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Jean-Baptiste Gouéré

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Monotonicity in first-passage percolation

Jean-Baptiste Gouéré*

Abstract

We consider standard first-passage percolation on \mathbb{Z}^d . Let e_1 be the first coordinate vector. Let $a(n)$ be the expected passage time from the origin to ne_1 . In this short paper, we note that $a(n)$ is increasing under some strong condition on the support of the distribution of the passage times on the edges.

1 Introduction and results

First passage percolation. We consider the graph \mathbb{Z}^d , $d \geq 2$, obtained by taking \mathbb{Z}^d as vertex set and by putting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d.r.v. $\tau = (\tau(e))_{e \in \mathcal{E}}$ indexed by the set of edges \mathcal{E} of the graph. We interpret $\tau(e)$ as the time needed to travel along the edge e (the graph is unoriented).

If a and b are two vertices of \mathbb{Z}^d , we call path from a to b any finite sequence of vertices $r = (a = x_0, \dots, x_k = b)$ such that, for all $i \in \{0, \dots, k-1\}$, the vertices x_i et x_{i+1} are linked by an edge. We denote by $\mathcal{C}(a, b)$ the set of such paths. The time needed to travel along a path $r = (x_0, \dots, x_k)$ is defined by:

$$\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

Then, the time needed to go from a to b is defined by:

$$T(a, b) = \inf\{\tau(r) : r \in \mathcal{C}(a, b)\}.$$

Let e_1, \dots, e_d denote the canonical basis vectors of \mathbb{R}^d . We are interested in the sequence $(a(n))$ defined by :

$$a(n) = E(T(0, ne_1)).$$

We write $T'(0, ne_1)$ and $a'(n)$ for the passage times and expected passage times obtained when the paths are restricted to $\{(x_1, \dots, x_d) : 0 \leq x_i \leq n\}$.

**Postal address:* Université d'Orléans MAPMO B.P. 6759 45067 Orléans Cedex 2 France *E-mail:* jbgouere@univ-orleans.fr

Main result and related results. We denote by S_- the infimum of the support of the distribution of the $\tau(e)$. We denote by S_+ the supremum of the support.

Theorem 1 *Assume $0 < S_-$ and $S_+ \leq 2S_-$. Then the sequence (a_n) is non-decreasing. More precisely, we have:*

$$a(n) \geq a(n-1) + S_- \left[1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

As soon as the distribution of the $\tau(e)$ is not a Dirac distribution there exists, with probability one, infinitely many random N such that $T(0, (N-1)e_1) > T(0, Ne_1)$. However, monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in [3]. In [2], Alm and Wierman proved the monotonicity for $\mathbb{Z} \times \mathbb{N}$ and other 2 dimensional models. In [1], Ahlberg made a detailed study of first passage percolation on essentially one-dimensional graphs, an example of which is $\mathbb{Z} \times \{0, \dots, K\}^{d-1}$. In particular, he proved the existence of a constant n_0 , depending on the graph, such that $n \geq n_0$ implies $a(n) \geq a(n-1)$. In [4], Howard proved the monotocity for an Euclidean first-passage percolation model. We are not aware of any other positive results.

On the other hand, van den Berg proved in [7] that, when $d = 2$, one has $a'(2) < a'(1)$ when $\tau(e) = 1$ with small probability and $\tau(e) = 0$ otherwise. Note that we still have $a'(2) < a'(1)$ if, instead of setting $\tau(e) = 0$ we set $\tau(e) = \varepsilon$ for a small enough ε . A related result was given by Joshi in [6].

We refer to the review by Howard [5] for a more detailed account.

Further remarks.

- The same result holds for the $a'(n)$.
- The proof gives that $T(0, ne_1)$ stochastically dominates the mean of n dependent copies of $T(0, (n-1)e_1)$ (see (8) and (2)).
- With the same strategy one can prove for example the following result:

$$a(n) \geq a(n-1) \text{ as soon as } S_- > 0 \text{ and } \left(\frac{a(n)n^{-1}}{S_-} - 1 \right) \left(\frac{E(\tau(e))}{S_-} - 1 \right) \leq \frac{1}{2} \quad (1)$$

where e is a fixed edge. We show how to adapt the proof of Theorem 1 to prove this result below the proof of Theorem 1. In particular, using the inequality $a(n) \leq nE(\tau(e))$, we get that a is non-decreasing as soon as :

$$S_- > 0 \text{ and } E(\tau(e)) \leq (1 + 2^{-1/2})S_-.$$

This gives a sufficient condition with no assumption on S_+ which can be infinite. However, this sufficient condition is still strong and we do not see how to give any significantly weaker condition.

1. Let us sketch a proof. Fix a and b such that $S_- < a < b < S_+$. For each n , consider a box $\{n-C, \dots, n\} \times \{-D, \dots, D\}^{d-1}$. Let A_n be the following event: $\tau(e) \leq a$ for edges inside the boundary of the box and $\tau(e) \geq b$ for edges inside the box. For suitably chosen large C and D and for $n > C$, we have $T(0, (n-1)e_1) > T(0, ne_1)$ as soon as A_n occurs. As the A_n are local event of fixed positive probability, the result follows.

2. Indeed, $a'(1)$ can only increase while $a'(2)$ increases by at most $\varepsilon E(N)$ where N is the length of a geodesic for the initial passage times. Using $T'(0, 2e_1) \leq 2$ one can check that any geodesic must remain in a random box of subgeometrical height. Therefore $E(N)$ is finite and the result follows.

- Fix the distribution of $\tau(e)$. Assume $S_- > 0$ and $E\tau(e) < \infty$. Then the conditions in (1) are true for large enough n and d . This is due to the fact that $a(n)n^{-1}$ can be made arbitrarily close to S_- .

2 Proofs

Proof of Theorem 1. For all i we consider the following sets of edges:

- H^i : the set of edges $(x, x + e_1)$ where $x = (x_1, \dots, x_d)$ is such that $x_1 = i$.
- V^i : the set of edges $(x, x + e_k)$ where $x_1 = i$ and k belongs to $\{2, \dots, d\}$.

We define new passage times $\tau^i(e)$ as follows:

- If e belongs to H^i then $\tau^i(e) = 0$.
- If e belongs to V^i then $\tau^i(e) = +\infty$.
- Otherwise, $\tau^i(e) = \tau(e)$.

We denote by $T^i(a, b)$ the time needed to travel from a to b with the passage times $\tau^i(e)$. Note, for all $n \geq 1$ and all $i \in \{0, n-1\}$, the following:

$$T^i(0, ne_1) \text{ and } T(0, (n-1)e_1) \text{ have the same distribution.} \quad (2)$$

We now compare $T^i(0, ne_1)$ and $T(0, ne_1)$. Let π be a path from 0 to ne_1 such that $\tau(\pi) = T(0, ne_1)$. We modify this path as follows. Each time the path goes, in this order, through an edge $(x, y) \in V^i$, we replace this part of the path by $(x, x + e_1, y + e_1, y)$. We denote by π^i the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

where, for example, $\text{card}(\pi \cap H_i)$ denotes the number of edges of H_i used by π . The term involving H_i is due to the time saved by the modification of the passage times. The term involving V_i is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i) \quad (3)$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + (S_+ - S_-) \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i). \quad (4)$$

Note

$$\sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq n, \quad (5)$$

as π is a path from 0 to ne_1 . But

$$\begin{aligned} T(0, ne_1) &= \tau(\pi) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- n \end{aligned} \quad (6)$$

and, moreover,

$$\begin{aligned} T(0, ne_1) &\leq \tau(0, e_1, 2e_2, \dots, ne_1) \\ &\leq nS_+. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) &\leq \frac{T(0, ne_1) - nS_-}{S_-} \\ &\leq \frac{nS_+ - nS_-}{S_-}. \end{aligned} \quad (7)$$

From (4) and (7) we get:

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-}. \quad (8)$$

Taking expectations and using (2) we get:

$$na(n-1) \leq na(n) - nS_- \left[1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

The proof follows. \square

Proof of (1). The proof is essentially the same. The main difference lies in the definition of the new passage times $\tau^i(e)$. We let $\tilde{\tau}$ be an independent copy of τ . We then set:

- If e belongs to H^i then $\tau^i(e) = 0$.
- If e belongs to V^i then $\tau^i(e) = +\infty$.
- If e belongs to V^{i+1} then $\tau^i(e) = \tilde{\tau}^i(e)$.
- Otherwise, $\tau^i(e) = \tau(e)$.

Instead of (3) we can write, after taking conditional expectation w.r.t. τ :

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (E(\tau(e)) - S_-) \text{card}(\pi \cap V_i) + (E(\tau(e)) - S_-) \text{card}(\pi \cap V_{i+1}). \quad (9)$$

Instead of (4) we can get :

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + 2(E(\tau(e)) - S_-) \sum_{i=0}^n \text{card}(\pi \cap V_i). \quad (10)$$

Using (5), an equality similar to (6) and taking expectation, we get:

$$na(n-1) \leq na(n) - S_- n + 2(E(\tau(e)) - S_-) \frac{a(n) - nS_-}{S_-}$$

and thus:

$$a(n-1) \leq a(n) - S_- \left(1 - 2 \left(\frac{E(\tau(e))}{S_-} - 1 \right) \left(\frac{a(n)n^{-1}}{S_-} - 1 \right) \right).$$

The proof follows. \square

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