo  $\Gamma$ ; and for the first identification, we need to make  $\Gamma$  torsion-free (by passing to a finite index subgroup or to  $\mathrm{PSL}_2$  of the ring of integers). Consider the map  $\alpha$  induced on homology when attaching the boundary into the Borel–Serre compactification of  $\Gamma \backslash \mathcal{H}$ . Passing to the elements in the cusp stabiliser in  $\Gamma$  which identify the endpoints of an edge of  $\mathcal{F}$  to yield a 1-cell of  $\mathbf{T}$ , we conclude the following from theorem 2.

**Corollary 5.** *From each cusp stabiliser  $\Gamma_{cusp}$ , precisely one generator of the Abelianization  $\Gamma_{cusp}^{ab}$  is in the kernel of  $\alpha$  :* 
$$\coprod_{\text{orbits of cusps}} \Gamma_{cusp}^{ab} \longrightarrow \Gamma^{ab}.$$

**Proposition 6.** *For any Bianchi group with units  $\{\pm 1\}$ , the vertical 2-cell of constant imaginary part zero (above the real axis) in the boundary of the Bianchi fundamental polyhedron admits, after being projected to the quotient space, a boundary consisting only of the cycle in the attached infinitely far torus, coming from the top edge of this 2-cell.*

*Proof.* By the natural inclusion of  $\mathrm{SL}_2(\mathbb{Z})$  into any Bianchi group, we obtain a natural inclusion of the modular tree for  $\mathrm{SL}_2(\mathbb{Z})$  into the cell complex of the Bianchi group, when including the upper half-plane into three-dimensional upper half-space at constant imaginary part zero (above the real axis). The boundary of the fundamental domain for  $\mathrm{SL}_2(\mathbb{Z})$  in the upper half-plane vanishes when passing to the quotient of the action. The considered inclusion maps this fundamental domain to the 2-cell in question, so the assertion follows.  $\square$

**Corollary 7.** *The cycle obtained from this top edge by the matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  identifying its endpoints, generates the kernel of the map induced on first degree homology by sending the 2-torus into the compactified quotient space, replacing the cusp at infinity.*

**Corollary 8.** *The matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  does not pass to the Abelianization of any Bianchi group.*

The latter statement can also be obtained from the decomposition of this matrix as a product of finite order matrices in  $\mathrm{SL}_2(\mathbb{Z})$ , which is included naturally into every Bianchi group.

Together with theorem 2, we obtain the following corollary, which has already been obtained by Serre [1, page 519].

**Corollary 9.** *The matrix  $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ , with  $\omega$  an element giving the ring of integers the form  $\mathbb{Z}[\omega]$ , passes to the Abelianization of the Bianchi group  $\mathrm{SL}_2(\mathbb{Z}[\omega])$ .*

**The long exact sequence.**

**Corollary 10.** *Let  $\Gamma$  be a torsion-free subgroup of finite index in a Bianchi group. Then the long exact sequence induced on integral homology by the map  $\alpha$  concentrates in*

$$\begin{array}{ccccccc}
 \mathrm{H}_3(\Gamma) = 0 & \rightarrow & \langle \cup_i \mathbf{T}_i \rangle & \longrightarrow & \bigoplus_i \langle \mathbf{T}_i \rangle & \xrightarrow{\alpha_2} & \bigoplus_i \langle \mathbf{T}_i \rangle / \langle \cup_i \mathbf{T}_i \rangle \oplus \mathrm{H}_2^{cusp} \\
 & & & & & \swarrow & \\
 \mathrm{H}_2^{cusp} \oplus \langle x_i \rangle & \longrightarrow & \bigoplus_i \langle x_i, y_i \rangle & \xrightarrow{\alpha_1} & \bigoplus_i \langle y_i \rangle \oplus \mathrm{H}_1^{cusp} & & \\
 & & & \swarrow & & & \\
 \mathrm{H}_1^{cusp} \oplus \{ \text{vanishing ideal} \} & \longrightarrow & \bigoplus_i \mathbb{Z} & \xrightarrow[\text{augmentation}]{\alpha_0} & \mathbb{Z} & \rightarrow & 0
 \end{array}$$

where  $x_i$  and  $y_i$  denote the cycles generating  $\mathrm{H}_1(\mathbf{T}_i)$ , and the maps without labels are the obvious restriction maps making the sequence exact.

*Proof.* We take into account that the virtual cohomological dimension of the Bianchi groups is 2, and apply corollary 4.  $\square$

Note that the cuspidal homology satisfies equality of ranks between  $\mathrm{H}_2^{cusp}$  and  $\mathrm{H}_1^{cusp}$ , because the naive Euler-Poincaré characteristic vanishes.

**Example. The case  $\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$ .** There are fifteen orbits of edges, labelled  $(s, s')$ ,  $(s', s'')$ ,  $(r, \infty)$ ,  $(\infty, \infty')$ ,  $(\infty', \infty'')$ ,  $(b, a)$ ,  $(a, s')$ ,  $(a, u)$ ,  $(u, v)$ ,  $(a'', v)$ ,  $(b, o)$ ,  $(o', b')$ ,  $(u, r')$ ,  $(r'', v)$  and  $(o, r')$ . The remaining edges of the Bianchi fundamental polyhedron, which we assemble from figures 1 and 3, are subject to identifications with those edges by  $\Gamma = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$ .

The boundary of the quotient of the Bianchi fundamental polyhedron is  $\mathbf{T}_\infty + \mathbf{T}_s$ . The kernel of  $\partial_2$  is the direct sum of  $\langle \mathbf{T}_\infty \rangle$  and  $\langle \mathbf{T}_s \rangle$ . Therefore,  $H_2(\Gamma; \mathbb{Q}) \cong \mathbb{Q}$ . The kernel of  $\partial_1$  is

$$\begin{aligned} & \langle (s, s') \rangle \\ & \oplus \langle (s', s'') \rangle \\ & \oplus \langle (a, u) + (u, v) + (v, a'') \rangle \\ & \oplus \langle (\infty, \infty') \rangle \\ & \oplus \langle (\infty', \infty'') \rangle \\ & \oplus \langle (r', u) + (u, v) + (v, r'') \rangle \\ & \oplus \langle (b, o) + (o', b') \rangle \\ & \oplus \langle (o, r') + (r', u) + (u, a) + (a, b) + (b, o) \rangle. \end{aligned}$$

The image of  $\partial_2$  is

$$\begin{aligned} & \langle (s, s') \rangle \\ & \oplus \langle (a, u) + (u, v) + (v, a'') - (s', s'') \rangle \\ & \oplus \langle (\infty, \infty') \rangle \\ & \oplus \langle (r', u) + (u, v) + (v, r'') - (\infty', \infty'') \rangle \\ & \oplus \langle (o, r') + (r', u) + (u, a) + (a, b) + (b, o) \rangle \\ & \oplus \langle -(o, r') + (a'', v) + (v, r'') - (a, b) + (o', b') \rangle, \end{aligned}$$

whence we see that  $H_1(\Gamma; \mathbb{Q}) \cong \mathbb{Q}^2$ .

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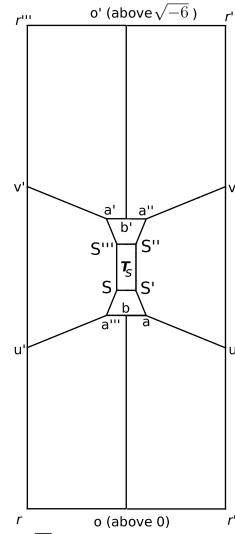


FIGURE 3. Bottom facets for  $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-6})})$