



**HAL**  
open science

## Monotonicity in first-passage percolation

Jean-Baptiste Gouéré

► **To cite this version:**

| Jean-Baptiste Gouéré. Monotonicity in first-passage percolation. 2012. hal-00669299v1

**HAL Id: hal-00669299**

**<https://hal.science/hal-00669299v1>**

Preprint submitted on 12 Feb 2012 (v1), last revised 3 Oct 2012 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Monotonicity in first-passage percolation

Jean-Baptiste Gouéré\*

## Abstract

We consider standard first-passage percolation on  $\mathbb{Z}^d$ . Let  $e_1$  be the first coordinate vector. Let  $a(n)$  be the expected passage time from the origin to  $ne_1$ . In this short paper, we note that  $a(n)$  is increasing under some strong condition on the support of the law of the passage times on the edges.

## 1 Introduction and results

**First passage percolation.** We consider the graph  $\mathbb{Z}^d$ ,  $d \geq 2$ , obtained by taking  $\mathbb{Z}^d$  as vertex set and by putting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d.r.v.  $\tau = (\tau(e))_{e \in \mathcal{E}}$  indexed by the set of edges  $\mathcal{E}$  of the graph. We interpret  $\tau(e)$  as the time needed to travel along the edge  $e$  (the graph is unoriented).

If  $a$  and  $b$  are two vertices of  $\mathbb{Z}^d$ , we call path from  $a$  to  $b$  any finite sequence of vertices  $r = (a = x_0, \dots, x_k = b)$  such that, for all  $i \in \{0, \dots, k-1\}$ , the vertices  $x_i$  et  $x_{i+1}$  are linked by an edge. We denote by  $\mathcal{C}(a, b)$  the set of such paths. The time needed to travel along a path  $r = (x_0, \dots, x_k)$  is defined by:

$$\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

Then, the time needed to go from  $a$  to  $b$  is defined by:

$$T(a, b) = \inf\{\tau(r) : r \in \mathcal{C}(a, b)\}.$$

Let  $e_1, \dots, e_d$  denote the canonical basis vectors of  $\mathbb{R}^d$ . We are interested in the sequence  $(a(n))$  defined by :

$$a(n) = E(T(0, ne_1)).$$

We write  $a'(n)$  for the expected passage times obtained when the paths are restricted to  $\{(x_1, \dots, x_d) : 0 \leq x_1 \leq n\}$ .

---

\*Postal address: Université d'Orléans MAPMO B.P. 6759 45067 Orléans Cedex 2 France *E-mail*: jbgouere@univ-orleans.fr

**Main result and related results.** We denote by  $S_-$  the infimum of the support of the law of the  $\tau(e)$ . We denote by  $S_+$  the supremum of the support.

**Theorem 1** *Assume  $0 < S_-$  and  $S_+ \leq 2S_-$ . Then the sequence  $(a_n)$  is non-decreasing. More precisely, we have:*

$$a(n) \geq a(n-1) + S_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

Monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in [2]. In [1], Alm and Wierman proved the monotonicity for  $\mathbb{Z} \times \mathbb{N}$  and other 2 dimensional models. In [3], Howard proved the monotonicity for an Euclidean first-passage percolation model. We are not aware of any other positive result.

On the other hand, van den Berg prove in [6] that, when  $d = 2$ , one has  $a'(2) < a'(1)$  when  $\tau(e) = 1$  with small probability and  $\tau(e) = 0$  otherwise. A related result was given by Joshi in [5].

We refer to the review by Howard [4] for a more detailed account.

**Further remarks.**

- The same result holds for the  $a'(n)$ .
- The proof gives that  $T(0, ne_1)$  stochastically dominates the mean of  $n$  dependent copies of  $T(0, (n-1)e_1)$  (see (4) and (1)).

## 2 Proof of Theorem 1

For all  $i$  we consider the following sets of edges:

- $H^i$ : the set of edges  $(x, x + e_1)$  where  $x = (x_1, \dots, x_d)$  is such that  $x_1 = i$ .
- $V^i$ : the set of edges  $(x, x + e_k)$  where  $x_1 = i$  and  $k$  belongs to  $\{2, \dots, d\}$ .

We define new passage times  $\tau^i(e)$  as follows:

- If  $e$  belongs to  $H^i$  then  $\tau^i(e) = 0$ .
- If  $e$  belongs to  $V^i$  then  $\tau^i(e) = +\infty$ .
- Otherwise,  $\tau^i(e) = \tau(e)$ .

We denote by  $T^i(a, b)$  the time needed to travel from  $a$  to  $b$  with the passage times  $\tau^i(e)$ . Note, for all  $n \geq 1$  and all  $i \in \{0, n-1\}$ , the following:

$$T^i(0, ne_1) \text{ and } T(0, (n-1)e_1) \text{ have the same distribution.} \tag{1}$$

We now compare  $T^i(0, ne_1)$  and  $T(0, ne_1)$ . Let  $\pi$  be a path from 0 to  $ne_1$  such that  $\tau(\pi) = T(0, ne_1)$ . We modify this path as follows. Each time the path goes, in this order, through an edge  $(x, y) \in V^i$ , we replace this part of the path by  $(x, x + e_1, y + e_1, y)$ . We denote by  $\pi^i$  the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

where, for example,  $\text{card}(\pi \cap H_i)$  denotes the number of edges of  $H_i$  used by  $\pi$ . The term involving  $H_i$  is due to the time saved by the modification of the passage times. The term involving  $V_i$  is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + (S_+ - S_-) \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i). \quad (2)$$

Note

$$\sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq n,$$

as  $\pi$  is a path from 0 to  $ne_1$ . But

$$\begin{aligned} T(0, ne_1) &= \tau(\pi) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- n \end{aligned}$$

and, moreover,

$$\begin{aligned} T(0, ne_1) &\leq \tau(0, e_1, 2e_2, \dots, ne_1) \\ &\leq nS_+. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) &\leq \frac{T(0, ne_1) - nS_-}{S_-} \\ &\leq \frac{nS_+ - nS_-}{S_-}. \end{aligned} \quad (3)$$

From (2) and (3) we get:

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-}. \quad (4)$$

Taking expectations and using (1) we get:

$$na(n-1) \leq na(n) - nS_- \left[ 1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

The proof follows.

## References

- [1] Sven Erick Alm and John C. Wierman. Inequalities for means of restricted first-passage times in percolation theory. *Combin. Probab. Comput.*, 8(4):307–315, 1999. Random graphs and combinatorial structures (Oberwolfach, 1997).

- [2] J. M. Hammersley and D. J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif*, pages 61–110. Springer-Verlag, New York, 1965.
- [3] C. Douglas Howard. Differentiability and monotonicity of expected passage time in Euclidean first-passage percolation. *J. Appl. Probab.*, 38(4):815–827, 2001.
- [4] C. Douglas Howard. Models of first-passage percolation. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 125–173. Springer, Berlin, 2004.
- [5] V. M. Joshi. First-passage percolation on the plane square lattice. *Sankhyā Ser. A*, 39(2):206–209, 1977.
- [6] J. van den Berg. A counterexample to a conjecture of J. M. Hammersley and D. J. A. Welsh concerning first-passage percolation. *Adv. in Appl. Probab.*, 15(2):465–467, 1983.