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BOUNDARIES OF LEVI-FLAT HYPERSURFACES: SPECIAL HYPERBOLIC POINTS

PIERRE DOLBEAULT

ABSTRACT. Let $S \subset \mathbb{C}^n$, $n \geq 3$ be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is S , possibly as a current. Our goal is to get examples of such S containing at least one special 1-hyperbolic point: sphere with two horns; elementary models and their gluing. The particular cases of graphs are also described.

1. INTRODUCTION

Let $S \subset \mathbb{C}^n$, be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface $M \subset \mathbb{C}^n \setminus S$ such that $dM = S$ (i.e. whose boundary is S , possibly as a current). The case $n = 2$ has been intensively studied since the beginning of the eighties, in particular by Bedford, Gaveau, Klingenberg; Shcherbina, Chirka, G. Tomassini, Slodkowski, Gromov, Eliashberg; it needs global conditions: S has to be contained in the boundary of a strictly pseudoconvex domain.

We consider the case $n \geq 3$; results on this case has been obtained since 2005 by Dolbeault, Tomassini and Zaitsev, local necessary conditions recalled in section 2 have to be satisfied by S , the singular CR points on S are supposed to be elliptic and the solution M is obtained in the sense of currents [DTZ05, DTZ10]. More recently a regular solution M has been obtained when S satisfies a supplementary global condition as in the case $n = 2$ [DTZ09], the singular CR points on S still supposed to be elliptic.

The problem we are interested in is to get examples of such S containing at least one special 1-hyperbolic point (section 2.4). The CR-orbits near a special 1-hyperbolic point are large and, assuming them compact, a careful examination has to be done (sections 2.6, 2.7). As a topological preliminary, we need a generalization of a theorem of Bishop on the difference of the numbers of special elliptic and 1-hyperbolic points (section 2.8); this result is a particular case of a theorem of Hon-Fei Lai [Lai72].

The first considered example is the sphere with two horns which has one special 1-hyperbolic point and three special elliptic points (section 3.4). Then we consider elementary models and their gluing to obtain more complicated examples (section 3.5). Results have been announced in [Dol08], and

in more precise way in [Dol11]; the first aim of this paper is to give complete proofs. Finally, we recall in detail and extend the results of [DTZ09] on regularity of the solution when S is a graph satisfying a supplementary global condition, as in the case $n = 2$, to the case of existence of special 1-hyperbolic points, and to gluing of elementary smooth models (section 4).

2. PRELIMINARIES: LOCAL AND GLOBAL PROPERTIES OF THE BOUNDARY

2.1. Definitions. A smooth, connected, CR submanifold $M \subset \mathbb{C}^n$ is called *minimal* at a point p if there does not exist a submanifold N of M of lower dimension through p such that $HN = HM|_N$. By a theorem of Sussman, all possible submanifolds N such that $HN = HM|_N$ contain, as germs at p , one of the minimal possible dimension, defining a so called CR *orbit* of p in M whose germ at p is uniquely determined.

Let S be a smooth compact connected oriented submanifold of dimension $2n - 2$. S is said to be a *locally flat boundary* at a point p if it locally bounds a Levi-flat hypersurface near p . Assume that S is CR in a small enough neighborhood U of $p \in S$. If all CR orbits of S are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ05]:

- (i) S is a locally flat boundary on U ;
- (ii) S is nowhere minimal on U .

2.2. Complex points of S . (i.e. singular CR points on S) [DTZ05].

At such a point $p \in S$, $T_p S$ is a complex hyperplane in $T_p \mathbb{C}^n$. In suitable local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p , with $w = z_n$ and $z = (z_1, \dots, z_{n-1})$, S is locally given by the equation

$$(1) \quad w = \varphi(z) = Q(z) + O(|z|^3), \quad Q(z) = \sum_{1 \leq i, j \leq n-1} (a_{ij} z_i z_j + b_{ij} z_i \bar{z}_j + c_{ij} \bar{z}_i \bar{z}_j)$$

S is said *flat* at a complex point $p \in S$ if $\sum b_{ij} z_i \bar{z}_j \in \lambda \mathbf{R}, \lambda \in \mathbb{C}$. We also say that p is *flat*.

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with a complex point p . Then p is flat.

By making the change of coordinates $(z, w) \mapsto (z, \lambda^{-1} w)$, we get $\sum b_{ij} z_i z_j \in \mathbf{R}$ for all z . By a change of coordinates $(z, w) \mapsto (z, w + \sum a'_{ij} z_i z_j)$ we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form Q real-valued.

We say that S is in a *flat normal form* at p if the coordinates (z, w) as in (1) are chosen such that $Q(z) \in \mathbf{R}$ for all $z \in \mathbb{C}^{n-1}$.

2.2.1. Properties of Q . Assume that S is in a flat normal form; then, the quadratic form Q is real valued. If Q is positive definite or negative definite, the point $p \in S$ is said to be *elliptic*; if the point $p \in S$ is not elliptic, and if Q is non degenerate, p is said to be *hyperbolic*. From section 2.4, we will only consider particular cases of the quadratic form Q .

2.3. Elliptic points.

2.3.1. Properties of Q .

Proposition 1. ([DTZ05, DTZ10]). *Assume that $S \subset \mathbb{C}^n$, ($n \geq 3$) is nowhere minimal at all its CR points and has an elliptic flat complex point p . Then there exists a neighborhood V of p such that $V \setminus \{p\}$ is foliated by compact real $(2n - 3)$ -dimensional CR orbits diffeomorphic to the sphere \mathbb{S}^{2n-3} and there exists a smooth function ν , having the CR orbits as the level surfaces.*

Sketch of Proof. (see [DTZ10]). In the case of a quadric S_0 ($w = Q(z)$), the CR orbits are defined by $w_0 = Q(z)$, where w_0 is constant. Using (1), we approximate the tangent space to S by the tangent space to S_0 at a point with the same coordinate z ; the same is done for the tangent spaces to the CR orbits on S and S_0 ; then we construct the global CR orbit on S through any given point close enough to p . \square

2.4. Special flat complex points. From [Bis65], for $n = 2$, in suitable local holomorphic coordinates centered at 0, $Q(z) = (z\bar{z} + \lambda \operatorname{Re} z^2)$, $\lambda \geq 0$, under the notations of [BK91]; for $0 \leq \lambda < 1$, p is said to be *elliptic*, and for $1 < \lambda$, it is said to be *hyperbolic*. The parabolic case $\lambda = 1$, not generic, will be omitted [BK91]. When $n \geq 3$, the Bishop's reduction cannot be generalized.

We say that the flat complex point $p \in S$ is *special* if in convenient holomorphic coordinates centered at 0,

$$(2) \quad Q(z) = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \operatorname{Re} z_j^2), \quad \lambda_j \geq 0$$

Let $z_j = x_j + iy_j$, x_j, y_j real, $j = 1, \dots, n-1$, then:

$$(3) \quad Q(z) = \sum_{l=1}^{n-1} ((1 + \lambda_l)x_l^2 + (1 - \lambda_l)y_l^2).$$

A flat point $p \in S$ is said to be *special elliptic* if $0 \leq \lambda_j < 1$ for any j .

A flat point $p \in S$ is said to be *special k -hyperbolic* if $1 < \lambda_j$ for $j \in J \subset \{1, \dots, n-1\}$ and $0 \leq \lambda_j < 1$ for $j \in \{1, \dots, n-1\} \setminus J \neq \emptyset$, where k denotes the number of elements of J .

Special elliptic (resp. special k -hyperbolic) points are elliptic (resp. hyperbolic).

2.5. Special hyperbolic points. S being given by (1), let S_0 be the quadric of equation $w = Q(z)$.

Lemma 2. *Suppose that S_0 is flat at 0 and that 0 is a special k -hyperbolic point. Then, in a neighborhood of 0, and with the above local coordinates, S_0 is CR and nowhere minimal outside 0, and the CR orbits of S_0 are the $(2n - 3)$ -dimensional submanifolds given by $w = \operatorname{const.} \neq 0$.*

Proof. The submanifolds $w = \text{const.} \neq 0$ have the same complex tangent space as S_0 and are of minimal dimension among submanifolds having this property, so they are CR orbits of codimension 1, and from the end of section 2.1, S_0 is nowhere minimal outside 0.

The section $w = 0$ of S_0 is a real quadratic cone Σ'_0 in \mathbf{R}^{2n} whose vertex is 0 and, outside 0, it is a CR orbit Σ_0 in the neighborhood of 0. We will improperly call Σ'_0 a *singular CR orbit*. \square

2.6. Foliation by CR-orbits in the neighborhood of a special 1-hyperbolic point. We first mimic and transpose the beginning of the proof of Proposition 1, i.e. of 2.4.2. in ([DTZ05, DTZ09]).

2.6.1. Local 2-codimensional submanifolds. In order to use simple notations, we will assume $n = 3$.

In \mathbb{C}^3 , consider the 4-dimensional submanifold S locally defined by the equation

$$(1) \quad w = \varphi(z) = Q(z) + O(|z|^3)$$

and the 4-dimensional submanifold S_0 of equation

$$(4) \quad w = Q(z)$$

with

$$Q = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2$$

having a special 1-hyperbolic point at 0, ($\lambda_1 > 1, 0 \leq \lambda_2 < 1$), and the cone Σ'_0 whose equation is: $Q = 0$. On S_0 , a CR orbit is the 3-dimensional submanifold \mathcal{K}_{w_0} whose equation is $w_0 = Q(z)$. If $w_0 > 0$, \mathcal{K}_{w_0} does not cut the line $L = \{x_1 = x_2 = y_2 = 0\}$; if $w_0 < 0$, \mathcal{K}_{w_0} cuts L at two points.

Lemma 3. $\Sigma_0 = \Sigma'_0 \setminus 0$ has two connected components in a neighborhood of 0.

Proof. The equation of $\Sigma'_0 \cap \{y_1 = 0\}$ is

$(\lambda_1 + 1)x_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2 = 0$ whose only zero, in the neighborhood of 0, is $\{0\}$: the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively. \square

2.6.2. CR-orbits. By differentiating (1), we get for the tangent spaces the following asymptotics

$$(5) \quad T_{(z, \varphi(z))}S = T_{(z, Q(z))}S_0 + O(|z|^2), \quad z \in \mathbb{C}^2$$

Here both $T_{(z, \varphi(z))}S$ and $T_{(z, Q(z))}S_0$ depend continuously on z near the origin.

Consider

(i) the hyperboloid $H_- = \{Q = -1\}$, (then $Q(\frac{z}{(-Q(z))^{1/2}}) = -1$), and the projection:

$$\pi_- : \mathbb{C}^3 \setminus \{z = 0\} \rightarrow H_-, \quad (z, w) \mapsto \frac{z}{(-Q(z))^{1/2}},$$

(ii) for every $z \in H_-$, a real orthonormal basis $e_1(z), \dots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_z H_-, \quad e_3(z) \in T_z H_-,$$

where HH_- is the complex tangent bundle to H_- .

Locally such a basis can be chosen continuously depending on z . For every $(z, w) \in \mathbb{C}^3 \setminus \{z = 0\}$, consider the basis $e_1(\pi_-(z, w)), \dots, e_6(\pi_-(z, w))$. The unit vectors $e_1(\pi_-(z, w_0)), e_2(\pi_-(z, w_0)), e_3(\pi_-(z, w_0))$ are tangent to the CR orbit \mathcal{K}_{w_0} in (z, w_0) for $w_0 < 0$. Then, from (5), we have:

$$(6) \quad H_{(z, \varphi(z))} S = H_{(z, Q(z))} S_0 + O(|z|^2), \quad z \neq 0, \quad z \rightarrow 0.$$

As in [DTZ10], in the neighborhood of 0, denote by $E(q)$, $q \in S \setminus \{0\}$, $w < 0$ the tangent space to the local CR orbit \mathcal{K} on S through q , and by $E_0(q_0)$, $q_0 \in S_0 \setminus \{0\}$, $w < 0$ the analogous object for S_0 . We have :

$$(7) \quad E(z, \varphi(z)) = E_0(z, Q(z)) + O(|z|^2), \quad z \neq 0, \quad z \rightarrow 0$$

Given $\underline{q} \in S$, by integration of $E(q)$, $q \in S$, we get, locally, the CR orbit (the leaf), on S through \underline{q} ; given $\underline{q}_0 \in S_0$, by integration of $E_0(q_0)$, $q_0 \in S_0$, we get, locally, the CR orbit (the leaf), on S_0 through \underline{q}_0 (theorem of Sussman). On S_0 , a leaf is the 3-dimensional submanifold $\mathcal{K}_{\underline{q}_0} = \mathcal{K}_{w_0} = \mathcal{K}_0$ whose equation is $w_0 = Q(z)$, with $\underline{q} = (z_0, w_0 = Q(z_0))$. $d\pi_-$ projects each $E_0(q)$, $q \in S_0$, $w < 0$, bijectively onto $T_{\pi(q)} H_-$, then $\pi_-|_{\mathcal{K}_0}$ is a diffeomorphism onto H_- ; this implies, from (7), that, in a suitable neighborhood of the origin, the restriction of π_- to each local CR orbit of S is a local diffeomorphism.

We have: $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$.

2.6.3. Behaviour of local CR orbits. Follow the construction of $E(z, \varphi(z))$; compare with $E_0(z, Q(z))$. We know the integral manifold, the orbit of $E_0(z, Q(z))$; deduce an evaluation of the integral manifold \mathcal{K} of $E(z, \varphi(z))$.

Lemma 4. *Under the above hypotheses, the local orbit Σ corresponding to Σ_0 has two connected components in the neighborhood of 0.*

Proof. Using the real coordinates, as for Lemma 3, consider $\Sigma' \cap \{y_1 = 0\}$. Locally, the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively, from formula (1). \square

We will improperly call $\Sigma' = \bar{\Sigma}$ a *singular CR orbit* and a *singular leaf of the foliation*.

- We intend to prove: 1) \mathcal{K} does not cross the singular leaf through 0;
2) the only separatrix is the singular leaf through 0.

From the orbit \mathcal{K}_0 , construct the differential equation defining it, and using (7), construct the differential equation defining \mathcal{K} .

In \mathbb{C}^3 , we use the notations: $x = x_1, y = y_1, u = x_2, v = y_2$; it suffices to consider the particular case: $Q = 3x^2 - y^2 + u^2 + v^2$. On S_0 , the orbit \mathcal{K}_0

issued from the point $(c, 0, 0, 0)$ is defined by: $3x^2 - y^2 - u^2 + v^2 = 3c^2$, i.e., for $x \geq 0$, $x = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{\frac{1}{2}} = A(y, u, v)$; the local coordinates on the orbit are (y, u, v) . \mathcal{K}_0 satisfies the differential equation: $dx = dA$. From (7), the orbit \mathcal{K} , issued from $(c, 0, 0, 0)$, satisfies $dx = dA + \Psi$ with $\Psi(y, u, v; c) = O(|z|^2)$; hence $\Psi = d\Phi$, then $x = A + \Phi$, with $\Phi = O(|z|^3)$. More explicitly, \mathcal{K} is defined by:

$$x = x_{\mathcal{K},c} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{\frac{1}{2}} + \Phi(y, u, v; c), \quad \Phi(y, u, v; c) = O(|z|^3)$$

The cone Σ'_0 whose equation is: $Q = 0$ is a separatrix for the orbits \mathcal{K}_0 . The corresponding object $\Sigma' = \{\varphi(z) = 0\}$ for S has the singular point 0 and for $x > 0, y > 0, u > 0, v > 0$ is defined by the differential equation $dx = d(A + \Phi)$, with $c = 0$, i.e. the local equation of Σ' is

$$x = x_{\mathcal{K},0} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2)^{\frac{1}{2}} + \Phi(y, u, v; 0), \quad \Phi(y, u, v; 0) = O(|z|^3)$$

For given (y, u, v) , $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} + \Phi(y, u, v; c) - \Phi(y, u, v; 0)$. But $x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(1)$ and $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$.

As a consequence, for $x > 0, y > 0, u > 0, v > 0$, locally, Σ' is a separatrix for the orbits \mathcal{K} , and the only one. Same result for $x < 0$.

2.6.4. What has been done from the hyperboloid $H_- = \{Q = -1\}$ can be repeated from the hyperboloid $H_+ = \{Q = 1\}$.

As at the beginning of the section 2.6.2, we consider

(i) the hyperboloid $H_+ \{Q = 1\}$ and the projection:

$$\pi_+ : \mathbb{C}^3 \setminus \{z = 0\} \rightarrow H_+, \quad (z, w) \mapsto \frac{z}{(Q(z))^{1/2}},$$

(ii) for every $z \in H_+$, a real orthonormal basis $e_1(z), \dots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_z H_+, \quad e_3(z) \in T_z H_+,$$

where HH_+ is the complex tangent bundle to H_+ .

2.6.5.

Lemma 5. *Given φ , there exists $R > 0$ such that, in $B(0, R) \cap \{x > 0, y > 0, u > 0, v > 0\} \subset \mathbb{C}^2$, the CR orbits \mathcal{K} have Σ' as unique separatrix.*

Proof. When c tends to zero, $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K}_0,c} - x_{\mathcal{K}_0,0} = O(|z|)$, $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$. For $\varphi(z) = Q(z) + \Phi(z)$ with $\Phi(z) = O(|z|^3)$ given, in (7), $E(z, \varphi(z)) - E_0(z, Q(z)) = O(|z|^2)$ and $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$ are also given. Then there exists R such that, for $|z| < R$, $x_{\mathcal{K},c} - x_{\mathcal{K},0} > 0$. \square

2.7. CR-orbits near a subvariety containing a special 1-hyperbolic point.

2.7.1. In the section 2.7, we will impose conditions on S and give a local property in the neighborhood of a compact $(2n - 3)$ -subvariety of S .

Assume that $S \subset \mathbb{C}^n$ ($n \geq 3$), is a locally closed $(2n - 2)$ -submanifold, nowhere minimal at all its CR points, which has a unique 1-hyperbolic flat complex point p , and such that:

(i) Σ being the orbit whose closure Σ' contains p , then Σ' is compact.

Let $q \in S$, $q \neq p$; then, in a neighborhood U of q disjoint from p , S is CR, $\text{CR-dim } S = n - 2$, S is non minimal and Σ is 1-codimensional. To show that the CR orbits constitute a foliation on S whose separatrix is Σ' : this is true in U since $\Sigma \cap U$ is a leaf. Moreover, let U_0 the ball $B(0, R)$ centered in $p = 0$ in Lemma 5, if $U \cap U_0 \neq \emptyset$, the leaves in U glue with the leaves in U_0 on $U \cap U_0$. Since Σ' is compact, there exists a finite number of points $q_j \in \Sigma'$, $j = 0, 1, \dots, J$, and open neighborhoods U_j , as above, such that $(U_j)_{j=0}^J$ is an open covering of Σ' . Moreover the leaves on U_j glue respectively with the leaves on U_k if $U_j \cap U_k \neq \emptyset$.

2.7.2.

Proposition 6. *Assume that $S \subset \mathbb{C}^n$ ($n \geq 3$), is a locally closed $(2n - 2)$ -submanifold, nowhere minimal at all its CR points, which has a unique special 1-hyperbolic flat complex point p , and such that:*

(i) Σ being the orbit whose closure Σ' contains p , then Σ' is compact;

(ii) Σ has two connected components σ_1, σ_2 , whose closures are homeomorphic to spheres of dimension $2n - 3$.

Then, there exists a neighborhood V of Σ' such that $V \setminus \Sigma'$ is foliated by compact real $(2n - 3)$ -dimensional CR orbits whose equation, in a neighborhood of p is (3), and, the $w(= x_n)$ -axis being assumed to be vertical, each orbit is diffeomorphic to

the sphere \mathbf{S}^{2n-3} above Σ' ,

the union of two spheres \mathbf{S}^{2n-3} under Σ' ,

and there exists a smooth function ν , having the CR orbits as the level surfaces.

Proof. From subsection 2.7.1 and the following remark:

When x_n tends to 0, the orbits tends to Σ' , and because of the geometry of the orbits near p , they are diffeomorphic to a sphere above Σ' , and to the union of two spheres under Σ' . The existence of ν is proved as in Proposition 1, namely, consider a smooth curve $\gamma : [0, \varepsilon) \rightarrow S$ such that $\gamma(0) = p$, and γ is a diffeomorphism onto its image $\Gamma = \gamma([0, \varepsilon))$. Let $\nu = \gamma^{-1}$ on the image of γ , then, close enough to p , every CR orbit cuts $\gamma([0, \varepsilon))$ at a unique point $q(t)$, $t \in [0, \varepsilon)$. Hence there is a unique extension of ν from $\gamma([0, \varepsilon))$ to a neighborhood of Σ' having CR orbits as its level surfaces. ν being smooth away from p , it is smooth on the orbit Σ and on a neighborhood of $\Sigma \cup \{p\} = \Sigma'$. \square

2.8. Geometry of the complex points of S . The results of section 2.8 are particular cases of theorems of H-F Lai [Lai72], that I learnt from F. Forstneric in July 2011.

In [BK91] E. Bedford & W. Klingenberg cite the following theorem of E. Bishop [Bis65][section 4, p.15]: *On a 2-sphere embedded in \mathbb{C}^2 , the difference between the numbers of elliptic points and of hyperbolic points is the Euler-Poincaré characteristic, i.e. 2.* For the proof, Bishop uses a theorem of ([CS 51], section 4).

We extend the result for $n \geq 3$ and give proofs which are essentially the same than in the general case of [Lai72, Lai74] but simpler.

2.8.1. Let S be a smooth compact connected oriented submanifold of dimension $2n - 2$. Let G be the manifold of the oriented real linear $(2n - 2)$ -subspaces of \mathbb{C}^n . The submanifold S of \mathbb{C}^n has a given orientation which defines an orientation $o(p)$ of the tangent space to S at any point $p \in S$. By mapping each point of S into its oriented tangent space, we get a smooth Gauss map

$$t : S \rightarrow G$$

Denote $-t(p)$ the tangent space to S at p with opposite orientation $-o(p)$.

2.8.2. Properties of G . (a) $\dim G = 2(2n - 2)$.

Proof. G is a two-fold covering of the Grassmannian $M_{m,k}$, of the linear k -subspaces of \mathbb{R}^m [Ste99][Part, section 7.9], for $m = 2n$ and $k = 2n - 2$; they have the same dimension. We have:

$$M_{m,k} \cong O_m/O_k \times O_{m-k}$$

But $\dim O_k = \frac{1}{2}k(k - 1)$, hence $\dim M_{m,k} = \frac{1}{2}(m(m - 1) - k(k - 1) - (m - k)(m - k - 1)) = k(m - k)$.

(b) G has the complex structure of a smooth quadric of complex dimension $(2n - 2)$ of $\mathbb{C}P^{2n-1}$ [Lai74], [Pol08].

(c) There exists a canonical isomorphism $h : G \rightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$.

(d) Homology of G (cf [Pol08]): Let S_1, S_2 be generators of $H_{2n-2}(G, \mathbb{Z})$; we assume that S_1 and S_2 are fundamental cycles of complex projective subspaces of complex dimension $(n - 1)$ of the complex quadric G . We also denote S_1, S_2 the ordered two factors $\mathbb{C}P^{n-1}$, so that $h : G \rightarrow S_1 \times S_2$. □

2.8.3.

Proposition 7. *For $n \geq 2$, in general, S has isolated complex points.*

Proof. Let $\pi \in G$ be a complex hyperplane of \mathbb{C}^n whose orientation is induced by its complex structure; the set of such π is $H = G_{n-1,n}^{\mathbb{C}} = \mathbb{C}P^{n-1*} \subset G$, as real submanifold. If p is a complex point of S , then $t(p) \in H$ or $-t(p) \in H$. The set of complex points of S is the inverse image by t of

the intersections $t(S) \cap H$ and $-t(S) \cap H$ in G . Since $\dim t(S) = 2n - 2$, $\dim H = 2(n - 1)$, $\dim G = 2(2n - 2)$, the intersection is 0-dimensional, in general. \square

2.8.4. Denoting also S , the fundamental cycle of the submanifold S and t_* the homomorphism defined by t , we have:

$$t_*(S) \sim u_1 S_1 + u_2 S_2$$

where \sim means *homologous to*.

2.8.5.

Lemma 8 (proved for $n = 2$ in [CS51]). *With the above notations, we have: $u_1 = u_2$; $u_1 + u_2 = \chi(S)$, Euler-Poincaré characteristic of S .*

The proof for $n = 2$ works for any $n \geq 3$, namely:

Let G' be the manifold of the oriented real linear 2-subspaces of \mathbb{C}^n . Let $\alpha : G \rightarrow G'$ map each oriented $2(n - 1)$ -subspace R onto its normal 2-subspace R' oriented so that R, R' determine the orientation of \mathbb{C}^n . α is a canonical isomorphism. Let $n : S \rightarrow G'$ the map defined by taking oriented normal planes; then: $n = \alpha t$ and $t = \alpha^{-1} n$, hence the mapping $h\alpha h^{-1} : S_1 \times S_2 \rightarrow S_1 \times S_2$. Let (x, y) be a point of $S_1 \times S_2$, then (\dagger) $h\alpha h^{-1}(x, y) = (x, -y)$.

Over G , there is a bundle V of spheres obtained by considering as fiber over a real oriented linear $(2n - 2)$ -subspace of \mathbb{C}^n through 0 the unit sphere S^{2n-3} of this subspace. Let Ω be the characteristic class of V , and let Ω_t , Ω_n denote the characteristic classes of the tangent and normal bundles of S . Then $t^* \Omega = \Omega_t$, $n^* \Omega = \Omega_n$.

V is the Stiefel manifold of ordered pairs of orthogonal unit vectors through in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Let $f : V \rightarrow G$ the projection.

From the Gysin sequence, we see that the kernel of $f^* : H^{2n-2}(G) \rightarrow H^{2n-2}(V)$ is generated by Ω . To find the kernel of f^* , we determine the morphism $f_* : H_{2n-2}(V) \rightarrow H_{2n-2}(G)$. A generating $2n - 2$ -cycle of in V is $S^2 \times e$ where $S^2 \cong \mathbb{C}P^{n-1}$ and e is a point. Let z be any point of S^2 , then from (\dagger) , we have

$$hf(z, e) = (z, -z)$$

Therefore, we see that $f_*(S^2 \times e) = S_1 - S_2$. Then, the kernel of f^* is \mathbb{Z} -generated by $S_1^* + S_2^*$.

With convenient orientation for the fibre of the bundle V , we get: $\Omega = S_1^* + S_2^*$. For convenient orientation of S , we get $\Omega_t.S = \chi_S = \text{Euler characteristic of } S$. We have

$$\Omega_t = t^*(S_1^* + S_2^*) = t^* S_1^* + t^* S_2^*$$

$$\Omega_n = n^*(S_1^* + S_2^*) = t^* \alpha^*(S_1^* + S_2^*) = t^*(S_1^* - S_2^*) = t^* S_1^* - t^* S_2^*$$

Since $\Omega_n = 0$, we get:

$$(t^* S_1^*).S = (t^* S_2^*).S = \frac{1}{2}\chi_S$$

2.8.6. *Local intersection numbers of H and $t(S)$ when all complex points are flat and special.* H is a complex linear $(n-1)$ -subspace of G , then is homologous to one of the S_j , $j = 1, 2$, say S_2 when G has its structure of complex quadric. The intersection number of H and S_1 is 1 and the intersection number of H and S_2 is 0. So, the intersection number of H and $u_1 S_1 + u_2 S_2$ is u_1 .

In the neighborhood of a complex point 0, S is defined by equation (1), with $w = z_n$ and

$$(1') \quad Q(z) = \sum_{j=1}^{n-1} \mu_j (z_j \bar{z}_j + \lambda_j \operatorname{Re} z_j^2), \quad \mu_j > 0, \lambda_j \geq 0$$

Let $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, with real x_l . Let e_l the unit vector of the x_l axis, $l = 1, \dots, 2n$.

For simplicity assume $n = 3$: $Q(z) = \mu_1(z_1 \bar{z}_1 + \lambda_1 \operatorname{Re} z_1^2) + \mu_2(z_2 \bar{z}_2 + \lambda_2 \operatorname{Re} z_2^2)$, with $\mu_1 = \mu_2 = 1$.

Then, up to higher order terms, S is defined by:

$$z_1 = x_1 + ix_2; \quad z_2 = x_3 + ix_4; \quad z_3 = (1 + \lambda_1)x_1^2 + (1 - \lambda_1)x_2^2 + (1 + \lambda_2)x_3^2 + (1 - \lambda_2)x_4^2.$$

In the neighborhood of 0, the tangent space to S is defined by the four independent vectors

$$\begin{aligned} \nu_1 &= e_1 + 2(1 + \lambda_1)x_1 e_5; \quad \nu_2 = e_2 + 2(1 - \lambda_1)x_2 e_5; \quad \nu_3 = e_3 + 2(1 + \lambda_2)x_3 e_5; \\ \nu_4 &= e_4 + 2(1 - \lambda_2)x_4 e_5 \end{aligned}$$

Then, if 0 is special elliptic or special k -hyperbolic with k even, the tangent plane at 0 has the same orientation; if 0 is special elliptic or special k -hyperbolic with k odd the tangent space has opposite orientation.

2.8.7.

Proposition 9 (known for $n = 2$ [Bis65], here for $n \geq 3$). *Let S be a smooth, oriented, compact, 2-codimensional, real submanifold of \mathbb{C}^n whose all complex points are flat and special elliptic or special 1-hyperbolic. Then, on S , \sharp (special elliptic points) - \sharp (special 1-hyperbolic points) = $\chi(S)$. If S is a sphere, this number is 2.*

Proof. Let $p \in S$ be a complex point and π be the tangent hyperplane to S at p . Assume that

(**) *the orientation of S induces, on π , the orientation given by its complex structure,*

then $\pi \in H$.

If p is elliptic, the intersection number of H and $t(S)$ is 1; if p is 1-hyperbolic, the intersection number of H and $t(S)$ is -1 at p .

From the beginning of section 2.8.6, the sum of the intersection numbers of H and $t(S)$ at complex points p satisfying $(**)$ is u_1 . Reversing the condition $(**)$, and using Lemma 8, we get the Proposition. \square

3. PARTICULAR CASES: HORNED SPHERE; ELEMENTARY MODELS AND THEIR GLUING

3.1. We recall the following Harvey-Lawson theorem with real parameter to be used later.

3.1.1. Let $E \cong \mathbf{R} \times \mathbb{C}^n$, and $k : \mathbf{R} \times \mathbb{C}^n \rightarrow \mathbf{R}$ be the projection. Let $N \subset E$ be a compact, (oriented) CR subvariety of \mathbb{C}^{n+1} of real dimension $2n - 2$ and CR dimension $n - 2$, ($n \geq 3$), of class C^∞ , with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n - 2)$ -dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold). Let τ' be the set of all points $z \in N$ such that either $z \in \tau$ or $z \in N \setminus \tau$ and N is not transversal to the complex hyperplane $k^{-1}(k(z))$ at z . Assume that N , as a current of integration, is d -closed and satisfies:

(H) there exists a closed subset $L \subset \mathbb{R}_{x_1}$ with $H^1(L) = 0$ such that for every $x \in k(N) \setminus L$, the fiber $k^{-1}(x) \cap N$ is connected and does not intersect τ' .

3.1.2.

Theorem 10 ([DTZ10] (see also [DTZ05])). *Let N satisfy (H) with L chosen accordingly. Then, there exists, in $E' = E \setminus k^{-1}(L)$, a unique C^∞ Levi-flat $(2n - 1)$ -subvariety M with negligible singularities in $E' \setminus N$, foliated by complex $(n - 1)$ -subvarieties, with the properties that M simply (or trivially) extends to E' as a $(2n - 1)$ -current (still denoted M) such that $dM = N$ in E' . The leaves are the sections by the hyperplanes $E_{x_1^0}$, $x_1^0 \in k(N) \setminus L$, and are the solutions of the “Harvey-Lawson problem” for finding a holomorphic subvariety in $E_{x_1^0} \cong \mathbb{C}^n$ with prescribed boundary $N \cap E_{x_1^0}$.*

3.1.3.

Remark 11. *Theorem 10 is valid in the space $E \cap \{\alpha_1 < x_1 < \alpha_2\}$, with the corresponding condition (H). Moreover, since N is compact, for convenient coordinate x_1 , we can assume $x_1 \in [0, 1]$.*

3.2. *To solve the boundary problem by Levi-flat hypersurfaces, S has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.*

3.3. Sphere with one special 1-hyperbolic point (sphere with two horns): Example.

3.3.1. In \mathbb{C}^3 , let (z_j) , $j = 1, 2, 3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbf{R}^6 \cong \mathbb{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) S defined by:

$$\begin{aligned} y_3 &= 0 \\ 0 \leq x_3 \leq 1; \quad &x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + \\ &4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0 \\ -1 \leq x_3 \leq 0; \quad &x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 \end{aligned}$$

The singular set of S is the 3-dimensional section $x_3 = 0$ along which the tangent space is not everywhere (uniquely) defined. S being in the real hyperplane $\{y_3 = 0\}$, the complex tangent spaces to S are $\{x_3 = x^0\}$ for convenient x^0 .

3.3.2. The tangent space to the hypersurface $f(x_1, y_1, x_2, y_2, x_3) = 0$ in \mathbb{R}^5 is

$$X_1 f'_{x_1} + Y_1 f'_{y_1} + X_2 f'_{x_2} + Y_2 f'_{y_2} + X_3 f'_{x_3} = 0,$$

Then, the tangent space to S in the hyperplane $\{y_3 = 0\}$ is:

for $0 \leq x_3$,

$$\begin{aligned} 2x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)]X_1 + 2y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)]Y_1 \\ + 2x_2[x_3 + (1 - x_3)(2x_2^2 + 1)]X_2 + 2y_2[x_3 + (1 - x_3)(2y_2^2 + 1)]Y_2 \\ + [(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 3x_3^2 - 1) \\ - (x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2)]X_3 = 0; \end{aligned}$$

for $x_3 \leq 0$,

$$4(x_1^2 + 2)x_1X_1 + 4(y_1^2 - 1)y_1Y_1 + 2(2x_2^2 + 1)x_2X_2 + 2(2y_2^2 + 1)y_2Y_2 - X_3 = 0.$$

3.3.3. The complex points of S are defined by the vanishing of the coefficients of X_j , $j=1,2,3,4$ in the equation of the tangent spaces for $0 \leq x_3 \leq 1$,

$$\begin{aligned} x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)] &= 0, \\ y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)] &= 0, \\ x_2[x_3 + (1 - x_3)(2x_2^2 + 1)] &= 0, \\ y_2[x_3 + (1 - x_3)(2y_2^2 + 1)] &= 0. \end{aligned}$$

We have the solutions

h : $x_j = 0, y_j = 0$, ($j = 1, 2$), $x_3 = 0$;

e_3 : $x_j = 0, y_j = 0$, ($j = 1, 2$), $x_3 = 1$.

for $x_3 \leq 0$,

$$\begin{aligned} (x_1^2 + 2)x_1 &= 0, \\ (y_1^2 - 1)y_1 &= 0, \\ (2x_2^2 + 1)x_2 &= 0, \\ (2y_2^2 + 1)y_2 &= 0. \end{aligned}$$

We have the solutions

$$\begin{aligned} h: & x_j = 0, y_j = 0, (j = 1, 2), x_3 = 0; \\ e_1, e_2: & x_1 = 0, y_1 = \pm 1, x_2 = 0, y_2 = 0, x_3 = -1. \end{aligned}$$

Remark that the tangent space to S at h is well defined. Moreover, the set S will be smoothed along its section by the hyperplane $\{x_3 = 0\}$ by a small deformation leaving h unchanged. In the following S will denote this smooth submanifold.

3.3.4.

Lemma 12. *The points e_1, e_2, e_3 are special elliptic; the point h is special $\{1\}$ -hyperbolic.*

Proof. Point e_3 : Let $x'_3 = 1 - x_3$, then the equation of S in the neighborhood of e_3 is:

$$(1 - x'_3)(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3'^2 - 2x'_3) - x'_3(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0, \text{ i.e.}$$

$$2x'_3 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + O(|z|^3), \text{ or } w = z\bar{z} + O(|z|^3)$$

then e_3 is special elliptic.

Points e_1, e_2 : Let $y'_1 = y_1 \pm 1$, $x'_3 = x_3 + 1$, then the equation of S in the neighborhood of e_1, e_2 is:

$$\begin{aligned} x'_3 - 1 &= x_1^4 + (y'_1 \mp 1)^4 + x_2^4 + y_2^4 + 4x_1^2 - 2(y'_1 \mp 1)^2 + x_2^2 + y_2^2 \\ &= x_1^4 + y_1^4 \mp 4y_1^3 + 6y_1^2 \mp 4y_1 + 1 + x_2^4 + y_2^4 + 4x_1^2 - 2(y'_1 \mp 1)^2 + x_2^2 + y_2^2, \end{aligned}$$

then

$$x'_3 = x_1^4 + y_1^4 \mp 4y_1^3 + 4y_1^2 + x_2^4 + y_2^4 + 4x_1^2 + x_2^2 + y_2^2, \text{ i.e.}$$

$$x'_3 = 4x_1^2 + 4y_1^2 + x_2^2 + y_2^2 + O(|z|^3), \text{ or } w = 4z_1\bar{z}_1 + z_2\bar{z}_2,$$

then e_1, e_2 are special elliptic.

Point h : The equation of S in the neighborhood of h is:

for $x_3 \geq 0$,

$$\begin{aligned} & x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) \\ & + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0 \end{aligned}$$

for $x_3 \leq 0$,

$$x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2, \text{ i.e.}$$

$x_3 = 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 + O(|z|^3)$, in both cases, up to the third order terms, i.e.: $w = z_1\bar{z}_1 + z_2\bar{z}_2 + 3\operatorname{Re} z_1^2$,

then h is special $\{1\}$ -hyperbolic. \square

3.3.5. *Section $\Sigma' = S \cap \{x_3 = 0\}$. Up to a small smooth deformation, its equation is:*

$$x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0, \text{ in } \{x_3 = 0\}.$$

The tangent cone to Σ' at 0 is: $4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0$.

Locally, the section of S by the coordinate 3-space

$$x_1, y_1, x_3 \text{ is: } x_3 = 4x_1^2 - 2y_1^2 + O(|z|^3)$$

x_2, y_2, x_3 is: $x_3 = x_2^2 + y_2^2 + O(|z|^3)$

3.3.1'. *Shape of $\Sigma' = S \cap \{x_3 = 0\}$ in the neighborhood of the origin 0 of \mathbb{C}^3 .*

Lemma 13. *Under the above hypotheses and notations,*

(i) $\Sigma = \Sigma' \setminus 0$ has two connected components σ_1, σ_2 .

(ii) *The closures of the three connected components of $S \setminus \Sigma'$ are submanifolds with boundaries and corners.*

Proof. (i) The only singular point of Σ' is 0. We work in the ball $B(0, A)$ of \mathbb{C}^2 (x_1, y_1, x_2, y_2) for small A and in the 3-space $\pi_\lambda = \{y_2 = \lambda x_2\}$, $\lambda \in \mathbb{R}$. For λ fixed, $\pi_\lambda \cong \mathbb{R}^3(x_1, y_1, x_2)$, and $\Sigma' \cap \pi_\lambda$ is the cone of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$ with vertex 0 and basis in the plane $x_2 = x_2^0$ the hyperboloid H_λ of equation $4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0$; the curves H_λ have no common point outside 0. So, when λ varies, the surfaces $\Sigma' \cap \pi_\lambda$ are disjoint outside 0. The set Σ' is clearly connected; $\Sigma' \cap \{y_1 = 0\} = \{0\}$, the origin of \mathbb{C}^3 ; from above: $\sigma_1 = \Sigma \cap \{y_1 > 0\}$; $\sigma_2 = \Sigma \cap \{y_1 < 0\}$.

(ii) The three connected components of $S \setminus \Sigma'$ are the components which contain, respectively e_1, e_2, e_3 and whose boundaries are $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_1 \cup \bar{\sigma}_2$; these boundaries have corners as shown in the first part of the proof. \square

The connected component of $\mathbb{C}^2 \times \mathbb{R} \setminus S$ containing the point $(0, 0, 0, 0, 1/2)$ is the Levi-flat solution, the complex leaves being the sections by the hyperplanes $x_3 = x_3^0$, $-1 < x_3^0 < 1$.

The sections by the hyperplanes $x_3 = x_3^0$ are diffeomorphic to a 3-sphere for $0 < x_3^0 < 1$ and to the union of two disjoint 3-spheres for $-1 < x_3^0 < 0$, as can be shown intersecting S by lines through the origin in the hyperplane $x_3 = x_3^0$; Σ' is homeomorphic to the union of two 3-spheres with a common point.

3.4. Sphere with one special 1-hyperbolic point (sphere with two horns). The example of section 3.3 shows that the necessary conditions of section 2 can be realised. Moreover, from Proposition 2.8.7, the hypothesis on the number of complex points is meaningful.

3.4.1.

Proposition 14. *[cf [Dol08][Proposition 2.6.1]] Let $S \subset \mathbb{C}^n$ be a compact connected real 2-codimensional manifold such that the following holds:*

(i) *S is a topological sphere; S is nonminimal at every CR point;*

(ii) *every complex point of S is flat; there exist three special elliptic points $e_j, j = 1, 2, 3$ and one special 1-hyperbolic point h ;*

(iii) *S does not contain complex manifolds of dimension $(n - 2)$;*

(iv) *the singular CR orbit Σ' through h on S is compact and $\Sigma' \setminus \{h\}$ has two connected components σ_1 and σ_2 whose closures are homeomorphic to spheres of dimension $2n - 3$;*

(v) *the closures S_1, S_2, S_3 of the three connected components S'_1, S'_2, S'_3 of $S \setminus \Sigma'$ are submanifolds with (singular) boundary.*

Then each $S_j \setminus \{e_j \cup \Sigma'\}$, $j = 1, 2, 3$ carries a foliation \mathcal{F}_j of class C^∞ with 1-codimensional CR orbits as compact leaves.

Proof. From conditions (i) and (ii), S satisfying the hypotheses of Proposition 1, near any elliptic flat point e_j , and of Proposition 6 near Σ' , all CR orbits being diffeomorphic to the sphere \mathbf{S}^{2n-3} . The assumption (iii) guarantees that all CR orbits in S must be of real dimension $2n - 3$. Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of Σ' , we obtain, from $S \setminus \Sigma'$, three compact manifolds S_j'' , $j = 1, 2, 3$, with boundary and with the foliation \mathcal{F}_j of codimension 1 given by its CR orbits whose first cohomology group with values in \mathbf{R} is 0, near e_j . It is easy to show that this foliation is transversely oriented. \square

3.4.2. Recall the Thurston's Stability Theorem ([CaC], Theorem 6.2.1).

Proposition 15. *Let (M, \mathcal{F}) be a compact, connected, transversely-orientable, foliated manifold with boundary or corners, of codimension 1, of class C^1 . If there is a compact leaf L with $H^1(L, \mathbf{R}) = 0$, then every leaf is homeomorphic to L and M is homeomorphic to $L \times [0, 1]$, foliated as a product,*

Then, from the above theorem, S_j'' is homeomorphic to $\mathbf{S}^{2n-3} \times [0, 1]$ with CR orbits being of the form $\mathbf{S}^{2n-3} \times \{x\}$ for $x \in [0, 1]$. Then the full manifold S_j is homeomorphic to a half-sphere supported by \mathbf{S}^{2n-2} and \mathcal{F}_j extends to S_j ; S_3 having its boundary pinched at the point h . \square

3.4.3.

Theorem 16. *Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying the conditions (i) to (v) of Proposition 14. Then there exists a Levi-flat $(2n - 1)$ -subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary \tilde{S} (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between \tilde{S} and S outside the complex points of S .*

Proof. By Proposition 1, for every e_j , a continuous function ν'_j , C^∞ outside e_j , can be constructed in a neighborhood U_j of e_j , $j = 1, 2, 3$, and by Proposition 6, we have an analogous result in a neighborhood of Σ' . Furthermore, from Proposition 15, a smooth function ν''_j whose level sets are the leaves of \mathcal{F}_j can be obtained globally on $S'_j \setminus \{e_j \cup \Sigma'\}$. With the functions ν'_j and ν''_j , and analogous functions near Σ' , then using a partition of unity, we obtain a global smooth function $\nu_j : S_j \rightarrow \mathbf{R}$ without critical points away from the complex points e_j and from Σ' .

Let σ_1 , resp. σ_2 be the two connected, relatively compact components of $\Sigma \setminus \{h\}$, according to condition (iv); $\bar{\sigma}_1$, resp. $\bar{\sigma}_2$ are the boundary of S_1 , resp. S_2 , and $\bar{\sigma}_1 \cup \bar{\sigma}_2$ the boundary of S_3 . We can assume that the three

functions ν_j are finite valued and get the same values on $\bar{\sigma}_1$ and $\bar{\sigma}_2$. Hence a function $\nu : S \rightarrow \mathbf{R}$.

The submanifold S being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{S} = N = \text{gr } \nu = \{(\nu(z), z) : z \in S\}$. Let $S_s = \{e_1, e_2, e_3, \overline{\sigma_1 \cup \sigma_2}\}$.

$\lambda : S \rightarrow \tilde{S}$ ($z \mapsto \nu((z), z)$) is bicontinuous; $\lambda|_{S \setminus S_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on S . Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(S_s)$.

At every point of $S \setminus S_s$, $d_{x_1} \nu \neq 0$, then condition (H) (section 3.1.1) is satisfied at every point of $N \setminus \tau$.

Then all the assumptions of Theorem 10 being satisfied by $N = \tilde{S}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$ -variety (with negligible singularities) \tilde{M} in $\mathbf{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. \square

3.5. Generalizations: elementary models and their gluing.

3.5.1. The examples and the proofs of the theorems when S is homeomorphic to a sphere (sections 3.4) suggest the following definitions.

3.5.2. Definitions. Let T' be a smooth, locally closed (i.e. closed in an open set), connected submanifold of \mathbb{C}^n , $n \geq 3$. We assume that T' has the following properties:

- (i) T' is relatively compact, non necessarily compact, and of codimension 2.
- (ii) T' is nonminimal at every CR point.
- (iii) T' does not contain complex manifold of dimension $(n-2)$.
- (iv) T' has exactly 2 complex points which are flat and either special elliptic or special 1-hyperbolic.
- (v) If $p \in T'$ is special 1-hyperbolic, the singular orbit Σ' through p is compact, $\Sigma' \setminus p$ has two connected components σ_1, σ_2 , whose closures are homeomorphic to spheres of dimension $2n-3$.
- (vi) If $p \in T'$ is special 1-hyperbolic, in the neighborhood of p , with convenient coordinates, the equation of T' , up to third order terms is

$$z_n = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \text{Re } z_j^2); \quad \lambda_1 > 1; \quad 0 \leq \lambda_j < 1 \quad \text{for } j \neq 1$$

or in real coordinates x_j, y_j with $z_j = x_j + iy_j$,

$$x_n = ((\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2) + \sum_{j=2}^{n-1} ((1 + \lambda_j)x_j^2 + (1 - \lambda_j)y_j^2) + O(|z|^3)$$

- (vii) the closures, in T' , T_1, T_2, T_3 of the three connected components T'_1, T'_2, T'_3 of $T' \setminus \Sigma'$ are submanifolds with (singular) boundary. Let T''_j , $j = 1, 2, 3$ be neighborhoods of the T'_j in T' .

up- and down- 1-hyperbolic points. Let τ be the $(2n - 2)$ -submanifold with (singular) boundary contained into T' such that either $\bar{\sigma}_1$ (resp. $\bar{\sigma}_2$) is the boundary of τ near p , or Σ' is the boundary of τ near p . In the first case, we say that p is 1-up, (resp. 2-up), in the second that p is down. If T' is contained in a small enough neighborhood of Σ' in \mathbb{C}^n , such a T' will be called a *local elementary model*, more precisely it defines a *germ of elementary model around Σ* .

The union T of T_1 , T_2 , T_3 and of the germ of elementary model around the singular orbit at every special 1-hyperbolic point is called an *elementary model*. T behaves as a locally closed submanifold still denoted T .

3.5.3. Examples of elementary models. We will say that T is a *elementary model of type*:

- (a) if it has: two elliptic points;
- (b) if it has: one special elliptic point and one down- $\{1\}$ -hyperbolic point;
- (c₁) if it has: one special elliptic point and one 1-up- $\{1\}$ -hyperbolic point;
- (c₂) if it has: one special elliptic point and one 2-up- $\{1\}$ -hyperbolic point;
- (d₁) if it has: two special 1-up- $\{1\}$ -hyperbolic points;
- (d₂) if it has: two special 2-up- $\{1\}$ -hyperbolic points;
- (e) if it has: two special down- $\{1\}$ -hyperbolic points;

Other configurations are easily imagined.

The prescribed boundary of a Levi-flat hypersurface of \mathbb{C}^n in [DTZ05] and [DTZ10], whose complex points are flat and elliptic, is an elementary model of type (a).

3.5.4. Properties of elementary models. For instance, T is 1-up and has one special elliptic point, we solve the boundary problem as in S_1 in the proof of Theorem 16.

Proposition 17. *Let T be a local elementary model. Then, T carries a foliation \mathcal{F} of class C^∞ with 1-codimensional CR orbits as compact leaves.*

Proof. From the definition at the end of section 3.5.2 and Proposition 6. \square

3.5.5.

Theorem 18. *Let T be the elementary model there exists an open neighborhood T'' in T' carrying a smooth function $\nu : T'' \rightarrow \mathbb{R}$ whose level sets are the leaves of a smooth foliation.*

Proof. By removing small connected open saturated neighborhoods of every special elliptic point, and of Σ' , the singular orbit through every special 1-hyperbolic point p , we obtain, from $T \setminus \Sigma'$, three manifolds T_j'' , $j = 1, 2, 3$, with boundary,

- (a) T_1 and T_2 containing one special elliptic point e or one special 1-hyperbolic point with the foliations \mathcal{F}_1 , \mathcal{F}_2 , from Propositions 1 and 17,

(b) T_3'' with the foliation \mathcal{F}_3 of codimension 1 given by its CR orbits whose first cohomology group with values in \mathbf{R} is 0, near e , or p . It is easy to show that this later foliation is transversely oriented.

From the Thurston's Stability Theorem (see section 3.4.2), T_3'' is homeomorphic to $\mathbf{S}^{2n-3} \times [0, 1]$, foliated as a product, with CR orbits being of the form $\mathbf{S}^{2n-3} \times \{x\}$ for $x \in [0, 1]$; hence smooth functions ν_1, ν_2, ν_3 , whose level sets are the leaves of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively, and using a partition of unity the desired function ν on T .

□

3.6.

Theorem 19. *Let T be an elementary model. Then there exists a Levi-flat $(2n-1)$ -subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary \tilde{T} (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between \tilde{T} and T outside the complex points of T .*

Proof. The submanifold T being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{T} = N = \text{gr } \nu = \{(\nu(z), z) : z \in S\} \subset E \cong \mathbf{R} \times \mathbb{C}^{n-1}$. Let T_s be the union of the flat complex points of T .

$\lambda : T \rightarrow \tilde{T}$ ($z \mapsto \nu((z), z)$) is bicontinuous; $\lambda|_{T \setminus T_s}$ is a diffeomorphism; moreover λ is a CR map. Choose an orientation on T . Then N is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(T_s)$.

Using Remark 11, at every point of $T \setminus T_s$, $d_{x_1} \nu \neq 0$, we see that condition (H) (section 3.1.1) is satisfied at every point of $N \setminus \tau$.

Then all the assumptions of Theorem 10 being satisfied by $N = \tilde{T}$, in a particular case, we conclude that N is the boundary of a Levi-flat $(2n-2)$ -variety (with negligible singularities) \tilde{M} in $\mathbf{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. □

3.7. Gluing of elementary models.

3.7.1. The gluing happens between two compatible elementary models along boundaries, for instance down and 1-up. Remark that the gluing can only be made at special 1-hyperbolic points. More precisely, it can be defined as follows.

The assumed properties of the submanifold S in section 2 in \mathbb{C}^n have a meaning in any complex analytic manifold X of complex dimension $n \geq 3$, and are kept under any holomorphic isomorphism.

We will define a submanifold S' of X obtained by gluing of elementary models by induction on the number m of models. An elementary model T in X is the image of an elementary model T_0 in \mathbb{C}^n by an analytic isomorphism of a neighborhood of T_0 in \mathbb{C}^n into X .

3.7.2. Let S' be a closed smooth real submanifold of X of dimension $2n - 2$ which is non minimal at every CR point. Assume that S' is obtained by gluing of m elementary models.

a) S' has a finite number of flat complex points, some special elliptic and the others special 1-hyperbolic;

b) for every special 1-hyperbolic p' , there exists a CR-isomorphism h induced by a holomorphic isomorphism of the ambient space \mathbb{C}^n from a neighborhood of p in T' onto a neighborhood of p' in S' .

c) for every CR-orbit $\Sigma_{p'}$ whose closure contains a special 1-hyperbolic point p' , there exists a CR-isomorphism h induced by a holomorphic isomorphism of the ambient space \mathbb{C}^n from a neighborhood of $\Sigma_p = \Sigma'_p \setminus p$ in T' onto a neighborhood V of $\Sigma_{p'}$ in S' .

Every special 1-hyperbolic point of S' which belongs to only one elementary model in S' will be called *free*.

We will define the gluing of one more elementary model to S' .

3.7.3. *Gluing an elementary model T of type (d_1) to a free down-1-hyperbolic point of S' .* Let h_1 be a CR-isomorphism from a neighborhood V_1 of $\bar{\sigma}'_1$ induced by a holomorphic isomorphism of the ambient space \mathbb{C}^n onto a neighborhood of σ_1 in S' . Let k_1 be a CR-isomorphism from a neighborhood T''_1 of T'_1 into X such that $k_1|V_1 = h_1$.

3.7.4.

Theorem 20. *The compact manifold or the manifold with singular boundary S' , obtained by the gluing of a finite number of elementary models, is the boundary of a Levi-flat hypersurface of X in the sense of currents.*

Proof. From Theorem 19 and the definition of gluing. \square

3.8. Examples of gluing. Denoting the gluing of the two models of type (d_1) and (d_2) to a free down-1-hyperbolic point of S' by: $\rightarrow (d_1) - (d_2)$, and the converse by: $(d_1) - (d_2) \rightarrow$, and, also, analogous configurations in the same way, we get:

torus: $(b) \rightarrow (d_1) - (d_2) \rightarrow (b)$; the Euler-Poincaré characteristic of a torus is $\chi(\mathbf{T}^k) = 0$: 2 special elliptic and 2 special 1-hyperbolic points.

bitorus: $(b) \rightarrow (d_1) - (d_2) \rightarrow (e) \rightarrow (d_1) - (d_2) \rightarrow (b)$.

4. CASE OF GRAPHS

(see [DTZ09] for the case of elliptic points only, and dropping the property of the function solution to be Lipschitz).

4.1. We want to add the following hypothesis: S is embedded into the boundary of a strictly pseudoconvex domain of \mathbb{C}^n , $n \geq 3$, and more precisely, let (z, w) be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \dots, z_{n-1})$, $w = u + iv = z_n$, let Ω be a strictly pseudoconvex domain of $\mathbb{C}^{n-1} \times \mathbb{R}_u$ (i.e. the second fundamental form of the boundary $b\Omega$ of Ω is everywhere positive

definite); let S be the graph $gr(g)$ of a smooth function $g : b\Omega \rightarrow \mathbb{R}_v$. notice that $b\Omega \times \mathbb{R}_v$ contains S and is strictly pseudoconvex.

Assume that S is a *horned sphere* (section 3.4), *satisfying the hypotheses* of Theorem 16. Denote by p_j , $j = i, \dots, 4$ the complex points of S . Our aim is to prove

4.2.

Theorem 21. *Let S be the graph of a smooth function $g : b\Omega \rightarrow \mathbb{R}_v$. Let $Q = (q_1, \dots, q_4) \in b\Omega$ be the projections of the complex points $P = (p_1, \dots, p_4)$ of S , respectively. Then, there exists a continuous function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f|_{b\Omega} = g$, and $M_0 = \text{graph}(f) \setminus S$ is a smooth Levi flat hypersurface of \mathbb{C}^n . Moreover, each complex leaf of M_0 is the graph of a holomorphic function $\phi : \Omega' \rightarrow \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and ϕ is smooth on $\overline{\Omega}'$.*

The natural candidate to be the graph M of f is $\pi(\tilde{M})$ where \tilde{M} and π are as in Theorem 16. We prove that this is the case proceeding in several steps.

4.3. Behaviour near S .

4.3.1. *Assume that D is a strictly pseudoconvex domain and that $S \subset bD$.*

Recall ([HL75][Theorem 10.4]: *Let D be a strictly pseudoconvex domain of \mathbb{C}^n , $n \geq 3$ with boundary bD , $\Sigma \subset bD$ be a compact connected maximally complex smooth $(2d-1)$ -submanifold with $d \geq 2$. Then, Σ is the boundary of a uniquely determined relatively compact subset $V \subset \overline{D}$ such that $\overline{V} \setminus \Sigma$ is a complex analytic subset of D with finitely many singularities of pure dimension $\leq d-1$, and near Σ , \overline{V} is a d -dimensional complex manifold with boundary.*

V is said to be *the solution of the boundary problem for Σ* .

4.3.2.

Lemma 22 ([DTZ09]). *Let Σ_1, Σ_2 be compact connected maximally complex $(2d-1)$ -submanifolds of bD . Let V_1, V_2 be the corresponding solutions of the boundary problem. If $d \geq 2$, $2d \geq n+1$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$, then $V_1 \cap V_2 = \emptyset$.*

Let Σ be a CR orbit of the foliation of $S \setminus P$. Then Σ is a compact maximally complex $(2n-3)$ -dimensional real submanifold of \mathbb{C}^n contained in bD . Let $V = V_\Sigma$ be the solution of the boundary problem corresponding to Σ . From Theorem 16, $V = \pi(\tilde{V})$, where $\tilde{V} = (\tilde{M} \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{x\})$ for suitable $x \in (0, 1)$, the projection on the x -axis being finite, we can always assume that it lies into $(0, 1)$. Moreover $\pi|_{\tilde{V}}$ is a biholomorphism $\tilde{V} \cong V$ and $M \setminus S \subset D$.

Let Σ_1, Σ_2 be two distinct orbits of the foliation of $S \setminus P$, and $\overline{V}_1, \overline{V}_2$ the corresponding leaves, then, from Lemma 22, $\overline{V}_1 \cap \overline{V}_2 = \emptyset$.

4.3.3. Assume that S satisfies the full hypotheses of Theorem 21.

Set $m_1 = \min_S g$, $m_2 = \max_S g$ and $r \gg 0$ such that

$$D = \Omega \times [m_1, m_2] \subset \subset \mathbf{B}(\mathbf{r}) \cap (\Omega \times i\mathbb{R}_v)$$

where $\mathbf{B}(\mathbf{r})$ is the ball $\{|(z, w)| < r\}$.

4.3.4.

Lemma 23. Let $p \in S$ be a CR point. Then, near p , M is the graph of a function ϕ on a domain $U \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_u$ which is smooth up to the boundary of U .

Proof. Near p , each CR orbit Σ is smooth and can be represented as the graph of a CR function over a strictly pseudoconvex hypersurface and V_Σ as the graph of the local holomorphic extension of this function. From Hopf lemma, V is transversal to the strictly pseudoconvex hypersurface $d\Omega \times i\mathbb{R}_v$ near p . Hence the family of the V_Σ , near p , forms a smooth real hypersurface with boundary on S that is the graph of a smooth function ϕ from a relative open neighborhood U of p on $\overline{\Omega}$ into \mathbb{R}_v . Finally, Lemma 22 guarantees that this family does not intersect any other leaf V from M . \square

4.3.5.

Corollary 24. If $p \in S$ is a CR point, each complex leaf V of M , near p , is the graph of a holomorphic function on a domain $\Omega_V \subset \mathbb{C}_z^{n-1}$, which is smooth up to the boundary of Ω_V .

4.4. Solution as a graph of a continuous function.

4.4.1. Recall results of Shcherbina [Shc93] from:

(a) the Main Theorem:

Let G be a bounded strictly convex domain in $\mathbb{C}_z \times \mathbb{R}_u$ ($z \in \mathbb{C}$) and $\varphi : bG \rightarrow \mathbb{R}_v$ be a continuous function. Then the following properties hold, where $\Gamma = gr$, and $\hat{\Gamma}(\varphi)$ means polynomial hull of $\Gamma(\varphi)$:

(a_i) the set $\hat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is the union of a disjoint family of complex discs $\{D_\alpha\}$;

(a_{ii}) for each α , there is a simply connected domain $\Omega_\alpha \subset \mathbb{C}_z$ and a holomorphic function $w = f_\alpha$, defined on Ω_α , such that D_α is the graph of f_α .

(a_{iii}) For each f_α , there exists an extension $f_\alpha^* \in C(\overline{\Omega}_\alpha)$ and $bD_\alpha = \{(z, w) \in b\Omega_\alpha \times \mathbb{C}_w : w = f_\alpha^*(z)\}$.

(b)

Lemma 25. Let $\{G_n\}_{n=0}^\infty$, $G_n \subset \mathbb{C}_z \times \mathbb{R}_u$, be a sequence of bounded strictly convex domains such that $G_n \rightarrow G_0$. Let $\{\varphi_n\}_{n=0}^\infty$, $\varphi_n : \partial G_n \rightarrow \mathbb{R}_v$ be a sequence of continuous functions such that $\Gamma(\varphi_n) \rightarrow \Gamma(\varphi_0)$ in the Hausdorff metric. Then, if Φ_n is the continuous function : $\overline{G}_n \rightarrow \mathbb{R}_v$ such that $\hat{\Gamma}(\varphi) = \Gamma(\Phi)$, we have $\Gamma(\Phi_n) \rightarrow \Gamma(\Phi_0)$ in the Hausdorff metric.

(c)

Lemma 26. *Let \mathcal{U} be a smooth connected surface which is properly embedded into some convex domain $G \subset \mathbb{C}_z \times \mathbb{R}_u$. Suppose that near each point of this surface, it can be defined locally by the equation $u = u(z)$. Then the surface \mathcal{U} can be represented globally as a graph of some function $u = U(z)$, defined on some domain $\Omega \subset \mathbb{C}_z$.*

4.4.2.

Proposition 27. *M is the graph of a continuous function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$.*

Proof. We will intersect the graph S with a convenient affine subspace of real dimension 4 to go back to the situation of Shcherbina.

Fix $a \in (\mathbb{C}_z^{n-1} \setminus 0)$ and, for a given point $(\zeta, \xi) \in \Omega$, with $\zeta \in \mathbb{C}_z^{n-1}$ and $\xi \in \mathbb{R}_u$, let $H_{(\zeta, \xi)} \subset \mathbb{C}_z^{n-1} \times \{\xi\}$ be the complex line through (ζ, ξ) in the direction $(a, 0)$. Set:

$$L_{(\zeta, \xi)} = H_{(\zeta, \xi)} + \mathbb{R}_u(0, 1), \quad \Omega_{(\zeta, \xi)} = L_{(\zeta, \xi)} \cap \Omega, \quad S_{(\zeta, \xi)} = (H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap S$$

Then $S_{(\zeta, \xi)}$ is contained in the strictly convex cylinder

$$(H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap (b\Omega \times i\mathbb{R}_v)$$

and is the graph of $g|_{b\Omega_{(\zeta, \xi)}}$.

From (a_{ii}) , the polynomial hull of $S_{(\zeta, \xi)}$ is a continuous graph over $\overline{\Omega}_{(\zeta, \xi)}$. Consider $M = \pi(\tilde{M})$ and set

$$M_{\zeta, \xi} = (H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap M.$$

It follows that $M_{\zeta, \xi}$ is contained in the polynomial hull $\hat{S}_{(\zeta, \xi)}$. From (a_{iii}) , $\hat{S}_{(\zeta, \xi)}$ is a graph over $\overline{\Omega}_{(\zeta, \xi)}$ foliated by analytic discs, so $M_{\zeta, \xi}$ is a graph over a subset U of $\overline{\Omega}_{(\zeta, \xi)}$.

Every analytic disc Δ of $\hat{S}_{(\zeta, \xi)}$ had its boundary on $S_{(\zeta, \xi)}$. Since all the complex points of S are isolated, $b\Delta$ contains a CR point p of S ; from Lemma 23, near p , $M_{\zeta, \xi}$ is a graph over $\overline{\Omega}_{(\zeta, \xi)}$. Near p , Δ is contained in $M_{\zeta, \xi}$, then in a closed complex analytic leaf V_Σ of M ; so $\Delta \subset V_\Sigma \subset M$; but $\Delta \subset H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)$; then: $\Delta \subset M_{\zeta, \xi}$. Consequently, near p , $M_{\zeta, \xi} = \hat{S}_{(\zeta, \xi)}$.

It follows that M is the graph of a function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$.

One proves, using (b), that f is continuous on Ω , whence on $\overline{\Omega} \setminus Q$, by Lemma 23. Then continuity at every q_j is proved using the *Kontinuitätsatz* on the domain of holomorphy $\Omega \times i\mathbb{R}_v$. \square

4.5. Regularity. The property: $M \setminus P = (p_1, \dots, p_4)$ is a smooth manifold with boundary results from:

4.5.1.

Lemma 28. *Let U be a domain of $\mathbb{C}_z^{n-i} \times \mathbb{R}_u$, $n \geq 2$, $f : U \rightarrow \mathbb{R}_v$ a continuous function. Let $A \subset \text{graph}(f)$ be a germ of complex analytic set of codimension 1. Then A is a germ of complex manifold which is a graph of over \mathbb{C}_z^{n-i} .*

Proof. Assume that A is a germ at 0. Let $h \in \mathcal{O}_{n+1}$, $h \neq 0$ such that $A = \{h = 0\}$. For $\varepsilon \ll 1$, let \mathbf{D}_ε be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$, then $A \cap \mathbf{D}_\varepsilon = \{0\}$, i.e. A is w -regular.

Let $\pi : \mathbb{C}_{z,w}^n \rightarrow \mathbb{C}_z^{n-1}$ be the projection. The local structure theorem for analytic sets gives:

for some neighborhood U of 0 in \mathbb{C}_z^{n-1} , there exists an analytic hypersurface $\Delta \subset U$ such that: $A_\Delta = A \cap ((U \setminus \Delta) \times \mathbf{D}_\varepsilon)$ is a manifold;

$\pi/A_\Delta \rightarrow U \setminus \Delta$ is a $d(\in \mathbb{N})$ -sheeted covering.

It is easy to show that the covering $\pi : A_\Delta \rightarrow U \setminus \Delta$ is trivial.

Then we may define d holomorphic functions $\tau_1, \dots, \tau_d : U \setminus \Delta \rightarrow \mathbb{C}$ such that A_Δ is the union of the graphs of the τ_j . By the Riemann extension theorem, the functions τ_j extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$. Suppose that $\tau_j \neq \tau_k$, for $j \neq k$, then for some disc $\mathbf{D} \subset U$ centered at 0, we have $\tau_j|_{\mathbf{D}} \neq \tau_k|_{\mathbf{D}}$, then $(\tau_j - \tau_k)|_{\mathbf{D}}$ vanishes only at 0. But, from the hypothesis, in restriction to \mathbf{D} , $\{Re(\tau_j - \tau_k) = 0\} \subset \{\tau_j - \tau_k = 0\}|_{\mathbf{D}} = \{0\}$, impossible. \square

4.6.

Proof of the Theorem 21. Consider the foliation of $S \setminus P$ given by the level sets of the smooth function $\nu : S \rightarrow [0, 1]$ (sections 2.3 and 2.7) and set $L_t = \{\nu = t\}$ for $t \in (0, 1)$. Let $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$ be the complex leaf of M bounded by L_t .

By Proposition 27, M is the graph of a continuous function over Ω , and, by Lemma 28, each leaf V_t is a complex smooth hypersurface and $\pi|_{V_t}$ is a submersion.

Since Ω is strictly convex, as in Shcherbina (see 4.4.1, c)), $\pi|_{V_t}$ is 1-1, then, by Corollary 24, π sends V_t onto a domain $\Omega_t \subset \mathbb{C}_z^{n-1}$ with smooth boundary. Let

$$\pi_u : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \rightarrow \mathbb{R}_u$$

$$\pi_v : (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \rightarrow \mathbb{R}_v$$

then $\pi_u|_{L_t} = a_t \cdot \pi|_{L_t}$ and $\pi_v|_{L_t} = b_t \cdot \pi|_{L_t}$ where a_t, b_t are smooth functions on $b\Omega_t$. Moreover $b\Omega_t, a_t, b_t$ depend smoothly on t .

If $(z_t, w_t) \in M$, then w_t varies on V_t , so w_t is the holomorphic extension of $a_t + ib_t$ to Ω_t . In particular u_t and v_t are smooth in (z, t) , from the Bochner-Martinelli formula.

$\frac{\partial u_t}{\partial t}$ is harmonic on Ω_t for each t and has a smooth extension on $b\Omega_t$.

From Lemma 23 and Corollary 24, $\frac{\partial u_t}{\partial t}$ does not vanish on $b\Omega_t$. Since the CR orbits L_t are connected from Proposition 14, $b\Omega_t$ is also connected,

hence $\frac{\partial u_t}{\partial t}$ has constant sign on $b\Omega_t$. Then, by the maximum principle, also $\frac{\partial u_t}{\partial t}$ on Ω_t and, in particular does not vanish. This implies that $M \setminus S$ is the graph of a smooth function over Ω which smoothly extends to $\overline{\Omega} \setminus Q$.

From Proposition 27, M is the graph of a continuous function over $\overline{\Omega}$. \square

4.7. Elementary smooth models.

4.7.1. Definition. An elementary smooth model in \mathbb{C}^n is an elementary model in the sense of section 3.5.2 and satisfying the further condition which makes sense from Theorem 21:

(G) Let (z, w) be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \dots, z_{n-1})$, $w = u + iv = z_n$, let Ω be a strictly pseudoconvex domain of $\mathbb{C}^{n-1} \times \mathbb{R}_u$; assume that T' is the graph of a smooth function $g : b\Omega \rightarrow \mathbb{R}_v$.

4.7.2.

Theorem 29. Let T be an elementary smooth model. Then, there exists a continuous function $f : \overline{\Omega} \rightarrow \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus Q$ and such that $f|_{b\Omega} = g$, and $M_0 = \text{graph}(f) \setminus S$ is a smooth Levi flat hypersurface of \mathbb{C}^n ; in particular, S is the boundary of the hypersurface $M = \text{graph}(f)$

Proof. similar to the proof of Theorem 21. \square

4.7.3. Gluing of elementary smooth models. In an open set of \mathbb{C}^n , a coordinate system (z, w) of $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$ defines an $(n-1, 1)$ -frame.

To define the gluing of elementary models (section 3.7) we considered a CR-isomorphism from an open set of \mathbb{C}^n induced by a holomorphic isomorphism of the ambient space \mathbb{C}^n onto an open set of \mathbb{C}^n . To define the gluing of elementary smooth models, we have to consider a holomorphic isomorphism of the ambient space \mathbb{C}^n onto an open set of \mathbb{C}^n sending an $(n-1, 1)$ -frame of $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$ onto an $(n-1, 1)$ -frame of $\mathbb{C}_{z'}^{n-1} \times \mathbb{R}_{u'}$.

As in section 3.7.1, we will define a submanifold S' of X obtained by gluing of elementary smooth models by induction on the number m of models. An elementary smooth model T in X is the image of an elementary smooth model T_0 of \mathbb{C}^n by an analytic isomorphism of a neighborhood of T_0 in \mathbb{C}^n into X .

Gluing an elementary smooth model T of type (d_1) to a free down-1-hyperbolic point of S' .

Every elementary smooth model is contained in a cylinder $b\Omega \times \mathbb{R}_v$ determined by Ω and an $(n-1, 1)$ -frame. Two sets Ω are *compatible* if either they coincide or one is part of the other.

The announced gluing is defined in the following way: there exists a CR-isomorphism h_1 from a neighborhood V_1 of $\overline{\sigma}'_1$ induced by a holomorphic isomorphism of the ambient space \mathbb{C}^n onto a neighborhood of σ_1 in S' . Let k_1 be a CR-isomorphism from a neighborhood T''_1 of T'_1 into X such that $k_1|_{V_1} = h_1$, and there exists a common $(n-1, 1)$ -frame on which the

corresponding sets Ω are compatible. The existence of such a situation is possible as the example of the horned (almost everywhere) smooth sphere shows (Theorem 21).

Remark that the gluing implies that the obtained submanifold S' is C^0 and smooth except at the complex points.

Other gluings are obtained in a similar way. Hence:

Theorem 30. *The manifold S' obtained by gluing of elementary smooth models is of class C^0 , and smooth except at the complex points.*

Corollary 31. *The manifold S' is the boundary of a manifold M of class C^0 whose interior is a Levi-flat smooth hypersurface.*

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