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L^∞ estimates and uniqueness results for nonlinear parabolic equations with gradient absorption terms

Marie Françoise BIDAUT-VERON*

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Abstract

We study the nonnegative solutions of the viscous Hamilton-Jacobi problem

$$\begin{cases} u_t - \nu \Delta u + |\nabla u|^q = 0, \\ u(0) = u_0, \end{cases}$$

in $Q_{\Omega,T} = \Omega \times (0, T)$, where $q > 1, \nu \geq 0, T \in (0, \infty]$, and $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded domain, and $u_0 \in L^r(\Omega), r \geq 1$, or $u_0 \in \mathcal{M}_b(\Omega)$. We show L^∞ decay estimates, valid for *any weak solution, without any conditions as $|x| \rightarrow \infty$, and without uniqueness assumptions*. As a consequence we obtain new uniqueness results, when $u_0 \in \mathcal{M}_b(\Omega)$ and $q < (N + 2)/(N + 1)$, or $u_0 \in L^r(\Omega)$ and $q < (N + 2r)/(N + r)$. We also extend some decay properties to quasilinear equations of the model type

$$u_t - \Delta_p u + |u|^{\lambda-1} u |\nabla u|^q = 0$$

where $p > 1, \lambda \geq 0$, and u is a signed solution.

Keywords Viscous Hamilton-Jacobi equation; quasilinear parabolic equations with gradient terms; regularity; decay estimates; regularizing effects; uniqueness results.

A.M.S. Subject Classification 35K15, 35K55, 35B33, 35B65, 35D30

1 Introduction

In this article we study a class of heat equations involving a nonlinear gradient absorption term. We are mainly concerned by the nonnegative solutions of the viscous parabolic Hamilton-Jacobi equation

$$u_t - \nu \Delta u + |\nabla u|^q = 0 \tag{1.1}$$

in $Q_{\Omega,T} = \Omega \times (0, T)$, $T \leq \infty$, where $q > 1, \nu \geq 0$, and $\Omega = \mathbb{R}^N$, or Ω is a smooth bounded domain of \mathbb{R}^N .

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We study the Cauchy problem in \mathbb{R}^N and the Cauchy-Dirichlet problem when Ω is bounded, with initial data $u(\cdot, 0) = u_0 \geq 0$, where $u_0 \in L^r(\Omega)$, $r \geq 1$, or u_0 is a bounded Radon measure on Ω .

We also consider the (signed) solutions of quasilinear equations of the type

$$u_t - \nu \Delta_p u + |u|^{\lambda-1} u |\nabla u|^q = 0 \quad (1.2)$$

where $p > 1$ and Δ_p is the p -Laplacian, and more generally

$$u_t - \operatorname{div}(A(x, t, u, \nabla u)) + g(x, u, \nabla u) = 0 \quad (1.3)$$

with natural growth conditions on the function A , and nonnegativity conditions

$$A(x, t, u, \eta) \cdot \eta \geq \nu |\eta|^p, \quad g(x, u, \eta) u \geq \gamma |u|^{\lambda+1} |\nabla u|^q \quad \gamma \geq 0, \nu \geq 0, \lambda \geq 0, \quad (1.4)$$

without monotonicity assumption.

In the sequel we give some *decay estimates, under very few assumptions on the solutions*. Then from Moser's technique, we deduce *regularizing effects* : L^∞ estimates, in terms of u_0 , and universal estimates when Ω is bounded. We show that *two types* of regularizing effect can occur: the first one is due to the gradient term $|\nabla u|^q$ (when $\gamma > 0$), the second one is due to the operator itself (when $\nu > 0$).

A part of these estimates are well known for equation (1.1) when the solutions can be approximated by smooth solutions, or satisfy growth conditions as $|x| \rightarrow \infty$ when $\Omega = \mathbb{R}^N$, for example semi-group solutions. Our approach is different, and our results are valid for *all the solutions* of the equation *in a weak sense*: in the sense of distributions for equation (1.1), in the renormalized sense for equation (1.3). And we make *no assumption of uniqueness*. In the case of equation (1.1) in \mathbb{R}^N , we require *no condition as $|x| \rightarrow \infty$, all our assumptions are local*.

As a consequence we deduce *new uniqueness results* for equation (1.1) in \mathbb{R}^N or in a bounded domain Ω .

2 Main results

We denote by $\mathcal{M}_b(\Omega)$ the set of bounded Radon measures in Ω , and $\mathcal{M}_b^+(\Omega)$ the cone of nonnegative ones.

We set $Q_{\Omega, s, \tau} = \Omega \times (s, \tau)$, for any $0 \leq s < \tau \leq \infty$, thus $Q_{\Omega, T} = Q_{\Omega, 0, T}$.

As usual, for any $\theta \geq 1$ we note by $\theta' = \theta/(\theta - 1)$ the conjugate of θ .

In **Section 3**, we give some key tools for obtaining regularizing properties. The main one is an iteration property based of Moser's method, inspired by [38]:

Lemma 2.1 *Let $m > 1$, $\theta > 1$ and $\lambda \in \mathbb{R}$ and $C_0 > 0$. Let $v \in C([0, T]; L_{loc}^1(\Omega))$ be nonnegative, and $v_0 = v(x, 0) \in L^r(\Omega)$ for some $r \geq 1$ such that*

$$r > \theta'(1 - m - \lambda). \quad (2.1)$$

If $r > 1$ we assume that for any $0 \leq s < t < T$ and any $\alpha \geq r - 1$, there holds

$$\frac{1}{\alpha + 1} \int_{\Omega} v^{\alpha+1}(\cdot, t) dx + \frac{C_0}{\beta^m} \int_s^t \left(\int_{\Omega} v^{\beta m \theta}(\cdot, \tau) dx \right)^{\frac{1}{\theta}} d\tau \leq \frac{1}{\alpha + 1} \int_{\Omega} v^{\alpha+1}(\cdot, s) dx, \quad (2.2)$$

where $\beta = \beta(\alpha) = 1 + (\alpha + \lambda) / m$, and the right-hand side **can be infinite**.

If $r = 1$ we make one of the two following assumptions:

(H₁) (2.2) holds for any $\alpha \geq 0$,

(H₂) $\int_{\Omega} v(\cdot, t) dx \leq \int_{\Omega} v_0 dx$ for any $t \in (0, T)$, and $v_0 \in L^{\rho}(\Omega)$ for some $\rho > 1$ such that $\rho \theta'(1 - m - \lambda) < 1$ and (2.2) holds for any $\alpha \geq \rho - 1$.

Then there exists $C > 0$, depending on N, m, r, λ, C_0 , and possibly ρ , such that for any $t \in (0, T)$,

$$\|v(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C t^{-\sigma_{r,m,\lambda,\theta}} \|v_0\|_{L^r(\Omega)}^{\varpi_{r,m,\lambda,\theta}}, \quad (2.3)$$

where

$$\sigma_{r,m,\lambda,\theta} = \frac{1}{\frac{r}{\theta'} + \lambda + m - 1} = \frac{\theta'}{r} \varpi_{r,m,\lambda,\theta}. \quad (2.4)$$

This Lemma allows to obtain L^{∞} estimates for the solutions of equation (1.1), when $q \leq N$, or $2 \leq N$, and for equation (1.2) when $p \leq N$. In the other cases the L^{∞} estimates follow from the Gagliardo-Nirenberg inequality, see Lemma 3.4. Moreover we deduce *universal* L^{∞} estimates when Ω is bounded, see Lemma 3.3.

In **Section 4** we study the Cauchy Hamilton-Jacobi problem in \mathbb{R}^N :

$$\begin{cases} u_t - \nu \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\mathbb{R}^N, T}, \\ u(x, 0) = u_0 \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (2.5)$$

This equation is the object of a huge literature, see [2], [12], [7], [15], [36], and the references therein, and also [7], [14], [28].

The first studies concern smooth initial data $u_0 \in C_b^2(\mathbb{R}^N)$. From [2], (2.5) has a unique global solution $u \in C^{2,1}(\mathbb{R}^N \times [0, \infty))$, and u satisfies decay properties:

$$\begin{aligned} \|u(\cdot, t)\|_{L^{\infty}(\mathbb{R}^N)} &\leq \|u_0\|_{L^{\infty}(\mathbb{R}^N)}, \\ \|\nabla u(\cdot, t)\|_{L^{\infty}(\mathbb{R}^N)} &\leq \|\nabla u_0\|_{L^{\infty}(\mathbb{R}^N)}. \end{aligned}$$

Estimates of the gradient have been obtained *for this solution*, by using the Bernstein technique, which consists in computing the equation satisfied by $|\nabla u|^2$: first from [31],

$$\|\nabla u(\cdot, t)\|_{L^{\infty}(\mathbb{R}^N)}^q \leq t^{-1} \|u_0\|_{L^{\infty}(\mathbb{R}^N)},$$

then from [12], when $\nu > 0$,

$$\|\nabla(u^{\frac{1}{q}})(\cdot, t)\|_{L^{\infty}(\mathbb{R}^N)} \leq C(q, \nu) t^{-\frac{1}{2}} \|u_0\|_{L^{\infty}(\mathbb{R}^N)}^{\frac{1}{q}}, \quad (2.6)$$

$$\left\| \nabla(u^{\frac{1}{q}})(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{(q-1)^{\frac{1}{q}}}{q} t^{-\frac{1}{q}}, \quad \text{that is} \quad |\nabla u(\cdot, t)|^q \leq \frac{t^{-1}u(\cdot, t)}{q-1}, \quad \text{a.e. in } \mathbb{R}^N. \quad (2.7)$$

If one only assumes $u_0 \in C_b(\mathbb{R}^N)$, then (2.5) still has a unique solution u such that $u \in C^{2,1}(Q_{\mathbb{R}^N, \infty})$ and $u \in C(\mathbb{R}^N \times [0, \infty) \cap L^\infty(\mathbb{R}^N \times (0, \infty))$ see [29], and estimates (2.6) and (2.7) are still valid, from [7].

In case of rough initial data $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ or $u \in L^r(\mathbb{R}^N)$, $r \geq 1$, assuming $\nu > 0$, the solutions have been searched in an integral form

$$u(\cdot, t) = e^{t\Delta}u_0(\cdot) - \nu \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds, \quad (2.8)$$

involving the semi-group of heat equation $e^{t\Delta}$. Existence results hold in corresponding classes of solutions, involving integral conditions on the gradient in space and time, of *global type*:

- If $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and $1 < q < (N+2)/(N+1)$, the existence of a solution $u \in C^{2,1}(Q_{\mathbb{R}^N, \infty})$ is proved in [12] by approximation, and independently in [15], from the Banach fixed point theorem.

- If $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$, existence holds for any $q \leq 2$ from [15]. When $q > 2$, it is required that u_0 is a limit of a monotone sequence of continuous functions, and existence is not known in the general case.

In those classes, decay properties and a regularizing effect follow directly from the semigroup $e^{t\Delta}$, since $u(\cdot, t) \leq e^{t\Delta}u_0$. Our first main results shows that *decay properties and L^∞ estimates* are valid for any weak solution, for any $\nu \geq 0$, without any condition as $|x| \rightarrow \infty$:

Theorem 2.2 *Let $u \in L_{loc}^1(Q_{\mathbb{R}^N, T})$, with $|\nabla u| \in L_{loc}^q(Q_{\mathbb{R}^N, T})$, be any nonnegative solution of equation (1.1) in $\mathcal{D}'(Q_{\mathbb{R}^N, T})$.*

(i) *Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$. Assume that $u \in C([0, T]; L_{loc}^r(\mathbb{R}^N))$ and $u(\cdot, 0) = u_0$. Then $u \in C([0, T]; L^r(\mathbb{R}^N))$; and for any $t \in (0, T)$, $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$ and*

$$\|u(\cdot, t)\|_{L^r(\mathbb{R}^N)} \leq \|u_0\|_{L^r(\mathbb{R}^N)}, \quad (2.9)$$

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \begin{cases} Ct^{-\sigma_{r,q,N}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\varpi_{r,q,N}}, & C = C(N, q, r), & \text{if } q \neq N, \\ C_\varepsilon t^{-(1+\varepsilon)\sigma_{r,N,N}} \|u_0\|_{L^r(\mathbb{R}^N)}^{(1+\varepsilon)\varpi_{r,q,N}}, & \forall \varepsilon > 0, \quad C_\varepsilon = C(N, q, r, \varepsilon), & \text{if } q = N, \end{cases} \quad (2.10)$$

where

$$\sigma_{r,q,N} = \frac{1}{\frac{rq}{N} + q - 1} = \frac{N}{rq} \varpi_{r,q,N}. \quad (2.11)$$

And if $\nu > 0$, then

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \begin{cases} Ct^{-\frac{N}{2r}} \|u_0\|_{L^r(\mathbb{R}^N)}, & C = C(N, r, \nu), & \text{if } N \neq 2, \\ C_\varepsilon t^{-\frac{1+\varepsilon}{r}} \|u_0\|_{L^r(\mathbb{R}^N)}, & \forall \varepsilon > 0, \quad C_\varepsilon = C(N, r, \nu, \varepsilon), & \text{if } N = 2. \end{cases} \quad (2.12)$$

(ii) *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Assume that $u(\cdot, t)$ converges weakly * to u_0 as $t \rightarrow 0$. Then $u \in C((0, T); L^1(\mathbb{R}^N))$, and for any $t \in (0, T)$, the conclusions above with $r = 1$ are still valid with $\|u_0\|_{L^1(\mathbb{R}^N)}$ replaced by $\int_{\mathbb{R}^N} du_0$.*

Note that estimates (2.9) are not valid for *any weak subsolution* of the heat equation. Here we prove that the result of (2.9) is *essentially due to the gradient term* $|\nabla u|^q$, which has a main regularizing effect on the equation. And then a second regularizing effect holds, due to the Laplacian, when $\nu > 0$.

For any $q \leq 2$, we deduce estimates of the gradient, obtained from (2.6). As a consequence we deduce new uniqueness results, where the assumptions are only of *local type*:

Theorem 2.3 (i) *Let $1 < q < (N + 2)/(N + 1)$, and $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then there exists a unique nonnegative function $u \in L_{loc}^1(Q_{\mathbb{R}^N, T})$, such that $|\nabla u| \in L_{loc}^q(Q_{\mathbb{R}^N, T})$, solution of equation (1.1) in $\mathcal{D}'(Q_{\mathbb{R}^N, T})$ such that*

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(\cdot, t) \psi dx = \int_{\mathbb{R}^N} \psi du_0, \quad \forall \psi \in C_c(\mathbb{R}^N).$$

(ii) *Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$ and $1 < q < (N + 2r)/(N + r)$. Then there exists a unique nonnegative solution u as above, such that $u \in C([0, T]; L_{loc}^r(\mathbb{R}^N))$ and $u(\cdot, 0) = u_0$.*

This improves the former uniqueness results of [12] and [15, Theorem 4.1], given in classes of semigroup solutions, satisfying conditions up to $t = 0$ for the gradient: $|\nabla u|^q \in L_{loc}^1([0, T]; L^1(\mathbb{R}^N))$ in case (i), and $|\nabla u|^q \in L_{loc}^1([0, T]; L^r(\mathbb{R}^N))$ in case (ii).

We also find again in a shorter way the existence result of [15, Theorem 4.1], see Proposition 4.26. Finally we improve the estimate (2.9) when $q < (N + 2r)/(N + r)$, see Theorem 4.28.

In **Section 5** we study the Cauchy-Dirichlet problem in a bounded domain Ω :

$$\begin{cases} u_t - \nu \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega, T}, \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0 \geq 0, \end{cases} \quad (2.13)$$

Here also the problem is the object of many works, such as [22], [8], [37], [9], [33].

If $u_0 \in C_0^1(\overline{\Omega})$, from [22], (2.13) admits a unique nonnegative solution $u \in C^{2,1}(\Omega \times (0, \infty)) \cap C(\overline{\Omega} \times [0, \infty))$, such that $|\nabla u| \in C(\overline{\Omega} \times [0, \infty))$. Universal a priori estimates hold: there exist $C = C(N, q, \Omega) > 0$ and a function $D \in C((0, \infty))$ such that

$$u(\cdot, t) \leq C(1 + t^{-\frac{1}{q-1}})d(x, \partial\Omega), \quad |\nabla u(\cdot, t)| \leq D(t), \quad (2.14)$$

see [16, Remark 2.8]. The estimate on u is based on the construction of supersolutions, and the estimate of the gradient is deduced from the first one by the Bernstein technique.

In case of rough initial data, a notion of mild solutions has been introduced by [8] (see definition 5.8). Such solutions satisfy $|\nabla u|^q \in L_{loc}^1([0, T]; L^1(\Omega))$.

• If $u_0 \in \mathcal{M}_b^+(\Omega)$ and $1 < q < (N + 2)/(N + 1)$, there is a unique nonnegative mild solution, see [8], [1]. If $u_0 \in L^1(\Omega)$, and $1 < q \leq 2$, there exists at least a solution, such that $u \in C([0, T]; L^1(\Omega))$.

• If $1 < q < (N + 2r)/(N + r)$ uniqueness holds in the class of mild solutions such that $u \in C([0, T]; L^r(\Omega)) \cap L^q_{loc}([0, T]; W^{qr}(\Omega))$.

Next we give decay properties and regularizing effects valid for *any weak solution* of the problem, in particular the universal estimate

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{q-1}} \quad \text{in } (0, T),$$

where $C = C(N, q)$, see Theorem 5.12. As above we deduce uniqueness results:

Theorem 2.4 *Assume that Ω is bounded.*

(i) *Let $1 < q < (N + 2)/(N + 1)$, and $u_0 \in \mathcal{M}_b^+(\Omega)$. Then there exists a unique nonnegative function $u \in C((0, T); L^1(\Omega)) \cap L^q_{loc}((0, T); W_0^{1,q}(\Omega))$, solution of equation (1.1) in $\mathcal{D}'(Q_{\Omega, T})$, such that*

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \psi dx = \int_{\Omega} \psi du_0, \quad \forall \psi \in C_b(\Omega).$$

(ii) *Let $u_0 \in L^r(\Omega)$, $r \geq 1$, $u_0 \geq 0$, and $1 < q < (N + 2r)/(N + r)$. Then there exists a unique nonnegative solution u as above, such that $u \in C([0, T]; L^r(\Omega))$ and $u(\cdot, 0) = u_0$.*

This improves the results of [8], which required assumptions up to $t = 0$ for the gradient: $|\nabla u|^q \in L^1_{loc}([0, T]; L^1(\Omega))$ in case (i), $|\nabla u| \in L^q_{loc}([0, T]; L^r(\Omega))$ in case (ii).

Finally we show the existence of weak solutions for any $u_0 \in L^r(\Omega)$, $r \geq 1$, such that $u \in C([0, T]; L^r(\Omega))$, see Proposition 5.17.

In **Section 6** we extend some results of Section 5 to the case of the quasilinear equations (1.3), with initial data $u_0 \in L^r(\Omega)$ or $u_0 \in \mathcal{M}_b(\Omega)$, and u may be a signed solution. In the case of equation

$$u_t - \Delta_p u = 0,$$

several local or global L^∞ estimates and Harnack properties have been obtained in the last decades, see for example [38], [24], [25], [30], and [23], [20] and references therein. Regularizing properties for equation (1.2) are given in [33] in a Hilbertian context in case $g = 0$ or $p = 2$.

Here we combine our iteration method of Section 3 with a notion of *renormalized solution*, developed by many authors [18], [32],[35], well adapted to rough initial data. We do not require that $u(\cdot, t) \in L^2(\Omega)$, but we only assume that the truncates $T_k(u)$ of u by $\pm k$ ($k > 0$) lie in $L^p((0, T); W^{1,p}(\Omega))$. We prove decay and L^∞ estimates of the following type: if $\gamma > 0$, for any $r \geq 1$, $p > 1$ and (for simplicity) $q \neq N$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\sigma} \|u_0\|_{L^r(\Omega)}, \quad \sigma = \frac{1}{\frac{rq}{N} + \lambda + q - 1} = \frac{N}{rq} \varpi,$$

If $\nu > 0$, then for any $r \geq 1$, and $p \neq N$ such that $p > 2N/(N + 2)$,

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\tilde{\sigma}} \|u_0\|_{L^r(\Omega)}, \quad \tilde{\sigma} = \frac{1}{\frac{rp}{N} + p - 2} = \frac{N}{rp} \tilde{\varpi}.$$

And we deduce universal estimates as before:

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{q-1+\lambda}} \quad \text{if } \gamma > 0; \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{p-2}} \quad \text{if } \nu > 0 \text{ and } p > 2.$$

Such methods can also be extended to porous media equations, and doubly nonlinear equations involving operators of the form $u \mapsto -\Delta_p(|u|^{m-1}u)$.

3 Regularization lemmas

We begin by a simple bootstrap property, used for example in [38]. We recall the proof for simplicity:

Lemma 3.1 *Let $\omega \in (0, 1)$ and $\sigma > 0$, and $K, M > 0$. Let y be any positive function on $(0, T)$ such that $y(t) \leq Mt^{-\sigma}$, and for any $0 < s < t < T$,*

$$y(t) \leq K(t-s)^{-\sigma}y^\omega(s),$$

Then y satisfies an estimate independent of M : for any $t \in (0, T)$,

$$y(t) \leq 2^{\sigma(1-\omega)^{-2}}(Kt^{-\sigma})^{(1-\omega)^{-1}} \quad (3.1)$$

Proof. We get by induction, for any $n \geq 1$

$$y^{\omega^{n-1}}(t/2^{n-1}) \leq K^{\omega^{n-1}}2^{n\sigma\omega^{n-1}}t^{-\sigma\omega^{n-1}}y^{\omega^n}(t/2^n),, \quad y^{\omega^n}(t/2^n) \leq 2^{n\sigma\omega^n}t^{-\sigma\omega^n}M^{\omega^n}.$$

Then

$$y(t) \leq K^{\sum_{k=0}^{n-1}\omega^k}t^{-\sigma\sum_{k=0}^n\omega^k}2^{\sigma\sum_{k=0}^n(k+1)\omega^k}M^{\omega^{n+1}},$$

implying (3.1) as $n \rightarrow \infty$, since $\lim_{n \rightarrow \infty} M^{\omega^{n+1}} = 1$. ■

Next we show the Moser's type property:

Proof of Lemma 2.1. (i) Let α be any real such that $\alpha \geq r - 1$, and $v(\cdot, s) \in L^{\alpha+1}(\Omega)$. From (2.2), $\int_{\Omega} v^{\alpha+1}(t)dx$ is decreasing for $t > s$. And $\int_{\Omega} v^{\beta m \theta}(\cdot, \xi)dx$ is finite for almost any $\xi \in (s, t)$, hence for a sequence (ξ_n) decreasing to s ,

$$\int_{\Omega} v^{\alpha+1}(\cdot, t)dx + \frac{C_0(\alpha+1)}{\beta^m} \int_{\xi_n}^t \left(\int_{\Omega} v^{\beta m \theta}(\cdot, \xi)dx \right)^{\frac{1}{\theta}} d\xi \leq \int_{\Omega} v^{\alpha+1}(\cdot, \xi_n)dx \leq \int_{\Omega} v^{\alpha+1}(\cdot, s)dx.$$

From (2.1), there holds $\beta m \theta > r$. Applying again (2.2) with $\beta m \theta - 1$ instead of α , and ξ_n instead of s , we deduce that $\int_{\Omega} v^{\beta m \theta}(t)dx$ is decreasing for $t > s$, thus

$$\int_{\Omega} v^{\alpha+1}(\cdot, t)dx + \frac{C_0(\alpha+1)}{\beta^m} (t - \xi_n) \left(\int_{\Omega} v^{\beta m \theta}(\cdot, \xi_n)dx \right)^{\frac{1}{\theta}} \leq \int_{\Omega} v^{\alpha+1}(\cdot, s)dx.$$

As $n \rightarrow \infty$, $v(\cdot, \xi_n) \rightarrow v(\cdot, s)$ in $L^1_{loc}(\Omega)$, and after extraction, a.e. in Ω . Then from the Fatou lemma,

$$\int_{\Omega} v^{\alpha+1}(\cdot, t)dx + \frac{C_0(\alpha+1)}{\beta^m} (t - s) \left(\int_{\Omega} v^{\beta m \theta}(\cdot, s)dx \right)^{\frac{1}{\theta}} \leq \int_{\Omega} v^{\alpha+1}(\cdot, s)dx.$$

Hence

$$\|v(t)\|_{L^{\beta m \theta}(\Omega)}^{\beta m \theta} \leq \left(\frac{\beta^m}{C_0(\alpha+1)} \frac{1}{t-s} \|v(s)\|_{L^{\alpha+1}(\Omega)}^{\alpha+1} \right)^{\theta}, \quad (3.2)$$

• **Case $r > 1$.** We start from $s = 0$, we have $v_0 \in L^r(\Omega)$. We take $\alpha_0 = r - 1$, thus $\int_{\Omega} v^{\alpha_0+1}(t)dx$ is finite, and set $\beta_0 = 1 + (\alpha_0 + \lambda) / m$. We define sequences $(t_n), (\alpha_n), (r_n), (\beta_n)$, by $t_0 = 0, r_0 = r$ and for any $n \geq 1$,

$$t_n = t(1 - \frac{1}{2^n}), \quad r_n = \alpha_n + 1, \quad \beta_n = 1 + \frac{\alpha_n + \lambda}{m}, \quad r_{n+1} = \beta_n m \theta = (r_n + \lambda + m - 1)\theta,$$

hence $(r_n), (\beta_n)$ are increasing, since $r_1 > r$ from (2.1). In (3.2), we replace $s, t, \alpha, \beta m \theta$, by $t_n, t_{n+1} r_n, r_{n+1}$, and get

$$\|v(t_{n+1})\|_{L^{r_{n+1}}(\Omega)} \leq \left(\frac{1}{C_0(m\theta)^m} \frac{r_{n+1}^m}{r_n} \frac{1}{t_{n+1} - t_n} \right)^{\frac{\theta}{r_{n+1}}} \|v(t_n)\|_{L^{r_n}(\Omega)}^{\frac{\theta \cdot r_n}{r_{n+1}}}. \quad (3.3)$$

From (2.2), it follows that

$$\|v(t)\|_{L^{r_{n+1}}(\Omega)} \leq \|v(t_{n+1})\|_{L^{r_{n+1}}(\Omega)} \leq I_n J_n L_n \|v_0\|_{L^r(\Omega)}^{\frac{\theta^{n+1} \cdot r}{r_{n+1}}}, \quad (3.4)$$

where

$$I_n = \prod_{k=1}^{n+1} \left(\frac{r_k^m}{r_{k-1}} \right)^{\frac{\theta^{n+2-k}}{r_{n+1}}}, \quad J_n = \prod_{k=1}^{n+1} \left(\frac{1}{t_k - t_{k-1}} \right)^{\frac{\theta^{n+2-k}}{r_{n+1}}}, \quad L_n = (C_0(m\theta)^m)^{-\sum_{k=1}^{n+1} \frac{\theta^{n+2-k}}{r_{n+1}}}.$$

Since $r_n = \theta^n(r + (\lambda + m - 1)\theta'(1 - \theta^{-n}))$, we find

$$\lim_{n \rightarrow \infty} \frac{\theta^{n+1} r}{r_{n+1}} = \varpi_{r,m,\lambda,\theta}, \quad \lim_{n \rightarrow \infty} \frac{1}{r_{n+1}} \sum_{k=1}^{n+1} \theta^{n+2-k} = \sigma_{r,m,\lambda,\theta}, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{n+1} k \theta^{1-k} = \theta'^2 \quad (3.5)$$

As a consequence

$$\lim_{n \rightarrow \infty} J_n = 2^{-\frac{\varpi_{r,m,\lambda,\theta}}{r} \theta'^2} t^{-\sigma_{r,m,\lambda,\theta}}, \quad \lim_{n \rightarrow \infty} L_n = (C_0(m\theta)^m)^{-\sigma_{r,m,\lambda,\theta}}. \quad (3.6)$$

Otherwise

$$\ln I_n = \frac{m}{r_{n+1}} \sum_{k=1}^{n+1} \theta^{n+2-k} \ln r_k - \frac{1}{r_{n+1}} \sum_{k=0}^n \theta^{n+1-k} \ln r_k = \frac{\theta^{n+1}}{r_{n+1}} (m \sum_{k=1}^{n+1} \theta^{-k} \ln r_k - \sum_{k=0}^n \theta^{-k} \ln r_k)$$

and the sum $S = \sum_{k=0}^{\infty} \theta^{-k} \ln r_k$ is finite, since $r_k \leq \theta^k(r + |\lambda + m - 1|\theta')$. Then I_n has a finite limit $\ell = \ell(N, m, r, \lambda, \theta) = \exp(r^{-1} \varpi_{r,m,\lambda,\theta} ((m\theta - 1)S - m\theta \ln r))$. Thus we can go to the limit in (3.4), and the conclusion follows.

• **Case $r = 1$.** If (H_1) holds we can take $\alpha_0 = r - 1 = 0$ and the proof is done. Next assume (H_2) . Then we obtain, for any $0 \leq s < t < T$, and a constant C as before,

$$\begin{aligned} \|v(\cdot, t)\|_{L^\infty(\Omega)} &\leq C(t-s)^{-\sigma_{\rho,m,\lambda,\theta}} \|v(\cdot, s)\|_{L^\rho(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}} \\ &\leq C(t-s)^{-\sigma_{\rho,m,\lambda,\theta}} \|v(\cdot, s)\|_{L^\infty(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}(\rho-1)/\rho} \|v(\cdot, s)\|_{L^1(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho} \\ &\leq C \|v_0\|_{L^1(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho} (t-s)^{-\sigma_{\rho,m,\lambda,\theta}} \|v(\cdot, s)\|_{L^\infty(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}(\rho-1)/\rho} \end{aligned}$$

Let $y(t) = \|v(\cdot, t)\|_{L^\infty(\Omega)}$. We can apply Lemma 3.1 to y , with

$$\sigma = \sigma_{\rho,m,\lambda,\theta}, \quad \omega = \frac{\varpi_{\rho,m,\lambda,\theta}}{\rho'}, \quad K = C \|v_0\|_{L^1(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho}, \quad M = C \|v_0\|_{L^\rho(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho}.$$

Indeed $\omega < 1$ since $\rho\theta'(1 - m - \lambda) < 1$. Then there holds

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq 2^{\sigma(1-\omega)^{-2}} (Kt^{-\sigma})^{(1-\omega)^{-1}} = 2^{\sigma(1-\omega)^{-2}} C^{(1-\omega)^{-1}} t^{-\sigma(1-\omega)^{-1}} \|v_0\|_{L^1(\Omega)}^{\varpi_{\rho,q,\lambda,\theta}/\rho((1-\omega))}.$$

Noticing that $\sigma(1 - \omega)^{-1} = \sigma_{1,m,\lambda,\theta}$ and $\varpi_{\rho,m,\lambda,\theta}/\rho((1 - \omega)) = \varpi_{1,m,\lambda,\theta}$, we deduce

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\sigma_{1,m,\lambda,\theta}} \|v_0\|_{L^1(\Omega)}^{\varpi_{1,m,\lambda,\theta}}, \quad (3.7)$$

with a new constant C , now depending on ρ . ■

Remark 3.2 *This lemma can be compared with the result of [33, Theorem 2.1] obtained by the Stampacchia's method. In order to obtain decay estimates for the solutions u of a parabolic equation such as (1.1) or (1.3), the Moser's method consists to take as test functions powers $|u|^{\alpha-1}u$ of u ; the Stampacchia's method uses test functions of the form $(u - k)^+ \text{sign}u$. If one applies to sufficiently smooth solutions, both techniques lead to decay estimates of the same type. In the case of weaker solutions, the Stampacchia method supposes that the functions $(u - k)^+$ are admissible in the equation, which leads to assume that $u(\cdot, t) \in W^{1,2}(\Omega)$, see [33]. In the sequel we combine Moser's method with regularization or truncature of u , in order to admit powers as test functions. So we do not need to make this assumption, thus the Moser's method appears to be more performant.*

Such type of L^∞ estimates as (2.3) may imply a universal one, that means independent of the initial data, in case Ω is bounded. This was observed for example in [38]:

Lemma 3.3 *Let Ω be bounded. (i) Let $v \in C([0, T]; L^1_{loc}(\Omega))$ be nonnegative, and $v_0 = v(x, 0) \in L^1(\Omega)$, such that for some $C > 0$, for any $0 \leq s < t < T$,*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - s)^{-\sigma} \|v(\cdot, s)\|_{L^1(\Omega)}^{\varpi},$$

where $\sigma > 0, \varpi \in (0, 1)$. Then there exists $M = M(C, \sigma, \varpi, |\Omega|)$ such that for any $t \in (0, T)$,

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq Mt^{-\frac{\sigma}{1-\varpi}}. \quad (3.8)$$

(ii) As a consequence, if v satisfies (2.3), with $m - 1 + \lambda > 0$, then

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq Mt^{-\frac{1}{m-1+\lambda}}. \quad (3.9)$$

Proof. (i) For any $0 < s < t < T$,

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - s)^{-\sigma} \|v(\cdot, s)\|_{L^r(\Omega)}^{\varpi} \leq C(t - s)^{-\sigma} |\Omega|^\varpi \|v(\cdot, s)\|_{L^\infty(\Omega)}^{\varpi}$$

Since $\varpi < 1$, (3.8) follows from Lemma 3.1: for any $t \in (0, T)$,

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq 2^{\sigma(1-\varpi)^{-2}} (C|\Omega|^\varpi t^{-\sigma})^{(1-\varpi)^{-1}}.$$

(ii) If v satisfies (2.3), with $m - 1 + \lambda > 0$, we take $\sigma = \sigma_{r,m,\lambda,\theta}$, and $\varpi = \varpi_{r,m,\lambda,\theta}$ defined at (2.4), then $\varpi = (1 + (m - 1 + \lambda)\theta'/r)^{-1} < 1$ and $\sigma((1 - \varpi)^{-1}) = (m - 1 + \lambda)^{-1}$, which proves (3.9). ■

In the sequel Lemma 2.1 is applied in situations where (2.2) comes from an estimate of v in a Sobolev Space $W^{1,m}(Q_{\Omega,s,t})$, when $1 < m < N$, with $\theta = N/(N - m)$, or $m = N$ and $\theta > 1$ is arbitrary.

In the case $m > N$, where Lemma 2.1 does not bring information, we use in the sequel a limit form of Gagliardo-Nirenberg inequality, see the proof of Theorems 4.16, 5.12 and 6.7:

Lemma 3.4 *Let $m > N$, and $r \geq 1$. Let Ω be any domain in \mathbb{R}^N . Then there exists $C = C(N, m, r) > 0$ such that for any $w \in L^r(\Omega) \cap W_0^{1,m}(\Omega)$,*

$$\|w\|_{L^\infty(\Omega)} \leq C \|\nabla w\|_{L^m(\Omega)}^k \|w\|_{L^r(\Omega)}^{1-k}, \quad \frac{1}{k} = 1 + r\left(\frac{1}{N} - \frac{1}{m}\right).$$

Proof. By extension by 0 outside of Ω , we can assume $\Omega = \mathbb{R}^N$. Since $m > N$, for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$, and any $x \in \mathbb{R}^N$,

$$|\varphi(x)| \leq C(N, m) \left(\left| \int_{B(x,1)} \varphi dx \right| + \|\nabla \varphi\|_{L^m(B(x,1))} \right) \leq C(N, m, r) \left(\|\varphi\|_{L^r(\mathbb{R}^N)} + \|\nabla \varphi\|_{L^m(\mathbb{R}^N)} \right);$$

by density, there holds

$$\|w\|_{L^\infty(\mathbb{R}^N)} \leq C \left(\|w\|_{L^r(\mathbb{R}^N)} + \|\nabla w\|_{L^m(\mathbb{R}^N)} \right)$$

for any $w \in L^r(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N)$. Setting $w_t(x) = w(tx)$ for any $t > 0$, we find

$$\|w\|_{L^\infty(\mathbb{R}^N)} = \|w_t\|_{L^\infty(\mathbb{R}^N)} \leq C \left(t^{-\frac{N}{r}} \|w\|_{L^r(\mathbb{R}^N)} + t^{\frac{m-N}{m}} \|\nabla w\|_{L^m(\mathbb{R}^N)} \right);$$

the result follows by taking $t = (\|w\|_{L^r(\mathbb{R}^N)} / \|\nabla w\|_{L^m(\mathbb{R}^N)})^{1/(1-N/m+N/r)}$. \blacksquare

4 The Hamilton-Jacobi equation in \mathbb{R}^N

4.1 Different notions of solution

In this section we study the Cauchy problem (2.5).

Here we consider the solutions *in a weak sense*, which does not use any formulation in terms of semigroups:

Definition 4.1 *We say that a nonnegative function u is a **weak solution** (resp. **subsolution**) of equation of (1.1) in $Q_{\mathbb{R}^N, T}$, if $u \in L_{loc}^1(Q_{\mathbb{R}^N, T})$, and $|\nabla u| \in L_{loc}^q(Q_{\mathbb{R}^N, T})$, and*

$$\int_0^T \int_{\Omega} (-u\varphi_t - u\Delta\varphi + |\nabla u|^q \varphi) dx dt = 0, \quad (\text{resp. } \leq), \quad \forall \varphi \in \mathcal{D}(Q_{\mathbb{R}^N, T}), \varphi \geq 0. \quad (4.1)$$

Remark 4.2 *From [16], any nonnegative weak solution satisfies*

$$u \in L_{loc}^\infty(Q_{\mathbb{R}^N, T}), \quad \nabla u \in L_{loc}^2(Q_{\mathbb{R}^N, T}), \quad u \in C((0, T); L_{loc}^\rho(\mathbb{R}^N)) \quad \forall \rho \geq 1. \quad (4.2)$$

Hence (4.1) is equivalent to:

$$\int_0^T \int_{\Omega} (-u\varphi_t + \nabla u \cdot \nabla \varphi + |\nabla u|^q \varphi) dx dt = 0, \quad \forall \varphi \in \mathcal{D}(Q_{\mathbb{R}^N, T}), \quad (4.3)$$

and there holds, for any $s, \tau \in (0, T)$,

$$\int_{\mathbb{R}^N} u(\cdot, \tau) \varphi(\cdot, \theta) dx - \int_{\mathbb{R}^N} u(\cdot, s) \varphi(\cdot, s) dx + \int_s^\tau \int_{\mathbb{R}^N} (-u\varphi_t + \nabla u \cdot \nabla \varphi + |\nabla u|^q \varphi) dx dt = 0; \quad (4.4)$$

and for any $\psi \in C_c^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} u(\cdot, \tau) \psi dx - \int_{\mathbb{R}^N} u(\cdot, s) \psi dx + \int_s^\tau \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \psi + |\nabla u|^q \psi) dx dt = 0. \quad (4.5)$$

Definition 4.3 Let $u_0 \in L^r_{loc}(\mathbb{R}^N)$, $r \geq 1$.

We say that u is a **weak L^r_{loc} solution** if u is a weak solution of (1.1) and the extension of u by u_0 at time 0 satisfies $u \in C([0, T]; L^r_{loc}(\mathbb{R}^N))$.

We say that u is a **weak r solution** of problem (2.5) if it is a weak solution of equation (1.1) such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u^r(\cdot, t) \psi dx = \int_{\mathbb{R}^N} u_0^r \psi dx, \quad \forall \psi \in C_c(\mathbb{R}^N). \quad (4.6)$$

Definition 4.4 Let u_0 be any nonnegative Radon measure in \mathbb{R}^N , we say that u is a **weak \mathcal{M}_{loc} solution** of problem (2.5) if it is a weak solution of (1.1) such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(\cdot, t) \psi dx = \int_{\mathbb{R}^N} \psi du_0, \quad \forall \psi \in C_c(\mathbb{R}^N). \quad (4.7)$$

Remark 4.5 Obviously, any weak L^r_{loc} solution is a weak r solution. When $r = 1$, the notions of weak 1-solution and weak \mathcal{M}_{loc} solution coincide. When $r > 1$, it can be easily checked that u is a weak L^r_{loc} solution if and only if it is a weak r solution and

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(\cdot, t) \psi dx = \int_{\mathbb{R}^N} u_0 \psi dx, \quad \forall \psi \in C_c(\mathbb{R}^N). \quad (4.8)$$

Other types of solutions using the semigroup of the heat equation have been introduced in ([15]):

Definition 4.6 Let $u_0 \in L^r(\mathbb{R}^N)$. A function u is called **mild L^r solution** of problem (2.5) if $u \in C([0, T]; L^r(\mathbb{R}^N))$, and $|\nabla u|^q \in L^1_{loc}([0, T]; L^r(\mathbb{R}^N))$ and

$$u(\cdot, t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^r(\mathbb{R}^N);$$

here $e^{t\Delta}$ is the semi-group of the heat equation acting on $L^r(\mathbb{R}^N)$.

Definition 4.7 Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. A function u is called **mild \mathcal{M} solution** of (2.5) if $u \in C_b([0, T]; L^1(\mathbb{R}^N))$ and $|\nabla u|^q \in L^1_{loc}([0, T]; L^1(\mathbb{R}^N))$, and for any $0 < t < T$,

$$u(\cdot, t) = e^{t\Delta} u_0(\cdot) - \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^1(\mathbb{R}^N), \quad (4.9)$$

where $e^{t\Delta}$ is defined on $\mathcal{M}_b^+(\mathbb{R}^N)$ as the adjoint of the operator $e^{t\Delta}$ on $C_0(\mathbb{R}^N)$, the space of continuous functions on \mathbb{R}^N which tend to 0 as $|x| \rightarrow \infty$.

Remark 4.8 Every mild L^r solution is a weak L^r_{loc} solution. Any mild \mathcal{M} solution is a weak \mathcal{M}_{loc} solution. Indeed for any $0 < \epsilon < t < T$, we find

$$u(\cdot, t) = e^{(t-\epsilon)\Delta} u(\cdot, \epsilon) - \int_\epsilon^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^1(\mathbb{R}^N);$$

and $u(\cdot, \epsilon) \in L^1(\mathbb{R}^N)$, thus u is a weak solution on (ϵ, T) , then on $(0, T)$. As $t \rightarrow 0$, $u(\cdot, t) - e^{t\Delta} u_0(\cdot)$ converges to 0 in $L^1(\mathbb{R}^N)$, then weakly $*$, and $e^{t\Delta} u_0(\cdot) \rightarrow u_0$ weakly $*$, hence (4.7) holds.

Another definition of solution with initial data measure was given in ([12]):

Definition 4.9 Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. A function u is called **weak semi-group solution** if $u \in C((0, T); L^1(\mathbb{R}^N))$ and $|\nabla u|^q \in L_{loc}^1([0, T]; L^1(\mathbb{R}^N))$ and for any $0 < \epsilon < t < T$,

$$u(\cdot, t) = e^{(t-\epsilon)\Delta} u(\cdot, \epsilon) - \int_{\epsilon}^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^1(\mathbb{R}^N), \quad (4.10)$$

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(\cdot, t) \varphi dx = \int_{\mathbb{R}^N} \varphi du_0, \quad \forall \varphi \in C_b(\mathbb{R}^N), \quad (4.11a)$$

In fact the two definitions coincide, see the proof in the Appendix:

Lemma 4.10 Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then

$$u \text{ is a mild } \mathcal{M} \text{ solution of (2.5)} \iff u \text{ is a weak semi-group solution of (2.5)}.$$

Remark 4.11 All these definitions of semi-group solutions assume a global in space condition: $|\nabla u|^q \in L_{loc}^1([0, T]; L^1(\mathbb{R}^N))$ or $|\nabla u|^q \in L_{loc}^1([0, T]; L^r(\mathbb{R}^N))$. Observe also that (4.11a) is assumed for any $\varphi \in C_b(\mathbb{R}^N)$. On the contrary, our definitions of weak solutions are **local in space and time**, they do not require such global properties.

Finally we mention another weaker form of semi-group solutions, given in ([15]), which will be used in the sequel:

Definition 4.12 Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then u is a pointwise mild solution of (2.5) if $u \in L_{loc}^1(Q_{\mathbb{R}^N, T})$, and $|\nabla u|^q \in L_{loc}^1(Q_{\mathbb{R}^N, T})$, and

$$u(x, t) = (e^{t\Delta} u_0)(x) - \int_0^t \int_{\mathbb{R}^N} g(x-y, t-s) |\nabla u(y, s)|^q dy ds \quad \text{for a.e. } (x, t) \in Q_{\mathbb{R}^N, T},$$

where g is the heat kernel.

Remark 4.13 For $r \geq 1$, it is clear that every mild L^r solution is a pointwise mild solution. If $u_0 \in L^1(\mathbb{R}^N)$ every pointwise mild solution is a mild L^1 solution; if $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$, every pointwise mild solution, is a mild \mathcal{M} solution. see [15, Proposition 1.1 and Remark 1.2].

4.2 Decay of the norms

Next we show a *decay result* for the solutions of Hamilton Jacobi equations, which is valid for any $q > 1$, and for *all the weak solutions, with no condition of boundedness at infinity*.

When $q \leq 2$, any weak solution u of equation (1.1) is smooth: $u \in C^{2,1}(Q_{\mathbb{R}^N, T})$, from [16, Theorem 2.9]. Since it may be false for $q > 2$, we regularize u by convolution, setting

$$u_\epsilon = u * \varrho_\epsilon,$$

where $(\varrho_\epsilon)_{\epsilon > 0}$ is a sequence of mollifiers. We recall that for given $0 < s < \tau < T$, and ϵ small enough, u_ϵ is a *subsolution* of equation (1.1), see [16]:

$$(u_\epsilon)_t - \nu \Delta u_\epsilon + |\nabla u_\epsilon|^q \leq 0, \quad \text{in } Q_{\mathbb{R}^N, s, \tau}. \quad (4.12)$$

Theorem 4.14 Assume $q > 1$. Let $r \geq 1$. Let $u_0 \in L^r(\mathbb{R}^N)$ be nonnegative. Let u be any non-negative weak r solution of problem (2.5).

(i) Then $u(\cdot, t) \in L^r(\mathbb{R}^N)$ for any $t \in (0, T)$, and

$$\int_{\mathbb{R}^N} u^r(\cdot, t) dx \leq \int_{\mathbb{R}^N} u_0^r dx. \quad (4.13)$$

(ii) Moreover $u^{r-1}|\nabla u|^q \in L^1_{loc}([0, T]; L^1(\mathbb{R}^N))$; and $u^{r-2}|\nabla u|^2 \in L^1_{loc}([0, T]; L^1(\mathbb{R}^N))$ if $r > 1$ and $\nu > 0$. For any $t \in (0, T)$, we have the equalities

$$\int_{\mathbb{R}^N} u^r(\cdot, t) dx + r \int_0^t \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q dx dt + r(r-1)\nu \int_0^t \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 dx dt = \int_{\mathbb{R}^N} u_0^r dx, \quad \text{if } r > 1, \quad (4.14)$$

$$\int_{\mathbb{R}^N} u(\cdot, t) dx + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q dx dt = \int_{\mathbb{R}^N} u_0 dx, \quad \text{if } r = 1, \quad (4.15)$$

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u^r(\cdot, t) dx = \int_{\mathbb{R}^N} u_0^r dx. \quad (4.16)$$

(iii) $u^{q-1+r} \in L^1_{loc}([0, T]; W^{1,1}(\mathbb{R}^N))$; and if $\nu > 0$, then $u^{r/2} \in L^2_{loc}([0, T]; W^{1,2}(\mathbb{R}^N))$.

(iv) If u is a weak L^r_{loc} solution, then $u \in C([0, T]; L^r(\mathbb{R}^N))$.

Proof. (i) **First step: case** $q' > N/r$. That means $r \geq N$ or q is small enough: $1 < q < N/(N-r)$. Let $0 < s < \tau < T$. Take $\varepsilon > 0$ small enough such that (4.12) holds. Let $\delta > 0$, and $u_{\varepsilon, \delta} = u_\varepsilon + \delta$, so that $u_{\varepsilon, \delta}^{r-2}$ is well defined for $r < 2$. For any $R > 0$, we consider $\xi(x) = \xi_R(x) = \psi(x/R)$, where $\psi(x) \in [0, 1]$, $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$. Multiplying (4.12) by $u_{\varepsilon, \delta}^{r-1} \xi^\lambda$ where $\lambda > 0$, we get for any $t \in [s, \tau]$,

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{r} \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \xi^\lambda dx \right) + (r-1)\nu \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-2} |\nabla u_{\varepsilon, \delta}|^2(\cdot, t) \xi^\lambda dx + \int_{\mathbb{R}^N} |\nabla u_{\varepsilon, \delta}|^q u_{\varepsilon, \delta}^{r-1} \xi^{\lambda-1} \xi^\lambda dx \\ & \leq -\lambda \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-1} \xi^{\lambda-1} \xi^{\lambda-1} \nabla u_{\varepsilon, \delta} \cdot \nabla \xi dx, \end{aligned}$$

and from the Hölder inequality, with $C = C(q, \lambda)$

$$\begin{aligned} \lambda \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-1} |\nabla u_{\varepsilon, \delta}|(\cdot, t) \xi^{\lambda-1} |\nabla \xi| dx & \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{\varepsilon, \delta}|^2(\cdot, t) u_{\varepsilon, \delta}^{r-1} \xi^\lambda dx + C \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-1}(\cdot, t) \xi^{\lambda-q'} |\nabla \xi|^{q'} dx, \\ \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-1}(\cdot, t) \xi^{\lambda-q'} |\nabla \xi|^{q'} dx & \leq \left(\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \xi^\lambda dx \right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}^N} \xi^{\lambda-rq'} |\nabla \xi|^{r q'} dx \right)^{\frac{1}{r}}. \end{aligned}$$

Choosing $\lambda = r q'$ we deduce

$$\frac{d}{dt} \left(\left(\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \xi^\lambda dx \right)^{\frac{1}{r}} \right) \leq C R^{\frac{N}{r}-q'},$$

where $C = C(N, q, r, \psi)$. By integration,

$$\left(\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \xi^\lambda dx \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, s) \xi^\lambda dx \right)^{\frac{1}{r}} + C \tau R^{\frac{N}{r}-q'}.$$

with a new constant C as above. Let $R_0 > 0$ be fixed and take $R > R_0$, thus

$$\left(\int_{B_{R_0}} u_{\varepsilon, \delta}^r(\cdot, t) dx \right)^{\frac{1}{r}} \leq \left(\int_{B_{2R}} u_{\varepsilon, \delta}^r(\cdot, s) \xi^\lambda dx \right)^{1/r} + C\tau R^{\frac{N}{r} - q'}.$$

As $\delta \rightarrow 0$, and then as $\varepsilon \rightarrow 0$, we deduce that

$$\left(\int_{B_{R_0}} u(\cdot, t)^r dx \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u(\cdot, s)^r \xi^\lambda dx \right)^{\frac{1}{r}} + C\tau R^{\frac{N}{r} - q'} \quad (4.17)$$

for any $0 < s < t < T$; from (4.6) we obtain, as $s \rightarrow 0$,

$$\left(\int_{B_{R_0}} u(\cdot, t)^r dx \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u_0^r \xi^\lambda dx \right)^{\frac{1}{r}} + C\tau R^{\frac{N}{r} - q'} \leq \left(\int_{\mathbb{R}^N} u_0^r dx \right)^{\frac{1}{r}} + C\tau R^{\frac{N}{r} - q'}.$$

Finally (4.13) follows as $R \rightarrow \infty$ and then as $R_0 \rightarrow \infty$.

Second step: case $q' \leq N/r$. Then $r < N$ and $q \geq N/(N-r) > 1$. Let us fix some $k \in (1, N/(N-r))$. For any $\eta \in (0, 1)$, we have $\eta |\nabla u|^k \leq \eta + |\nabla u|^q$, hence the function

$$w_\eta = \eta^{1/(k-1)}(u - \eta t)$$

satisfies

$$(w_\eta)_t - \nu \Delta w_\eta + |\nabla w_\eta|^k \leq 0$$

in the weak sense. Thanks to Kato's inequality, see [21], [5], we deduce that

$$(w_\eta^+)_t - \nu \Delta w_\eta^+ + |\nabla w_\eta^+|^k \leq 0, \quad (4.18)$$

in $\mathcal{D}'(Q_{\mathbb{R}^N, T})$. And w_η^+ has the same regularity as u . Moreover it satisfies an analogous property to (4.6):

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} (w_\eta^+)^r(\cdot, t) \psi dx = \int_{\mathbb{R}^N} (\eta^{1/(k-1)} u_0)^r \psi dx, \quad \forall \psi \in C_c(\mathbb{R}^N). \quad (4.19)$$

Indeed

$$\begin{aligned} \left| \int_{\mathbb{R}^N} ((u - \eta t)^+)^r - u^r(\cdot, t) \psi dx \right| &\leq \int_{\{u \geq \eta t\}} |(u(\cdot, t) - \eta t)^r - u^r(\cdot, t)| \psi dx + \int_{\{u \leq \eta t\}} u^r(\cdot, t) \psi dx \\ &\leq r\eta t \int_{\mathbb{R}^N} u^{r-1}(\cdot, t) \psi dx + C(\psi)t^r \\ &\leq r\eta t \left(\int_{\mathbb{R}^N} u^r(\cdot, t) \psi dx \right)^{1/r'} \left(\int_{\mathbb{R}^N} \psi dx \right)^{1/r} + C(\psi)t^r \end{aligned}$$

then

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} ((u - \eta t)^+)^r - u^r(\cdot, t) \psi dx = 0,$$

and (4.19) follows from (4.6) applied to $\eta^{1/(k-1)}u$. Since $k' > N/r$, we can apply the first step to w_η^+ ; we deduce that $w_\eta^+(t) \in L^r(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (w_\eta^+)^r(\cdot, t) dx \leq \eta^{\frac{r}{k-1}} \int_{\mathbb{R}^N} u_0^r dx.$$

Then $\|(u - \eta t)^+\|_{L^r(\mathbb{R}^N)} \leq \|u_0\|_{L^r(\mathbb{R}^N)}$. Since $u \leq \eta t + (u - \eta t)^+$, we find, for any $R > 0$,

$$\|u(\cdot, t)\|_{L^r(B_R)} \leq \|u_0\|_{L^r(\mathbb{R}^N)} + \eta t |B_R|^{\frac{1}{r}}$$

As $\eta \rightarrow 0$ we get $\|u(\cdot, t)\|_{L^r(B_R)} \leq \|u_0\|_{L^r(\mathbb{R}^N)}$, then as $R \rightarrow \infty$ we deduce that $u(\cdot, t) \in L^r(\mathbb{R}^N)$, and (4.13) holds.

(ii) Consider again $0 < s < \tau < T$ and $u_{\varepsilon, \delta}$ as above. Setting $F_\varepsilon = |\nabla u|^q * \varrho_\varepsilon$, there holds

$$(u_{\varepsilon, \delta})_t - \nu \Delta u_{\varepsilon, \delta} + F_\varepsilon = 0.$$

Then for any $t \in [s, \tau]$,

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \xi^\lambda dx \right) + r(r-1)\nu \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-2} |\nabla u_{\varepsilon, \delta}|^2(\cdot, t) \xi^\lambda dx + r \int_{\mathbb{R}^N} F_\varepsilon u_{\varepsilon, \delta}^{r-1}(\cdot, t) \xi^\lambda dx \\ &= -r\nu \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-1} \nabla u_{\varepsilon, \delta}(\cdot, t) \cdot \nabla(\xi^\lambda) dx = \nu \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \Delta(\xi^\lambda) dx \end{aligned}$$

thus

$$\begin{aligned} & \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, t) \xi^\lambda dx + r \int_s^t \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-1} F_\varepsilon \xi^\lambda dx dt \\ &+ r(r-1)\nu \int_s^t \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^{r-2} |\nabla u_{\varepsilon, \delta}|^2 \xi^\lambda dx = \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r(\cdot, s) \xi^\lambda dx + \nu \int_s^t \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^r \Delta(\xi^\lambda) dx \end{aligned}$$

First we go to the limit as $\varepsilon \rightarrow 0$, because $u \in L_{loc}^\infty(Q_{\mathbb{R}^N, T})$, and $|\nabla u|^2 \in L_{loc}^1(Q_{\mathbb{R}^N, T})$, and F_ε converges to $|\nabla u|^q$ in $L_{loc}^1(Q_{\mathbb{R}^N, T})$. Setting $v_\delta = u + \delta$, we obtain for almost any s, t , and by continuity for any $0 < s < t \leq \tau$,

$$\begin{aligned} & \int_{\mathbb{R}^N} v_\delta^r(\cdot, t) \xi^\lambda dx + r \int_s^t \int_{\mathbb{R}^N} v_\delta^{r-1} |\nabla u|^q \psi dx dt \\ &+ r(r-1)\nu \int_s^t \int_{\mathbb{R}^N} v_\delta^{r-2} |\nabla u|^2 \xi^\lambda dx = \int_{\mathbb{R}^N} v_\delta^r(\cdot, s) \xi^\lambda dx + \nu \int_s^t \int_{\mathbb{R}^N} v_\delta^r \Delta(\xi^\lambda) dx \end{aligned}$$

Next we go to the limit as $\delta \rightarrow 0$: from the Fatou Lemma, $\int_s^t \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q \psi dx dt$ and $(r-1)\nu \int_s^t \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 \xi^\lambda dx$ are finite, and then from the dominated convergence theorem,

$$\begin{aligned} & \int_{\mathbb{R}^N} u^r(\cdot, t) \xi^\lambda dx + r \int_s^t \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q \xi^\lambda dx dt \\ &+ r(r-1)\nu \int_s^t \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 \xi^\lambda dx = \int_{\mathbb{R}^N} u^r(\cdot, \sigma) \xi^\lambda dx + \nu \int_s^t \int_{\mathbb{R}^N} u^r \Delta(\xi^\lambda) dx. \end{aligned}$$

As $s \rightarrow 0$, from (4.6), we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^N} u^r(\cdot, t) \xi^\lambda dx + r \int_0^t \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q \xi^\lambda dx dt \\ &+ r(r-1) \int_0^t \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 \xi^\lambda dx = \int_{\mathbb{R}^N} u_0^r(\cdot, \sigma) \xi^\lambda dx + \nu \int_\sigma^t \int_{\mathbb{R}^N} u^r \Delta(\xi^\lambda) dx \end{aligned}$$

Now $u(\cdot, t) \in L^r(\mathbb{R}^N)$ for any $t \in [s, \tau]$, and

$$\int_{\sigma}^t \int_{\mathbb{R}^N} u^r \Delta(\xi^\lambda) dx \leq \frac{C}{R^2} \tau \int_{\mathbb{R}^N} u^r(\sigma) dx,$$

thus we can make $R \rightarrow \infty$. Then $\int_0^\tau \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q dx dt$ and $(r-1)\nu \int_0^\tau \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 dx dt$ are finite and, from the dominated convergence theorem,

$$\int_{\mathbb{R}^N} u^r(\cdot, t) dx + r \int_0^t \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q dx dt + r(r-1)\nu \int_0^t \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 dx = \int_{\mathbb{R}^N} u_0^r dx \quad (4.22)$$

Hence (4.14) and (4.15) follow, implying (4.16).

(iii) Setting $v = u^b$ with $b = (q-1+r)/q \leq r$, there holds $|\nabla v|^q \in L^1_{loc}([0, T]; L^1(\mathbb{R}^N))$, and $v \in L^\infty((0, T); L^{r/b}(\mathbb{R}^N))$. From the Gagliardo-Nirenberg inequality,

$$\|v(\cdot, t)\|_{L^q(\mathbb{R}^N)} \leq C(N, q, r) \|v(\cdot, t)\|_{L^{r/b}(\mathbb{R}^N)}^{1-k} \|\nabla v(\cdot, t)\|_{L^q(\mathbb{R}^N)}^k, \quad \frac{1}{k} = 1 + \frac{rq'}{N}. \quad (4.23)$$

By integration, for any $0 < \tau < T$, we get, from Hölder inequality, with $C = C((\tau, N, q, r))$,

$$\int_0^\tau \int_{\mathbb{R}^N} v^q(\cdot, t) dx dt = \int_0^\tau \int_{\mathbb{R}^N} u^{q-1+r}(\cdot, t) dx dt \leq C \|v\|_{L^\infty((0, \tau); L^{r/m}(\mathbb{R}^N))}^{(1-k)q} \left(\int_0^\tau \int_{\mathbb{R}^N} |\nabla v|^q dx dt \right)^k.$$

Then $u \in L^{q-1+r}(Q_{\mathbb{R}^N, \tau})$, and $v^q = u^{q-1+r} \in L^1((0, \tau); W^{1,1}(\mathbb{R}^N))$, $v \in L^q((0, \tau); W^{1,q}(\mathbb{R}^N))$. If $\nu > 0$, we also have $u^{r-2} |\nabla u|^2 = |\nabla(u^{r/2})|^2 \in L^1(Q_{\mathbb{R}^N, \tau})$, and $u^{r/2} \in L^2(Q_{\mathbb{R}^N, \tau})$, then $u^{r/2} \in L^2((0, \tau); W^{1,2}(\mathbb{R}^N))$.

(iv) Here $u \in C([0, T]; L^r_{loc}(\mathbb{R}^N))$. We only need to prove that $\lim_{t \rightarrow 0} \|u(\cdot, t) - u_0\|_{L^r(\mathbb{R}^N)} = 0$. From a diagonal procedure, there exists $t_n \rightarrow 0$ such that $(u(\cdot, t_n))$ converges to u_0 a.e. in \mathbb{R}^N . First assume $r > 1$; since the convergence holds weakly in $L^r(\mathbb{R}^N)$, and $\lim_{n \rightarrow \infty} \|u(\cdot, t_n)\|_{L^r(\mathbb{R}^N)} = \|u_0\|_{L^r(\mathbb{R}^N)}$ from (4.16). Then it holds from any sequence, and $u \in C([0, T]; L^r(\mathbb{R}^N))$. Next assume $r = 1$. We have for any $p > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} |u(t_n) - u_0| dx &\leq \int_{B_p} |u(t_n) - u_0| dx + \int_{\mathbb{R}^N \setminus B_p} |u(t_n) - u_0| dx \\ &\leq \int_{B_p} |u(t_n) - u_0| dx + \int_{\mathbb{R}^N \setminus B_p} u(t_n) dx + \int_{\mathbb{R}^N \setminus B_p} u_0 dx \\ &= \int_{B_p} |u(t_n) - u_0| dx + \int_{\mathbb{R}^N} u(t_n) dx - \int_{B_p} u_0 dx \\ &\quad - \int_{B_p} (u(t_n) - u_0) dx + \int_{\mathbb{R}^N \setminus B_p} u_0 dx \\ &\leq 2 \int_{B_p} |u(t_n) - u_0| dx + \int_{\mathbb{R}^N} u(t_n) dx - \int_{\mathbb{R}^N} u_0 dx + 2 \int_{\mathbb{R}^N \setminus B_p} u_0 dx. \end{aligned}$$

The result follows from (4.16), because $u_0 \in L^1(\mathbb{R}^N)$. ■

The decay result is also available for initial data measures, where we do not assume that $q < (N+2)/(N+1)$:

Theorem 4.15 Assume $q > 1$. Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and u be any non-negative weak \mathcal{M}_{loc} solution of equation (2.5) in $Q_{\mathbb{R}^N, T}$. Then $u(\cdot, t) \in L^1(\mathbb{R}^N)$ for any $t > 0$, and

$$\int_{\mathbb{R}^N} u(\cdot, t) dx \leq \int_{\mathbb{R}^N} du_0. \quad (4.24)$$

Moreover $u \in C((0, T); L^1(\mathbb{R}^N))$, $|\nabla u|^q \in L_{loc}^1([0, T]; L^1(\mathbb{R}^N))$ and

$$\int_{\mathbb{R}^N} u(\cdot, t) dx + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q dx dt = \int_{\mathbb{R}^N} du_0, \quad (4.25)$$

and

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(\cdot, t) \varphi dx = \int_{\mathbb{R}^N} \varphi du_0, \quad \forall \varphi \in C_b(\mathbb{R}^N). \quad (4.26)$$

Proof. If $q' < N$, we obtain in the same way (4.17) with $r = 1$, and we go to the limit as $s \rightarrow 0$ from (4.7), then

$$\int_{B_{R_0}} u(\cdot, t) dx \leq \int_{\mathbb{R}^N} \xi^\lambda du_0 + C\tau R^{N-q'} \leq \int_{\mathbb{R}^N} du_0 + C\tau R^{N-q'}.$$

Going to the limit as $R \rightarrow \infty$, and then as $R_0 \rightarrow \infty$, we deduce (4.24). If $q' \geq N$, we proceed as in the second step of Theorem 4.14, and get again (4.24). Then (4.25) follows. And $u \in C((0, T); L^1(\mathbb{R}^N))$, from the dominated convergence theorem, because $u \in C((0, T); L_{loc}^1(\mathbb{R}^N))$, and $u \in L^\infty((0, T); L^1(\mathbb{R}^N))$.

Let us show (4.26): let $\varphi \in C_b(\mathbb{R}^N)$ be nonnegative, we can assume that φ takes its values in $[0, 1]$. Let $t_n \rightarrow 0$. We know that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} u(\cdot, t_n) dx = \int_{\mathbb{R}^N} du_0$. Let $\psi_p \in \mathcal{D}(\mathbb{R}^N)$ with values in $[0, 1]$, $\psi_p(x) = 1$ if $|x| \leq p$, 0 if $|x| \geq 2p$. Then $\lim_{p \rightarrow \infty} \int_{\mathbb{R}^N} (1 - \psi_p) du_0 = 0$, from the dominated convergence Theorem. Thus for any $\eta > 0$, one can choose p_η such that $\int_{\mathbb{R}^N} (1 - \psi_{p_\eta}) du_0 \leq \eta$; and

$$\begin{aligned} & \overline{\lim} \left| \int_{\mathbb{R}^N} u(\cdot, t_n) \varphi dx - \int_{\mathbb{R}^N} \varphi du_0 \right| \\ & \leq \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} u(\cdot, t_n) \varphi \psi_{p_\eta} dx - \int_{\mathbb{R}^N} \varphi \psi_{p_\eta} du_0 \right| + \int_{\mathbb{R}^N} \varphi (1 - \psi_{p_\eta}) du_0 + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} u(\cdot, t_n) \varphi (1 - \psi_{p_\eta}) dx \leq \eta, \end{aligned}$$

hence the conclusion follows. \blacksquare

4.3 Regularizing effects

Here we deduce of the decay estimates a regularizing effect *without any condition at ∞* , ending the proof of Theorem 2.2.

Theorem 4.16 Let $q > 1$. Let $r \geq 1$ and $u_0 \in L^r(\mathbb{R}^N)$. Let u be any non-negative weak L_{loc}^r solution of problem (2.5) in $Q_{\mathbb{R}^N, T}$ (4.6).

Then $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$ for any $t \in (0, T)$ and u satisfies the estimates (2.10), where $\sigma_{r,q,N}, \varpi_{r,q,N}$ are given by (2.11).

Moreover if $\nu > 0$, then u satisfies the estimates (2.12). If $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$, the same results hold, where $\|u_0\|_{L^1(\mathbb{R}^N)}$ is replaced by $\int_{\mathbb{R}^N} du_0$.

Proof. Since u is a weak L_{loc}^r solution, then $u \in C([0, T]; L^r(\mathbb{R}^N))$, from Theorem 4.14. Thus for any $0 \leq s < T$, u is a weak r solution in $Q_{\mathbb{R}^N, s, T}$; and $\int_{\mathbb{R}^N} u^r(s) dx < \infty$ with $r \geq 1$. For any $0 < s \leq t < T$, and any $\alpha \geq r - 1$ such that $\int_{\mathbb{R}^N} u^{\alpha+1}(s) dx < \infty$, we can apply Theorem 4.14 to u starting at point s , because of (4.2). Denoting $\beta = 1 + \alpha/q$, we have

$$\frac{1}{\alpha + 1} \int_{\mathbb{R}^N} u^{\alpha+1}(\cdot, t) dx + \frac{1}{\beta^q} \int_s^t \int_{\mathbb{R}^N} |\nabla(u^\beta)|^q dx dt \leq \frac{1}{\alpha + 1} \int_{\mathbb{R}^N} u^{\alpha+1}(\cdot, s) dx, \quad (4.27)$$

and $u^\beta(\cdot, t) \in L^q(\mathbb{R}^N)$ for almost any $t \in (0, T)$.

(i) Proof of (2.10).

First assume $q < N$. From the Sobolev injection of $W^{1,q}(\mathbb{R}^N)$ into $L^{Nq/(N-q)}(\mathbb{R}^N)$, there holds

$$\frac{1}{\alpha + 1} \int_{\mathbb{R}^N} u^{\alpha+1}(\cdot, t) dx + \frac{C(N, q)}{\beta^q} \int_s^t \left(\int_{\mathbb{R}^N} u^{\beta \frac{Nq}{N-q}}(\cdot, t) dx \right)^{\frac{N-q}{N}} dt \leq \frac{1}{\alpha + 1} \int_{\mathbb{R}^N} u^{\alpha+1}(\cdot, s) dx;$$

thus Lemma 2.1 applies with $m = q$ and $\theta = N/(N - q)$. We obtain

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C(t - s)^{-\sigma_{r,q,N}} \|u(\cdot, s)\|_{L^r(\mathbb{R}^N)}^{\varpi_{r,q,N}}, \quad C = C(N, q, r),$$

and deduce (2.10) as s goes to 0.

If $q = N$, we deduce (2.10) from Lemma 2.1 with $\theta > 1$ arbitrary, since $W^{1,N}(\mathbb{R}^N) \subset L^{N\theta}(\mathbb{R}^N)$.

Next assume $q > N$. We straight away obtain, for any $t \in (0, T)$,

$$\int_0^t \int_{\mathbb{R}^N} |\nabla(u^\beta)|^q dx dt \leq \frac{1}{r} \int_{\mathbb{R}^N} u_0^r dx, \quad (4.28)$$

with $\beta = 1 + (r - 1)/q$. From the Sobolev injection $W^{1,q}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$, $u(\cdot, s) \in L^\infty(\mathbb{R}^N)$ for almost any $s \in (0, T)$, hence $u(\cdot, s) \in L^\rho(\mathbb{R}^N)$ for any $\rho \geq r$, and $u \in C([s, t], L^\rho(\mathbb{R}^N))$ from (4.2). In turn $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$ for any $t \in (0, T)$ and $t \mapsto \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}$ is nonincreasing, thus

$$rt \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^{q+r-1} \leq C(N, q) \int_{\mathbb{R}^N} u_0^r dx.$$

This does not give the optimal estimate (2.11). However from Lemma 3.4, $v = u^\beta$ satisfies the Gagliardo-Nirenberg inequality, for almost any $t \in (0, T)$,

$$\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C \|v(\cdot, t)\|_{L^{\frac{r}{\beta}}(\mathbb{R}^N)}^{1-k} \|\nabla v(\cdot, t)\|_{L^q(\mathbb{R}^N)}^k,$$

where $1/k = 1 + (1/N - 1/q)r/\beta$ and $C = C(N, q, r)$. Then

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{\beta q}{k}} \leq C \|u(\cdot, t)\|_{L^r(\mathbb{R}^N)}^{\frac{\beta q(1-k)}{k}} \int_{\mathbb{R}^N} |\nabla(u^\beta)|^q dx dt \leq C \|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{\beta q(1-k)}{k}} \int_{\mathbb{R}^N} |\nabla(u^\beta)|^q dx dt.$$

By integration, using (4.28), we find

$$rt \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{\beta q}{k}} \leq C \|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{\beta q(1-k)}{k} + r},$$

which gives precisely (2.10), since $k/\beta q = \sigma_{r,q,N}$ and $(1-k) + kr/\beta q = \varpi_{r,q,N}$.

(ii) Proof of (2.12).

First assume $N > 2$. For any $\alpha \geq r - 1$ such that $\int_{\mathbb{R}^N} u^{\alpha+1}(s) dx < \infty$,

$$\frac{1}{\alpha+1} \int_{\mathbb{R}^N} u^{\alpha+1}(t) dx + \frac{\alpha}{\tilde{\beta}^2} \nu \int_s^t \int_{\mathbb{R}^N} |\nabla(u^{\tilde{\beta}})|^2 dx dt \leq \frac{1}{\alpha+1} \int_{\mathbb{R}^N} u^{\alpha+1}(s) dx$$

where $\tilde{\beta} = (\alpha+1)/2$; and $u^{\tilde{\beta}} \in L_{loc}^2((0, \tau); W^{1,2}(\mathbb{R}^N))$. From the Sobolev injection of $W^{1,2}(\mathbb{R}^N)$ into $L^{2N/(N-2)}(\mathbb{R}^N)$, we get

$$\frac{1}{\alpha+1} \int_{\mathbb{R}^N} u^{\alpha+1}(t) dx + \frac{\alpha C(N)}{\tilde{\beta}^2} \nu \int_s^t \left(\int_{\mathbb{R}^N} u^{\tilde{\beta} \frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \leq \frac{1}{\alpha+1} \int_{\mathbb{R}^N} u^{\alpha+1}(s) dx.$$

In case $r > 1$, Lemma 2.1 applies with $C_0 = (r-1)C(N)\nu$, $q = 2$, $\theta = N/(N-2)$ and $\lambda = -1$, $\tilde{\beta} = 1 + (\alpha-1)/2$, since $r > N(1-2+1)/2$; and (2.12) follows.

In case $r = 1$, then $u \in C([0, T]; L^1(\mathbb{R}^N)) \cap L_{loc}^\infty((0, T); L^\infty(\mathbb{R}^N))$ because of estimate (2.10). Hence $C([0, T]; L^\rho(\mathbb{R}^N))$ for any $\rho > 1$, for example with $\rho = 2$, and $\|u(\cdot, t)\|_{L^1(\mathbb{R}^N)}$ is nonincreasing, from Theorem 4.14. Therefore Lemma 2.1 applies on (ϵ, t) for $0 < \epsilon < t < T$:

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C(t-\epsilon)^{-\frac{N}{2}} \|u(\cdot, \epsilon)\|_{L^1(\mathbb{R}^N)} \leq C(t-\epsilon)^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)},$$

with $C = C(N, q, r, \nu)$, hence (2.12) follows as $\epsilon \rightarrow 0$.

If $N = 2$, we proceed as above to conclude. Next assume $N = 1$. In case $r > 1$, there holds, for any $t \in (0, T)$,

$$4(r-1)\nu \int_0^t \int_{\mathbb{R}^N} |\nabla(u^{\frac{r}{2}})|^2 dx dt \leq \int_{\mathbb{R}^N} u_0^r dx;$$

and, from Lemma 3.4, applied to $v = u^{r/2}$, with $m = 2 = 1/k$,

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}^{2r} \leq C \|u(\cdot, t)\|_{L^r(\mathbb{R})}^r \int_{\mathbb{R}} |\nabla(u^{\frac{r}{2}})|^2 dx dt \leq C \|u_0\|_{L^r(\mathbb{R}^N)}^r \int_{\mathbb{R}} |\nabla(u^{\frac{r}{2}})|^2 dx dt;$$

by integration, we get, with a new constant $C = C(r, \nu)$,

$$t \|u(\cdot, t)\|_{L^\infty(\mathbb{R})}^{2r} \leq C \|u_0\|_{L^r(\mathbb{R})}^{2r},$$

which proves (2.12). In case $r = 1$, taking $\rho = 2$ as above, we obtain, for any $0 < \epsilon < t < T$,

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C(\nu)(t-\epsilon)^{-\frac{1}{4}} \|u(\cdot, \epsilon)\|_{L^2(\mathbb{R})} \leq C(\nu)(t-\epsilon)^{-\frac{1}{4}} \|u(\cdot, \epsilon)\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \|u(\cdot, \epsilon)\|_{L^1(\mathbb{R})}^{\frac{1}{2}}.$$

From Lemma 3.1, we deduce

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C(\nu)(t-\epsilon)^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R})},$$

and we conclude as $\epsilon \rightarrow 0$.

If $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$, we apply the estimates on (ϵ, T) and go to the limit as $\epsilon \rightarrow 0$. ■

Remark 4.17 As a consequence, for any $k \geq 1$, and for example $q \neq N, N \neq 2$,

$$\|u(\cdot, t)\|_{L^{kr}(\mathbb{R}^N)} \leq Ct^{-\frac{\sigma_{r,q,N}}{k'}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{\varpi_{r,q,N}}{k'} + \frac{1}{k}}, \quad (4.29)$$

$$\|u(\cdot, t)\|_{L^{kr}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2rk'}} \|u_0\|_{L^r(\mathbb{R}^N)}, \quad \text{if } \nu > 0. \quad (4.30)$$

Indeed it follows from (4.13) and (2.10), (2.12) by interpolation:

$$\|u(\cdot, t)\|_{L^{kr}(\mathbb{R}^N)} \leq \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^{1/k'} \|u(\cdot, t)\|_{L^r(\mathbb{R}^N)}^{1/k}.$$

Remark 4.18 If $q \leq 2$, then $u \in C^{2,1}(Q_{\mathbb{R}^N, T})$, thus we do not need to introduce the regularization by u_ε ; we only need to introduce $u + \delta$, when $r > 1$ and make $\delta \rightarrow 0$.

Remark 4.19 Up to now, the decay estimate (4.13) and the L^∞ estimate of u were proved for $u_0 \in C_b(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, and for the **unique bounded solution** u of problem (2.5), and based on the estimate (2.7) given in [15, Theorem 5.6]; indeed from the classical inequality

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C(N, r) \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{N}{N+r}} \|u(\cdot, t)\|_{L^r(\mathbb{R}^N)}^{\frac{r}{N+r}},$$

and (2.7), there holds, with $C = C(N, q, r)$,

$$\begin{aligned} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &\leq Ct^{-\frac{N}{q(N+r)}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{N}{q(N+r)}} \|u(\cdot, t)\|_{L^r(\mathbb{R}^N)}^{\frac{r}{N+r}}, \\ \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &\leq Ct^{-\sigma_{r,q,N}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\varpi_{r,q,N}}. \end{aligned} \quad (4.31)$$

4.4 Further estimates and convergence results for $q \leq 2$.

Here we consider the case $1 < q \leq 2$. From the L^∞ estimates above, and the interior regularity of u , we deduce new local estimates and convergence results:

Corollary 4.20 Assume $1 < q \leq 2$.

(i) Any nonnegative weak L^r_{loc} solution (resp. \mathcal{M}_{loc} solution) u of problem (2.5) with initial data $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$ (resp. $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$) satisfies $u \in C^{2,1}(Q_{\mathbb{R}^N, T}) \cap L^\infty_{loc}((0, T); C_b(\mathbb{R}^N))$.

(ii) Let $(u_{0,n})$ be any bounded sequence in $L^r(\mathbb{R}^N)$, $r \geq 1$ (resp. in $\mathcal{M}_b^+(\mathbb{R}^N)$). For any $n \in \mathbb{N}$, let u_n be any nonnegative weak L^r_{loc} solution (resp. \mathcal{M}_{loc} solution) of problem (2.5) with initial data $u_{0,n}$. Then one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N, T})$ to a weak solution u of (1.1) in $Q_{\mathbb{R}^N, T}$.

Proof. From [16, Theorem 2.9] there exists $\gamma \in (0, 1)$ such that for any nonnegative weak solution of equation (1.1) u in $Q_{\mathbb{R}^N, T}$ and any ball $B_R \subset \mathbb{R}^N$, and $0 < s < \tau < T$,

$$\|u\|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(Q_{B_R, s, \tau})} \leq C\Phi(\|u\|_{L^\infty(Q_{B_{2R}, \frac{s}{2}, \tau})}).$$

where $C = C(N, q, R, s, \tau)$ and Φ is a continuous increasing function. From estimates (2.10), we deduce that $u \in L_{loc}^\infty((0, T); C_b(\mathbb{R}^N))$ and

$$\|u\|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(Q_{B_R, s, \tau})} \leq C\Phi(\|u_0\|_{L^r(\mathbb{R}^N)}), \quad (\text{resp. } \|u\|_{C^{2+\gamma, 1+\frac{\gamma}{2}}(Q_{B_R, s, \tau})} \leq C\Phi(\int_{\mathbb{R}^N} du_0)) \quad (4.32)$$

and the conclusions follow. \blacksquare

We also deduce global gradient estimates in \mathbb{R}^N :

Corollary 4.21 *Assume $\nu > 0$, $1 < q \leq 2$. (i) Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$. Then any weak L_{loc}^r solution u of problem (2.5) satisfies for $q \neq N$*

$$\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\vartheta_{r, q, N}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\varkappa_{r, q, N}}, \quad (4.33)$$

$$\vartheta_{r, q, N} = \frac{N+r}{rq + N(q-1)}, \quad \varkappa_{r, q, N} = \frac{r}{rq + N(q-1)};$$

and $|\nabla u|^q \in L_{loc}^\infty((0, T); L^r(\mathbb{R}^N))$, and

$$\int_{\mathbb{R}^N} |\nabla u(\cdot, t)|^{qr} dx \leq Ct^{-r(\frac{q}{2} + \sigma_{r, q, N}(q-1))} \|u_0\|_{L^r(\mathbb{R}^N)}^{(1 + \varpi_{r, q, N}(q-1))r} \quad (4.34)$$

where $\sigma_{r, q, N}$, $\varpi_{r, q, N}$ are defined at (2.11), and $C = C(N, q, r, \nu)$. For $N \neq 2$, then

$$\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{1}{q}(\frac{N}{2r} + 1)} \|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{1}{q}}; \quad (4.35)$$

$$\int_{\mathbb{R}^N} |\nabla u(\cdot, t)|^{qr} dx \leq Ct^{-r(\frac{q}{2} + \frac{N}{2r}(q-1))} \|u_0\|_{L^r(\mathbb{R}^N)}^{qr}. \quad (4.36)$$

If $N = 2$, estimates hold up to an $\varepsilon > 0$. Moreover if $q < 2$, u is a pointwise mild solution.

(ii) Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then any weak \mathcal{M}_{loc} solution of (2.5) satisfies the same estimates as in case $r = 1$, with $\|u_0\|_{L^r(\mathbb{R}^N)}$ replaced by $\int_{\mathbb{R}^N} du_0$.

Proof. (i) Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$. Then for any $\epsilon > 0$, $u(\cdot, \epsilon) \in C_b(\mathbb{R}^N)$, from Corollary 4.20. From [29], u is the unique solution v such that $v \in C^{2,1}(\mathbb{R}^N \times (\epsilon, T)) \cap C_b(\mathbb{R}^N \times [\epsilon, T])$, and $v(\cdot, \epsilon) = u(\cdot, \epsilon)$; since $v \in C_b^2(\mathbb{R}^N \times (\epsilon, T))$, we deduce that $u \in C_b^2(\mathbb{R}^N \times (0, T))$; and for any $\epsilon \leq t < T$,

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u(\cdot, \epsilon)\|_{L^\infty(\mathbb{R}^N)}, \quad \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla u(\cdot, \epsilon)\|_{L^\infty(\mathbb{R}^N)},$$

and from (2.7),

$$|\nabla u(\cdot, t)|^q \leq C(q)(t - \epsilon)^{-1} u(\cdot, t), \quad \text{a.e. in } \mathbb{R}^N. \quad (4.37)$$

From the decay estimates, we also have $\|u(\cdot, \epsilon)\|_{L^r(\mathbb{R}^N)} \leq \|u_0\|_{L^r(\mathbb{R}^N)}$. And $u(\cdot, \epsilon) \in \tilde{L}^{\tilde{r}}(\mathbb{R}^N)$ for any $\tilde{r} \in [r, \infty]$, and $u \in C([\epsilon, T]; \tilde{L}^{\tilde{r}}(\mathbb{R}^N))$. Going to the limit in (4.37) as $\epsilon \rightarrow 0$, we deduce (4.33) from (2.10), and (4.35) from (2.12), if $q \neq N$ or $N \neq 2$. Moreover $|\nabla u|^q \in L_{loc}^\infty((0, T); L^r(\mathbb{R}^N))$, since

$$\|\nabla u(\cdot, t)\|_{L^{qr}(\mathbb{R}^N)} \leq C(q)t^{-\frac{1}{q}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{1}{q}}.$$

More precisely we get from estimate (2.6),

$$\|\nabla(u^{\frac{1}{q'}}(\cdot, t))\|_{L^\infty(\mathbb{R}^N)} \leq C(t - \epsilon)^{-\frac{1}{2}} \|u(\cdot, \epsilon)\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{q'}}$$

with $C = C(q, \nu)$; then from estimate (2.12), for any $t \in (0, T)$, with other constants $C = C(q, \nu)$,

$$\|\nabla(u^{\frac{1}{q'}}(\cdot, t))\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{1}{2}} \|u(\cdot, \frac{t}{2})\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{q'}}$$

$$|\nabla u(\cdot, t)|^q \leq Ct^{-\frac{q}{2}} \|u(\cdot, \frac{t}{2})\|_{L^\infty(\mathbb{R}^N)}^{q-1} u(\cdot, t),$$

then from estimate (2.10) we get

$$\int_{\mathbb{R}^N} |\nabla u(\cdot, t)|^{qr} dx \leq C \|u_0\|_{L^r(\mathbb{R}^N)}^{\varpi_{r,q,N}(q-1)r} t^{-r(\frac{q}{2} + \sigma_{r,q,N}(q-1))} \int_{\mathbb{R}^N} u(\cdot, t)^r dx;$$

then (4.34) follows. And (4.36) follows from (2.12). If $N = 2$, in particular if $q = N$, the same estimates hold up to an $\varepsilon > 0$, from (2.10) and (2.12).

Next we prove that u is a pointwise mild solution as $q < 2$. From [29, Theorem 6], $u(\cdot, t) \in C_b^2(\mathbb{R}^N)$ for any $t \in (\epsilon, T)$, in particular $u(\cdot, 2\epsilon) \in C_b^2(\mathbb{R}^N)$, then for any $t \geq \epsilon$, and any $x \in \mathbb{R}^N$,

$$u(x, t) = e^{(t-2\epsilon)\Delta} u(x, 2\epsilon) - \int_{2\epsilon}^t \int_{\mathbb{R}^N} g(x-y, t-s) |\nabla u(y, s)|^q dy ds, \quad (4.38)$$

see for example [7, Proposition 4.2]. But $u(x, 2\epsilon)$ converges to u_0 in $L^r(\mathbb{R}^N)$, and then $e^{(t-2\epsilon)\Delta} u(\cdot, \epsilon)$ converges to $e^{t\Delta} u_0$ in $L^r(\mathbb{R}^N)$. Then we can go to the limit as $\epsilon \rightarrow 0$ in (4.38), for a.e. $x \in \mathbb{R}^N$: the integral is convergent, then the conclusion follows.

(ii) For Theorem 4.15, we have $u(\cdot, t) \in L^1(\mathbb{R}^N)$ for $t \geq \epsilon > 0$, which gives from (i)

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C(t - \epsilon)^{-\sigma_{1,q,N}} \|u(\cdot, \epsilon)\|_{L^1(\mathbb{R}^N)}^{\varpi_{1,q,N}} \leq C(t - \epsilon)^{-\sigma_{1,q,N}} \left(\int_{\mathbb{R}^N} du_0 \right)^{\varpi_{1,q}}.$$

As $\epsilon \rightarrow 0$, we obtain (4.33), (4.35), (4.34) and (4.36) hold with $r = 1$ and $\|u_0\|_{L^1(\mathbb{R}^N)}$ replaced by

$\int_{\mathbb{R}^N} du_0$. And

$$\|\nabla u(\cdot, t)\|_{L^q(\mathbb{R}^N)} \leq Ct^{-\frac{1}{q}} \left(\int_{\mathbb{R}^N} du_0 \right)^{\frac{1}{q}},$$

thus $|\nabla u|^q \in L_{loc}^\infty((0, T); L^1(\mathbb{R}^N))$. ■

Remark 4.22 *As a consequence, under the assumptions of Corollary 4.21, there holds $u(\cdot, t) \in C_b^1(\mathbb{R}^N)$, for any $t \in (0, T)$, then u can be extended to a global solution of problem (2.5) on $Q_{\mathbb{R}^N, \infty}$, see for example [36].*

4.5 Existence and uniqueness results for $q \leq 2$

Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$. We first consider the "subcritical" case

$$1 < q < \frac{N+2r}{N+r}, \quad \text{equivalently} \quad q < 2 \text{ and } r > \frac{N(q-1)}{2-q}. \quad (4.39)$$

Theorem 4.23 *Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$. Suppose (4.39), and $\nu > 0$. Then any weak L_{loc}^r solution u of problem (2.5) satisfies*

$$|\nabla u|^q \in L_{loc}^1([0, T]; L^r(\mathbb{R}^N)). \quad (4.40)$$

And

$$u \text{ is a weak } L_{loc}^r \text{ solution} \iff u \text{ is a mild } L^r \text{ solution.}$$

Proof. Let u be any weak L_{loc}^r solution. Then from (4.34),

$$\int_0^\tau \|\nabla u(\cdot, t)\|_{L^{qr}(\mathbb{R}^N)}^q dt = \int_0^\tau \left(\int_{\mathbb{R}^N} |\nabla u(\cdot, t)|^{qr} dx \right)^{\frac{1}{r}} dt \leq C \int_0^\tau t^{-(\frac{q}{2} + \sigma_{r,q,N}(q-1))} dt$$

with $C = C_q \|u_0\|_{L^r(\mathbb{R}^N)}^{(1+\varpi_{r,q,N}(q-1))r}$, and (4.39) is equivalent to $q/2 + \sigma_{r,q,N}(q-1) < 1$. Since $\nu > 0$, the estimate (4.36) leads to the same conclusion, because (4.39) is also equivalent to $q/2 + (q-1)N/2r < 1$. Then (4.40) holds. Moreover from Corollary 4.21, u is a mild pointwise solution:

$$u(\cdot, t) = e^{t\Delta} u_0(\cdot) - \nu \int_0^t \int_{\mathbb{R}^N} g(x-y, t-s) |\nabla u(y, s)|^q dy ds. \quad (4.41)$$

Otherwise $u \in C([0, T]; L^r(\mathbb{R}^N))$ from Theorem 4.14, and $f = |\nabla u|^q \in L_{loc}^1([0, T]; L^r(\mathbb{R}^N))$, thus the relation (4.41) holds in $L^r(\mathbb{R}^N)$,

$$u(\cdot, t) = (e^{t\Delta} u_0) - \nu \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q(s) ds \quad \text{in } L^r(\mathbb{R}^N), \quad (4.42)$$

that means u is a mild L^r solution. The converse is clear. \blacksquare

Next we deduce the uniqueness results of Theorem 2.3.

Theorem 4.24 *Let $u_0 \in L^r(\mathbb{R}^N)$. Assume (4.39) or $q = 2$, and $\nu > 0$. Then there exists a unique weak L_{loc}^r solution u of problem (2.5). In the first case, $u \in C((0, T); W^{1,qr}(\mathbb{R}^N))$.*

Proof. (i) Case $1 < q < (N+2r)/(N+r)$. From [15, Theorem 2.1], there exists a mild L^r solution u , and it is unique in the class of mild L^r solutions such that $u \in L_{loc}^\infty((0, T); W^{1,qr}(\mathbb{R}^N))$, see [15, Lemma 2.2 and Remark 2.5]. Then u is a L_{loc}^r solution. Let v be any weak L_{loc}^r solution, thus u is a mild L^r solution, from Theorem 4.23. From Theorem 4.14, Corollary 4.20, and Theorem 4.23, $v \in L^\infty((0, T); L^r(\mathbb{R}^N)) \cap L_{loc}^\infty((0, T); C_b(\mathbb{R}^N))$, and $|\nabla v| \in L_{loc}^\infty((0, T); L^{qr}(\mathbb{R}^N))$ from Theorem 4.23. Then $v \in L_{loc}^\infty((0, T); W^{1,qr}(\mathbb{R}^N))$, then $v = u$, and we reach the conclusion. Moreover $u \in C((0, T); W^{1,qr}(\mathbb{R}^N))$, from [15, Theorem 2.1].

(ii) Case $q = 2$. From [15, Theorem 4.2] there exists a unique solution u such that $u \in C([0, T]; L^r(\mathbb{R}^N)) \cap u \in C^{2,1}((Q_{\mathbb{R}^N, \infty}))$ solution of (1.1) at each point. Then it is a weak L_{loc}^r solution. Reciprocally any weak L_{loc}^r solution u satisfies the conditions above, from Theorem 4.14 and [16]. \blacksquare

Theorem 4.25 Assume $1 < q < (N + 2)/(N + 1)$, $\nu > 0$. Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then there exists a unique weak \mathcal{M}_{loc} solution of problem (2.5).

Proof. The existence of a weak semi-group solution was obtained in [12] by approximation. The existence of a mild \mathcal{M} solution was proved in [15, Theorem 2.2], and the two notions are equivalent from Lemma 4.10. In any case the solution is a weak \mathcal{M}_{loc} solution. Next consider any solution \mathcal{M}_{loc} solution u . Then $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$ for any $t \in (\epsilon, T)$ by applying Theorem 4.16 on $(\epsilon/2, T)$. Then again we deduce $u(\cdot, \epsilon) \in C_b(\mathbb{R}^N)$, and then (4.37) holds. From Theorem 4.14 we still obtain that $u \in L_{loc}^\infty((0, T); W^{1,q}(\mathbb{R}^N))$. And from the uniqueness on (ϵ, T) , we have $u \in C((\epsilon, T); W^{1,q}(\mathbb{R}^N))$ from Theorem 4.24. Then $u \in C((0, T); W^{1,q}(\mathbb{R}^N))$. And u satisfies (4.26), from Theorem 4.15. Then u is a weak semi-group solution, thus a mild \mathcal{M} solution from Lemma 4.10. Therefore u belongs to the class of uniqueness of [15, Theorem 2.2]. We can also prove the uniqueness directly: if u_1, u_2 are two solutions, they are mild \mathcal{M} solutions, thus

$$(u_1 - u_2)(\cdot, t) = \nu \int_0^t e^{(t-s)\Delta} (|\nabla u_1(\cdot, s)|^q - |\nabla u_2(\cdot, s)|^q) ds$$

and we know that $|\nabla u_j|^q \in C((0, T); L^1(\mathbb{R}^N))$, hence

$$\begin{aligned} \|\nabla(u_1 - u_2)(\cdot, t)\|_{L^q(\mathbb{R}^N)} &\leq \nu \int_0^t \left\| \nabla(e^{(t-s)\Delta} \cdot) \right\|_{L^1(\mathbb{R}^N)} \| |\nabla u_1(\cdot, s)|^q - |\nabla u_2(\cdot, s)|^q \|_{L^{qr}(\mathbb{R}^N)} ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}} \max_{j=1,2} \|\nabla u_j(\cdot, s)\|_{L^\infty(\mathbb{R}^N)}^{q-1} \|\nabla(u_1 - u_2)(\cdot, s)\|_{L^q(\mathbb{R}^N)} ds \\ &\leq C \int_0^t (t-s)^{-\frac{1}{2}} s^{-(q-1)\vartheta_{1,q,N}} \|\nabla(u_1 - u_2)(\cdot, s)\|_{L^{qr}(\mathbb{R}^N)} ds. \end{aligned}$$

thus we can apply the singular Gronwall Lemma when $(q-1)\vartheta_{1,q,N} < 1/2$, which means precisely $q < (N+2)/(N+1)$. Then $\nabla(u_1 - u_2)(\cdot, t) = 0$ in $L^q(\mathbb{R}^N)$, hence $u_1 = u_2$. \blacksquare

Finally we give a short proof of the existence result of [15, Theorem 4.1].

Proposition 4.26 Let $\nu > 0$, $1 < q < 2$. For any nonnegative $u_0 \in L^r(\mathbb{R}^N)$, $r \geq 1$, there exists a mild pointwise solution u of problem (2.5), and $u \in C([0, T]; L^r(\mathbb{R}^N))$.

Proof. Let $u_{0,n} = \min(u_0, n)$. Then $u_{0,n} \in L^\rho(\mathbb{R}^N)$ for any $\rho \geq r$. We choose $\rho > N(q-1)/(2-q)$, that means $q < (N+2\rho)/(N+\rho)$. From [15, Theorem 2.1], there exists a mild L^ρ solution u_n with initial data $u_{0,n}$, and $u_n \in C((0, T); C_b^2(\mathbb{R}^N)) \cap C^{2,1}(Q_{\mathbb{R}^N, T})$. The sequence (u_n) is nondecreasing from the comparison principle, and $u_n(\cdot, t) \leq e^{t\Delta} u_0 \leq Ct^{-N/2r} \|u_0\|_{L^r(\mathbb{R}^N)}$. From Corollary 4.20, (u_n) converges in $C_{loc}^{2,1}(Q_{\mathbb{R}^N, T})$ to a weak solution u of (1.1) in $Q_{\mathbb{R}^N, T}$, and $u(\cdot, t) \leq e^{t\Delta} u_0$. Moreover $(|\nabla u_n|^q)$ is bounded in $L_{loc}^1([0, T]; L_{loc}^1(\mathbb{R}^N))$: indeed for any $\xi \in \mathcal{D}(\mathbb{R}^N)$, with values in $[0, 1]$, and any $0 < s < t < T$,

$$\begin{aligned} \int_{\mathbb{R}^N} u_n(t, \cdot) \xi^{q'} dx + \nu \int_s^t \int_{\mathbb{R}^N} |\nabla u_n|^q \xi^{q'} dx &\leq -q' \nu \int_s^t \int_{\mathbb{R}^N} \xi^{\frac{1}{q-1}} \nabla u_n \cdot \nabla \xi dx + \int_{\mathbb{R}^N} u_n(s, \cdot) \xi^{q'} dx \\ &\leq \frac{\nu}{2} \int_s^t \int_{\mathbb{R}^N} |\nabla u_n|^q \xi^{q'} dx + Ct \int_{\mathbb{R}^N} |\nabla \xi|^{q'} dx + \int_{\mathbb{R}^N} u_n(s, \cdot) \xi^{q'} dx, \end{aligned}$$

and $u_n \in C([0, T]; L^p(\mathbb{R}^N))$; thus we can go to the limit as $s \rightarrow 0$:

$$\int_{\mathbb{R}^N} u_n(t, \cdot) \xi^{q'} dx + \frac{1}{2} \int_s^t \int_{\mathbb{R}^N} |\nabla u_n|^q \xi^{q'} dx \leq Ct \int_{\mathbb{R}^N} |\nabla \xi|^{q'} dx + \int_{\mathbb{R}^N} u_0 \xi^{q'} dx.$$

Thus $|\nabla u|^q \in L^1_{loc}([0, T]; L^1_{loc}(\mathbb{R}^N))$, hence, from [16, Proposition 2.15], u admits a trace as $t \rightarrow 0$: there exists a Radon measure μ_0 in \mathbb{R}^N , such that $u(\cdot, t)$ converges weakly* to μ_0 . Otherwise $e^{t\Delta} u_0$ converges to u_0 in $L^r(\mathbb{R}^N)$, thus $\mu_0 \in L^1_{loc}(\mathbb{R}^N)$ and $0 \leq \mu_0 \leq u_0$; and $u_n \leq u$, thus $u_{0,n} \leq \mu_0$, hence $\mu_0 = u_0$. Moreover there exists a function $g \in L^r(\mathbb{R}^N)$ such that $u(\cdot, t) \leq g$ for small t . Then the nonnegative function $e^{t\Delta} u_0 - u(\cdot, t)$ converges weakly* to 0, and then in $L^1_{loc}(\mathbb{R}^N)$. Hence $u(\cdot, t)$ converges to u_0 in $L^1_{loc}(\mathbb{R}^N)$, then in $L^r(\Omega)$ from the dominated convergence theorem. Thus $u \in C([0, T]; L^r(\mathbb{R}^N))$. In particular u is a weak L^r_{loc} solution, then a pointwise mild solution, from Corollary 4.21. \blacksquare

Remark 4.27 *The uniqueness of the solution is still an open problem when $u_0 \in L^r(\mathbb{R}^N)$ and $q \geq (N + 2r)/(N + r)$.*

4.6 More decay estimates for $q < (N + 2r)/(N + r)$

Here, we exploit theorem 4.14 to obtain a better decay estimate of the L^r norm when $u_0 \in L^r(\mathbb{R}^N)$ in the subcritical case (4.39), which appears to be new for $r > 1$. In case $r = 1$ we find again the result of [3], proved *under the assumption that the energy relation (4.25) holds*.

Theorem 4.28 *Let $r \geq 1$ and assume (4.39), $\nu > 0$. Let u be any non-negative weak r solution of problem (2.5) in $Q_{\mathbb{R}^N, \infty}$, with initial data $u_0 \in L^r(\mathbb{R}^N)$. Then there exists $C = C(N, q, r)$ such that, for any $t > 0$,*

$$\int_{\mathbb{R}^N} u^r(\cdot, t) dx \leq C \left(\int_{\{|x| > \sqrt{t}\}} u_0^r(x) dx + t^{-\frac{ar-N}{2}} \right), \quad a = \frac{2-q}{q-1}. \quad (4.43)$$

As a consequence, $\lim_{t \rightarrow \infty} \|u(t)\|_{L^r(\mathbb{R}^N)} = 0$ and

$$r \int_0^\infty \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q dx dt + r(r-1)\nu \int_0^\infty \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 dx dt = \int_{\mathbb{R}^N} u_0^r dx.$$

Proof. We still consider $v = u^b$ with $b = (q-1+r)/q < r$, and set $E(s) = \int_{\mathbb{R}^N} u^r(\cdot, s) dx$. Then $E \in W^{1,1}((0, T))$, from the energy relation (4.14), and for almost any $s \in (0, T)$,

$$E'(s) = -r(r-1)\nu \int_{\mathbb{R}^N} |\nabla u|^2 u^{r-2}(\cdot, s) dx - \int_{\mathbb{R}^N} |\nabla u|^q u^{r-1}(\cdot, s) dx \leq 0.$$

Next, we set $E = E_1 + E_2$ with

$$E_1(s) = \int_{\{|x| < 2R\}} u^r(x, s) dx, \quad E_2(s) = \int_{\{|x| \geq 2R\}} u^r(x, s) dx.$$

From the Gagliardo-Nirenberg inequality (4.23), we obtain successively, with $C = C(N, q, r)$,

$$\begin{aligned} E_1(s) &= \int_{\{|x| < 2R\}} v^{\frac{r}{b}}(x, s) dx \leq \left(\int_{\{|x| < 2R\}} v^q(x, s) dx \right)^{\frac{r}{bq}} (2R)^{1 - \frac{r}{bq}} \\ &\leq C \|\nabla v(s)\|_{L^q(\mathbb{R}^N)}^{\frac{kr}{b}} \|v(s)\|_{L^{r/b}(\mathbb{R}^N)}^{\frac{(1-k)r}{b}} R^{N(1 - \frac{r}{bq})} \\ &\leq \frac{1}{2} \|v(s)\|_{L^{r/b}(\mathbb{R}^N)}^{\frac{r}{b}} + C \|\nabla v(s)\|_{L^q(\mathbb{R}^N)}^{\frac{kr}{b}} R^{\frac{N}{k}(1 - \frac{r}{bq})}, \end{aligned}$$

thus

$$E(s) \leq C (\|\nabla v(s)\|_{L^q(\mathbb{R}^N)}^{\frac{r}{b}} R^{\frac{N}{k}(1 - \frac{r}{bq})} + 2E_2(s)). \quad (4.44)$$

Let $\eta \in \mathcal{D}(\mathbb{R}^N)$ with values in $[0, 1]$, such that $\varphi = 1$ in B_1 , with support in $\overline{B_2}$, and set $\eta = 1 - \varphi$, and $\varphi_l(x) = \varphi(\frac{x}{l})$, $\eta_R(x) = \eta(\frac{x}{R})$. Observe that our assumption on q implies $q' > N/r$. As in the first step of theorem 4.14, we obtain for any $0 < \sigma < s < t < T$, and $l > 2R$,

$$\left(\int_{\mathbb{R}^N} u^r(\cdot, s) \varphi_l^\lambda \eta_R^\lambda dx \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u^r(\cdot, \sigma) \varphi_l^\lambda \eta_R^\lambda dx \right)^{\frac{1}{r}} + C(s - \sigma)(R^{\frac{N}{r} - q'} + l^{\frac{N}{r} - q'}), \quad (4.45)$$

with $\lambda = rq'$, and $C = C(N, q, r, \eta)$. As $\sigma \rightarrow 0$ and $l \rightarrow \infty$. we deduce

$$\left(\int_{\mathbb{R}^N} u^r(x, s) \eta_R dx \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u_0^r(x) \eta_R dx \right)^{\frac{1}{r}} + CsR^{\frac{N}{r} - q'}.$$

Taking $R = \sqrt{t}$, and setting

$$\rho = r + \frac{N - rq'}{2} = \frac{(N + 2r) - q(N + r)}{2(q - 1)} = \frac{ar - N}{2},$$

we find, with a constant C as above,

$$E_2(s) \leq A(t) = C \left(\int_{\{|x| > \sqrt{t}\}} u_0^r(x) dx + t^{-\rho} \right),$$

Next, we consider $F(s) = E(s) - 2A(t)$. If there exists $t_0 \in (0, t)$ such that $F(t_0) \leq 0$, then $F(s) \leq 0$, $\forall s \in (t_0, t)$; thus $E(t) \leq 2A(t)$, by continuity, hence (4.43) holds. Next assume that $F(s) > 0$, for any $s \in (0, t)$. Since

$$-F'(s) \geq \nu \int_{\mathbb{R}^N} |\nabla u|^q u^{r-1}(x, s) dx = \nu \int_{\mathbb{R}^N} |\nabla v(x, s)|^q dx, \quad (4.46)$$

we find $F(s) \leq C(-F'(s))^{r/bq} t^{(1-r/bq)N/2k}$ from (4.44). By integration we get

$$C(t - s)t^{-\frac{N}{2k}(1 - \frac{r}{bq})} \leq F(t)^{-\frac{q-1}{r}} - F(s)^{\frac{q-1}{r}}.$$

As $s \rightarrow 0$ we deduce that $F(t) \leq Ct^{-\rho}$, since $\rho = r/(q - 1) - N/2k$, and (4.43) still holds. \blacksquare

Remark 4.29 *The case $r = 1$ has been the object of many works, assuming that $u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. There holds*

$$\lim_{t \rightarrow \infty} \|u(t)\|_{L^1(\mathbb{R}^N)} = 0 \iff q \leq (N+2)/(N+1),$$

see [2], [12], [4], [28]. When $q < (N+2)/(N+1)$, the absorption plays a role in the asymptotics. From [10], if $\lim_{|x| \rightarrow \infty} |x|^a u_0(x) = 0$, where $a = (2-q)/(q-1)$, then $u(\cdot, t)$ converges as $t \rightarrow \infty$ to the very singular solution constructed in [34], [13]; then $\int_{\mathbb{R}^N} u(\cdot, t) dx$ behaves like $t^{-(a-N)/2}$ for large t , and estimate (4.43) is sharp. When $q > (N+2)/(N+1)$, and $u_0 \in L^1(\mathbb{R}^N)$, then $u(\cdot, t)$ behaves as the fundamental solution of heat equation, see [10].

Our result is new when $u_0 \in L^r(\mathbb{R}^N)$, $r > 1$. When $q > (N+2)/(N+1)$, and u_0 is bounded and behaves like $|x|^{-b}$ as $|x| \rightarrow \infty$ with $b \in (a, N)$, it has been shown that $u(\cdot, t)$ behaves as the selfsimilar solution of the heat equation with initial data $|x|^{-b}$, see [17]. In that case $u_0 \in L^r(\mathbb{R}^N)$ for any $r > N/b$ and $\int_{\mathbb{R}^N} u^r(\cdot, t) dx$ behaves like $t^{-(br-N)/2}$. Thus (4.43) is sharp as $b \rightarrow a$.

5 The Dirichlet problem in $Q_{\Omega, T}$

Here we study equation (1.1) in case of a regular bounded domain Ω , with Dirichlet conditions on $\partial\Omega \times (0, T)$, with $\nu > 0$; by homothety we can assume $\nu = 1$:

$$(D_{\Omega, T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega, T}, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (5.1)$$

As in section 4, we study the problem with rough initial data, and introduce different notions of solutions.

5.1 Solutions of the heat equation with L^1 data

The regularization method used at Section 4 does not provide estimates up to the boundary. In this section we use another argument: the notion of *entropy solution*, introduced in [35], for the problem

$$\begin{cases} u_t - \Delta u = f, & \text{in } Q_{\Omega, s, \tau}, \\ u = 0 & \text{on } \partial\Omega \times (s, \tau), \\ u(\cdot, s) = u_s \geq 0 \end{cases} \quad (5.2)$$

when f and u_s are integrable, that we recall now. For any $k > 0$ and $\theta \in \mathbb{R}$, we define as usual the truncation function T_k and a primitive Θ_k by

$$T_k(\theta) = \max(-k, \min(k, \theta)), \quad \Theta_k(s) = \int_0^s T_k(\theta) d\theta. \quad (5.3)$$

Definition 5.1 *Let $s, \tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega, s, \tau})$ and $u_s \in L^1(\Omega)$. A function $u \in C([s, \tau]; L^1(\Omega))$ is an entropy solution of the problem (5.2) if $T_k(u) \in L^2((s, \tau); W_0^{1,2}(\Omega))$ for any $k > 0$, and*

$$\begin{aligned} \int_{\Omega} \Theta_k(u - \varphi)(\cdot, \tau) dx - \int_{\Omega} \Theta_k(u_s - \varphi(\cdot, s)) dx + \int_s^\tau \langle \varphi_t, T_k(u - \varphi) \rangle dt \\ + \int_s^\tau \int_{\Omega} (\nabla u \cdot \nabla T_k(u - \varphi) - f T_k(u - \varphi)) dx dt \leq 0 \end{aligned} \quad (5.4)$$

for any $\varphi \in L^2((s, \tau); W^{1,2}(\Omega)) \cap L^\infty(Q_{\Omega, \tau})$ such that $\varphi_t \in L^2((s, \tau); W^{-1,2}(\Omega))$.

Other notions of solutions have been used for this problem, see [8], recalled below. In fact they are equivalent: here $e^{t\Delta}$ denotes the semi-group of the heat equation with Dirichlet conditions acting on $L^1(\Omega)$,

Lemma 5.2 *Let $-\infty < s < \tau < \infty$, $f \in L^1(Q_{\Omega,s,\tau})$, $u_s \in L^1(\Omega)$ and $u \in C([s, \tau]; L^1(\Omega))$, $u(\cdot, s) = u_s$. Then the three properties are equivalent:*

(i) $u \in L^1((s, \tau); W_0^{1,1}(\Omega))$, such that

$$u_t - \Delta u = f, \quad \text{in } \mathcal{D}'(Q_{\Omega,s,\tau}); \quad (5.5)$$

(ii) u is a mild solution of (5.2), that means, for any $t \in [s, \tau]$,

$$u(\cdot, t) = e^{(t-s)\Delta} u_s + \int_s^t e^{(t-\sigma)\Delta} f(\sigma) d\sigma \quad \text{in } L^1(\Omega); \quad (5.6)$$

(iii) u is an entropy solution of (5.2).

Such a solution exists, is unique, and will be called weak solution of (5.2).

Proof. It follows from the existence and uniqueness of the solutions of (i) from [5, Lemma 3.4], as noticed in [8], and of the entropy solutions, see [18]. \blacksquare

As a consequence, when u is bounded, we can admit test functions of the form u^α :

Lemma 5.3 *Let $s, \tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and u be any nonnegative **bounded** weak solution in $Q_{\Omega,s,\tau}$ of (5.2).*

Then, for any $\alpha > 0$, there holds $u^{\alpha-1} |\nabla u|^2 \in L^1(Q_{\Omega,s,\tau})$ and

$$\frac{1}{\alpha+1} \int_{\Omega} u^{\alpha+1}(\cdot, \tau) dx + \alpha \int \int_{Q_{\Omega,s,\tau}} u^{\alpha-1} |\nabla u|^2 dx dt = \frac{1}{\alpha+1} \int_{\Omega} u^{\alpha+1}(\cdot, s) dx + \int_s^\tau \int_{\Omega} f u^\alpha dx dt. \quad (5.7)$$

Proof. We have $u \in L^2((s, \tau); W_0^{1,2}(\Omega)) \cap L^\infty(Q_{\Omega,s,\tau})$, and $u_t \in L^2((s, \tau); W^{-1,2}(\Omega)) + L^1(Q_{\Omega,s,\tau})$. Then any function $\varphi \in L^2((s, \tau); W_0^{1,2}(\Omega)) \cap L^\infty(Q_{\Omega,s,\tau})$ is admissible in equation (5.5). In particular for any $\alpha > 0$, we can take $\varphi = M_{\alpha,\delta}(u) = (u + \delta)^\alpha - \delta^\alpha$, with $\delta > 0$. Integrating on $[s, \tau]$ we deduce that

$$\int_s^\tau \langle u_t, \varphi \rangle + \alpha \int \int_{Q_{\Omega,s,\tau}} (u + \delta)^{\alpha-1} |\nabla u|^2 dx dt = \int_s^\tau \int_{\Omega} f M_{\alpha,\delta}(u) dx dt.$$

Let $k > 0$ such that $\sup_{Q_{\Omega,s,\tau}} u \leq k$, thus $u = T_k(u)$. The function $\theta \mapsto M(\theta) = (T_k(\theta) + \delta)^\alpha - \delta^\alpha$ is continuous on \mathbb{R}^+ and piecewise C^1 such that $M(0) = 0$ and M' has a compact support. Denoting $\mathcal{M}_{\alpha,\delta}(r) = (u + \delta)^{\alpha+1}/(\alpha+1) - \delta^{\alpha+1}$, we can integrate by parts from [27, Lemma 7.1], and deduce that

$$\int_{\Omega} \mathcal{M}_{\alpha,\delta}(u)(\cdot, \tau) dx - \int_{\Omega} \mathcal{M}_{\alpha,\delta}(u)(\cdot, s) dx + \alpha \int \int_{Q_{\Omega,s,\tau}} (u + \delta)^{\alpha-1} |\nabla u|^2 dx dt = \int_s^\tau \int_{\Omega} f \mathcal{M}_{\alpha,\delta}(u) dx dt;$$

We can go to the limit as $\delta \rightarrow 0$ from the Fatou Lemma, and then from the dominated convergence theorem. Thus (5.7) holds for $\alpha > 0$. \blacksquare

Remark 5.4 *From [27], the notion of entropy solution of (5.2) is also equivalent to the notion of renormalized solution, that we develop in Section 6. Lemma 5.3 is a special case of a much more general property of the truncates when u is not necessarily bounded, see Lemma 6.3.*

5.2 Different notions of solutions of problem $(D_{\Omega,T})$

Definition 5.5 We say that u is a **weak solution** of the problem $(D_{\Omega,T})$ if $u \in C((0,T); L^1(\Omega)) \cap L^1_{loc}((0,T); W_0^{1,1}(\Omega))$, such that $|\nabla u|^q \in L^1_{loc}((0,T); L^1(\Omega))$ and u satisfies

$$u_t - \Delta u + |\nabla u|^q = 0, \quad \text{in } \mathcal{D}'(Q_{\Omega,T}). \quad (5.8)$$

Next we study the Cauchy-Dirichlet problem

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \geq 0 \end{cases} \quad (5.9)$$

with $u_0 \in L^r(\Omega)$, $r \geq 1$, or only $u_0 \in \mathcal{M}_b^+(\Omega)$. Here in any case $u_0 \in \mathcal{M}_b^+(\Omega)$.

Definition 5.6 If $u_0 \in L^r(\Omega)$, $r \geq 1$, we say that u is a **weak L^r solution** of problem (5.9) if it is a weak solution of $(D_{\Omega,T})$, such that the extension of u by u_0 at $t = 0$ satisfies $u \in C([0,T]; L^r(\Omega))$.

Definition 5.7 For any $u_0 \in \mathcal{M}_b^+(\Omega)$, we say that u is a **weak \mathcal{M} solution** of problem (5.9) if it is a weak solution of $(D_{\Omega,T})$, such that

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \psi dx = \int_{\Omega} \psi du_0, \quad \forall \psi \in C_b(\Omega). \quad (5.10)$$

Semi-group type solutions have been introduced in [8], see also [1]. For $u_0 \in \mathcal{M}_b^+(\Omega)$, we set $e^{t\Delta}u_0 = \int_{\Omega} g_{\Omega}(\cdot, y, t) du_0(y)$, where g_{Ω} is the heat kernel with Dirichlet conditions on $\partial\Omega$.

Definition 5.8 For any $u_0 \in \mathcal{M}_b^+(\Omega)$, a function u is a **mild solution** of problem (5.9) if $u \in C((0,T); L^1(\Omega))$, and $|\nabla u|^q \in L^1_{loc}([0,T]; L^1(\Omega))$ and

$$u(\cdot, t) = e^{t\Delta}u_0(\cdot) - \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^1(\Omega), \quad (5.11)$$

Remark 5.9 As it was shown in [8, p.1420], from Lemma 5.2,

$$u \text{ is a mild solution} \iff u \text{ is a weak } \mathcal{M} \text{ solution such that } |\nabla u|^q \in L^1_{loc}([0,T]; L^1(\Omega));$$

and then $u \in L^1_{loc}([0,T]; W_0^{1,1}(\Omega))$.

Remark 5.10 As in Remark 4.11, the definition of mild solution requires an integrability property of the gradient up to $t = 0$, namely $|\nabla u|^q \in L^1_{loc}([0,T]; L^1(\Omega))$. The definition of weak solution only assumes that $|\nabla u|^q \in L^1_{loc}((0,T); L^1(\Omega))$.

5.3 Decay and regularizing effect

Here Ω is bounded, then the situation is simpler than in \mathbb{R}^N : indeed we take benefit of the regularizing effect of the semi-group $e^{t\Delta}$ associated with the first eigenvalue λ_1 of the Laplacian, and also of the inclusion $L^r(\Omega) \subset L^1(\Omega)$.

Lemma 5.11 *Let $q > 1$, and $u_0 \in L^r(\Omega)$, $r \geq 1$. 1) Let u be any non-negative weak L^r solution of problem (5.9).*

(i) *Then $u(\cdot, t) \in L^\infty(\Omega)$ for any $t \in (0, T)$, and*

$$\|u(\cdot, t)\|_{L^r(\Omega)} \leq C e^{-\lambda_1 t} \|u_0\|_{L^r(\Omega)}, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C t^{-\frac{N}{2r}} e^{-\lambda_1 t} \|u_0\|_{L^r(\Omega)}. \quad (5.12)$$

(ii) *Moreover $|\nabla u|^q \in L^1_{loc}([0, T]; L^1(\Omega))$, and*

$$\int_{\Omega} u(\cdot, t) dx + \int_0^t \int_{\Omega} |\nabla u|^q dx dt \leq \int_{\Omega} u_0 dx. \quad (5.13)$$

If $r > 1$, then $u^{r-1}|\nabla u|^q \in L^1_{loc}([0, T]; L^1(\Omega))$ and $u^{r-2}|\nabla u|^2 \in L^1_{loc}([0, T]; L^1(\Omega))$, and

$$\frac{1}{r} \int_{\Omega} u^r(\cdot, t) dx + \int_0^t \int_{\Omega} u^{r-1} |\nabla u|^q dx dt + (r-1) \int_0^t \int_{\Omega} u^{r-2} |\nabla u|^2 dx dt = \frac{1}{r} \int_{\Omega} u_0^r dx, \quad (5.14)$$

As a consequence, $u^{q-1+r} \in L^1_{loc}([0, T]; W_0^{1,1}(\Omega))$.

2) *Let $u_0 \in \mathcal{M}_b^+(\Omega)$ and u be any non-negative weak \mathcal{M} solution of problem (5.9). Then (5.12) and (5.13) still hold as in case $u_0 \in L^1(\Omega)$, where the norm $\|u_0\|_{L^1(\Omega)}$ is replaced by $\int_{\Omega} du_0$. In particular u is a mild solution.*

Proof. 1) (i) Let $0 < \epsilon < \tau < T$. Since u is a weak solution of $(D_{\Omega, T})$, we can apply Lemma 5.2 with $f = -|\nabla u|^q$ in $Q_{\Omega, \epsilon, \tau}$. Thus u is a mild solution of the problem in $Q_{\Omega, \epsilon, \tau}$: for any $t \in [\epsilon, \tau]$,

$$u(\cdot, t) = e^{(t-\epsilon)\Delta} u(\cdot, \epsilon) - \int_{\epsilon}^t e^{(t-\sigma)\Delta} |\nabla u|^q d\sigma \quad \text{in } L^1(\Omega).$$

therefore $u(\cdot, t) \leq e^{(t-\epsilon)\Delta} u(\cdot, \epsilon)$. From our assumptions $u \in C([0, T]; L^r(\Omega))$, we deduce that $u(\cdot, t) \leq e^{t\Delta} u_0$ as $\epsilon \rightarrow 0$. Then (5.12) follows from the properties of the semi-group $e^{t\Delta}$.

(ii) The function u is bounded in $Q_{\Omega, s, \tau}$, thus from Lemma 5.3, for any $\rho > 1$,

$$\frac{1}{\rho} \int_{\Omega} u^\rho(\cdot, t) dx + \int_{\epsilon}^t \int_{\Omega} u^{\rho-1} |\nabla u|^q dx dt + (\rho-1) \int_{\epsilon}^t \int_{\Omega} u^{\rho-2} |\nabla u|^2 dx dt = \frac{1}{\rho} \int_{\Omega} u^\rho(\cdot, \epsilon) dx. \quad (5.15)$$

As $\rho \rightarrow 1$, we deduce that $|\nabla u|^q \in L^1(Q_{\Omega, \epsilon, \tau})$ from the Fatou Lemma, and

$$\int_{\Omega} u(\cdot, t) dx + \int_{\epsilon}^t \int_{\Omega} |\nabla u|^q dx dt \leq \int_{\Omega} u(\cdot, \epsilon) dx.$$

As $\epsilon \rightarrow 0$ we deduce that $|\nabla u|^q \in L^1(Q_{\Omega, \tau})$ and (5.13) holds. If $r > 1$, we can take $\rho = r$ in (5.15) and obtain (5.14) as $\epsilon \rightarrow 0$. Then $u^{q-1+r} \in L^1_{loc}([0, T]; W_0^{1,1}(\Omega))$ as in the case of \mathbb{R}^N .

2) The same estimates hold because $\lim_{\epsilon \rightarrow 0} \|u(\cdot, \epsilon)\|_{L^1(\Omega)} = \int_{\Omega} du_0$. ■

Theorem 5.12 *Let $q > 1$ and $u_0 \in L^r(\Omega)$, $r \geq 1$. 1) Let u be any non-negative weak L^r solution of problem (5.9). Then*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \begin{cases} Ct^{-\sigma_{r,q,N}} \|u_0\|_{L^r(\Omega)}^{\varpi_{r,q,N}}, & C = C(N, q, r), \quad \text{if } q \neq N, \\ C_\varepsilon t^{-(1+\varepsilon)\sigma_{r,N,N}} \|u_0\|_{L^r(\Omega)}^{(1+\varepsilon)\varpi_{r,q,N}}, \quad \forall \varepsilon > 0, \quad C_\varepsilon = C(N, q, r, \varepsilon), & \text{if } q = N, \end{cases} \quad (5.16)$$

where $\sigma_{r,q,N}, \varpi_{r,q,N}$ are given at (2.11).

2) Any non-negative weak solution u of $(D_{\Omega,T})$ satisfies the universal estimate, where $C = C(N, q, |\Omega|)$,

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{q-1}}. \quad (5.17)$$

Proof. 1) First assume $q < N$. For any $\alpha > 0$, setting $\rho = 1 + \alpha$, and $0 < \epsilon \leq s < t < T$, setting $\beta = 1 + \alpha/q$, we obtain, from (5.15),

$$\frac{1}{\alpha + 1} \int_{\Omega} u^{\alpha+1}(\cdot, t) dx + \frac{1}{\beta^q} \int_s^t \int_{\Omega} |\nabla(u^\beta)|^q dx dt \leq \frac{1}{\alpha + 1} \int_{\Omega} u^{\alpha+1}(\cdot, s) dx.$$

Then $u^\beta(\cdot, t) \in W_0^{1,q}(\Omega)$, since $u(\cdot, t) \in L^\infty(Q_{\Omega,s,\tau}) \cap W_0^{1,1}(\Omega)$. From the Sobolev injection of $W_0^{1,q}(\Omega)$ into $L^{Nq/(N-q)}(\Omega)$,

$$\frac{1}{\alpha + 1} \int_{\Omega} u^{\alpha+1}(\cdot, t) dx + \frac{C(N, q)}{\beta^q} \int_s^t \left(\int_{\Omega} u^{\beta \frac{Nq}{N-q}}(\cdot, \sigma) dx \right)^{\frac{N-q}{N}} dt \leq \frac{1}{\alpha + 1} \int_{\Omega} u^{\alpha+1}(\cdot, s) dx.$$

From Lemma 2.1 on $[\epsilon, T)$ with $m = q$ and $\theta = N/(N - q)$, we obtain estimates for $\epsilon < t < T$:

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - \epsilon)^{-\sigma_{r,q,N}} \|u(\cdot, \epsilon)\|_{L^r(\Omega)}^{\varpi_{r,q,N}}, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - \epsilon)^{-\frac{1}{q-1}}.$$

and we deduce (5.16) and (5.17) as $\epsilon \rightarrow 0$. In the case $q = N$ the same conclusion follows from Lemma 2.1 with any $\theta > 1$. If $q > N$ we proceed as in Theorem 4.16 by applying Lemma 3.4.

2) Let u be any weak solution of $(D_{\Omega,T})$. Since $u \in C([\epsilon, T); L^1(\Omega))$ for $\epsilon > 0$, we find, for any $t \in [\epsilon, T)$,

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - \epsilon)^{-\frac{1}{q-1}}$$

with $C = C(N, q)$, and deduce (5.17) for any $t \in (0, T)$ as $\epsilon \rightarrow 0$. \blacksquare

Remark 5.13 *In particular we find again estimate (5.17) obtained in [33] in case $q < 2$, for solutions u such that $u \in C((0, T); L^2(\Omega)) \cap L^2((0, T); W_0^{1,2}(\Omega))$, and $(u - k)^+$ is admissible as a test function in the equation; those conditions imply integrability properties of $u|\nabla u|^q$. Our result is valid without any of these conditions.*

5.4 Existence and uniqueness results for $q \leq 2$

From estimate (5.17), we deduce new convergence results when $q \leq 2$:

Corollary 5.14 *Assume $1 < q \leq 2$. Then*

(i) *any weak solution u of problem $(D_{\Omega,T})$ satisfies $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0, T))$;*

(ii) *for any sequence of weak solutions (u_n) of $(D_{\Omega,T})$, one can extract a subsequence converging in $C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0, T))$ to a weak solution u of $(D_{\Omega,T})$.*

Proof. (i) From [16, Theorem 2.9], any weak solution u of $(D_{\Omega,T})$ such that $u \in L_{loc}^\infty((0,T); L^\infty(\Omega))$ satisfies $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0,T))$. And we obtain precisely $u \in L_{loc}^\infty((0,T); L^\infty(\Omega))$, at Theorem 5.12,3.

(ii) Moreover (u_n) is uniformly bounded in $L_{loc}^\infty(0,T); L^\infty(\Omega)$. From [16], there exists $v \in (0,1)$ such that, for any $0 < s < \tau < T$,

$$\|u_n\|_{C(\overline{\Omega} \times [s,\tau])} + \|\nabla u_n\|_{C^{v,v/2}(\overline{\Omega} \times [s,\tau])} \leq C\Phi(\|u_n\|_{L^\infty(Q_{\Omega,s/2,\tau})}) \quad (5.18)$$

where $C = C((N, q, \Omega, s, \tau, v))$, and Φ is an increasing function. The conclusion follows. \blacksquare

Theorem 5.15 *Suppose $1 < q < (N+2)/(N+1)$. For any $u_0 \in \mathcal{M}_b^+(\Omega)$, problem (5.9) admits a unique weak \mathcal{M} solution.*

Proof. From [8, Theorem 3.2], [1], for any (possibly signed) $u_0 \in \mathcal{M}_b(\Omega)$, problem (5.9) has a unique mild \mathcal{M} solution, and it is nonnegative when $u_0 \in \mathcal{M}_b^+(\Omega)$. From Lemma 5.11, any weak \mathcal{M} solution is a mild \mathcal{M} solution, thus uniqueness holds in this class. \blacksquare

Next assume that $u_0 \in L^r(\Omega)$ and consider the subcritical case (4.39). In [8, Theorem 3.3], it is proved that there exists a weak L^r solution such that $u \in L_{loc}^q([0,T]; W_0^{1,qr}(\Omega))$, and *it is unique in this space*. The local existence and uniqueness in an interval $(0, T_1)$ is obtained by the Banach fixed point theorem in a ball of radius K_1 of the space

$$X_{K_1}(T_1) = \left\{ u \in C((0, T_1], W_0^{1,qr}(\Omega)) : \sup_{(0, t_1]} t^\theta (\|u(\cdot, t)\|_{L^{qr}(\Omega)} + t^{\frac{1}{2}} \|\nabla u(\cdot, t)\|_{L^{qr}(\Omega)}) < \infty \right\}$$

where $\theta = N/2rq'$, under the condition

$$\|u_0\|_{L^r(\Omega)} + K_1^q T_1^\gamma \leq CK_1, \quad \text{where } \gamma = 1 - q(\theta + 1/2) \quad \text{and } C = C(N, q, r, \Omega). \quad (5.19)$$

We prove the uniqueness *with no condition of integrability*:

Theorem 5.16 *Assume that $u_0 \in L^r(\Omega)$ and $1 < q < (N+2r)/(N+r)$. Then problem (5.9) admits a unique weak L^r solution.*

Proof. Let $\epsilon > 0$. From Theorem 5.12, u is bounded on (ϵ, T) for any $\epsilon \in (0, T)$. Then $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0, T))$ because $q < 2$, from [16, Theorem 2.10]. From (2.14), there exists a function $D \in C((0, \infty))$ such that for any $\epsilon > 0$ and for $t \geq \epsilon$

$$\|\nabla u(\cdot, t)\|_{L^\infty(\Omega)} \leq D(t - \epsilon).$$

Then $|\nabla u|$ is bounded in $Q_{\epsilon, T, \Omega}$ for any $\epsilon > 0$. Thus $u \in C((0, T), W_0^{1,qr}(\Omega))$. The problem with initial data $u(\cdot, \epsilon)$ at time 0 has a unique solution v_ϵ such that $v_\epsilon \in C((0, T - \epsilon), W_0^{1,qr}(\Omega))$, then $v_\epsilon(\cdot, t) = u(\cdot, t + \epsilon)$. Let K_1 and T_1 such that (5.19) holds. Since $\|u(\cdot, \epsilon)\|_{L^r(\Omega)} \leq \|u_0\|_{L^r(\Omega)}$, we also have $\|v_\epsilon(0)\|_{L^r(\Omega)} + K_1^q T_1^\gamma \leq CK_1$, thus for any $t \in (0, T_1)$

$$t^\theta (\|v_\epsilon(\cdot, t)\|_{L^{qr}(\Omega)} + t^{\frac{1}{2}} \|\nabla v_\epsilon(\cdot, t)\|_{L^{qr}(\Omega)}) \leq K_1.$$

Going to the limit as $\epsilon \rightarrow 0$ from the Fatou Lemma, we obtain

$$t^\theta (\|u(\cdot, t)\|_{L^{qr}(\Omega)} + t^{\frac{1}{2}} \|\nabla u(\cdot, t)\|_{L^{qr}(\Omega)}) \leq K_1.$$

Uniqueness follows in $(0, T_1)$, and by induction on $(0, T)$. \blacksquare

Finally we give existence results for any $u_0 \in L^r(\Omega)$, $r \geq 1$, extending the results of [8, Theorem 3.4] for $u_0 \in L^1(\Omega)$, see also [32] for more general operators. We proceed as in Proposition 4.26.

Proposition 5.17 *Let $1 < q \leq 2$. For any nonnegative $u_0 \in L^r(\Omega)$, $r \geq 1$, there exists a weak L^r solution of problem (5.9). And it is unique if $q = 2$.*

Proof. (i) Case $q < 2$. Let $u_{0,n} = \min(u_0, n)$. Then for $\rho > N(q-1)/(2-q)$, from [8, Theorem 3.3], there exists a mild solution u_n with initial data $u_{0,n}$, and $u_n \in C([0, T]; L^\rho(\Omega)) \cap L^q((0, T); W_0^{1,q\rho}(\Omega) \cap C^{2,1}(Q_{\Omega,T}))$. Then $u_n(\cdot, t) \leq e^{t\Delta}u_0$, and (u_n) is nondecreasing and $|\nabla u_n|^q$ is bounded in $L_{loc}^1([0, T]; L^1(\Omega))$ from (5.13). From Corollary 4.20, (u_n) converges in $C_{loc}^{2,1}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$. As a consequence, $u(\cdot, t) \leq e^{t\Delta}u_0$ and $|\nabla u|^q \in L_{loc}^1([0, T]; L^1(\Omega))$. From [16, Proposition 2.11], $u(\cdot, t)$ converges weakly* to some Radon measure μ_0 on Ω . And $e^{t\Delta}u_0$ converges to u_0 in $L^r(\Omega)$, thus $\mu_0 \in L_{loc}^1(\Omega)$ and $0 \leq \mu_0 \leq u_0$. Since $u_n \leq u$, there holds $u_{0,n} \leq \mu_0$, hence $\mu_0 = u_0 \in L^r(\Omega)$. Also there exists a function $g \in L^r(\Omega)$ such that $u(\cdot, t) \leq g$ for small t . Then $e^{t\Delta}u_0 - u(\cdot, t)$ converges weakly* to 0, and then in $L_{loc}^1(\Omega)$. Hence $u(\cdot, t)$ converges to u_0 in $L_{loc}^1(\Omega)$, then in $L^r(\Omega)$ from the dominated convergence theorem. Thus $u \in C([0, T]; L^r(\Omega))$.

(ii) Case $q = 2$. As in [15, Theorem 4.2], using the classical transformation $v = 1 - e^{-u}$, it can be shown that there exists a unique solution u such that $u \in C([0, T]; L^r(\Omega)) \cap C^{2,1}(Q_{\Omega,T}) \cap C^1(\overline{\Omega} \times (0, T))$. Then it is a weak L^r solution. Reciprocally any weak L^r solution u satisfies the conditions above, from Corollary 5.14 and [16, Theorem 2.17]. \blacksquare

6 Regularizing effects for quasilinear Dirichlet problems

Here we extend some results of section 5 to a general quasilinear problem, where u may be a signed solution. In this section, we suppose Ω is a smooth bounded domain in \mathbb{R}^N .

Let $p > 1$ and A be a Caratheodory function on $Q_{\Omega,\infty} \times \mathbb{R} \times \mathbb{R}^N$ such that for any $(u, \eta) \in \mathbb{R} \times \mathbb{R}^N$, and a.e. $(x, t) \in Q_{\Omega,\infty}$,

$$|A(x, t, u, \eta)| \leq C(|\eta|^{p-1} + b(x, t)), \quad C > 0, \quad b \in L^{p'}(Q_{\Omega,\infty}), \quad (6.1)$$

and A is nonnegative operator:

$$A(x, t, u, \eta) \cdot \eta \geq \nu |\eta|^p \quad \nu \geq 0, \quad (6.2)$$

with no monotonicity assumption.

Let $q > 1$ and g be a Caratheodory function on $Q_{\Omega,\infty} \times \mathbb{R}^+ \times \mathbb{R}^N$, such that

$$g(x, t, u, \eta)u \geq \gamma |u|^{\lambda+1} |\eta|^q, \quad \lambda \geq 0, \quad \gamma \geq 0. \quad (6.3)$$

We say that A is **coercive** if (6.2) holds with $\nu > 0$, and g is **coercive** if (6.3) holds with $\gamma > 0$.

We consider the solutions of the Dirichlet problem

$$(P_{\Omega,T}) \begin{cases} u_t - \operatorname{div}(A(x,t,u,\nabla u)) + g(x,t,u,\nabla u) = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0, \end{cases} \quad (6.4)$$

where $u_0 \in L^r(\Omega)$, $r \geq 1$ or only $u_0 \in \mathcal{M}_b(\Omega)$.

6.1 Solutions of quasilinear heat equation with L^1 data

First consider the problem in $Q_{\Omega,s,\tau}$

$$\begin{cases} u_t - \operatorname{div}(A(x,t,u,\nabla u)) = f, & \text{in } Q_{\Omega,s,\tau}, \\ u = 0, & \text{on } \partial\Omega \times (s,\tau), \\ u(x,s) = u_s \end{cases} \quad (6.5)$$

Let us recall the notion of renormalized solution introduced in [18] for this problem with L^1 data, where the truncations T_k are defined by (5.3):

Definition 6.1 *Let $s, \tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u_s \in L^1(\Omega)$. A function $u \in L^\infty((s,\tau); L^1(\Omega))$ is a renormalized solution in $Q_{\Omega,s,\tau}$ of (6.5) if $T_k(u) \in L^p((s,\tau); W_0^{1,p}(\Omega))$ for any $k \geq 0$, and for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support,*

$$(S(u))_t - \operatorname{div}(A(x,t,u,\nabla u)S'(u)) + S''(u)(A(x,t,u,\nabla u) \cdot \nabla u - S'(u)f) = 0 \quad \text{in } \mathcal{D}'(Q_{\Omega,s,\tau}), \quad (6.6)$$

and $u(s) = u_s$, and

$$\lim_{n \rightarrow \infty} \int \int_{Q_{\Omega,s,\tau} \cap \{n \leq u \leq n+1\}} |\nabla u|^p dx dt = 0, \quad (6.7)$$

Remark 6.2 *The initial condition takes sense from [18], because $S(u)$ lies in the set*

$$E = \left\{ \varphi \in L^p((0,T); W_0^{1,p}(\Omega)) : \varphi_t \in L^{p'}((0,T); W^{-1,p'}(\Omega)) + L^1(Q_{\Omega,T}) \right\} \quad (6.8)$$

and $E \subset C([0,T]; L^1(\Omega))$. Any function $\varphi \in L^p((0,T); W_0^{1,p}(\Omega)) \cap L^\infty(Q_{\Omega,T})$ can be chosen as a test function in equation (6.6). Moreover, from [27, Lemma 7.1], $v = S(u)$ satisfies for any $\psi \in C^\infty([s,\tau] \times \bar{\Omega})$ the integration formula

$$\int_s^\tau \langle v_t, M(v)\psi \rangle = \int_\Omega \mathcal{M}(v(\cdot, \tau))\psi(\cdot, \tau) dx - \int_\Omega \mathcal{M}(v(\cdot, s))\psi(\cdot, s) dx - \int_s^\tau \int_\Omega \psi_t \mathcal{M}(v) dx dt, \quad (6.9)$$

for any function M continuous and piecewise C^1 such that $M(0) = 0$ and M' has a compact support, where $\mathcal{M}(r) = \int_0^r M(\theta) d\theta$.

A main point in the sequel is the choice of test functions: here we approximate $|u|^{\alpha-1}u$ for $\alpha > 0$ by truncation. In the following lemma, we solve some technical difficulties arising because the truncates are not smooth enough to apply the integration formula, and moreover we do not assume $\alpha \geq 1$.

Lemma 6.3 Let $s, \tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega, s, \tau})$. Let $u \in C([s, \tau]; L^1(\Omega))$ be any nonnegative renormalized solution in $Q_{\Omega, s, \tau}$ of (6.5), with $u_s = u(\cdot, s)$. For any $\alpha > 0$ and $k > 0$, we set

$$\mathcal{T}_{k, \alpha}(r) = \int_0^r |T_k(\theta)|^{\alpha-1} T_k(\theta) d\theta.$$

Then $|T_k(u)|^{\alpha-1} A(x, t, u, \nabla u) \cdot \nabla(T_k(u)) \in L^1(Q_{\Omega, s, \tau})$ and

$$\begin{aligned} & \int_{\Omega} \mathcal{T}_{k, \alpha}(u)(\cdot, \tau) dx + \alpha \int \int_{Q_{\Omega, s, \tau}} |T_k(u)|^{\alpha-1} A(x, t, u, \nabla u) \cdot \nabla(T_k(u)) dx dt \\ &= \int_{\Omega} \mathcal{T}_{k, \alpha}(u)(\cdot, s) dx + \int_s^{\tau} \int_{\Omega} f |T_k(u)|^{\alpha-1} T_k(u) dx dt. \end{aligned} \quad (6.10)$$

Proof. Let $\alpha > 0, k > 0$ be fixed, and for any $n \geq 2$, and $\theta \in \mathbb{R}$,

$$S_n(\theta) = \int_0^{\theta} (1 - |T_1(s - T_n(s))|) ds, \quad n \geq 2.$$

This function, introduced in [18], is still a truncation, smoother than T_{n+1} , such that $0 \leq S_n(\theta) \leq T_{n+1}(\theta)\theta$, $\text{supp } S'_n \subset [-(n+1), n+1]$, $S''_n = \chi_{(-n, -n-1) \cup (n, n+1)}$, and $S_n(T_k(\theta)) = T_k(\theta)$ for any $n > k$. Let $\delta \in (0, \min(1, k))$, and $n > k$. We set

$$T_{\delta, k, \alpha}(\theta) = ((T_k(|\theta|) + \delta)^\alpha - \delta^\alpha) \text{sign} \theta, \quad \mathcal{T}_{\delta, k, \alpha}(r) = \int_0^r T_{\delta, k, \alpha}(\theta) d\theta.$$

We can take in (6.6) $S = S_n$ and $\varphi = T_{\delta, k, \alpha}(u) = T_{\delta, k, \alpha}(S_n(u))$ as a test function. We obtain

$$\begin{aligned} & \int_s^t \langle (S_n(u))_t, \varphi \rangle + \int_s^t \int_{\Omega} S'_n(u) A(x, t, u, \nabla u) \cdot \nabla \varphi dx dt \\ &= \int_s^t \int_{\Omega} S'_n(u) f \varphi dx dt - \int_s^t \int_{\Omega} S''_n(u) (A(x, t, u, \nabla u) \cdot \nabla u) \varphi dx dt. \end{aligned}$$

then from (6.9), we deduce

$$\begin{aligned} & \alpha \int \int_{Q_{\Omega, s, \tau}} (T_k(|u|) + \delta)^{\alpha-1} A(x, t, u, \nabla u) \cdot \nabla(T_k(u)) dx dt \\ &= \int_{\Omega} \mathcal{T}_{\delta, k, \alpha}(S_n(u)(\cdot, s)) dx - \int_{\Omega} \mathcal{T}_{\delta, k, \alpha}(S_n(u)(\cdot, \tau)) dx \\ &+ \int_s^{\tau} \int_{\Omega} S'_n(u) f \varphi dx dt - \int_s^t \int_{\Omega} S''_n(u) (A(x, t, u, \nabla u) \cdot \nabla u) \varphi dx dt \end{aligned}$$

First we make $\delta \rightarrow 0$. Notice that $|\mathcal{T}_{\delta, k, \alpha}(\theta)| \leq (k+1)^\alpha |\theta|$ for any $\theta \in \mathbb{R}$, and $S_n(u) \in C([0, T]; L^1(\Omega))$, and S'_n is bounded. Thus we can go to the limit in the right hand side. In the left hand side, from the positivity of A , and the Fatou Lemma we deduce that

$$T_k(|u|)^{\alpha-1} A(x, u, \nabla u) \cdot \nabla T_k(u) \in L^1(Q_{\Omega, s, \tau}).$$

Moreover we can apply dominated convergence theorem. Indeed $A(x, u, \nabla u) \cdot \nabla T_k(u) \in L^1(Q_{\Omega, s, \tau})$ from (6.1), since $T_k(u) \in L^p((s, \tau); W_0^{1,p}(\Omega))$, and

$$(T_k(|u|) + \delta)^{\alpha-1} A(x, t, u, \nabla u) \cdot \nabla(T_k(u)) \leq \max(T_k^{\alpha-1}(|u|), (k+1)^{\alpha-1}) A(x, u, \nabla u) \cdot \nabla(T_k(u)).$$

Hence the same relation holds with $\delta = 0$, with $T_{0,k,\alpha}(r) = T_k^{\alpha-1}(|u|)T_k(u)$:

$$\begin{aligned} & \int_{\Omega} \mathcal{T}_{k,\alpha}(S_n(u)(\cdot, \tau)) dx - \int_{\Omega} \mathcal{T}_{k,\alpha}(S_n(u)(\cdot, s)) dx + \alpha \int_s^t \int_{\Omega} T_k^{\alpha-1}(|u|) A(x, u, \nabla u) \cdot \nabla (T_k(u)) dx dt \\ &= \int_s^{\tau} \int_{\Omega} S'_n(u) f T_{0,k,\alpha}(u) dx dt - \int_s^t \int_{\Omega} S''_n(u) (A(x, t, u, \nabla u) \cdot \nabla u) T_{0,k,\alpha}(u) dx dt. \end{aligned}$$

Then we make $n \rightarrow \infty$. Since $u \in C([0, T]; L^1(\Omega))$, for any $t \in [s, \tau]$, we find

$$\lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{T}_{k,\alpha}(S_n(u)(\cdot, t)) dx = \int_{\Omega} \mathcal{T}_{k,\alpha}(u(\cdot, t)) dx;$$

moreover

$$\lim_{n \rightarrow \infty} \int_s^t \int_{\Omega} S''_n(u) (A(x, t, u, \nabla u) \cdot \nabla u) T_{0,k,\alpha}(u) dx dt = 0,$$

from (6.7), (6.1). Finally

$$\lim_{n \rightarrow \infty} \int_s^{\tau} \int_{\Omega} S'_n(u) f T_{0,k,\alpha}(u) dx dt = \int_s^{\tau} \int_{\Omega} f T_{0,k,\alpha}(u) dx dt,$$

since $S'_n(u) \rightarrow 1$ a.e. and is uniformly bounded. Then (6.10) follows. \blacksquare

6.2 Notion of solutions of problem $(P_{\Omega, T})$

Definition 6.4 We say that u is a **renormalized solution** of problem $(P_{\Omega, T})$ if:

- (i) $u \in C((0, T); L^1(\Omega))$, $T_k(u) \in L^p_{loc}((0, T); W_0^{1,p}(\Omega))$ for any $k \geq 0$, and $g(x, u, \nabla u) \in L^1_{loc}((0, T); L^1(\Omega))$;
- (ii) for any $0 < s < \tau < T$, u is a renormalized solution of problem

$$\begin{cases} u_t - \operatorname{div}(A(x, t, u, \nabla u)) + g(x, t, u, \nabla u) = 0, & \text{in } Q_{\Omega, s, \tau}, \\ u = 0, & \text{on } \partial\Omega \times (0, T), \end{cases}$$

with initial data $u(\cdot, s)$;

- (iii) for $u_0 \in L^r(\Omega)$, the extension of u by u_0 at time 0 belongs to $C([0, T]; L^r(\Omega))$; for $u_0 \in \mathcal{M}_b(\Omega)$, there holds

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \psi dx = \int_{\Omega} \psi du_0, \quad \forall \psi \in C_b(\Omega). \quad (6.11)$$

Remark 6.5 Recall that ∇u is defined by $\nabla u = \nabla(T_k(u))$ on the set $|u| \leq k$. The assumption on g means that, for any $0 < s < \tau < T$,

$$\int_{Q_{\Omega, s, \tau}} |g(\cdot, u, \nabla u)| dx dt = \sum_{k=1}^{\infty} \int_{Q_{\Omega, s, \tau} \cap \{k-1 \leq |u| \leq k\}} |g(\cdot, u, \nabla(T_k(u)))| dx dt < \infty.$$

We first prove decay properties of the solutions.

Theorem 6.6 Let $p, q > 1$, and A and g satisfying (6.1) (6.2) and (6.3).

1) Let $u_0 \in L^r(\Omega)$, $r \geq 1$ and u be any renormalized solution of $(P_{\Omega, T})$. Then for any $t \in [0, T)$,

$$\int_{\Omega} |u|^r(., t) dx \leq \int_{\Omega} |u_0|^r dx. \quad (6.12)$$

Moreover if $r > 1$, or if g is coercive, then $\gamma |u|^{\lambda+r-1} |\nabla u|^q + \nu |u|^{r-2} |\nabla u|^p \in L^1_{loc}([0, T); L^1(\Omega))$, and

$$\int_{\Omega} |u|^r(., t) dx + r\gamma \int_0^t \int_{\Omega} |u|^{\lambda+r-1} |\nabla u|^q dx dt + r(r-1)\nu \int_0^t \int_{\Omega} |u|^{r-2} |\nabla u|^p dx dt \leq \int_{\Omega} |u_0|^r dx. \quad (6.13)$$

2) Let $u_0 \in \mathcal{M}_b^+(\Omega)$ and u be any nonnegative renormalized solution of $(P_{\Omega, T})$ of problem (5.9). Then the same conclusions hold as in case $u_0 \in L^1(\Omega)$, where the norm $\|u_0\|_{L^1(\Omega)}$ is replaced by $\int_{\Omega} du_0$.

Proof. 1) Let $0 < s < t < T$. Then for any $\alpha > 0$, any $k > 0$, from Lemma 6.3,

$$\begin{aligned} & \int_{\Omega} \mathcal{T}_{k, \alpha}(u)(., \tau) dx + \alpha \int_s^t \int_{\Omega} |T_k(u)|^{\alpha-1} A(x, t, u, \nabla u) \cdot \nabla(T_k(u)) dx dt \\ &= \int_{\Omega} \mathcal{T}_{k, \alpha}(u)(., s) dx - \int_s^t \int_{\Omega} |T_k(u)|^{\alpha-1} T_k(u) g(., u, \nabla u) dx dt. \end{aligned}$$

And $|T_k(u)|^{\alpha-1} T_k(u) g(., u, \nabla u) \geq \gamma |T_k(u)|^{\alpha+\lambda} |\nabla T_k(u)|^q$ from (6.3). Therefore $\int_{\Omega} \mathcal{T}_{k, \alpha}(u)(., t)$ is decreasing for any $k, \alpha > 0$, and

$$\begin{aligned} & \int_{\Omega} \mathcal{T}_{k, \alpha}(u)(., \tau) dx + \gamma \int_s^t \int_{\Omega} |T_k(u)|^{\alpha+\lambda} |\nabla T_k(u)|^q dx dt + \alpha\nu \int_s^t \int_{\Omega} |T_k(u)|^{\alpha-1} |\nabla T_k(u)|^p dx dt \\ & \leq \int_{\Omega} \mathcal{T}_{k, \alpha}(u)(., s) dx. \end{aligned} \quad (6.14)$$

• If $r > 1$, we can take $\alpha = r - 1 > 0$ in (6.14) and get

$$\begin{aligned} & \int_{\Omega} \mathcal{T}_{k, r-1}(u)(., t) dx + \gamma \int_s^t \int_{\Omega} |T_k(u)|^{r-1+\lambda} |\nabla T_k(u)|^q dx dt + \alpha\nu \int_s^t \int_{\Omega} |T_k(u)|^{r-2} |\nabla T_k(u)|^p dx dt \\ & \leq \int_{\Omega} \mathcal{T}_{k, r-1}(u)(., s) dx \leq \frac{1}{r} \int_{\Omega} |u|^r(., s) dx. \end{aligned} \quad (6.15)$$

Since $u \in C([0, T); L^r(\Omega))$ we can go to the limit as $k \rightarrow \infty$, and $s \rightarrow 0$; we obtain that $\gamma |u|^{r-1+\lambda} |\nabla u|^q$ and $\alpha\nu |u|^{r-2} |\nabla u|^p$ belong to $L^1_{loc}([0, T); L^1(\Omega))$; and for any $t \in (0, T)$,

$$\int_{\Omega} |u|^r(., t) dx + r\gamma \int_0^t \int_{\Omega} |u|^{r-1+\lambda} |\nabla u|^q dx dt + r(r-1)\nu \int_0^t \int_{\Omega} |u|^{r-2} |\nabla u|^p dx dt \leq \int_{\Omega} |u_0|^r dx.$$

• If $r = 1$, we take any $\alpha > 0$ in (6.14); notice that

$$\frac{|T_k(\theta)|^{\alpha+1}}{\alpha+1} \leq \mathcal{T}_{k, \alpha}(\theta) \leq k^\alpha |\theta|, \quad (6.16)$$

for any $\theta > 0$. Then

$$\int_{\Omega} |T_k(u)|^{\alpha+1}(\cdot, t) dx + (\alpha + 1)\gamma \int_s^t \int_{\Omega} |T_k(u)|^{\alpha+\lambda} |\nabla T_k(u)|^q dx dt \leq (\alpha + 1)k^\alpha \int_{\Omega} |u|(\cdot, s) dx.$$

Going to the limit as $\alpha \rightarrow 0$, we deduce

$$\int_{\Omega} |T_k(u)|(\cdot, t) dx + \gamma \int_s^t \int_{\Omega} |T_k(u)|^\lambda |\nabla T_k(u)|^q dx dt \leq \int_{\Omega} |u|(\cdot, s) dx; \quad (6.17)$$

and then as $s \rightarrow 0$ we find

$$\int_{\Omega} |T_k(u)|(\cdot, t) dx + \gamma \int_0^t \int_{\Omega} |T_k(u)|^\lambda |\nabla T_k(u)|^q dx dt \leq \int_{\Omega} |u_0| dx, \quad (6.18)$$

and finally as $k \rightarrow \infty$, we obtain that $\int_{\Omega} |u|(\cdot, t) dx \leq \int_{\Omega} |u_0| dx$. Moreover if $\gamma > 0$, we find

$$\int_{\Omega} |u|(\cdot, t) dx + \gamma \int_0^t \int_{\Omega} |u|^\lambda |\nabla u|^q dx dt \leq \int_{\Omega} |u_0| dx,$$

thus (6.13) still holds with $r = 1$.

2) We still find (6.17). And $\lim_{s \rightarrow 0} \int_{\Omega} u(\cdot, s) dx = \int_{\Omega} du_0$ from (6.11), hence the conclusion. \blacksquare

Next we deduce L^∞ estimates, in particular a universal one.

Theorem 6.7 *Let $p, q > 1$, and A and g satisfying (6.1) (6.2) and (6.3). Let $u_0 \in L^r(\Omega), r \geq 1$, and u be any renormalized solution of $(P_{\Omega, T})$.*

(i) *If g is coercive, then*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \begin{cases} Ct^{-\sigma_{r,q,\lambda}} \|u_0\|_{L^r(\Omega)}^{\varpi_{r,q,N,\lambda}}, & C = C(N, q, r, \lambda, \gamma), \quad \text{if } q \neq N, \\ C_\varepsilon t^{-(1+\varepsilon)\sigma_{r,n,\lambda}} \|u_0\|_{L^r(\Omega)}^{\varpi_{r,q,N,\lambda}}, & C_\varepsilon = C(N, q, r, \lambda, \gamma, \varepsilon), \quad \text{if } q = N, \end{cases} \quad (6.19)$$

where

$$\sigma_{r,q,N,\lambda} = \frac{1}{\frac{rq}{N} + \lambda + q - 1} = \frac{N}{rq} \varpi_{r,q,N,\lambda}.$$

Moreover

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{q-1+\lambda}}, \quad C = C(N, q, \lambda, |\Omega|). \quad (6.20)$$

(ii) *If A is coercive and $r > (2-p)N/p$, in particular if $p > 2N/(N+1)$, then*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \begin{cases} Ct^{-\sigma_{r,p,N,-1}} \|u_0\|_{L^r(\Omega)}^{\varpi_{r,p,N,-1}}, & C = C(N, p, r, \nu, \Omega), \quad \text{if } p \neq N, \\ C_\varepsilon t^{-(1+\varepsilon)\sigma_{r,p,N,-1}} \|u_0\|_{L^r(\Omega)}^{\varpi_{r,p,N,-1}}, & C_\varepsilon = C(N, p, r, \nu, \Omega, \varepsilon), \quad \text{if } p = N, \end{cases} \quad (6.21)$$

where

$$\sigma_{r,p,N,-1} = \frac{1}{\frac{rp}{N} + p - 2} = \frac{N}{rp} \varpi_{r,p,N,-1}.$$

Moreover if $p > 2$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{p-2}}, \quad C = C(N, p, |\Omega|). \quad (6.22)$$

(iii) *The same conclusions hold if u is nonnegative and $u_0 \in \mathcal{M}_b^+(\Omega)$, as in case $u_0 \in L^1(\Omega)$, where the norm $\|u_0\|_{L^1(\Omega)}$ is replaced by $\int_{\Omega} du_0$. In particular (6.22) holds for $p > 2$.*

Proof. (i) Let $0 < s < t < T$. Since g is coercive, from Theorem 6.6, for any $\alpha \geq 0$ such that $|u|^{\alpha+1}(\cdot, s) \in L^1(\Omega)$, there holds

$$\int_{\Omega} |u|^{\alpha+1}(\cdot, t) dx + (\alpha + 1)\gamma \int_s^t \int_{\Omega} |u|^{\lambda+\alpha} |\nabla u|^q dx dt \leq \int_{\Omega} |u|^{\alpha+1}(\cdot, s) dx,$$

from (6.13); in particular

$$\int_{\Omega} |T_k(u)|^{\alpha+1}(\cdot, t) dx + (\alpha + 1)\gamma \int_s^t \int_{\Omega} (T_k(u))^{\lambda+\alpha} |\nabla T_k(u)|^q dx dt \leq \int_{\Omega} |u|^{\alpha+1}(\cdot, s) dx.$$

And $|u|^{\lambda+\alpha} |\nabla u|^q = |\nabla(|u|^{\beta-1} u)|^q$ with $\beta = 1 + (\alpha + \lambda)/q \geq 1$. Then $|\nabla(|u|^{\beta-1} u)(\cdot, t)|$, and also $|\nabla(|T_k(u)|^{\beta-1} T_k(u))(\cdot, t)|$ belong to $L^q(\Omega)$ for almost any $t \in (0, T)$. Since $|T_k(u)|^{\beta-1} T_k(u)(\cdot, t) \in L^\infty(\Omega)$, it follows that $|T_k(u)|^{\beta-1} T_k(u)(\cdot, t) \in W^{1,q}(\Omega)$. Moreover $T_k(u)(\cdot, t) \in W_0^{1,p}(\Omega)$, hence $|T_k(u)|^{\beta-1} T_k(u)(\cdot, t) \in W_0^{1,q}(\Omega)$. If $q < N$, we deduce

$$\int_{\Omega} |T_k(u)|^{\alpha+1}(\cdot, t) dx + \gamma \frac{(\alpha + 1)C(N, q)}{\beta^q} \int_s^t \left(\int_{\Omega} |T_k(u)|^{\beta \frac{Nq}{N-q}}(\cdot, \sigma) dx \right)^{\frac{N-q}{N}} d\sigma \leq \int_{\Omega} |u|^{\alpha+1}(\cdot, s) dx.$$

Going to the limit as $k \rightarrow \infty$, we find

$$\int_{\Omega} |u|^{\alpha+1}(\cdot, t) dx + \gamma \frac{(\alpha + 1)C(N, q)}{\beta^q} \int_s^t \left(\int_{\Omega} |u|^{\beta \frac{Nq}{N-q}}(\cdot, \sigma) dx \right)^{\frac{N-q}{N}} d\sigma \leq \frac{1}{\alpha + 1} \int_{\Omega} |u|^{\alpha+1}(\cdot, s) dx.$$

Then we can apply Lemma 2.1 on $[\epsilon, T]$, with $m = q$ and $\theta = N/(N - q)$; indeed (2.1) is satisfied, since $\lambda \geq 0$; we deduce the estimate for $[\epsilon, T]$,

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - \epsilon)^{-\sigma_{r,q,N,\lambda}} \|u(\cdot, \epsilon)\|_{L^r(\Omega)}^{\varpi_{r,q,N,\lambda}},$$

with $C = C(N, q, r, \lambda, \gamma, \Omega)$. Going to the limit as $\epsilon \rightarrow 0$, we get (6.20), and (6.19) for $u_0 \in L^r(\Omega)$, and the analogous when $u_0 \in \mathcal{M}_b^+(\Omega)$. In case $q \geq N$ we proceed as in Theorem 5.12.

(ii) Assume that A is coercive. Then for any $\alpha > 0$,

$$\int_{\Omega} \mathcal{T}_{k,\alpha}(u)(\cdot, t) dx + \alpha\nu \int_s^t \int_{\Omega} |T_k(u)|^{\alpha-1} |\nabla T_k(u)|^p dx dt \leq \int_{\Omega} \mathcal{T}_{k,\alpha}(u)(\cdot, s) dx,$$

from (6.14). First assume $p < N$. From the Sobolev injection of $W_0^{1,p}(\Omega)$ into $L^{Np/(N-p)}(\Omega)$, we deduce

$$\frac{1}{\alpha + 1} \int_{\Omega} |u|^{\alpha+1}(\cdot, t) dx + \alpha\nu \frac{C(N, p)}{k^p} \int_s^t \left(\int_{\Omega} |u|^k(\cdot, \sigma) dx \right)^{\frac{N-p}{N}} dt \leq \frac{1}{\alpha + 1} \int_{\Omega} |u|^{\alpha+1}(\cdot, s) dx,$$

with $k = 1 + (\alpha - 1)/p$.

First suppose $r > 1$; then we start from $\alpha_0 = r - 1 > 0$, and we can apply Lemma 2.1 with $C_0 = (r - 1)\nu C(N, p)$, $m = p$, $\theta = N/(N - p)$ and $\lambda = -1$; indeed (2.1) is satisfied, since $r > N(2 - p)/p$.

Next suppose $r = 1$. Then $1 > (2 - p)N/p$, thus $p - 1 + p/N > 1$. For any $\alpha > 0$,

$$\int_{\Omega} |T_k(u)|^{\alpha+1}(\cdot, t) dx + \alpha(\alpha + 1)\nu \int_s^t \int_{\Omega} |T_k(u)|^{\alpha-1} |\nabla T_k(u)|^p dx dt \leq (\alpha + 1)k^\alpha \int_{\Omega} |u|(\cdot, s) dx.$$

Taking $\alpha = 1$, we get from (6.12),

$$\nu \int_s^t \int_{\Omega} |\nabla T_k(u)|^p dx dt \leq k \int_{\Omega} |u|(\cdot, s) dx \leq k \int_{\Omega} |u_0| dx.$$

And from (6.12), $u \in L^\infty((s, T); L^1(\Omega))$. From standard estimates, there holds $u \in L^\rho(Q_{\Omega, s, t})$ for any $\rho \in (1, p - 1 + p/N)$, see [19]. Then $|u|^\rho(\cdot, t) \in L^1(\Omega)$ for almost any $t \in (0, T)$. Hence we can apply Lemma 2.1 on $[\epsilon, T)$ for $\epsilon > 0$, with the same parameters, after fixing such a $\rho = \rho_{p, N}$ such that $\rho N(2 - p)/p < 1$. We obtain that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(t - \epsilon)^{-\sigma_{1, p, -1}} \|u(\cdot, \epsilon)\|_{L^1(\Omega)}^{\varpi_{1, p, -1}},$$

where $C = C(N, p\rho_{p, N}) = C(N, p)$; finally we go to the limit as $\epsilon \rightarrow 0$ because $u \in C([0, T]; L^1(\Omega))$. Estimate (6.22) follows, since $-1 + p - 1 > 0$.

If $p = N$, we proceed as above, applying Lemma 2.1 with $m = N$, $\lambda = -1$ and $\theta > 1$ arbitrary. Next assume $p > N$. In case $r > 1$, there holds, for any $t \in (0, T)$,

$$r(r - 1) \frac{\nu}{\kappa^p} \int_0^t \int_{\mathbb{R}^N} |\nabla(|u|^\kappa)|^p dx dt \leq \int_{\mathbb{R}^N} |u_0|^r dx,$$

where $\kappa = 1 + (r - 2)/p > 0$. From Lemma 3.4, applied to $v = |u|^\kappa$, with $m = p$, $1/k = 1 + r(p - N)/Np\kappa$, we obtain

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}^{\frac{\kappa p}{k}} \leq C \|u(\cdot, t)\|_{L^r(\mathbb{R})}^{\frac{\kappa((1-k)p}{k}} \int_{\mathbb{R}} |\nabla(|u|^\kappa)|^p dx dt;$$

and by integration, with a new constant $C = C(N, p, r, \nu)$,

$$t \|u(\cdot, t)\|_{L^\infty(\mathbb{R})}^{\frac{\kappa p}{k}} \leq C \|u_0\|_{L^r(\mathbb{R}^N)}^{r + \frac{\kappa((1-k)p}{k}},$$

which is precisely (6.21). In case $r = 1$, we choose $\rho = p \in (1, p - 1 + p/N)$, and obtain from above, for any $0 < \epsilon < s < t < T$,

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C(t - s)^{-\sigma_{1, p, p, -1}} \|u(\cdot, s)\|_{L^p(\mathbb{R})}^{\varpi_{1, p, N, -1}} \leq C(t - \epsilon)^{-\sigma_{1, p, p, -1}} \|u(\cdot, s)\|_{L^\infty(\mathbb{R})}^{\frac{\varpi_{1, p, N, -1}}{p}} \|u_0\|_{L^1(\mathbb{R})}^{\frac{\varpi_{1, p, N, -1}}{p}},$$

where $C = C(N, p, \nu)$. From Lemma 3.1, we deduce precisely

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C(t - \epsilon)^{-\sigma_{1, p, p, -1}} \|u_0\|_{L^1(\mathbb{R})}^{\varpi_{1, p, N, -1}},$$

and we conclude as $\epsilon \rightarrow 0$.

(iii) We obtain the estimates on (ϵ, T) as above and go to the limit as $\epsilon \rightarrow 0$. ■

Remark 6.8 *Our results apply in particular to the problem*

$$\begin{cases} u_t - \operatorname{div}(A(x, t, u, \nabla u)) = 0, & \text{in } Q_{\Omega, T}, \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0 \end{cases}$$

Thus we find again and improve the estimates of [33, Theorem 5.3], with less regularity on the solutions: those estimates were proved for solutions $u \in C([0, T]; L^r(\Omega))$ such that $u \in L^p((0, T); W_0^{1,p}(\Omega)) \cap C([0, T]; L^2(\Omega))$. The notion of renormalized solutions, equivalent to the notion of entropy solutions of [35] (see [27]), is weaker. Moreover our results in case $p > N$ are optimal.

Remark 6.9 *The extension of results of section 4 to the case of equation of type (1.2) in the case $\Omega = \mathbb{R}^N$ will be treated a further article.*

7 Appendix

Proof of Lemma 4.10. (i) Let u be a mild \mathcal{M} solution. Then clearly (4.10) holds. Moreover for any $\psi \in C_0(\mathbb{R}^N)$, from the assumption on the gradient,

$$\langle e^{t\Delta} u_0, \psi \rangle = \langle u_0, e^{t\Delta} \psi \rangle = \int_{\mathbb{R}^N} e^{t\Delta} \psi du_0 = \int_{\mathbb{R}^N} (u(\cdot, t) + \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds) \psi dx$$

The relation extends to any $\varphi \in C_b(\mathbb{R}^N)$: we can assume that $\varphi \geq 0$; from the Beppo-Levi theorem,

$$\begin{aligned} \int_{\mathbb{R}^N} e^{t\Delta} \varphi du_0 &= \int_{\mathbb{R}^N} u(\cdot, t) \varphi dx + \int_{\mathbb{R}^N} \left(\int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \right) \varphi dx \\ &= \int_{\mathbb{R}^N} u(\cdot, t) \varphi dx + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \varphi dx ds, \end{aligned}$$

since the measure is bounded. From the integrability of the gradient and the dominated convergence theorem in $L^1(\mathbb{R}^N, du_0)$, we deduce

$$\lim_{t \rightarrow 0} \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \varphi dx ds = 0, \quad \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} e^{t\Delta} \varphi d\mu_0 = \int_{\mathbb{R}^N} \varphi d\mu_0,$$

since $\|e^{t\Delta} \varphi\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^N)}$ and $e^{t\Delta} \varphi$ converges to φ everywhere as $t \rightarrow 0$; thus (4.11a) holds.

(ii) Let u be a weak semi-group solution. Then obviously $u \in C_b((0, T); L^1(\mathbb{R}^N))$. As $\epsilon \rightarrow 0$, we have

$$\lim_{\epsilon \rightarrow 0} \int_\epsilon^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds = \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^1(\mathbb{R}^N).$$

Then

$$\lim_{\epsilon \rightarrow 0} e^{(t-\epsilon)\Delta} u(\cdot, \epsilon) = u(\cdot, t) + \int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \quad \text{in } L^1(\mathbb{R}^N).$$

Moreover (4.11a) entails that that $u(\cdot, \epsilon) \rightarrow u_0$ in $\mathcal{S}'(\mathbb{R}^N)$ and

$$\lim_{\epsilon \rightarrow 0} e^{(t-\epsilon)\Delta} u(\cdot, \epsilon) = e^{t\Delta} u_0 \quad \text{in } \mathcal{S}'(\mathbb{R}^N); \quad (7.1)$$

indeed for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned} \left| \langle e^{(t-\epsilon)\Delta} u(\cdot, \epsilon) - e^{t\Delta} u_0, \varphi \rangle \right| &\leq \left| \langle e^{t\Delta} (u(\cdot, \epsilon) - u_0(\cdot)), \varphi \rangle \right| + \left| \int_{\mathbb{R}^N} (u(x, \epsilon) ((e^{(t-\epsilon)\Delta} - e^{t\Delta}) \varphi)(x) dx \right| \\ &\leq \left| \langle e^{t\Delta} (u(\cdot, \epsilon) - u_0(\cdot)), \varphi \rangle \right| \\ &\quad + \|u(\cdot, \epsilon)\|_{L^1(\mathbb{R}^N)} \left\| (e^{(t-\epsilon)\Delta} - e^{t\Delta}) \varphi \right\|_{L^\infty(\mathbb{R}^N)} \end{aligned}$$

and $e^{t\Delta}$ is continuous on $\mathcal{S}(\mathbb{R}^N)$. Hence, for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$, we get

$$\langle e^{t\Delta} u_0, \varphi \rangle = \int_{\mathbb{R}^n} u(\cdot, t) \varphi dx + \int_{\mathbb{R}^n} \left(\int_0^t e^{(t-s)\Delta} |\nabla u(\cdot, s)|^q ds \right) \varphi dx$$

which extends to any $\varphi \in C_0(\mathbb{R}^N)$ by density. Thus (4.9) follows. \blacksquare

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