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Monotonicity in first-passage percolation

Jean-Baptiste Gouéré*

Abstract

We consider standard first-passage percolation on \mathbb{Z}^d . Let e_1 be the first coordinate vector. Let $a(n)$ be the expected passage time from the origin to ne_1 . In this short paper, we note that $a(n)$ is increasing under some strong condition on the support of the law of the passage times on the edges.

1 Introduction and results

First passage percolation. We consider the graph \mathbb{Z}^d , $d \geq 2$, obtained by taking \mathbb{Z}^d as vertex set and by putting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d.r.v. $\tau = (\tau(e))_{e \in \mathcal{E}}$ indexed by the set of edges \mathcal{E} of the graph. We interpret $\tau(e)$ as the time needed to travel along the edge e (the graph is unoriented).

If a and b are two vertices of \mathbb{Z}^d , we call path from a to b any finite sequence of vertices $r = (a = x_0, \dots, x_k = b)$ such that, for all $i \in \{0, \dots, k-1\}$, the vertices x_i et x_{i+1} are linked by an edge. We denote by $\mathcal{C}(a, b)$ the set of such paths. The time needed to travel along a path $r = (x_0, \dots, x_k)$ is defined by:

$$\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

Then, the time needed to go from a to b is defined by:

$$T(a, b) = \inf\{\tau(r) : r \in \mathcal{C}(a, b)\}.$$

Let e_1, \dots, e_d denote the canonical basis vectors of \mathbb{R}^d . We are interested in the sequence $(a(n))$ defined by :

$$a(n) = E(T(0, ne_1)).$$

We write $a'(n)$ for the expected passage times obtained when the paths are restricted to $\{(x_1, \dots, x_d) : 0 \leq x_1 \leq n\}$.

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Main result and related results. We denote by S_- the infimum of the support of the law of the $\tau(e)$. We denote by S_+ the supremum of the support.

Theorem 1 *Assume $0 < S_-$ and $S_+ \leq 2S_-$. Then the sequence (a_n) is non-decreasing. More precisely, we have:*

$$a(n) \geq a(n-1) + S_- \left[1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

Monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in [2]. In [1], Alm and Wierman proved the monotonicity for $\mathbb{Z} \times \mathbb{N}$ and other 2 dimensional models. In [3], Howard proved the monotonicity for an Euclidean first-passage percolation model. We are not aware of any other positive result.

On the other hand, van den Berg prove in [6] that, when $d = 2$, one has $a'(2) < a'(1)$ when $\tau(e) = 1$ with small probability and $\tau(e) = 0$ otherwise. A related result was given by Joshi in [5].

We refer to the review by Howard [4] for a more detailed account.

Further remarks.

- The same result holds for the $a'(n)$.
- The proof gives that $T(0, ne_1)$ stochastically dominates the mean of n dependent copies of $T(0, (n-1)e_1)$ (see (4) and (1)).

2 Proof of Theorem 1

For all i we consider the following sets of edges:

- H^i : the set of edges $(x, x + e_1)$ where $x = (x_1, \dots, x_d)$ is such that $x_1 = i$.
- V^i : the set of edges $(x, x + e_k)$ where $x_1 = i$ and k belongs to $\{2, \dots, d\}$.

We define new passage times $\tau^i(e)$ as follows:

- If e belongs to H^i then $\tau^i(e) = 0$.
- If e belongs to V^i then $\tau^i(e) = +\infty$.
- Otherwise, $\tau^i(e) = \tau(e)$.

We denote by $T^i(a, b)$ the time needed to travel from a to b with the passage times $\tau^i(e)$. Note, for all $n \geq 1$ and all $i \in \{0, n-1\}$, the following:

$$T^i(0, ne_1) \text{ and } T(0, (n-1)e_1) \text{ have the same distribution.} \tag{1}$$

We now compare $T^i(0, ne_1)$ and $T(0, ne_1)$. Let π be a path from 0 to ne_1 such that $\tau(\pi) = T(0, ne_1)$. We modify this path as follows. Each time the path goes, in this order, through an edge $(x, y) \in V^i$, we replace this part of the path by $(x, x + e_1, y + e_1, y)$. We denote by π^i the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

where, for example, $\text{card}(\pi \cap H_i)$ denotes the number of edges of H_i used by π . The term involving H_i is due to the time saved by the modification of the passage times. The term involving V_i is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_- \text{card}(\pi \cap H_i) + (S_+ - S_-) \text{card}(\pi \cap V_i)$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) + (S_+ - S_-) \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i). \quad (2)$$

Note

$$\sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \geq n,$$

as π is a path from 0 to ne_1 . But

$$\begin{aligned} T(0, ne_1) &= \tau(\pi) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap H_i) \\ &\geq S_- \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) + S_- n \end{aligned}$$

and, moreover,

$$\begin{aligned} T(0, ne_1) &\leq \tau(0, e_1, 2e_2, \dots, ne_1) \\ &\leq nS_+. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=0}^{n-1} \text{card}(\pi \cap V_i) &\leq \frac{T(0, ne_1) - nS_-}{S_-} \\ &\leq \frac{nS_+ - nS_-}{S_-}. \end{aligned} \quad (3)$$

From (2) and (3) we get:

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-}. \quad (4)$$

Taking expectations and using (1) we get:

$$na(n-1) \leq na(n) - nS_- \left[1 - \frac{(S_+ - S_-)^2}{S_-^2} \right].$$

The proof follows.

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