



HAL
open science

A simple model of trees for unicellular maps

Guillaume Chapuy, Valentin Féray, Eric Fusy

► **To cite this version:**

Guillaume Chapuy, Valentin Féray, Eric Fusy. A simple model of trees for unicellular maps. 2012. hal-00669896v4

HAL Id: hal-00669896

<https://hal.science/hal-00669896v4>

Preprint submitted on 10 Sep 2012 (v4), last revised 24 Jul 2013 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A SIMPLE MODEL OF TREES FOR UNICELLULAR MAPS

GUILLAUME CHAPUY, VALENTIN FÉRAY AND ÉRIC FUSY

ABSTRACT. We consider unicellular maps, or polygon gluings, of fixed genus. A few years ago the first author gave a recursive bijection transforming unicellular maps into trees, explaining the presence of Catalan numbers in counting formulas for these objects. In this paper, we give another bijection that explicitly describes the “recursive part” of the first bijection. As a result we obtain a very simple description of unicellular maps as pairs made by a plane tree and a permutation-like structure. All the previously known formulas follow as an immediate corollary or easy exercise, thus giving a bijective proof for each of them, in a unified way. For some of these formulas, this is the first bijective proof, *e.g.* the Harer-Zagier recurrence formula, or the Lehman-Walsh/Goupil-Schaeffer formulas. Thanks to previous work of the second author, this also leads us to a new expression for Stanley character polynomials, which evaluate irreducible characters of the symmetric group.

1. INTRODUCTION

A unicellular map is a connected graph embedded in a surface in such a way that the complement of the graph is a topological disk. These objects have appeared frequently in combinatorics in the last forty years, in relation with the general theory of map enumeration, but also with the representation theory of the symmetric group, the study of permutation factorizations, the computation of matrix integrals or the study of moduli spaces of curves. All these connections have turned the enumeration of unicellular maps into an important research field (for the many connections with other areas, see [18] and references therein; for an overview of the results see the introductions of the papers [9, 2]). The counting formulas for unicellular maps that appear in the literature can be roughly separated into two types.

The first type deals with *colored* maps (maps endowed with a mapping from its vertex set to a set of q colors). This implies “summation” enumeration formulas (see [16, 25, 20] or paragraph 3.4 below). These formulas are often elegant, and different combinatorial proofs for them have been given in the past few years [19, 14, 25, 20, 2]. The issue is that some important topological information, such as the genus of the surface, is not apparent in these constructions.

Formulas of the second type keep track explicitly of the genus of the surface; they are either inductive relations, like the Harer-Zagier recurrence formula [16], or are explicit (but quite involved) closed forms, like the Lehman-Walsh [28] and the Goupil-Schaeffer [15] formulas. From a combinatorial point of view, these formulas are harder to understand. A step in this direction was done by the first author in

Key words and phrases. one-face map, Stanley character polynomial, bijection, Harer-Zagier formula, Rémy’s bijection.

First and third author partially supported by the ERC grant StG 208471 – ExploreMaps.

Second author partially support by ANR project PSYCO.

[9] (this construction is explained in subsection 2.2), which led to new induction relations and to new formulas. However the link with other formulas of the second type remained mysterious, and [9] left open the problem of finding combinatorial proofs of these formulas.

The goal of this paper is to present a new bijection between unicellular maps and surprisingly simple objects which we call *C-decorated trees* (these are merely plane trees equipped with a certain kind of permutation on their vertices). This bijection, presented in Section 2, is based on the previous work of the first author [9]: we explicitly describe the “recursive part” appearing in this work. As a consequence, not only can we reprove all the aforementioned formulas in a bijective way, thus giving the first bijective proof for several of them, but we do that in a unified way. Indeed, C-decorated trees are so simple combinatorial objects that all formulas follow from our bijection as an immediate corollary or easy exercise, as we will see in Section 3.

Another interesting application of this bijection, studied in Section 4, is a new explicit way of computing the so-called Stanley character polynomials. The latter are nothing but the evaluation of irreducible characters of the symmetric groups, properly normalized and parametrized. Indeed, in a previous work [11], the second author expressed these polynomials as a generating function of (properly weighted) unicellular maps. Although we do not obtain a “closed form” expression (there is no reason to believe that such a form exists!), we express Stanley character polynomials as the result of a term-substitution in free cumulants, which are another meaningful quantity in representation theory of symmetric groups.

In Section 5 we discuss the possibility of applying our tools to *m-constellations*. This notion is a generalization of the notion bipartite maps introduced in connection with the study of factorizations in the symmetric group. A remarkable formula by Poulalhon and Schaeffer [22] (proved with the help of algebraic tools) suggests the possibility of a combinatorial proof using technique similar to ours. Although our bijection does not apply to these objects, we present two partial results in this direction, in the case of 3-constellations. One of them is an enumeration formula for a related family of objects that we call *quasi-3-constellations*, that turns out to be surprisingly similar to the Poulalhon-Schaeffer formula.

2. THE MAIN BIJECTION

2.1. Unicellular maps and C-decorated trees. We first briefly review some standard denomination for maps.

A *map* M of genus $g \geq 0$ is a connected graph G embedded on a closed compact oriented surface S of genus g , such that $S \setminus G$ is a collection of topological disks, which are called the *faces* of M . Loops and multiple edges are allowed. The graph G is called the *underlying graph* of M and S its *underlying surface*. Two maps that differ only by an oriented homeomorphism between the underlying surfaces are considered the same. A *corner* of M is the angular sector between two consecutive edges around a vertex. A *rooted map* is a map with a marked corner, called the *root*; the vertex incident to the root is called the *root-vertex*. From now on, all maps are assumed to be rooted (note that the underlying graph of a rooted map is naturally vertex-rooted). A *unicellular map* is a map with a unique face. The classical Euler relation $|V| - |E| + |F| = 2 - 2g$ ensures that a unicellular map with n edges has $n + 1 - 2g$ vertices. A *plane tree* is a unicellular map of genus 0.

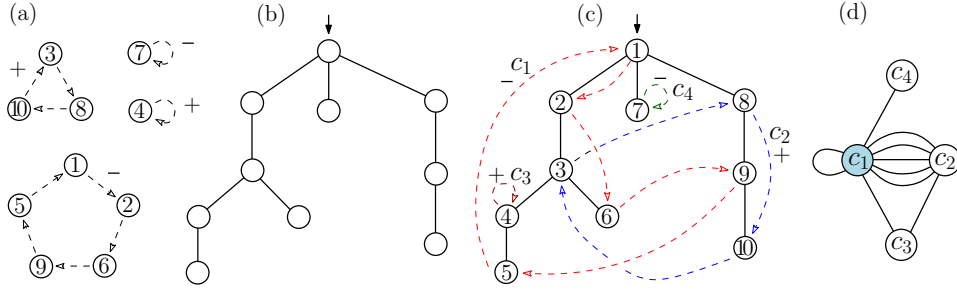


FIGURE 1. (a) A C-permutation σ . (b) A plane tree T . (c) The C-decorated tree (T, σ) . (d) The underlying graph of (T, σ) .

A *rotation system* on a connected graph G consists in a cyclic ordering of the half-edges of G around each vertex. Given a map M , its underlying graph G is naturally equipped with a rotation system given by the *clockwise ordering* of half-edges on the surface in a vicinity of each vertex. It is well-known that this correspondence is 1-to-1, i.e. a map can be considered as a connected graph equipped with a rotation system (thus, as a purely combinatorial object). We will take this viewpoint from now on.

We now introduce a new object called C-decorated tree.

A *cycle-signed* permutation is a permutation where each cycle carries a sign, either $+$ or $-$. A *C-permutation* is a cycle-signed permutation where all cycles have odd length, see Figure 1(a). For each C-permutation σ on n elements, the *rank* of σ is defined as $r(\sigma) = n - \ell(\sigma)$, where $\ell(\sigma)$ is the number of cycles of σ . Note that $r(\sigma)$ is even since all cycles have odd length. The *genus* of σ is defined as $r(\sigma)/2$. A *C-decorated tree* on n edges is a pair $\gamma = (T, \sigma)$ where T is a plane tree with n edges and σ is a C-permutation of $n + 1$ elements. The *genus* of γ is defined to be the genus of σ . Note that the $n + 1$ vertices of T can be canonically numbered from 1 to $n + 1$ (e.g., following a left-to-right depth-first traversal), hence σ can be seen as a permutation of the vertices of T , see Figure 1(c). The *underlying graph* of γ is the (vertex-rooted) graph G obtained from T by merging into a single vertex the vertices in each cycle of σ (so that the vertices of G correspond to the cycles of σ), see Figure 1(d).

Definition 1. For n, g nonnegative integers, denote by $\mathcal{E}_g(n)$ the set of unicellular maps of genus g with n edges; and denote by $\mathcal{T}_g(n)$ the set of C-decorated trees of genus g with n edges.

For two finite sets \mathcal{A} and \mathcal{B} , we denote by $\mathcal{A} + \mathcal{B}$ their disjoint union and by $k\mathcal{A}$ the set made of k disjoint copies of \mathcal{A} . Besides, we write $\mathcal{A} \simeq \mathcal{B}$ if there is a bijection between \mathcal{A} and \mathcal{B} . Our main result will be to show that $2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$, with a bijection which preserves the underlying graphs of the objects.

2.2. Recursive decomposition of unicellular maps. In this section, we briefly recall a combinatorial method developed in [9] to decompose unicellular maps.

Proposition 1 (Chapuy [9]). For $k \geq 1$, denote by $\mathcal{E}_g^{(2k+1)}(n)$ the set of maps from $\mathcal{E}_g(n)$ in which a set of $2k + 1$ vertices is distinguished. Then for $g > 0$ and

$n \geq 0$,

$$(1) \quad 2g \mathcal{E}_g(n) \simeq \mathcal{E}_{g-1}^{(3)}(n) + \mathcal{E}_{g-2}^{(5)}(n) + \mathcal{E}_{g-3}^{(7)}(n) + \cdots + \mathcal{E}_0^{(2g+1)}(n).$$

In addition, if M and (M', S') are in correspondence, then the underlying graph of M is obtained from the underlying graph of M' by merging the vertices in S' into a single vertex.

We now sketch briefly the construction of [9]. Although this is not really needed for the sequel, we believe that it gives a good insight into the objects we are dealing with (readers in a hurry may take Proposition 1 for granted and jump directly to subsection 2.3). We refer to [9] for proofs and details.

We first explain where the factor $2g$ comes from in (1). Let M be a rooted unicellular map of genus g with n edges. Then M has $2n$ corners, and we label them from 1 to $2n$ incrementally, starting from the root, and going clockwise around the (unique) face of M (Figure 2). Let v be a vertex of M , let k be its degree, and let (a_1, a_2, \dots, a_k) be the sequence of the labels of corners incident to it, read in *counterclockwise direction around v* starting from the minimal label $a_1 = \min a_i$. If for some $j \in \llbracket 1, k-1 \rrbracket$, we have $a_{j+1} < a_j$, we say that the corner of v labelled by a_{j+1} is a *trisection* of M . Figure 2(a) shows a map of genus two having four trisections. More generally we have:

Lemma 2 ([9]). *A unicellular map of genus g contains exactly $2g$ trisections. In other words, the set of unicellular maps of genus g with n edges and a marked trisection is isomorphic to $2g \mathcal{E}_g(n)$.*

Now, let τ be a trisection of M of label $\ell(\tau)$, and let v the vertex it belongs to. We denote by c the corner of v with minimum label and by c' the corner with minimum label among those which appear between c and τ clockwise around v and whose label is greater than $\ell(\tau)$. By definition of a trisection, c' is well defined. We then construct a new map M' , by *slicing* the vertex v into three new vertices using the three corners c, c', τ as in Figure 2(b). We say that the map M' is obtained from M by *slicing the trisection τ* . As shown in [9], the new map M' is a unicellular map of genus $g-1$. We can thus relabel the $2n$ corners of M' from 1 to $2n$, according to the procedure we already used for M . Among these corners, three of them, say c_1, c_2, c_3 are naturally inherited from the slicing of v , as on Figure 2(b). Let v_1, v_2, v_3 be the vertices they belong to, respectively. Then the following is true [9]: *In the map M' , the corner c_i has the smallest label around the vertex v_i , for $i \in \{1, 2\}$. For $i = 3$, either the same is true, or c_3 is a trisection of the map M' .*

We now finally describe the bijection promised in Proposition 1. It is defined recursively on the genus, as follows. Given a map $M \in \mathcal{E}_g(n)$ with a marked trisection τ , let M' be obtained from M by slicing τ , and let c_i, v_i be defined as above for $i \in \{1, 2, 3\}$. If c_3 has the minimum label at v_3 , set $\Psi(M, \tau) := (M', \{v_1, v_2, v_3\})$, which is an element of $\mathcal{E}_{g-1}^{(3)}(n)$. Otherwise, let $(M'', S) = \Psi(M', c_3)$, and set $\Psi(M, \tau) := (M'', S \cup \{v_1, v_2\})$. Note that this recursive algorithm necessarily stops, since the genus of the map decreases and since there are no trisections in unicellular maps of genus 0 (plane trees). Thus this procedure yields recursively a mapping that associates to a map M with a marked trisection τ another map M'' of a smaller genus, with a set S'' of marked vertices (namely the set of vertices which have been involved in a slicing at some point of the procedure). The set S'' of marked vertices

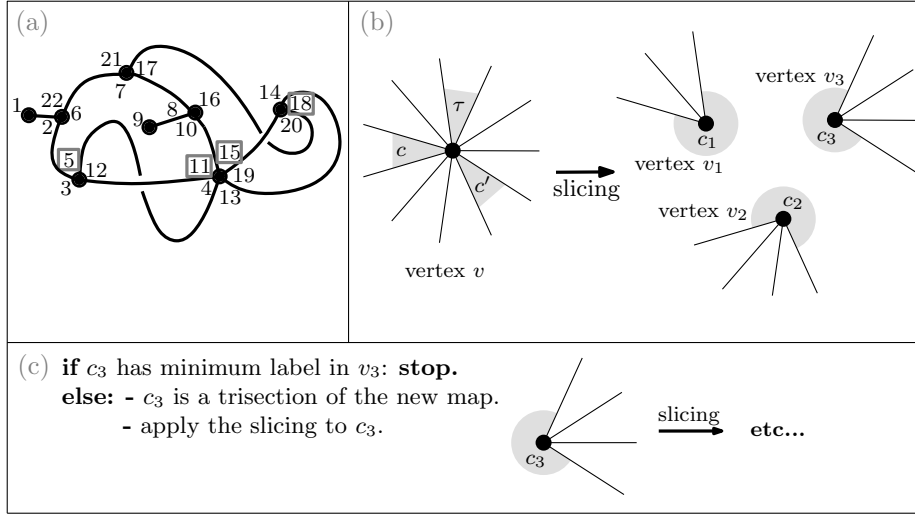


FIGURE 2. (a) A unicellular map of genus 2 equipped with its corner labelling. Labels corresponding to trisections are boxed. (b) Given a trisection τ , two other corners of interest c and c' are canonically defined (see text). “Slicing the trisection” then gives rise to three new vertices v_1, v_2, v_3 , with distinguished corners c_1, c_2, c_3 . (c) The recursive procedure of [9]: if c_3 is the minimum corner of v_3 , then stop; else, as shown in [9], c_3 is a trisection of the new map M' : in this case, iterate the slicing operation on (M', c_3) .

necessarily has odd cardinality, as easily seen by induction. Moreover, it is clear that the underlying graph of M coincides with the underlying graph of M'' in which the vertices of S'' have been identified together into a single vertex. One can show that Ψ is a bijection [9], with an explicit inverse mapping.

2.3. Recursive decomposition of C-decorated trees. We now propose a recursive method to decompose C-decorated trees, which can be seen as parallel to the decomposition of unicellular maps given in the previous section. Denote by $\mathcal{C}(n)$ (resp. $\mathcal{C}_g(n)$) the set of C-permutations on n elements (resp. on n elements and of genus g). A *signed sequence* of integers is a pair (ϵ, S) where S is an integer sequence and ϵ is a sign, either $+$ or $-$.

Lemma 3. *Let X be a finite non-empty set of positive integers. Then there is a bijection between signed sequences of distinct integers from X —all elements of X being present in the sequence— and C-permutations on the set X . In addition the C-permutation has one cycle if and only if the signed sequence has odd length and starts with its minimal element.*

Proof. Let γ be a signed sequence, e.g. $\gamma = +(4731562)$. If γ has odd length and starts with its minimal element, return γ seen as a unicyclic C-permutation (where the unique cycle is written sequentially). Otherwise cut γ as $\gamma = \gamma_1\gamma_2$, where γ_2 starts with the minimal element in γ (in our example, $\gamma_1 = +(473)$ and $\gamma_2 = (1562)$). If γ_2 has odd length, then “produce” the signed cycle $+\gamma_2$. If γ_2 has even length,

move the second element of γ_2 to the end of γ_1 , and “produce” the signed cycle $^- \gamma_2$. Then (in both cases), restart the same process on $\gamma = \gamma_1$, producing one (signed) cycle of odd length at each step, until γ is odd and starts with its minimal element, in which case one produces γ as the last signed cycle. (In our example, the signed cycles successively produced are $^-(162)$, $^-(3)$, and $^+(475)$.) The process clearly yields a collection of signed cycles of odd lengths, i.e., yields a C-permutation. The mapping is straightforward to invert, so it gives a bijection. \square

An element of a C-permutation is called *non-minimal* if it is not the minimum in its cycle. Non-minimal elements play the same role for C-permutations (and C-decorated trees) as trisections for unicellular maps. Indeed, a C-permutation of genus g has $2g$ non-minimal elements (compare with Lemma 2), and moreover we have the following analogue of Proposition 1:

Proposition 4. *For $k \geq 1$, denote by $\mathcal{T}_g^{(2k+1)}(n)$ the set of C-decorated trees from $\mathcal{T}_g(n)$ in which a set of $2k+1$ cycles is distinguished. Then for $g > 0$ and $n \geq 0$,*

$$2g \mathcal{T}_g(n) \simeq \mathcal{T}_{g-1}^{(3)} + \mathcal{T}_{g-2}^{(5)} + \mathcal{T}_{g-3}^{(7)} + \cdots + \mathcal{T}_0^{(2g+1)}.$$

In addition, if γ and (γ', S') are in correspondence, then the underlying graph of γ is obtained from the underlying graph of γ' by merging the vertices corresponding to cycles from S' into a single vertex.

Proof. For $k \geq 1$ let $\mathcal{C}_g^{(2k+1)}(n)$ be the set of C-permutations from $\mathcal{C}_g(n)$ where a subset of $2k+1$ cycles are marked. Let $\mathcal{C}_g^\circ(n)$ be the set of C-permutations from $\mathcal{C}_g(n)$ where a non-minimal element is marked. Note that $\mathcal{C}_g^\circ(n) \simeq 2g \mathcal{C}_g(n)$ since a C-permutation in $\mathcal{C}_g(n)$ has $2g$ non-minimal elements. Besides, one has $\mathcal{C}_g^\circ(n) \simeq \sum_{k=1}^g \mathcal{C}_{g-k}^{(2k+1)}(n)$.

Indeed take an element in $\mathcal{C}_g^\circ(n)$, write the signed cycle containing the marked element i as a signed sequence beginning by i and apply Lemma 3 to this signed sequence: this produces a collection of $(2k+1) \geq 3$ signed cycles of odd length, which we take as the marked cycles.

Finally $2g \mathcal{C}_g(n) \simeq \sum_{k=1}^g \mathcal{C}_{g-k}^{(2k+1)}(n)$. Since $\mathcal{T}_g(n) = \mathcal{E}_0(n) \times \mathcal{C}_g(n+1)$, we conclude that $2g \mathcal{T}_g(n) \simeq \sum_{k=1}^g \mathcal{T}_{g-k}^{(2k+1)}(n)$. The statement on the underlying graph just follows from the fact that the procedure in Lemma 3 merges the marked cycles into a unique cycle. \square

2.4. The main result.

Theorem 5. *For each non-negative integers n and g we have*

$$2^{n+1} \mathcal{E}_g(n) \simeq \mathcal{T}_g(n).$$

In addition the cycles of a C-decorated tree naturally correspond to the vertices of the associated unicellular map, in such a way that the respective underlying graphs are the same.

Proof. The proof is a simple induction on g , whereas n is fixed. The case $g = 0$ is obvious. Let $g > 0$. The induction hypothesis ensures that for each $g' < g$, $2^{n+1} \mathcal{E}_{g'}^{(2k+1)}(n) \simeq \mathcal{T}_{g'}^{(2k+1)}(n)$, where the underlying graphs (taking marked vertices into account) of corresponding objects are the same. Hence, by Propositions 1 and 4, we have $2g 2^{n+1} \mathcal{E}_g(n) \simeq 2g \mathcal{T}_g(n)$, where the underlying graphs

of corresponding objects are the same. Finally, one can extract from this $2g$ -to- $2g$ correspondence a 1-to-1 correspondence (think of extracting a perfect matching from a $2g$ -regular bipartite graph, which is possible according to Hall's marriage theorem). And obviously the extracted 1-to-1 correspondence, which realizes $2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$, also preserves the underlying graphs. \square

2.5. A fractional, or stochastic, formulation. Even if this does not hinder enumerative applications to be detailed in the next section, we do not know of an effective (polynomial-time) way to implement the bijection of Theorem 5; indeed the last step of the proof is to extract a perfect matching from a $2g$ -regular bipartite graph whose size is exponential in n .

What can be done effectively is a *fractional* formulation of the bijection. For a finite set X , let $\mathbb{C}\langle X \rangle$ be the set of linear combinations of the form $\sum_{x \in X} u_x \cdot x$, where the $x \in X$ are seen as independent formal vectors, and the coefficients u_x are in \mathbb{C} . Let $\mathbb{R}_1^+\langle X \rangle \subset \mathbb{C}\langle X \rangle$ be the subset of linear combinations where the coefficients are nonnegative and add up to 1. Denote by $\mathbf{1}_X$ the vector $\sum_{x \in X} x$. For two finite sets X and Y , a *fractional mapping* from X to Y is a linear mapping φ from $\mathbb{C}\langle X \rangle$ to $\mathbb{C}\langle Y \rangle$ such that the image of each $x \in X$ is in $\mathbb{R}_1^+\langle Y \rangle$; the set of elements of Y whose coefficients in $\varphi(x)$ are strictly positive is called the *image-support* of x . Note that $\varphi(x)$ identifies to a probability distribution on Y ; a “call to $\varphi(x)$ ” is meant as picking up $y \in Y$ under this distribution. A fractional mapping is *bijective* if $\mathbf{1}_X$ is mapped to $\mathbf{1}_Y$, and is *deterministic* if each $x \in X$ is mapped to some $y \in Y$. Note that, if there is a fractional bijection from X to Y , then $|X| = |Y|$ (indeed in that case the matrix of φ is bistochastic).

One can now formulate by induction on the genus an effective (the cost of a call is $O(gn)$) fractional bijection from $2^{n+1}\mathcal{E}_g[n]$ to $\mathcal{T}_g(n)$, and similarly from $\mathcal{T}_g[n]$ to $2^{n+1}\mathcal{E}_g(n)$. The crucial property is that, for $k \geq 1$ and E, F finite sets, if there is a fractional bijection Φ from kE to kF then one can *effectively* derive from it a fractional bijection $\tilde{\Phi}$ from E to F : for $x \in E$, just define $\tilde{\Phi}(x)$ as $\frac{1}{k}(\iota(\Phi(x_1)) + \dots + \iota(\Phi(x_k)))$, where x_1, \dots, x_k are the representatives of x in kE , and where ι is the projection from kF to F . In other words a call to $\tilde{\Phi}(x)$ consists in picking up a representative x_i of x in kE uniformly at random and then calling $\Phi(x)$. Hence by induction on g , Propositions 1 and 4 (where the stated combinatorial isomorphisms are effective) ensure that there is an effective fractional bijection from $2^{n+1}\mathcal{E}_g(n)$ to $\mathcal{T}_g[n]$ and similarly from $\mathcal{T}_g[n]$ to $2^{n+1}\mathcal{E}_g[n]$, such that if γ' is in the image-support of γ then the underlying graphs of γ and γ' are the same.

Note that, given an effective fractional bijection between two sets X and Y , and a uniform random sampling algorithm on the set X , one obtains immediately a uniform random sampling algorithm for the set Y . In the next section, we will use our bijection to prove several enumerative formulas for unicellular maps, starting from elementary results on the enumeration of trees or permutations. In all cases, we will also be granted with a uniform random sampling algorithm for the corresponding unicellular maps, though we will not emphasize this point in the rest of the paper.

3. COUNTING FORMULAS FOR UNICELLULAR MAPS

It is quite clear that C-decorated trees are much simpler combinatorial objects than unicellular maps. In this section, we use them to give bijective proofs of

several known formulas concerning unicellular maps. We focus on the Lehman-Walsh and Goupil-Schaeffer formulas, and the Harer-Zagier recurrence, of which bijective proofs were long-awaited. We also sketch a bijective proof of the Harer-Zagier summation formula (different bijective proofs are already known for such summation formulas). We insist on the fact that all these proofs are elementary consequences of our main bijection (Theorem 5).

3.1. Two immediate corollaries. The set $\mathcal{T}_g(n) = \mathcal{E}_0(n) \times \mathcal{C}_g(n+1)$ is the product of two sets that are easy to count. Precisely, let $\epsilon_g(n) = |\mathcal{E}_g(n)|$ and $c_g(n) = |\mathcal{C}_g(n)|$. Recall that $\epsilon_0(n) = \text{Cat}(n)$, where $\text{Cat}(n) := \frac{(2n)!}{n!(n+1)!}$ is the n th Catalan number. Therefore Theorem 5 gives $\epsilon_g(n) = 2^{-n-1} \text{Cat}(n) c_g(n+1)$.

One gets easily a closed form for $c_g(n+1)$ (by summing over all possible cycle types) and an explicit formula for the generating series, thereby recovering two classical results for the enumeration of unicellular maps.

Every partition of $n+1$ in $n+1-2g$ odd parts writes as $1^{n+1-2g-\ell} 3^{m_1} \dots (2k+1)^{m_k}$ for some partition $\gamma = (\gamma_1, \dots, \gamma_\ell) = 1^{m_1} \dots k^{m_k}$ of g . But the number $a_\gamma(n+1)$ of permutations of $n+1$ elements with cycle-type equal to $1^{n+1-2g-\ell} 3^{m_1} \dots (2k+1)^{m_k}$ is classically given by

$$a_\gamma(n+1) = \frac{(n+1)!}{(n+1-2g-\ell)! \prod_i m_i! (2i+1)^{m_i}},$$

and the number of C-permutations with this cycle-type is just $a_\gamma(n+1) 2^{n+1-2g}$ (since each cycle has 2 possible signs). Hence, we get the equality

$$c_g(n+1) = 2^{n+1-2g} \sum_{\gamma \vdash g} a_\gamma(n+1).$$

We thus recover:

Proposition 6 (Walsh and Lehman [28]). *The number $\epsilon_g(n)$ is given by*

$$\epsilon_g(n) = \frac{(2n)!}{n!(n+1-2g)! 2^{2g}} \sum_{\gamma \vdash g} \frac{(n+1-2g)_\ell}{\prod_i m_i! (2i+1)^{m_i}},$$

where $(x)_k = \prod_{j=0}^{k-1} (x-j)$, ℓ is the number of parts of γ , and m_i is the number of parts of length i in γ .

Define the exponential generating function

$$C(x, y) := \sum_{n, g} \frac{1}{(n+1)!} c_g(n+1) y^{n+1} x^{n+1-2g}$$

of C-permutations where y marks the number of elements, which are labelled, and x marks the number of cycles. Using the standard techniques of symbolic method (see *e.g.* [12, Part A] for a general presentation of this method and the page 120 for the application to permutations seen as sets of cycles), this generating series is given by

$$C(x, y) = \exp\left(2x \sum_{k \geq 1} \frac{y^{2k+1}}{2k+1}\right) - 1 = \exp\left(x \log\left(\frac{1+y}{1-y}\right)\right) - 1 = \left(\frac{1+y}{1-y}\right)^x - 1.$$

Since $c_0(1) = 2$ and $\frac{1}{(n+1)!} c_g(n+1) = \frac{2^{n+1} n!}{(2n)!} \epsilon_g(n) = \frac{2}{(2n-1)!!} \epsilon_g(n)$ for $n \geq 1$, we recover:

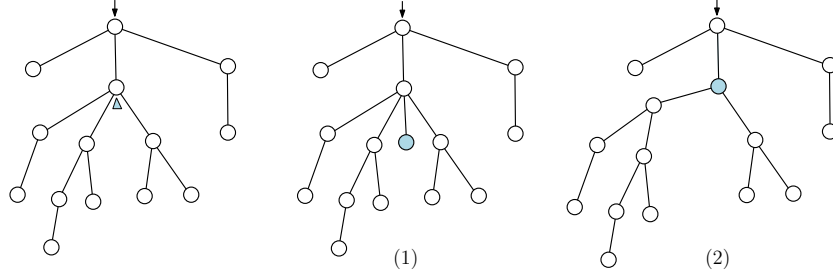


FIGURE 3. Rémy’s procedure gives two ways to obtain a plane tree with n edges and a marked vertex v from a plane tree with $n - 1$ edges and a marked corner: (1) in the first way (replacing the marked corner by a leg) v is a leaf, (2) in the second way (stretching an edge to carry the subtree on the left of the marked corner) v is a non-leaf.

Proposition 7 (Harer-Zagier series formula [16, 18]). *The generating function*

$$E(x, y) := 1 + 2xy + 2 \sum_{g \geq 0, n > 0} \frac{\epsilon_g(n)}{(2n - 1)!!} y^{n+1} x^{n+1-2g} \text{ is given by}$$

$$E(x, y) = \left(\frac{1 + y}{1 - y} \right)^x.$$

3.2. Harer-Zagier recurrence formula. Elementary algebraic manipulations on the expression of $E(x, y)$ yield a very simple recurrence satisfied by $\epsilon_g(n)$, known as the Harer-Zagier recurrence formula (stated in Proposition 10 hereafter). We now show that the model of C-decorated trees makes it possible to derive this recurrence directly from a combinatorial isomorphism, that generalizes Rémy’s beautiful bijection for plane trees [24].

It is convenient here to consider C-decorated trees as *unlabelled structures*: precisely we see a C-decorated tree as a plane tree where the vertices are partitioned into parts of odd size, where each part carries a sign $+$ or $-$, and such that the vertices in each part are cyclically ordered (the C-permutation can be recovered by numbering the vertices of the tree according to a left-to-right depth-first traversal), think of Figure 1(c) where the labels have been taken out. We take here the convention that a plane tree with n edges has $2n + 1$ corners, considering that the sector of the root has two corners, one on each side of the root.

We denote by $\mathcal{P}(n) = \mathcal{E}_0(n)$ the set of plane trees with n edges, and by $\mathcal{P}^\vee(n)$ (resp. $\mathcal{P}^c(n)$) the set of plane trees with n edges where a vertex (resp. a corner) is marked. Rémy’s procedure, shown in Figure 3, realizes the isomorphism $\mathcal{P}^\vee(n) \simeq 2\mathcal{P}^c(n - 1)$, or equivalently

$$(2) \quad (n + 1)\mathcal{P}(n) \simeq 2(2n - 1)\mathcal{P}(n - 1).$$

Let $\mathcal{T}_g^\vee(n)$ be the set of C-decorated trees from $\mathcal{T}_g(n)$ where a vertex is marked. Let \mathcal{A} (resp. \mathcal{B}) be the subset of objects in $\mathcal{T}_g^\vee(n)$ where the signed cycle containing the marked vertex has length 1 (resp. length greater than 1). Let $\gamma \in \mathcal{T}_g^\vee(n)$, with $n \geq 1$. If $\gamma \in \mathcal{A}$, record the sign of the 1-cycle containing v and then apply the Rémy’s procedure to the plane tree with respect to v (so as to delete v). This reduction, which does not change the genus, yields $\mathcal{A} \simeq 2 \cdot 2(2n - 1)\mathcal{T}_g(n - 1)$.

If $\gamma \in \mathcal{B}$, let c be the cycle containing the marked vertex v ; c is of the form $(v, v_1, v_2, \dots, v_{2k})$ for some $k \geq 1$. Move v_1 and v_2 out of c (the successor of v becomes the former successor of v_2). Then apply the Rémy's procedure twice, firstly with respect to v_1 (on a plane tree with n edges), secondly with respect to v_2 (on a plane tree with $n-1$ edges). This reduction, which decreases the genus by 1, yields $\mathcal{B} \simeq 2(2n-1)2(2n-3)\mathcal{T}_{g-1}^v(n-2)$, hence $\mathcal{B} \simeq 4(n-1)(2n-1)(2n-3)\mathcal{T}_{g-1}(n-2)$. Since $\mathcal{T}_g^v(n) = \mathcal{A} + \mathcal{B}$ and $\mathcal{T}_g^v(n) \simeq (n+1)\mathcal{T}_g(n)$, we finally obtain the isomorphism

$$(3) \quad (n+1)\mathcal{T}_g(n) \simeq 4(2n-1)\mathcal{T}_g(n-1) + 4(n-1)(2n-1)(2n-3)\mathcal{T}_{g-1}(n-2),$$

which holds for any $n \geq 1$ and $g \geq 0$ (with the convention $\mathcal{T}_g(n) = \emptyset$ if g or n is negative). Since $2^{n+1}\mathcal{E}_g(n) \simeq \mathcal{T}_g(n)$, we recover:

Proposition 8 (Harer-Zagier recurrence formula [16, 18]). *The coefficients $\epsilon_g(n)$ satisfy the following recurrence relation valid for any $g \geq 0$ and $n \geq 1$ (with $\epsilon_0(0) = 1$ and $\epsilon_g(n) = 0$ if $g < 0$ or $n < 0$):*

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2).$$

To the best of our knowledge this is the first proof of the Harer-Zagier recurrence formula that directly follows from a combinatorial isomorphism. The isomorphism (3) also provides a natural extension to arbitrary genus of Rémy's isomorphism (2).

3.3. Refined enumeration of bipartite unicellular maps. In this paragraph, we explain how to recover a formula due to A. Goupil and G. Schaeffer [15, Theorem 2.1] from our bijection. Let us first give a few definitions. A graph is *bipartite* if its vertices can be colored in black and white such that each edge connects a black and a white vertex. If the graph has a root-vertex v , then v is required to be black; thus, if the graph is also connected, then such a bicolouration of the vertices is unique. From now on, a connected bipartite graph with a root-vertex is assumed to be endowed with this canonical bicolouration.

The degree distribution of a map/graph is the sequence of the degrees of its vertices taken in decreasing order (it is a partition of $2n$, where n is the number of edges). If we consider a bipartite map/graph, we can consider separately the *white vertex degree distribution* and the *black vertex degree distribution*, which are two partitions of n .

Let ℓ, m, n be positive integers such that $n+1-\ell-m$ is even. Fix two partitions λ, μ of n of respective lengths ℓ and m . We call $\text{Bi}(\lambda, \mu)$ the number of bipartite unicellular maps, with white (resp. black) vertex degree distribution λ (resp. μ). The corresponding genus is $g = (n+1-\ell-m)/2$.

The purpose of this paragraph is to compute $\text{Bi}(\lambda, \mu)$. It will be convenient to change a little bit the formulation of the problem and to consider *labelled maps* instead of the usual non-labelled maps: a *labelled map* is a map whose vertices are labelled with integers $1, 2, \dots$. If the map is bipartite, we require instead that the white and black vertices are labelled separately (with respective labels w_1, w_2, \dots and b_1, b_2, \dots). The degree distribution(s) of a (bipartite) labelled map with n edges can be seen as a composition of $2n$ (resp. two compositions of n). For $\mathbf{I} = (i_1, \dots, i_\ell)$ and $\mathbf{J} = (j_1, \dots, j_m)$ two compositions of n , we denote by $\text{BiL}(\mathbf{I}, \mathbf{J})$ the number of labelled bipartite unicellular maps with white (resp. black) vertex degree distribution \mathbf{I} (resp. \mathbf{J}). The link between $\text{Bi}(\lambda, \mu)$ and $\text{BiL}(\mathbf{I}, \mathbf{J})$

is straightforward: $\text{BiL}(\mathbf{I}, \mathbf{J}) = m_1(\lambda)!m_2(\lambda)! \cdots m_1(\mu)!m_2(\mu)! \cdots \text{Bi}(\lambda, \mu)$, where λ and μ are the sorted versions of \mathbf{I} and \mathbf{J} . We now recover the following formula:

Proposition 9 (Goupil and Schaeffer [15, Theorem 2.1]).

$$(4) \quad \text{BiL}(\mathbf{I}, \mathbf{J}) = 2^{-2g} \cdot n \cdot \sum_{g_1+g_2=g} (\ell + 2g_1 - 1)!(m + 2g_2 - 1)! \\ \sum_{\substack{p_1+\dots+p_\ell=g_1 \\ q_1+\dots+q_m=g_2}} \prod_{r=1}^{\ell} \frac{1}{2p_r+1} \binom{i_r-1}{2p_r} \prod_{r=1}^m \frac{1}{2q_r+1} \binom{j_r-1}{2q_r}.$$

Proof. For $g = 0$ the formula is simply

$$(5) \quad \text{BiL}(\mathbf{I}, \mathbf{J}) = n(\ell - 1)!(m - 1)!,$$

which can easily be established by a bivariate version of the cyclic lemma, see also [13, Theorem 2.2]. (Note, that in that case, the cardinality only depends on the lengths of \mathbf{I} and \mathbf{J} .)

We now prove the formula for arbitrary g . Consider some lists $\mathbf{p} = (p_1, \dots, p_\ell)$ and $\mathbf{q} = (q_1, \dots, q_m)$ of nonnegative integers with total sum g : let $g_1 = \sum p_i$ and $g_2 = \sum q_i$. We say that a composition \mathbf{H} refines \mathbf{I} along \mathbf{p} if \mathbf{H} is of the form $(h_1^1, \dots, h_1^{2p_1+1}, \dots, h_\ell^1, \dots, h_\ell^{2p_\ell+1})$, with $\sum_{t=1}^{2p_r+1} h_r^t = i_r$ for all r between 1 and ℓ . Clearly, there are $\prod_{r=1}^{\ell} \binom{i_r-1}{2p_r}$ such compositions \mathbf{H} . One defines similarly a composition \mathbf{K} refining \mathbf{J} along \mathbf{q} .

Consider now the set of labelled bipartite plane trees of vertex degree distributions \mathbf{H} and \mathbf{K} , where \mathbf{H} (resp. \mathbf{K}) refines \mathbf{I} (resp. \mathbf{J}) along \mathbf{p} (resp. \mathbf{q}). By (5), there are $n \cdot (\ell + 2g_1 - 1)!(m + 2g_2 - 1)!$ trees for each pair (\mathbf{H}, \mathbf{K}) , so in total, with $\mathbf{I}, \mathbf{J}, \mathbf{p}$ and \mathbf{q} fixed, the number of such trees is:

$$(6) \quad n \cdot (\ell + 2g_1 - 1)!(m + 2g_2 - 1)! \prod_{r=1}^{\ell} \binom{i_r-1}{2p_r} \prod_{r=1}^m \binom{j_r-1}{2q_r}.$$

As the parts of \mathbf{H} (resp. \mathbf{K}) are naturally indexed by pairs of integers, we can see these trees as labelled by the set $\{w_r^t; 1 \leq r \leq \ell, 1 \leq t \leq 2p_r+1\} \sqcup \{b_r^t; 1 \leq r \leq m, 1 \leq t \leq 2q_r+1\}$. There is a canonical permutation of the vertices of the trees with cycles of odd sizes and which preserves the bicolouration: just send w_r^t to w_r^{t+1} (resp. b_r^t to b_r^{t+1}), where $t+1$ is meant modulo $2p_r+1$ (resp. $2q_r+1$). If we additionally put a sign on each cycle, we get a C-decorated tree (with labelled cycles) that corresponds to a labelled bipartite map with white (resp. black) vertex degree distribution \mathbf{I} (resp. \mathbf{J}). Conversely, to recover a labelled bipartite plane tree from such a C-decorated tree, one has to choose in each cycle which vertex gets the label w_r^1 or b_r^1 , and one has to forget the signs of the $(n+1-g)$ cycles. This represents a factor $2^{n+1-2g} \left(\prod_{r=1}^{\ell} (2p_r+1) \prod_{r=1}^m (2q_r+1) \right)^{-1}$.

Multiplying (6) by the above factor, and summing over all possible sequences \mathbf{p} and \mathbf{q} of total sum g , we conclude that the number of C-decorated trees associated with labelled bipartite unicellular maps of white (resp. black) vertex degree distribution \mathbf{I} (resp. \mathbf{J}), is equal to 2^{n+1} times the right-hand side of (4). By Theorem 5, this number is also equal to $2^{n+1} \text{BiL}(\mathbf{I}, \mathbf{J})$. This ends the proof of Proposition 9. \square

This is the first combinatorial proof of (4) (the proof by Goupil and Schaeffer involves representation theory of the symmetric group). Moreover, the authors of [15] found surprising that “the two partitions contribute independently to the genus”. With our approach, this is very natural, since the cycles are carried independently by white and black vertices.

3.4. Counting colored maps. In this paragraph, we deal with what was presented in the introduction as the *first type* of formulas. These formulas give an expression for a certain *sum* of coefficients counting unicellular maps, the expressions being usually simpler than those for the counting coefficients taken separately (like the Goupil-Schaeffer’s formula). These sums can typically be seen as counting formulas for *colored unicellular maps* (where the control is on the number of colors, which gives indirect access to the genus).

3.4.1. A summation formula for unicellular maps. We begin with Harer-Zagier’s summation formula [16, 18] (which can also be very easily derived from the expression of $E(x, y)$). In contrast to the formulas presented so far, this one has already been given combinatorial proofs [19, 14, 2] using different bijective constructions, but we want to insist on the fact that our construction gives bijective proofs for all the formulas in a unified way.

Proposition 10 (Harer-Zagier summation formula [16, 18]). *Let $A(v; n)$ be the number of unicellular maps with n edges and v vertices. Then for $n \geq 1$*

$$\sum_v A(v; n)x^v = (2n - 1)!! \sum_{r \geq 1} 2^{r-1} \binom{n}{r-1} \binom{x}{r}.$$

Proof. It suffices to prove that the number $A_r(n)$ of unicellular maps with n edges, each vertex having a color in $[1..r]$, and each color in $[1..r]$ being used at least once, is given by $A_r(n) = (2n - 1)!! 2^{r-1} \binom{n}{r-1}$. Our main bijection sends these objects onto C-decorated trees with n edges, where each (signed) cycle has a color in $[1..r]$, and such that each color in $[1..r]$ is used by at least one cycle. Each of the r colors yields a (non-empty) C-permutation, which can be represented as a signed sequence, according to Lemma 3. Then one can concatenate these r signed sequences into a unique sequence S of length $n + 1$, together with r signs and a subset of $r - 1$ elements among the n elements from position 2 to $n + 1$ in S (in order to recover from S the r signed sequences). For instance if $r = 3$ and if the signed sequences corresponding respectively to colors 1, 2, 3 are $+(3, 9, 4)$, $-(5, 8, 6, 2)$, and $-(1, 7)$, then the concatenated sequence is $(3, 9, 4, 5, 8, 6, 2, 1, 7)$, together with the 3 signs $(+, -, -)$ and the two selected elements $\{5, 1\}$. Hence the number of such C-decorated trees is $(n + 1)! 2^r \binom{n}{r-1}$, and by Theorem 5,

$$A_r(n) = 2^{-n-1} \text{Cat}(n)(n + 1)! 2^r \binom{n}{r-1} = (2n - 1)!! 2^{r-1} \binom{n}{r-1}. \quad \square$$

3.4.2. A summation formula for bipartite unicellular maps. By Theorem 5, a C-decorated tree associated to a bipartite unicellular map is a bipartite plane tree such that each signed cycle must contain only white (resp. black) vertices. Recall that the $n + 1$ vertices carry distinct labels from 1 to $n + 1$ (the ordering follows by convention a left-to-right depth-first traversal, see Figure 1(c)). Without loss of information the i black vertices (resp. j white vertices) can be relabelled from 1

to i (resp. from 1 to j) in the order-preserving way; we take here this convention for labelling the vertices of such a C-decorated tree. We now recover the following summation formula due to Jackson (different bijective proofs have been given in [25] and in [2]):

Proposition 11 (Jackson's summation formula [17]). *Let $B(v, w; n)$ be the number of bipartite unicellular maps with n edges, v black vertices and w white vertices. Then for $n \geq 1$*

$$\sum_{v,w} B(v, w; n) y^v z^w = n! \sum_{r,s \geq 1} \binom{n-1}{r-1, s-1} \binom{y}{r} \binom{z}{s}.$$

Proof. It suffices to prove that, for $r, s \geq 1$, the number $B_{r,s}(n)$ of bipartite unicellular maps with n edges, each black (resp. white) vertex having a so-called b -color in $[1..r]$ (resp. a so-called w -color in $[1..s]$), such that each b -color in $[1..r]$ (resp. w -color in $[1..s]$) is used at least once, is given by $B_{r,s}(n) = n! \binom{n-1}{r-1, s-1}$. For n, i, j such that $i + j = n + 1$, consider a bipartite C-decorated tree with n edges, i black vertices, j white vertices, where each black (resp. white) signed cycle has a b -color in $[1..r]$ (resp. a w -color in $[1..s]$), and each b -color in $[1..r]$ (resp. w -color in $[1..s]$) is used at least once. By the same argument as in Proposition 10, the C-permutation and b -colors on black vertices can be encoded by a sequence S_b of length i of distinct integers in $[1..i]$, together with r signs and a subset of $r - 1$ elements among the $i - 1$ elements at positions from 2 to i in S_b . And the C-permutation and w -colors on white vertices can be encoded by a sequence S_w of length j of distinct integers in $[1..j]$, together with s signs and a subset of $s - 1$ elements among the $j - 1$ elements at positions from 2 to j in S_w . Hence there are $\text{Nar}(i, j; n) 2^{r+s} i! \binom{i-1}{r-1} j! \binom{j-1}{s-1}$ such C-decorated trees, where $\text{Nar}(i, j; n)$ (called the *Narayana* number) is the number of bipartite plane trees with n edges, i black vertices and j white vertices, given by $\text{Nar}(i, j; n) = \frac{1}{n} \binom{n}{i} \binom{n}{j}$. By Theorem 5,

$$\begin{aligned} B_{r,s}(n) &= 2^{-n-1} 2^{r+s} \sum_{i+j=n+1} \text{Nar}(i, j; n) i! j! \binom{i-1}{r-1} \binom{j-1}{s-1} \\ &= n!(n-1)! \frac{2^{r+s-n-1}}{(r-1)!(s-1)!} \sum_{\substack{i+j=n+1 \\ i \geq r, j \geq s}} \frac{1}{(i-r)!(j-s)!}. \end{aligned}$$

But we have

$$\sum_{i+j=n+1} \frac{1}{(i-r)!(j-s)!} = \sum_{i+j=n+1-r-s} \frac{1}{i!j!} = \frac{2^{n+1-r-s}}{(n+1-r-s)!}.$$

Hence $B_{r,s}(n) = n! \binom{n-1}{r-1, s-1}$. \square

3.4.3. A refinement. A. Morales and E. Vassilieva [20] have established a very elegant summation formula for bipartite maps, counted with respect to their degree distributions, which can be viewed as a refinement of Jackson's summation formula (indeed, it is an easy exercise to recover Jackson's summation formula out of it):

Proposition 12 (Morales and Vassilieva [20, Theorem 1]). *Let m_λ and p_ρ be the monomial and power sum basis of the ring of symmetric functions and \mathbf{x} and \mathbf{y} two*

infinite sets of variables. Then, for any $n \geq 1$,

$$\sum_{\lambda, \mu \vdash n} \text{Bi}(\lambda, \mu) p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) = \sum_{\rho, \nu \vdash n} \frac{n(n - \ell(\rho))!(n - \ell(\nu))!}{(n + 1 - \ell(\rho) - \ell(\nu))!} m_\rho(\mathbf{x}) m_\nu(\mathbf{y}).$$

The original proof given in [20] goes through a complicated bijection with newly introduced objects called *thorn trees* by the authors. The bijective method in [2] (which is well adapted to summation formulas) also makes it possible to get the formula. And a short non-bijective proof has been given recently in [27] using characters of the symmetric groups and Schur functions. We explain here how this result can be recovered from our bijection. The proof is very similar to the one of Goupil-Schaeffer's formula.

Let us first recall that Proposition 12 can be reformulated in purely combinatorial terms (without symmetric functions) using colored maps.

By definition here, a bipartite unicellular map is *colored* by associating to each white (resp. black) vertex a color in $[1 \dots \ell_w]$ (resp. $[1 \dots \ell_b]$), each color between 1 and ℓ_w (resp. ℓ_b) being chosen at least once (note: we always think of the color r of a white vertex as *different* from the color r of a black vertex). To a colored bipartite map with n edges one can associate its *colored degree distribution*, that is, the pair (\mathbf{I}, \mathbf{J}) of compositions of n , where the k th part of \mathbf{I} (resp. of \mathbf{J}) is the sum of the degrees of the white (resp. black) vertices with color k .

We denote by $\text{BiC}(\mathbf{I}, \mathbf{J})$ the number of colored bipartite unicellular maps of colored degree distribution (\mathbf{I}, \mathbf{J}) . Then Proposition 12 is equivalent to the following statement [20, paragraph 2.4]:

Proposition 13. *For any compositions \mathbf{I} and \mathbf{J} of the same integer n which satisfy $\ell(\mathbf{I}) + \ell(\mathbf{J}) \leq n + 1$, one has*

$$\text{BiC}(\mathbf{I}, \mathbf{J}) = \frac{n(n - \ell(\mathbf{I}))!(n - \ell(\mathbf{J}))!}{(n + 1 - \ell(\mathbf{I}) - \ell(\mathbf{J}))!}.$$

Proof. Let us consider two compositions \mathbf{H} and \mathbf{K} which refine respectively \mathbf{I} and \mathbf{J} , and such that $\ell(\mathbf{H}) + \ell(\mathbf{K}) = n + 1$. A part of \mathbf{H} (resp. of \mathbf{K}) is said to have color r if it is contained in the r th part of \mathbf{I} (resp. of \mathbf{J}).

Consider a labelled bipartite tree T whose white vertex degrees (in the order given by the labels) follow the composition \mathbf{H} and whose black vertex degrees follow the composition \mathbf{K} . A black (resp. white) vertex is said to have color r if the corresponding part of \mathbf{H} (resp. of \mathbf{K}) has color r . Since the tree is labelled, the white (resp. black) vertices with the same color r are totally ordered as (w_r^1, w_r^2, \dots) (resp. (b_r^1, b_r^2, \dots)). Hence if we add the data of a sign per color ($2^{\ell(\mathbf{I}) + \ell(\mathbf{J})}$ choices for all signs), using Lemma 3, we can see the vertices with the same color as endowed with a C -permutation.

Putting all these C -permutations together, we obtain a C -permutation of the vertices of the tree T , which has the following property: the vertices in the same cycle always have the same color. Applying our main bijection (Theorem 5), we obtain a bipartite unicellular map. The vertices of this map have a canonical coloration, as each vertex corresponds to a cycle of the C -permutation. By construction, this colored map has colored degree distribution (\mathbf{I}, \mathbf{J}) .

To sum up, by Theorem 5 and the construction above, each colored bipartite unicellular map with colored degree distribution (\mathbf{I}, \mathbf{J}) can be obtained in 2^{n+1} different ways from

- a labelled bipartite tree T of white (resp. black) vertex degree given by \mathbf{H} (resp. \mathbf{K}) for *some* refinements \mathbf{H} and \mathbf{K} with $\ell(\mathbf{H}) + \ell(\mathbf{K}) = n + 1$;
- the assignment of a sign to each color.

The number of possible signs is always $2^{\ell(\mathbf{I})+\ell(\mathbf{J})}$, so this yields a constant factor. For given compositions \mathbf{H} and \mathbf{K} , the number of corresponding trees is

$$n(\ell(\mathbf{H}) - 1)!(\ell(\mathbf{K}) - 1)!$$

Thus we have to count the number of refinements \mathbf{H} (resp. \mathbf{K}) of \mathbf{I} (resp. \mathbf{J}) with a given value ℓ of $\ell(\mathbf{H})$ (resp. m of $\ell(\mathbf{K})$). It is easily seen to be equal to

$$\binom{n - \ell(\mathbf{I})}{\ell - \ell(\mathbf{I})} \text{ (resp. } \binom{n - \ell(\mathbf{J})}{m - \ell(\mathbf{J})} \text{)}.$$

Finally, by Theorem 5, we get:

$$2^{n+1}\text{BiC}(\mathbf{I}, \mathbf{J}) = 2^{\ell(\mathbf{I})+\ell(\mathbf{J})} \sum_{\substack{\ell+m=n+1 \\ \ell \geq \ell(\mathbf{I}), m \geq \ell(\mathbf{J})}} n(\ell - 1)!(m - 1)! \binom{n - \ell(\mathbf{I})}{\ell - \ell(\mathbf{I})} \binom{n - \ell(\mathbf{J})}{m - \ell(\mathbf{J})}.$$

Denoting $h = n + 1 - \ell(\mathbf{I}) - \ell(\mathbf{J})$ and setting $h_1 = \ell - \ell(\mathbf{I}), h_2 = m - \ell(\mathbf{J})$ in the summation index, the right hand side of the previous equation writes as:

$$2^{\ell(\mathbf{I})+\ell(\mathbf{J})} \sum_{h_1+h_2=h} n(\ell(\mathbf{I}) + h_1 - 1)!(\ell(\mathbf{J}) + h_2 - 1)! \binom{n - \ell(\mathbf{I})}{h_1} \binom{n - \ell(\mathbf{J})}{h_2}.$$

But the relation $\ell(\mathbf{I}) + \ell(\mathbf{J}) + h_1 + h_2 = n + 1$ implies that $(\ell(\mathbf{J}) + h_2 - 1)! \binom{n - \ell(\mathbf{I})}{h_1} = \frac{(n - \ell(\mathbf{I})!}{h_1!}$ and $(\ell(\mathbf{I}) + h_1 - 1)! \binom{n - \ell(\mathbf{J})}{h_2} = \frac{(n - \ell(\mathbf{J})!}{h_2!}$. Plugging this in the expression above, we get

$$\begin{aligned} 2^{n+1}\text{BiC}(\mathbf{I}, \mathbf{J}) &= 2^{\ell(\mathbf{I})+\ell(\mathbf{J})} \cdot n \cdot (n - \ell(\mathbf{I}))! \cdot (n - \ell(\mathbf{J}))! \sum_{h_1+h_2=h} \frac{1}{h_1!h_2!} \\ &= 2^{\ell(\mathbf{I})+\ell(\mathbf{J})} \cdot n \cdot (n - \ell(\mathbf{I}))! \cdot (n - \ell(\mathbf{J}))! \frac{2^h}{h!}. \end{aligned}$$

The powers of 2 cancel each other and we get the desired result. \square

3.5. Covered maps, shuffles, and an identity of [3]. *Covered maps* were introduced in [3] as an extension of the notion of tree-rooted map (map equipped with a spanning tree). A covered map of genus g is a rooted map M of genus g , non necessarily unicellular, equipped with a distinguished connected subgraph S (with the same vertex set as M) having the following property:

viewed as a map, S is a unicellular map, possibly of a different genus than M .

Here, in order to view S “as a map”, we equip it with the map structure induced by M : the clockwise ordering of half-edges of S around each vertex is defined as the restriction of the clockwise ordering in M (see [3] for details). The genus g_1 of S is an element of $\llbracket 0, g \rrbracket$. For example, $g_1 = 0$ if and only if S is a spanning tree of M . In general, we say that the covered map (M, S) has *type* (g, g_1) .

Covered maps have an interesting duality property that generalizes the existence of dual spanning trees in the planar case: namely, each covered map (M, S) of type (g, g_1) has a dual covered map (M^*, S') of type (g, g_2) with $g_1 + g_2 = g$. By extending ideas of Mullin [21], it is not difficult to describe the covered map M as a “shuffle” of the two unicellular maps S and S' , see [3]. It follows that the number

$\text{Cov}_{g_1, g_2}(n)$ of covered maps of type $(g_1 + g_2, g_1)$ with n edges can be expressed as the following shuffle-sum [3, eq. (6)]:

$$(7) \quad \text{Cov}_{g_1, g_2}(n) = \sum_{n_1 + n_2 = n} \binom{2n}{2n_1} \epsilon_{g_1}(n_1) \epsilon_{g_2}(n_2).$$

In the case $g_1 = g_2 = 0$, this sum simplifies thanks to the Chu-Vandermonde identity, and we have the remarkable result due to Mullin [21] (see [1] for a bijective proof):

$$(8) \quad \text{Cov}_{0,0}(n) = \text{Cat}(n)\text{Cat}(n+1).$$

The main enumerative result of the paper [3] is a generalisation of (8) to any genus, obtained via a difficult bijection:

Proposition 14 (Bernardi and Chapuy, [3]). *For all $n \geq 1$ and $g \geq 0$, the number $\text{Cov}_g(n) = \sum_{g_1 + g_2 = g} \text{Cov}_{g_1, g_2}(n)$ of covered maps of genus g with n edges is equal to:*

$$\text{Cov}_g(n) = \text{Cat}(n)\text{Bip}_g(n+1),$$

where $\text{Bip}_g(n+1)$ is the number of rooted bipartite unicellular maps of genus g with $n+1$ edges. Equivalently, the following identity holds:

$$(9) \quad \sum_{g_1 + g_2 = g} \sum_{n_1 + n_2 = n} \binom{2n}{2n_1} \epsilon_{g_1}(n_1) \epsilon_{g_2}(n_2) = \text{Cat}(n)\text{Bip}_g(n+1).$$

Proof. We denote as before by $c_g(m)$ the number of C-permutations of genus g of a set of m elements. By our main result, Theorem 5, the left-hand side of (9) can be rewritten as:

$$2^{-n-2} \sum_{g_1 + g_2 = g} \sum_{n_1 + n_2 = n} \binom{2n}{2n_1} c_{g_1}(n_1+1) c_{g_2}(n_2+1) \text{Cat}(n_1) \text{Cat}(n_2).$$

We now observe that:

$$\binom{2n}{2n_1} \text{Cat}(n_1) \text{Cat}(n_2) = \text{Cat}(n) \text{Nar}(n_1+1, n_2+1; n+1)$$

where as before the Narayana number $\text{Nar}(i, j; n)$ is the number of bipartite plane trees with n edges, i black vertices and j white vertices (this last equality follows directly from the explicit expressions of Catalan and Narayana numbers; an interpretation in terms of planar tree-rooted maps is given by the bijection of [1]). Therefore we have:

$$\text{Cov}_g(n) = 2^{-n-2} \text{Cat}(n) \sum_{g_1 + g_2 = g} \sum_{n_1 + n_2 = n} c_{g_1}(n_1+1) c_{g_2}(n_2+1) \text{Nar}(n_1+1, n_2+1; n+1).$$

Now, the double-sum in this equation is equal to the number of bipartite C-decorated trees (that is, bipartite trees equipped with a C-permutation of the vertices that stabilizes each color class) with $n+1$ edges and genus g : indeed in the double-sum, the quantities g_1 and n_1+1 can be interpreted respectively as the genus of the restriction of the C-permutation to black vertices of the tree, and as the number of black vertices in the tree. By our main result, Theorem 5, this double-sum is therefore equal to $2^{n+2}\text{Bip}_g(n+1)$, which proves (9). \square

The proof above and (7) also show the following fact. Let G_1 be the genus of the submap S in a covered map (M, S) of genus g with n edges chosen uniformly at random, and let G_\circ be the genus of the restriction to white vertices of the C -permutation in a bipartite C -decorated tree of genus g with $n + 1$ edges chosen uniformly at random. Then the random variables G_1 and G_\circ have the same distribution.

It is possible to prove that, when g is fixed and n tends to infinity, the variable G_\circ is close to a binomial random variable $B(g, 1/2)$: the idea behind this property is that a random bipartite tree with $n + 1$ edges has about $n/2 + O(\sqrt{n})$ vertices of each color with high probability, and that with high probability the C -permutation of its vertices is made of g cycles of length 3, that independently “fall” into each of the color classes with probability $1/2$. Giving a proper proof of these elementary statements would lead us too far from our main subject, so we leave to the reader the details of a proof along these lines of the following fact, which was proved in [3] with no combinatorial interpretation:

Proposition 15 ([3]). *Let $g \geq g_1 \geq 0$. When n tends to infinity, the probability that a covered map of genus g with n edges chosen uniformly at random has type (g, g_1) tends to $2^{-g} \binom{g}{g_1}$.*

To conclude this section, we mention that, in [3], refined results were given that take more parameters into account (*e.g.*, the number of vertices and faces of the covered map). These extensions can be proved exactly in the same way as Proposition 14, but we do not state them explicitly here, for the sake of brevity.

4. COMPUTING STANLEY CHARACTER POLYNOMIALS

4.1. Formulation of the problem. We now consider the following enumerative problem. For n a fixed integer, we would like to compute the generating series

$$F_n(p_1, p_2, \dots; q_1, q_2, \dots) = \sum_{(M, \varphi)} \text{wt}(M, \varphi)$$

of pairs (M, φ) where M is a rooted bipartite unicellular map with n edges, and φ is a mapping from the vertex set V_M of M to positive integers, satisfying the following *order condition*:

for each edge e of M , one has $\varphi(b_e) \geq \varphi(w_e)$, where b_e and w_e are respectively the black and white extremities of e .

The weight of such a pair is $\text{wt}(M, \varphi) := \prod_{v \in V_M^\circ} p_{\varphi(i)} \prod_{v \in V_M^\bullet} q_{\varphi(i)}$, where V_M^\bullet and V_M° are respectively the sets of black (resp. white) vertices of M .

Our motivation comes from representation theory of the symmetric group. This topic is linked to map enumeration by the following formula conjectured in [26] and proved in [11]. Let $\mathbf{p} = p_1, \dots, p_r$ and $\mathbf{q} = q_1, \dots, q_r$ be two finite lists of positive integers of the same length. Then the evaluation of the generating series considered above is equal to

$$(10) \quad F_n(p_1, \dots, p_r, 0, \dots; q_1, \dots, q_r, 0, \dots) = L(L-1) \cdots (L-n+1) \tilde{\chi}^\lambda((1 \ 2 \ \cdots \ n)),$$

where:

- λ is the partition with p_1 parts equal to $q_1 + \cdots + q_r$, p_2 parts equal to $q_2 + \cdots + q_r$, and so on...

- $L = \sum_{1 \leq i \leq j \leq r} p_i q_j$ is its number of boxes ;
- $\hat{\chi}^\lambda$ is the normalized character of the irreducible representation of S_L associated to λ ;
- $(1 \ 2 \ \cdots \ n)$ is an n -th cycle seen as a permutation of S_L (if $n > L$, it is not defined but, as the numerical factor is 0, it is not a problem).

Remark 1. In [26, 11], this formula is stated under a slightly different form. We call G_n the same generating series as F_n except that the order condition is replaced by the following *maximum condition*:

for each black vertex b , one has $\varphi(b) = \max \varphi(w)$, where the maximum is taken over all white neighbours w of b .

Then the main theorem of [11] states that

$$G_n(p'_1, \dots, p'_r, 0, \dots; q'_1, \dots, q'_r, 0, \dots) = L(L-1) \cdots (L-n+1) \hat{\chi}^\lambda((1 \ 2 \ \cdots \ n)),$$

where everything is defined as above except that

λ is the partition with p'_1 parts equal to q'_1 , p'_2 parts equal to q'_2 , and so on...

This result is clearly equivalent to (10) by setting:

$$\forall i \geq 1, \begin{cases} p_i = p'_i \\ q_i = q'_i - q'_{i+1} \end{cases}.$$

4.2. A new expression for F_n . Our main bijection allows us to express the generating series F_n in terms of the corresponding generating series for plane trees:

$$R_{n+1}(\mathbf{p}, \mathbf{q}) = \sum_{(T, \varphi)} \text{wt}(T, \varphi),$$

where the sum runs over all pairs (T, φ) , T being a plane tree and φ a function $V_T \rightarrow \mathbb{N}$ satisfying the order condition.

The strange notation R_{n+1} comes from the following fact: A. Rattan has proved [23] that this generating series is the $n+1$ -th free cumulant R_{n+1} of the transition measure of the Young diagram λ (λ states here for the Young diagram defined in terms on \mathbf{p} and \mathbf{q} in the previous paragraph). Free cumulants have become in the last few years an important tool in (asymptotic) representation theory of the symmetric groups, see for example the work of P. Biane [5].

Let us define an operator D by

$$D(x^k) := \sum_{g \geq 0} c_g(k) x^{k-2g} = k! \sum_{r=1}^k 2^r \binom{k-1}{r-1} \binom{x}{r},$$

D being extended multiplicatively to monomials in distinct variables, and then extended linearly to multivariate polynomials and series (in particular, series in the variables \mathbf{p} and \mathbf{q}).

Theorem 16. *For any $n \geq 1$, one has $2^{n+1} F_n = D(R_{n+1})$.*

Proof. A pair (M, φ) as above corresponds by the bijection of Theorem 5 to a bipartite C -decorated tree T , together with a function $\varphi : V_T \rightarrow \mathbb{N}$ which fulfills the order condition and such that all vertices in a given cycle have the same image by φ . Equivalently, we choose the tree T , a function $\varphi : V_T \rightarrow \mathbb{N}$ and then, for each

$i \geq 1$ a C -permutation of the white (resp. black) vertices of color i . The result follows directly. \square

The free cumulant R_{n+1} is the compositional inverse of an explicit series [23]. Hence Theorem 16 gives an efficient, easily implemented way of computing Stanley character polynomials F_n .

5. COUNTING 3-CONSTELLATIONS

5.1. Constellations and Poulalhon-Schaeffer formula. Constellations are a family of colored maps, depending on an integer parameter $m \geq 2$, that are in bijection with factorizations of a permutation into m factors. We refer to [18, Chapter 1] for a general discussion on constellations, and in particular to Section 1.6.2 of that book for the correspondence between the factorization viewpoint and the map-theoretic perspective that we adopt here (see also [4, Section 2], or the introduction of [6]). For $m = 2$, constellations are in bijection with bipartite maps, which are well-known to be in bijection with factorizations of permutations into 2 factors [10].

Definition 2. *An m -constellation is a map with circle and square vertices such that:*

- (i) *the circle vertices are colored with m colors $1, 2, \dots, m$;*
- (ii) *all edges have one circle and one square extremity;*
- (iii) *each square vertex is linked to exactly one circle vertex of each color;*
- (iv) *moreover, the circle vertices around each square vertex appear counterclockwise in the cyclic order $1, 2, \dots, m$.*

A constellation is rooted if we distinguish a corner of a circle vertex of color 1. Unless mentioned explicitly, all constellations considered will be rooted.

The size of an m -constellation is its number of square vertices.

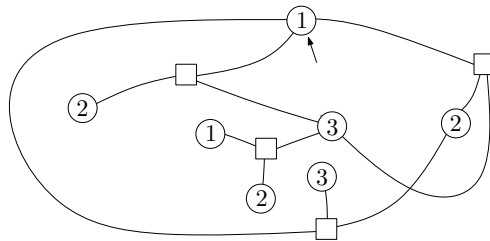


FIGURE 4. A rooted 3-constellation of size 4 (the root corner is pointed with an arrow). This 3-constellation has genus 1 and is unicellular.

Note that several papers, *e.g.* [6, 7, 8, 4], use a different but equivalent definition of constellations in terms of maps, where square vertices are replaced by "black faces" of degree m . For the purpose of using the bijection of Section 2.2, the definition we use here will be much more convenient.

Fix a m -constellation of size n . The sequence of the degrees of its circle vertices of color i , taken in decreasing order, forms a partition $\lambda^{(i)}$ of size n . The list $\lambda^{(1)}, \dots, \lambda^{(m)}$ is called the *multi-type* of the constellation.

For unicellular m constellations, Euler's formula links the genus and the multi-type (here, ℓ_i is the length of $\lambda^{(i)}$):

$$2g = n(m - 1) + 1 - \sum_{i=1}^m \ell_i.$$

Using algebraic tools, D. Poulalhon and G. Schaeffer have given a general formula for the number of unicellular m -constellations of size n [22, Theorem 1] with a given multi-type. Though explicit, their formula requires quite heavy notations to be stated, therefore we present here only the case $m = 3$, which is the only case we are able to attack with our combinatorial tools.

For a partition λ of length ℓ and a non-negative integer g , we denote

$$a(\lambda) = \prod m_i(\lambda)!;$$

$$S_g(\lambda) = (\ell + 2g - 1)! \sum_{p_1 + \dots + p_\ell = g} \prod_{i=1}^{\ell} \frac{1}{2p_i + 1} \binom{\lambda_i - 1}{2p_i}.$$

Theorem 17 (Poulalhon and Schaeffer, 2002). *Let $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ be three partitions of lengths ℓ_1 , ℓ_2 and ℓ_3 and of the same size n , such that*

$$g = 1/2 \cdot (2n + 1 - \ell_1 - \ell_2 - \ell_3)$$

is a non-negative integer. Then the number c_{λ} of 3-constellations of multi-type $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is given by the formula

$$c_{\lambda} = \frac{n^2}{2^{2g} a(\lambda^{(1)}) a(\lambda^{(2)}) a(\lambda^{(3)})} \sum_{g_0 + g_1 + g_2 + g_3 = g} \left[(g_0!)^2 \binom{n - \ell_1 - 2g_1}{g_0} \cdot \binom{n - \ell_2 - 2g_2}{g_0} \binom{n - \ell_3 - 2g_3}{g_0} S_{g_1}(\lambda^{(1)}) S_{g_2}(\lambda^{(2)}) S_{g_3}(\lambda^{(3)}) \right].$$

The planar case ($g = 0$) of this theorem:

$$(11) \quad c_{\lambda} = \frac{n^2}{a(\lambda^{(1)}) a(\lambda^{(2)}) a(\lambda^{(3)})} (\ell_1 - 1)! (\ell_2 - 1)! (\ell_3 - 1)!,$$

has been proved earlier by I. Goulden and D. Jackson [13, Theorem 3.2] and can be handled in a purely combinatorial way [4].

Finding a combinatorial proof of Theorem 17 for higher genus is an open problem (and seems difficult, due to the complexity of the formula). We did not succeed in solving this problem but we shall present two results in this direction.

5.2. Refined enumeration of quasi-constellations. Our bijection preserves the underlying multi-graph of a unicellular map, but not the cyclic order around vertices. Therefore the last condition in the definition of constellations is hard to handle with our method.

This paragraph is devoted to the proof of a refined enumeration formula for new objects that we call *3-quasi-constellations*, which are defined by the same conditions than constellation, except that condition (iv) is dropped.

The formula obtained is surprisingly close to the one for constellations, although we are not able to explain this phenomenon.

Proposition 18. *Let $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ be three partitions of lengths ℓ_1 , ℓ_2 and ℓ_3 and of the same size n , such that $g = 1/2 \cdot (2n + 1 - \ell_1 - \ell_2 - \ell_3)$ is a non-negative integer. Then the number \tilde{c}_λ of 3-quasi-constellations of multi-type $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is given by the formula*

$$\tilde{c}_\lambda = \frac{n^2}{2^{2g} a(\lambda^{(1)}) a(\lambda^{(2)}) a(\lambda^{(3)})} \sum_{g_0 + g_1 + g_2 + g_3 = g} \frac{n}{n - g_0} 2^{n - g_0} \left[(g_0!)^2 \binom{n - \ell_1 - 2g_1}{g_0} \cdot \binom{n - \ell_2 - 2g_2}{g_0} \binom{n - \ell_3 - 2g_3}{g_0} S_{g_1}(\lambda^{(1)}) S_{g_2}(\lambda^{(2)}) S_{g_3}(\lambda^{(3)}) \right].$$

As in the proof of Proposition 9, we prefer to work with *labelled* 3-quasi-constellations (circle vertices of each color are labelled separately, square vertices are *not* labelled). The multi-type of such an object is a triple of compositions with obvious definition. We denote by $\hat{c}(\mathbf{I}, \mathbf{J}, \mathbf{L})$ the number of labelled 3-quasi-constellations of multi-type $(\mathbf{I}, \mathbf{J}, \mathbf{L})$, where \mathbf{I} , \mathbf{J} and \mathbf{L} are three compositions of the same size n . From now on, ℓ_1 , ℓ_2 and ℓ_3 will denote the respective lengths of the compositions \mathbf{I} , \mathbf{J} and \mathbf{L} .

In genus 0 and size n , there are clearly 2^n times more 3-quasi-constellations than 3-constellations (indeed in the planar case, cyclically reordering the neighbours around a vertex keeps the property of being unicellular; this does not hold in higher genus). Hence, using Equation (11), we obtain in that case

$$(12) \quad \hat{c}(\mathbf{I}, \mathbf{J}, \mathbf{L}) = 2^n n^2 (\ell_1 - 1)! (\ell_2 - 1)! (\ell_3 - 1)!.$$

If we apply our main bijection to a 3-(quasi-)constellation, the tree object is not necessarily a planar 3-quasi-constellation. Indeed, one can get square vertices of degree 1. Therefore, we need to introduce the concept of prickly planar 3-quasi-constellations.

Definition 3. *A prickly planar 3-quasi-constellation is a rooted tree with circle and square vertices such that:*

- (i) *the circle vertices are colored with 3 colors 1, 2, 3;*
- (ii) *all edges have one circle and one square extremity;*
- (iii) *each square vertex is either a leaf or linked to exactly one circle vertex of each color;*
- (iv) *the number of square leaves linked to vertices of color 1, 2, 3 are the same (this number g_0 will be called the prickling number).*

The size n of such objects is one third of the number of edges. For a labelled object, its multi-type $(\mathbf{I}, \mathbf{J}, \mathbf{L})$ is defined as for (quasi-)constellations.

Planarity is equivalent to the relation:

$$\ell_1 + \ell_2 + \ell_3 = 2n - 2g_0 + 1$$

Lemma 19. *The number of labelled prickly planar 3-quasi-constellations of multi-type $(\mathbf{I}, \mathbf{J}, \mathbf{L})$ and prickling number g_0 is*

$$n(n - g_0) 2^n \binom{n - \ell_1}{g_0} \binom{n - \ell_2}{g_0} \binom{n - \ell_3}{g_0} (\ell_1 - 1)! (\ell_2 - 1)! (\ell_3 - 1)!$$

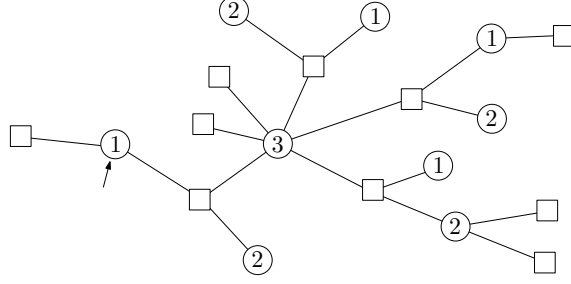


FIGURE 5. A prickly planar 3-quasi-constellation of size $n = 6$, with $l_1 = 4$, $l_2 = 4$, $l_3 = 1$. The prickling number is $g_0 = 2$.

Proof. It is easier in the proof of this lemma to work with *unrooted* labelled quasi-constellations. Note that a planar (vertex-)labelled tree has no symmetry, hence, dealing with unrooted or rooted objects only change the enumerations by constant explicit factors (this would not be true for higher genus objects!).

Consider an (unrooted) prickly planar 3-quasi-constellation of multi-type $(\mathbf{I}, \mathbf{J}, \mathbf{L})$ and prickling number g_0 . We denote by a_i (resp. b_i and c_i) the number of square leaves attached to the circle vertex labelled i of color 1 (resp. 2 and 3). If we erase these leaves we get an unrooted planar 3-quasi-constellation of size $n - g_0$ and multi-type $(\mathbf{I} - \mathbf{a}, \mathbf{J} - \mathbf{b}, \mathbf{L} - \mathbf{c})$, where, by definition, $\mathbf{I} - \mathbf{a} = (I_1 - a_1, \dots, I_{\ell_1} - a_{\ell_1})$ and similar definitions hold for $\mathbf{J} - \mathbf{b}$ and $\mathbf{L} - \mathbf{c}$.

The number of unrooted planar 3-quasi-constellation of multi-type $(\mathbf{I} - \mathbf{a}, \mathbf{J} - \mathbf{b}, \mathbf{L} - \mathbf{c})$ is, by equation (12) (beware of the rooting, which yields a factor $n - g_0$):

$$2^{n-g_0}(n-g_0)(\ell_1-1)!(\ell_2-1)!(\ell_3-1)!.$$

If we want to recover the prickly planar 3-quasi-constellation, one has to remember for each circle vertex of color 1 (resp. 2, 3) where to add the a_i (resp. b_i, c_i) square leaves. This gives

$$\prod_{i=1}^{\ell_1} \binom{I_i-1}{a_i} \prod_{i=1}^{\ell_2} \binom{I_i-1}{b_i} \prod_{i=1}^{\ell_3} \binom{I_i-1}{c_i}$$

choices (working with unrooted objects is crucial here). Finally the number of unrooted prickly planar 3-quasi-constellation of multi-type $(\mathbf{I}, \mathbf{J}, \mathbf{L})$ and prickling number g_0 is

$$2^{n-g_0}(n-g_0)(\ell_1-1)!(\ell_2-1)!(\ell_3-1)! \cdot \sum_{a_1+\dots+a_{\ell_1}} \prod_{i=1}^{\ell_1} \binom{I_i-1}{a_i} \cdot \sum_{b_1+\dots+b_{\ell_2}} \prod_{i=1}^{\ell_2} \binom{J_i-1}{b_i} \cdot \sum_{c_1+\dots+c_{\ell_3}} \prod_{i=1}^{\ell_3} \binom{K_i-1}{c_i}.$$

The first (resp. second, third) sum corresponds to the number of ways of choosing g_0 elements among $n - \ell_1$ (resp. $n - \ell_2$, $n - \ell_3$). This yields the formula of the lemma, because every unrooted object can be rooted in n different ways. \square

The end of the proof of Proposition 18 is now very similar to the one of Goupil-Schaeffer formula (Proposition 9). Therefore, we do not give all the details.

Sketch of proof of Proposition 18. Our main bijection sends 3-quasi-constellations of multi-type $(\mathbf{I}, \mathbf{J}, \mathbf{L})$ on C -decorated prickly planar 3-quasi-constellation of multi-type $(\mathbf{H}, \mathbf{K}, \mathbf{M})$, where \mathbf{H} , \mathbf{K} and \mathbf{M} are refinements of \mathbf{I} , \mathbf{J} and \mathbf{L} .

For given refinements \mathbf{H} , \mathbf{K} and \mathbf{M} of respective lengths $\ell_1 + 2g_1$, $\ell_2 + 2g_2$ and $\ell_3 + 2g_3$, the number of labelled prickly planar 3-quasi-constellation is given by Lemma 19:

$$n(n-g_0)2^n \binom{n-\ell_1-2g_1}{g_0} \binom{n-\ell_2-2g_2}{g_0} \binom{n-\ell_3-2g_3}{g_0} (\ell_1+2g_1-1)!(\ell_2+2g_2-1)!(\ell_3+2g_3-1)!,$$

where $g_0 = 1/2 \cdot (2n+1-\ell_1-2g_1-\ell_2-2g_2-\ell_3-2g_3)$ should be a non-negative integer and is the prickling number of the object.

We need the number of refinement \mathbf{H} of \mathbf{I} of length $\ell_1 + 2g_1$, where each part I_r of \mathbf{I} corresponds to an odd number $2p_r + 1$ of parts of \mathbf{H} . It is given by

$$\sum_{p_1+\dots+p_{\ell_1}=g_1} \prod_{r=1}^{\ell_1} \binom{I_r-1}{2p_r+1}$$

and similar formulas hold for the number of refinements \mathbf{K} and \mathbf{M} of \mathbf{J} and \mathbf{L} .

When, we apply our bijection to a quasi-constellation, we get a C -decorated tree. As in the proof of Proposition 9, to transform this tree into a labelled prickly 3-quasi-constellations, we need:

- to choose for each cycle of circle vertices one distinguish vertex (factor $\prod_r 2p_r + 1$ for vertices of color 1 and similar factors for other colors);
- to group the square leaves by triples and to choose an orientation for each triple (factor $(g_0!)^2 \cdot 2^{g_0}$);
- to forget the signs (factor $1/2^{\ell_1+\ell_2+\ell_3+n}$).

Finally, we get, by Theorem 5, that $2^{3n+1}\hat{c}(\mathbf{I}, \mathbf{J}, \mathbf{L})$ is

$$\begin{aligned} & \sum_{g_0+g_1+g_2+g_3=g} \frac{2^{\ell_1+\ell_2+\ell_3+n}}{(g_0!)^2 \cdot 2^{g_0} \prod_r 2p_r + 1} \\ & \cdot \sum_{\substack{p_1+\dots+p_{\ell_1}=g_1 \\ q_1+\dots+q_{\ell_2}=g_2 \\ s_1+\dots+s_{\ell_3}=g_3}} \prod_{r=1}^{\ell_1} \binom{I_r-1}{2p_r+1} \cdot \prod_{r=1}^{\ell_2} \binom{J_r-1}{2q_r+1} \cdot \prod_{r=1}^{\ell_3} \binom{L_r-1}{2s_r+1} \\ & \cdot \left[n(n-g_0)2^n \binom{n-\ell_1-2g_1}{g_0} \binom{n-\ell_2-2g_2}{g_0} \binom{n-\ell_3-2g_3}{g_0} (\ell_1+2g_1-1)!(\ell_2+2g_2-1)!(\ell_3+2g_3-1)! \right] \end{aligned}$$

Indeed, the term in the bracket is the number of labelled prickly planar 3-quasi-constellation of a given multi-type $(\mathbf{H}, \mathbf{K}, \mathbf{M})$, the factor in the second line the number of possible multi-types and the first line the ratio between numbers of labelled and C -decorated objects.

Using the fact that

$$\hat{c}(\mathbf{I}, \mathbf{J}, \mathbf{L}) = a(\lambda^{(1)})a(\lambda^{(2)})a(\lambda^{(3)}) \cdot \tilde{c}_{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}}$$

if $\lambda^{(1)}$, $\lambda^{(2)}$ and $\lambda^{(3)}$ are the sorted version of \mathbf{I} , \mathbf{J} and \mathbf{K} , this ends the proof of Proposition 18. \square

5.3. Enumeration of constellations taking only numbers of vertices into account. In the previous paragraph, we have explained that Theorem 5 is not suitable to count unicellular constellations, as it does not give any information on the rotation system of the map.

Nevertheless, it is possible to get some enumerative results on constellations not by using Theorem 5 directly, but by mimicking its proof: find a combinatorial induction for constellations and then simpler objects with the same induction.

In this paragraph, we give a combinatorial proof of the enumeration of 3-constellations with respect to their number of circle vertices of each color.

5.3.1. Combinatorial induction for 3-constellations. Let us first introduce some notation: define $\mathcal{C}_{(\ell_1, \ell_2, \ell_3)}(n)$ as the set of 3-constellations with ℓ_1 (resp. ℓ_2 and ℓ_3) circle vertices of color 1 (resp. 2 and 3). Its cardinality is denoted $c_{(\ell_1, \ell_2, \ell_3)}(n)$.

Let us apply Chapuy's bijection (recalled in paragraph 2.2) to a 3-constellation of genus g in $\mathcal{C}_{(\ell_1, \ell_2, \ell_3)}(n)$. Two things may happen:

- either the sliced vertex is a circle vertex, in which case we get a 3-constellation of genus $g - h$ with $2h + 1$ marked circle vertices of the corresponding color (for some $h \geq 1$);
- or the sliced vertex is a square vertex. In this case, as square vertices have degree 3, the resulting map has always genus $g - 1$ and has three square leaves. Erasing these leaves, we get a 3-constellation of size $n - 1$ and genus $g - 1$.

In the first case, Chapuy's inverse mapping works well and always produces a constellation in $\mathcal{C}_{(\ell_1, \ell_2, \ell_3)}(n)$.

In the second case, one has to choose where to add square leaves, that is to choose a corner c_i of a circle vertex of color i for $i = 1, 2, 3$ ($(n - 1)^3$ choices). Then, when we apply Chapuy's inverse mapping, it may happen that the newly created square vertex does not fulfill condition (iv) of the definition of constellations.

Lemma 20. *Chapuy's inverse mapping leads to a constellation if and only if the three chosen corners appear in the cyclic order (c_1, c_2, c_3) when we turn clockwise around the unique face of the map.*

Proof. This is direct by construction: let f, f', f'' be three leaves in a unicellular map and e, e', e'' the edges incident to them, respectively. Then (e, e', e'') appear in counterclockwise order around the new vertex created by the gluing of the three leaves if and only if f, f', f'' appear in this order clockwise around the face in the original map. See Paragraph 2.2 (or [9, Paragraph 4.2] for a direct description of the inverse mapping). \square

Lemma 21. *For each constellation of size $n - 1$ and genus $g - 1$, exactly $n(n - 1)^2 / 2$ triples of corners (c_1, c_2, c_3) over $(n - 1)^3$ satisfy the condition above.*

Proof. Observe that a unicellular 3-constellation of size $n - 1$ has $n - 1$ corners incident to circle vertices in each color class (1, 2, and 3), and that clockwise around the face the colors of these $3(n - 1)$ corners appear in the order $(1, 2, 3, 1, 2, 3, \dots, 1, 2, 3)$. From there it is a simple exercise to check the statement of the Lemma. \square

Finally, we get the following induction:

$$(13) \quad 2gc_{\ell_1, \ell_2, \ell_3}(n) = \frac{n(n-1)^2}{2}c_{\ell_1, \ell_2, \ell_3}(n-1) + \sum_{h \geq 1} \binom{\ell_1 + 2h}{2h+1} c_{\ell_1+2h, \ell_2, \ell_3}(n) \\ + \binom{\ell_2 + 2h}{2h+1} c_{\ell_1, \ell_2+2h, \ell_3}(n) + \binom{\ell_3 + 2h}{2h+1} c_{\ell_1, \ell_2, \ell_3+2h}(n),$$

where g is defined by $\ell_1 + \ell_2 + \ell_3 + 2g = 2n$.

We can define some refinement of the number $c_{\ell_1, \ell_2, \ell_3}(n)$ by the induction

$$2gc_{\ell_1, \ell_2, \ell_3}(n; g_0) = \frac{n(n-1)^2}{2}c_{\ell_1, \ell_2, \ell_3}(n-1; g_0-1) + \sum_{h \geq 1} \binom{\ell_1 + 2h}{2h+1} c_{\ell_1+2h, \ell_2, \ell_3}(n; g_0) \\ + \binom{\ell_2 + 2h}{2h+1} c_{\ell_1, \ell_2+2h, \ell_3}(n; g_0) + \binom{\ell_3 + 2h}{2h+1} c_{\ell_1, \ell_2, \ell_3+2h}(n; g_0)$$

and initial conditions

$$c_{\ell_1, \ell_2, \ell_3}(0; g_0) = \delta_{g_0, 0} c_{\ell_1, \ell_2, \ell_3}(0).$$

Then an immediate induction proves that

$$c_{\ell_1, \ell_2, \ell_3}(n) = \sum_{g_0 \geq 0} c_{\ell_1, \ell_2, \ell_3}(n; g_0).$$

5.3.2. *A planar object with (almost) the same induction.* Denote by $d_{\ell_1, \ell_2, \ell_3}(n; g_0)$ the number of planar 3-constellations of size $n - g_0$ endowed with:

- a C -permutation of its circle vertices of each color with respectively ℓ_1 , ℓ_2 and ℓ_3 cycles;
- An unordered set of g_0 triples of square leaves, such that the triples are mutually disjoint and each triple is of the form f_1, f_2, f_3 , with f_i connected to a circle vertex of color i .

Lemma 22. *These numbers satisfy the induction:*

$$2gd_{\ell_1, \ell_2, \ell_3}(n; g_0) = 2(n-1)^3 d_{\ell_1, \ell_2, \ell_3}(n-1; g_0-1) + \sum_{h \geq 1} \binom{\ell_1 + 2h}{2h+1} d_{\ell_1+2h, \ell_2, \ell_3}(n; g_0) \\ + \binom{\ell_2 + 2h}{2h+1} d_{\ell_1, \ell_2+2h, \ell_3}(n; g_0) + \binom{\ell_3 + 2h}{2h+1} d_{\ell_1, \ell_2, \ell_3+2h}(n; g_0).$$

Proof. First, we have

$$g_0 d_{\ell_1, \ell_2, \ell_3}(n; g_0) = (n-1)^3 d_{\ell_1, \ell_2, \ell_3}(n-1; g_0-1).$$

Indeed, the left-hand side count the same objects as above with a marked triple of square leaves. If we erase this triple, we get objects counted by $d_{\ell_1, \ell_2, \ell_3}(n-1; g_0-1)$. This can be inverted if we remember in which corners were attached the leaves ($(n-1)^3$ possibilities).

Second, the induction for C -permutations lead to

$$2(g-g_0)d_{\ell_1, \ell_2, \ell_3}(n; g_0) = \sum_{h \geq 1} \binom{\ell_1 + 2h}{2h+1} d_{\ell_1+2h, \ell_2, \ell_3}(n; g_0) \\ + \binom{\ell_2 + 2h}{2h+1} d_{\ell_1, \ell_2+2h, \ell_3}(n; g_0) + \binom{\ell_3 + 2h}{2h+1} d_{\ell_1, \ell_2, \ell_3+2h}(n; g_0).$$

□

Corollary 23. *For each $n, g_0, \ell_1, \ell_2, \ell_3$, we have:*

$$c_{\ell_1, \ell_2, \ell_3}(n; g_0) = \frac{n}{2^{2g_0}(n - g_0)} d_{\ell_1, \ell_2, \ell_3}(n; g_0).$$

Proof. Both families of numbers satisfy the same initial conditions and the same induction relations. □

By using the method of the paragraph on quasi-constellations, we can compute $d_{\ell_1, \ell_2, \ell_3}(n; g_0)$. So this provides a combinatorial proof for counting 3-constellations with respect to the number of vertices of each color.

If we could refine (13) so as to control the degree distribution, we could give a purely combinatorial proof of the Poulalhon-Schaeffer formula in the case $m = 3$.

5.4. Conclusion on constellations. We are unfortunately not able to give a combinatorial proof of the Poulalhon-Schaeffer formula (even in the case $m = 3$). Nevertheless, the work presented here suggests that the different elements in this formula have a combinatorial meaning. The case $m > 3$ seems even harder.

REFERENCES

- [1] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.*, 14(1):Research Paper 9, 36 pp. (electronic), 2007.
- [2] O. Bernardi. An analogue of the Harer-Zagier formula for unicellular maps on general surfaces. *Adv. Appl. Math.*, 48(1):164–180, 2012.
- [3] O. Bernardi and G. Chapuy. A bijection for covered maps, or a shortcut between Harer-Zagier’s and Jackson’s formulas. *J. Combin. Theory Ser. A*, 118(6):1718–1748, 2011.
- [4] O. Bernardi and A. Morales. Bijections and symmetries for the factorizations of the long cycle. arXiv preprint 1112.4970, 2011.
- [5] P. Biane. Representations of symmetric groups and free probability. *Adv. Math.*, 138(1):126–181, 1998.
- [6] M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.
- [7] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):Research Paper 69, 27, 2004.
- [8] G. Chapuy. Asymptotic enumeration of constellations and related families of maps on orientable surfaces. *Combin. Probab. Comput.*, 18(4):477–516, 2009.
- [9] G. Chapuy. A new combinatorial identity for unicellular maps, via a direct bijective approach. *Advances in Applied Mathematics*, 47(4):874 – 893, 2011.
- [10] R. Cori and A. Machì. Maps, hypermaps and their automorphisms: a survey. I, II, III. *Exposition. Math.*, 10(5):403–427, 429–447, 449–467, 1992.
- [11] V. Féray. Stanley’s formula for characters of the symmetric group. *Ann. Comb.*, 13(4):453–461, 2010.
- [12] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.
- [13] I. P. Goulden and D. M. Jackson. The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group. *European J. Combin.*, 13(5):357–365, 1992.
- [14] I. P. Goulden and A. Nica. A direct bijection for the Harer-Zagier formula. *J. Comb. Theory, Ser. A*, 111(2):224–238, 2005.
- [15] A. Goupil and G. Schaeffer. Factoring n -cycles and counting maps of given genus. *European J. Combin.*, 19(7):819–834, 1998.
- [16] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. *Invent. Math.*, 85:457–486, 1986.
- [17] D. M. Jackson. Some combinatorial problems associated with products of conjugacy classes of the symmetric group. *J. Comb. Theory, Ser. A*, 49(2):363–369, 1988.
- [18] S. K. Lando and A. K. Zvonkin. *Graphs on Surfaces and Their Applications*. Springer, 2004.

- [19] B. Lass. Démonstration combinatoire de la formule de Harer-Zagier. *C. R. Acad. Sci. Paris*, 333, Série I:155–160, 2001.
- [20] A. Morales and E. Vassilieva. Bijective enumeration of bicolored maps of given vertex degree distribution. *DMTCS Proceedings*, AK:661–672, 2009.
- [21] R. C. Mullin. On the enumeration of tree-rooted maps. *Canad. J. Math.*, 19:174–183, 1967.
- [22] D. Poulalhon and G. Schaeffer. Factorizations of large cycles in the symmetric group. *Disc. Math.*, 254(1):433–458, 2002.
- [23] A. Rattan. Stanley’s character polynomials and coloured factorizations in the symmetric group. *J. Combin. Theory Ser. A*, 114(4):535–546, 2008.
- [24] J.-L. Rémy. Un procédé itératif de dénombrement d’arbres binaires et son application à leur génération aléatoire. *RAIRO Inform. Théor.*, 19(2):179–195, 1985.
- [25] G. Schaeffer and E. A. Vassilieva. A bijective proof of Jackson’s formula for the number of factorizations of a cycle. *J. Comb. Theory, Ser. A*, 115(6):903–924, 2008.
- [26] R. P. Stanley. A conjectured combinatorial interpretation of the normalized irreducible character values of the symmetric group. *arxiv:math/0606467*, 2006.
- [27] E. A. Vassilieva. Explicit monomial expansions of the generating series for connection coefficients. arXiv:1111.6215, 2011.
- [28] T. R. S. Walsh and A. B. Lehman. Counting rooted maps by genus. I. *J. Combin. Theory Ser. B*, 13:192–218, 1972.

(G.C.) CNRS, LIAFA - UMR 7089, UNIVERSITÉ PARIS 7, 75205 PARIS CEDEX, FRANCE,
GUILLAUME.CHAPUY@LIAFA.UNIV-PARIS-DIDEROT.FR

(V.F.) CNRS, LABRI - UMR 5800, UNIVERSITÉ DE BORDEAUX, 33405 TALENCE CEDEX,
FRANCE, FERAY@LABRI.FR

(É.F.) CNRS, LIX - UMR 7161, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE,
FUSY@LIX.POLYTECHNIQUE.FR