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# Monotonicity in first-passage percolation

Jean-Baptiste Gouéré\*

## Abstract

We consider standard first-passage percolation on  $\mathbb{Z}^d$ . Let  $e_1$  be the first coordinate vector. Let  $a(n)$  be the expected passage time from the origin to  $ne_1$ . In this short paper, we note that  $a(n)$  is increasing under some strong condition on the support of the distribution of the passage times on the edges.

## 1 Introduction and results

**First passage percolation.** We consider the graph  $\mathbb{Z}^d$ ,  $d \geq 2$ , obtained by taking  $\mathbb{Z}^d$  as vertex set and by putting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d.r.v.  $\tau = (\tau(e))_{e \in \mathcal{E}}$  indexed by the set of edges  $\mathcal{E}$  of the graph. We interpret  $\tau(e)$  as the time needed to travel along the edge  $e$  (the graph is unoriented).

If  $a$  and  $b$  are two vertices of  $\mathbb{Z}^d$ , we call path from  $a$  to  $b$  any finite sequence of vertices  $r = (a = x_0, \dots, x_k = b)$  such that, for all  $i \in \{0, \dots, k-1\}$ , the vertices  $x_i$  et  $x_{i+1}$  are linked by an edge. We denote by  $\mathcal{C}(a, b)$  the set of such paths. The time needed to travel along a path  $r = (x_0, \dots, x_k)$  is defined by:

$$\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

Then, the time needed to go from  $a$  to  $b$  is defined by:

$$T(a, b) = \inf\{\tau(r) : r \in \mathcal{C}(a, b)\}.$$

Let  $e_1, \dots, e_d$  denote the canonical basis vectors of  $\mathbb{R}^d$ . We are interested in the sequence  $(a(n))$  defined by :

$$a(n) = E(T(0, ne_1)).$$

We write  $T'(0, ne_1)$  and  $a'(n)$  for the passage times and expected passage times obtained when the paths are restricted to  $\{(x_1, \dots, x_d) : 0 \leq x_i \leq n\}$ .

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**Main result and related results.** We denote by  $S_-$  the infimum of the support of the distribution of the  $\tau(e)$ . We denote by  $S_+$  the supremum of the support.

**Theorem 1** *Assume  $0 < S_-$  and  $S_+ \leq 2S_-$ . Then the sequence  $(a_n)$  is non-decreasing. More precisely, we have:*

$$a(n) \geq a(n-1) + S_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

As soon as the distribution of the  $\tau(e)$  is not a Dirac distribution there exists, with probability one, infinitely many random  $N$  such that  $T(0, (N-1)e_1) > T(0, Ne_1)$ . However, monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in [3]. In [2], Alm and Wierman proved the monotonicity for  $\mathbb{Z} \times \mathbb{N}$  and other 2 dimensional models. In [1], Ahlberg made a detailed study of first passage percolation on essentially one-dimensional graphs, an example of which is  $\mathbb{Z} \times \{0, \dots, K\}^{d-1}$ . In particular, he proved the existence of a constant  $n_0$ , depending on the graph, such that  $n \geq n_0$  implies  $a(n) \geq a(n-1)$ . In [4], Howard proved the monotonicity for an Euclidean first-passage percolation model. We are not aware of any other positive results.

On the other hand, van den Berg proved in [7] that, when  $d = 2$ , one has  $a'(2) < a'(1)$  when  $\tau(e) = 1$  with small probability and  $\tau(e) = 0$  otherwise. Note that we still have  $a'(2) < a'(1)$  if, instead of setting  $\tau(e) = 0$  we set  $\tau(e) = \varepsilon$  for a small enough  $\varepsilon$ . A related result was given by Joshi in [6].

We refer to the review by Howard [5] for a more detailed account.

#### Further remarks.

- The same result holds for the  $a'(n)$ .
- The proof gives that  $T(0, ne_1)$  stochastically dominates the mean of  $n$  dependent copies of  $T(0, (n-1)e_1)$  (see (8) and (2)).
- With the same strategy one can prove for example the following result:

$$a(n) \geq a(n-1) \text{ as soon as } S_- > 0 \text{ and } \left( \frac{a(n)n^{-1}}{S_-} - 1 \right) \left( \frac{E(\tau(e))}{S_-} - 1 \right) \leq \frac{1}{2} \quad (1)$$

where  $e$  is a fixed edge. We show how to adapt the proof of Theorem 1 to prove this result below the proof of Theorem 1. In particular, using the inequality  $a(n) \leq nE(\tau(e))$ , we get that  $a$  is non-decreasing as soon as :

$$S_- > 0 \text{ and } E(\tau(e)) \leq (1 + 2^{-1/2})S_-.$$

This gives a sufficient condition with no assumption on  $S_+$  which can be infinite. However, this sufficient condition is still strong and we do not see how to give any significantly weaker condition.

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1. Let us sketch a proof. Fix  $a$  and  $b$  such that  $S_- < a < b < S_+$ . For each  $n$ , consider a box  $\{n-C, \dots, n\} \times \{-D, \dots, D\}^{d-1}$ . Let  $A_n$  be the following event:  $\tau(e) \leq a$  for edges inside the boundary of the box and  $\tau(e) \geq b$  for edges inside the box. For suitably chosen large  $C$  and  $D$  and for  $n > C$ , we have  $T(0, (n-1)e_1) > T(0, ne_1)$  as soon as  $A_n$  occurs. As the  $A_n$  are local event of fixed positive probability, the result follows.

2. Indeed,  $a'(1)$  can only increase while  $a'(2)$  increases by at most  $\varepsilon E(N)$  where  $N$  is the length of a geodesic for the initial passage times. Using  $T'(0, 2e_1) \leq 2$  one can check that any geodesic must remain in a random box of subgeometrical height. Therefore  $E(N)$  is finite and the result follows.

- Fix the distribution of  $\tau(e)$ . Assume  $S_- > 0$  and  $E\tau(e) < \infty$ . Then the conditions in (1) are true for large enough  $n$  and  $d$ . This is due to the fact that  $a(n)n^{-1}$  can be made arbitrarily close to  $S_-$ .

## 2 Proofs

**Proof of Theorem 1.** For all  $i$  we consider the following sets of edges:

- $H^i$ : the set of edges  $(x, x + e_1)$  where  $x = (x_1, \dots, x_d)$  is such that  $x_1 = i$ .
- $V^i$ : the set of edges  $(x, x + e_k)$  where  $x_1 = i$  and  $k$  belongs to  $\{2, \dots, d\}$ .

We define new passage times  $\tau^i(e)$  as follows:

- If  $e$  belongs to  $H^i$  then  $\tau^i(e) = 0$ .
- If  $e$  belongs to  $V^i$  then  $\tau^i(e) = +\infty$ .
- Otherwise,  $\tau^i(e) = \tau(e)$ .

We denote by  $T^i(a, b)$  the time needed to travel from  $a$  to  $b$  with the passage times  $\tau^i(e)$ . Note, for all  $n \geq 1$  and all  $i \in \{0, n-1\}$ , the following:

$$T^i(0, ne_1) \text{ and } T(0, (n-1)e_1) \text{ have the same distribution.} \quad (2)$$

We now compare  $T^i(0, ne_1)$  and  $T(0, ne_1)$ . Let  $\pi$  be a path from 0 to  $ne_1$  such that  $\tau(\pi) = T(0, ne_1)$ . We modify this path as follows. Each time the path goes, in this order, through an edge  $(x, y) \in V^i$ , we replace this part of the path by  $(x, x + e_1, y + e_1, y)$ . We denote by  $\pi^i$  the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

where, for example,  $\text{card}(\pi \cap H_i)$  denotes the number of edges of  $H_i$  used by  $\pi$ . The term involving  $H_i$  is due to the time saved by the modification of the passage times. The term involving  $V_i$  is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i) \quad (3)$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + (S_+ - S_-) \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i). \quad (4)$$

Note

$$\sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq n, \quad (5)$$

as  $\pi$  is a path from 0 to  $ne_1$ . But

$$\begin{aligned} T(0, ne_1) &= \tau(\pi) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- n \end{aligned} \quad (6)$$

and, moreover,

$$\begin{aligned} T(0, ne_1) &\leq \tau(0, e_1, 2e_2, \dots, ne_1) \\ &\leq nS_+. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) &\leq \frac{T(0, ne_1) - nS_-}{S_-} \\ &\leq \frac{nS_+ - nS_-}{S_-}. \end{aligned} \quad (7)$$

From (4) and (7) we get:

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-}. \quad (8)$$

Taking expectations and using (2) we get:

$$na(n-1) \leq na(n) - nS_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

The proof follows.  $\square$

**Proof of (1).** The proof is essentially the same. The main difference lies in the definition of the new passage times  $\tau^i(e)$ . We let  $\tilde{\tau}$  be an independent copy of  $\tau$ . We then set:

- If  $e$  belongs to  $H^i$  then  $\tau^i(e) = 0$ .
- If  $e$  belongs to  $V^i$  then  $\tau^i(e) = +\infty$ .
- If  $e$  belongs to  $V^{i+1}$  then  $\tau^i(e) = \tilde{\tau}^i(e)$ .
- Otherwise,  $\tau^i(e) = \tau(e)$ .

Instead of (3) we can write, after taking conditional expectation w.r.t.  $\tau$  :

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (E(\tau(e)) - S_-) \text{card}(\pi \cap V_i) + (E(\tau(e)) - S_-) \text{card}(\pi \cap V_{i+1}). \quad (9)$$

Instead of (4) we can get :

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + 2(E(\tau(e)) - S_-) \sum_{i=0}^n \text{card}(\pi \cap V_i). \quad (10)$$

Using (5), an equality similar to (6) and taking expectation, we get:

$$na(n-1) \leq na(n) - S_- n + 2(E(\tau(e)) - S_-) \frac{a(n) - nS_-}{S_-}$$

and thus:

$$a(n-1) \leq a(n) - S_- \left( 1 - 2 \left( \frac{E(\tau(e))}{S_-} - 1 \right) \left( \frac{a(n)n^{-1}}{S_-} - 1 \right) \right).$$

The proof follows.  $\square$

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