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BOUNDARIES OF LEVI-FLAT HYPERSURFACES: SPECIAL HYPERBOLIC POINTS

PIERRE DOLBEAULT

Abstract. Let $S \subset \mathbb{C}^n$, $n \geq 3$ be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface whose boundary is $S$, possibly as a current. Our goal is to get examples of such $S$ containing at least one special 1-hyperbolic point: sphere with two horns; elementary models and their gluing. The particular cases of graphs are also described.

1. Introduction

Let $S \subset \mathbb{C}^n$, be a compact connected 2-codimensional submanifold having the following property: there exists a Levi-flat hypersurface $M \subset \mathbb{C}^n \setminus S$ such that $dM = S$ (i.e. whose boundary is $S$, possibly as a current). The case $n = 2$ has been intensively studied since the beginning of the eighties, in particular by Bedford, Gaveau, Klingenberg; Shcherbina, Chirka, G. Tomassini, Słodkowski, Gromov, Eliashberg; it needs global conditions: $S$ has to be contained in the boundary of a strictly pseudoconvex domain.

We consider the case $n \geq 3$; results on this case has been obtained since 2005 by Dolbeault, Tomassini and Zaitsev, local necessary conditions recalled in section 2 have to be satisfied by $S$, the singular CR points on $S$ are supposed to be elliptic and the solution $M$ is obtained in the sense of currents [DTZ05, DTZ10]. More recently a regular solution $M$ has been obtained when $S$ satisfies a supplementary global condition as in the case $n = 2$ [DTZ09], the singular CR points on $S$ still supposed to be elliptic.

The problem we are interested in is to get examples of such $S$ containing at least one special 1-hyperbolic point (section 2.4). The CR-orbits near a special 1-hyperbolic point are large and, assuming them compact, a careful examination has to be done (sections 2.6, 2.7). As a topological preliminary, we need a generalization of a theorem of Bishop on the difference of the numbers of special elliptic and 1-hyperbolic points (section 2.8); this result is a particular case of a theorem of Hon-Fei Lai [Lai72].

The first considered example is the sphere with two horns which has one special 1-hyperbolic point and three special elliptic points (section 3.4). Then we consider elementary models and their gluing to obtain more complicated examples (section 3.5). Results have been announced in [Dol08], and

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in more precise way in [Dol11]; the first aim of this paper is to give complete proofs. Finally, we recall in detail and extend the results of [DTZ09] on regularity of the solution when $S$ is a graph satisfying a supplementary global condition, as in the case $n = 2$, to the case of existence of special 1-hyperbolic points, and to gluing of elementary smooth models (section 4).

2. Preliminaries: Local and Global Properties of the Boundary

2.1. Definitions. A smooth, connected, CR submanifold $M \subset \mathbb{C}^n$ is called minimal at a point $p$ if there does not exist a submanifold $N$ of $M$ of lower dimension through $p$ such that $HN = HM|_N$. By a theorem of Sussman, all possible submanifolds $N$ such that $HN = HM|_N$ contain, as germs at $p$, one of the minimal possible dimension, defining a so called CR orbit of $p$ in $M$ whose germ at $p$ is uniquely determined.

Let $S$ be a smooth compact connected oriented submanifold of dimension $2n - 2$. $S$ is said to be a locally flat boundary at a point $p$ if it locally bounds a Levi-flat hypersurface near $p$. Assume that $S$ is CR in a small enough neighborhood $U$ of $p \in S$. If all CR orbits of $S$ are 1-codimensional (which will appear as a necessary condition for our problem), the following two conditions are equivalent [DTZ05]:

(i) $S$ is a locally flat boundary on $U$;
(ii) $S$ is nowhere minimal on $U$.

2.2. Complex Points of $S$. (i.e. singular CR points on $S$) [DTZ05].

At such a point $p \in S$, $T_pS$ is a complex hyperplane in $T_p\mathbb{C}^n$. In suitable local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at $p$, with $w = z_n$ and $z = (z_1, \ldots, z_{n-1})$, $S$ is locally given by the equation

$$w = Q(z) + O(|z|^3), \quad Q(z) = \sum_{1 \leq i, j \leq n-1} (a_{ij} z_i z_j + b_{ij} z_i \bar{z}_j + c_{ij} \bar{z}_i \bar{z}_j)$$

$S$ is said flat at a complex point $p \in S$ if $\sum b_{ij} z_i \bar{z}_j \in \lambda \mathbb{R}$, $\lambda \in \mathbb{C}$. We also say that $p$ is flat.

Let $S \subset \mathbb{C}^n$ be a locally flat boundary with a complex point $p$. Then $p$ is flat.

By making the change of coordinates $(z, w) \mapsto (z, \lambda^{-1} w)$, we get $\sum b_{ij} z_i z_j \in \mathbb{R}$ for all $z$. By a change of coordinates $(z, w) \mapsto (z, w + \sum a'_{ij} z_i z_j)$ we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form $Q$ real-valued.

We say that $S$ is in a flat normal form at $p$ if the coordinates $(z, w)$ as in (1) are chosen such that $\Re Q(z) \in \mathbb{R}$ for all $z \in \mathbb{C}^{n-1}$.

2.2.1. Properties of $Q$. Assume that $S$ is in a flat normal form; then, the quadratic form $Q$ is real valued. If $Q$ is positive definite or negative definite, the point $p \in S$ is said to be elliptic; if the point $p \in S$ is not elliptic, and if $Q$ is non degenerate, $p$ is said to be hyperbolic. From section 2.4, we will only consider particular cases of the quadratic form $Q$. 

2.3. Elliptic points.

2.3.1. Properties of $Q$.

**Proposition 1.** ([DTZ05, DTZ10]). Assume that $S \subset \mathbb{C}^n$, $(n \geq 3)$ is nowhere minimal at all its CR points and has an elliptic flat complex point $p$. Then there exists a neighborhood $V$ of $p$ such that $V \setminus \{p\}$ is foliated by compact real $(2n - 3)$-dimensional CR orbits diffeomorphic to the sphere $S^{2n-3}$ and there exists a smooth function $\nu$, having the CR orbits as the level surfaces.

**Sketch of Proof.** (see [DTZ10]). In the case of a quadric $S_0$ ($w = Q(z)$), the CR orbits are defined by $w_0 = Q(z)$, where $w_0$ is constant. Using (1), we approximate the tangent space to $S$ by the tangent space to $S_0$ at a point with the same coordinate $z$; the same is done for the tangent spaces to the CR orbits on $S$ and $S_0$; then we construct the global CR orbit on $S$ through any given point close enough to $p$. □

2.4. Special flat complex points. From [Bis65], for $n = 2$, in suitable local holomorphic coordinates centered at 0, $Q(z) = (z \bar{z} + \lambda \Re z^2)$, $\lambda \geq 0$, under the notations of [BK91]; for $0 \leq \lambda < 1$, $p$ is said to be elliptic, and for $1 < \lambda$, it is said to be hyperbolic. The parabolic case $\lambda = 1$, not generic, will be omitted [BK91]. When $n \geq 3$, the Bishop’s reduction cannot be generalized.

We say that the flat complex point $p \in S$ is special if in convenient holomorphic coordinates centered at 0,

$$Q(z) = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \Re z_j^2), \quad \lambda_j \geq 0$$

Let $z_j = x_j + iy_j$, $x_j, y_j$ real, $j = 1, \ldots, n - 1$, then:

$$Q(z) = \sum_{j=1}^{n-1} [(1 + \lambda_j)x_j^2 + (1 - \lambda_j)y_j^2] + O(|z|^3).$$

A flat point $p \in S$ is said to be *special elliptic* if $0 \leq \lambda_j < 1$ for any $j$.

A flat point $p \in S$ is said to be *special $k$-hyperbolic* if $1 < \lambda_j$ for $j \in J \subset \{1, \ldots, n - 1\}$ and $0 \leq \lambda_j < 1$ for $j \in \{1, \ldots, n - 1\} \setminus J \neq \emptyset$, where $k$ denotes the number of elements of $J$.

Special elliptic (resp. special $k$-hyperbolic) points are elliptic (resp. hyperbolic).

Special flat complex points

2.5. Special hyperbolic points. $S$ being given by (1), let $S_0$ be the quadric of equation $w = Q(z)$.

**Lemma 2.** Suppose that $S_0$ is flat at 0 and that 0 is a special $k$-hyperbolic point. Then, in a neighborhood of 0, and with the above local coordinates, $S_0$ is CR and nowhere minimal outside 0, and the CR orbits of $S_0$ are the $(2n - 3)$-dimensional submanifolds given by $w = \text{const.} \neq 0$. 
Proof. The submanifolds \( w = \text{const.} \neq 0 \) have the same complex tangent space as \( S_0 \) and are of minimal dimension among submanifolds having this property, so they are CR orbits of codimension 1, and from the end of section 2.1, \( S_0 \) is nowhere minimal outside 0.

The section \( w = 0 \) of \( S_0 \) is a real quadratic cone \( \Sigma'_0 \) in \( \mathbb{R}^{2n} \) whose vertex is 0 and, outside 0, it is a CR orbit \( \Sigma_0 \) in the neighborhood of 0. We will improperly call \( \Sigma'_0 \) a singular CR orbit. \( \square \)

2.6. Foliation by CR-orbits in the neighborhood of a special 1-hyperbolic point. We first mimic and transpose the begining of the proof of Proposition 1, i.e. of 2.4.2. in ([DTZ05, DTZ09]).

2.6.1. Local 2-codimensional submanifolds. In order to use simple notations, we will assume \( n = 3 \).

In \( \mathbb{C}^3 \), consider the 4-dimensional submanifold \( S \) locally defined by the equation
\[
(1) \quad w = \varphi(z) = Q(z) + O(|z|^3)
\]
and the 4-dimensional submanifold \( S_0 \) of equation
\[
(4) \quad w = Q(z)
\]
with
\[
Q = (\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2
\]
having a special 1-hyperbolic point at 0, \( (\lambda_1 > 1, 0 \leq \lambda_2 < 1) \), and the cone \( \Sigma'_0 \) whose equation is: \( Q = 0 \). On \( S_0 \), a CR orbit is the 3-dimensional submanifold \( K_{w_0} \) whose equation is \( w_0 = Q(z) \). If \( w_0 > 0 \), \( K_{w_0} \) does not cut the line \( L = \{ x_1 = x_2 = y_2 = 0 \} \); if \( w_0 < 0 \), \( K_{w_0} \) cuts \( L \) at two points.

**Lemma 3.** \( \Sigma_0 = \Sigma'_0 \setminus 0 \) has two connected components in a neighborhood of 0.

Proof. The equation of \( \Sigma'_0 \cap \{ y_1 = 0 \} \) is
\[
(\lambda_1 + 1)x_1^2 + (1 + \lambda_2)x_2^2 + (1 - \lambda_2)y_2^2 = 0
\]
whose only zero, in the neighborhood of 0, is \( \{ 0 \} \): the connected components are obtained for \( y_1 > 0 \) and \( y_1 < 0 \) respectively. \( \square \)

Local 2-codimensional submanifolds

2.6.2. CR-orbits. By differentiating (1), we get for the tangent spaces the following asymptotics
\[
(5) \quad T_{(z,\varphi(z))}S = T_{(z,Q(z))}S_0 + O(|z|^2), \quad z \in \mathbb{C}^2
\]
Here both \( T_{(z,\varphi(z))}S \) and \( T_{(z,Q(z))}S_0 \) depend continuously on \( z \) near the origin.

Consider
(i) the hyperboloid $H_\pm = \{Q = \mp 1\}$, (then $Q(\frac{z}{(-Q(z))^{1/2}}) = -1$), and the projection: $\pi_\pm : \mathbb{C}^3 \setminus \{z = 0\} \to H_\pm$, $(z, w) \mapsto \frac{z}{(-Q(z))^{1/2}}$.

(ii) for every $z \in H_-$, a real orthonormal basis $e_1(z), \ldots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

\[ e_1(z), e_2(z) \in H_z H_-, \quad e_3(z) \in T_z H_-, \]

where $HH_-$ is the complex tangent bundle to $H_-$.

Locally such a basis can be chosen continuously depending on $z$. For every $(z, w) \in \mathbb{C}^3 \setminus \{z = 0\}$, consider the basis $e_1(\pi_-(z, w)), \ldots, e_6(\pi_-(z, w))$. The unit vectors $e_1(\pi_-(z, w_0)), e_2(\pi_-(z, w_0)), e_3(\pi_-(z, w_0))$ are tangent to the CR orbit $\mathcal{K}_{w_0}$ in $(z, w_0)$ for $w_0 < 0$. Then, from (5), we have:

\[ H_{(z, \varphi(z))} S = H_{(z, Q(z))} S_0 + O(|z|^2), \quad z \neq 0, \quad z \to 0. \]

As in [DTZ10], in the neighborhood of 0, denote by $E(q), q \in S \setminus \{0\}, w < 0$ the tangent space to the local CR orbit $\mathcal{K}$ on $S$ through $q$, and by $E_0(q_0), q_0 \in S_0 \setminus \{0\}, w < 0$ the analogous object for $S_0$. We have:

\[ E(z, \varphi(z)) = E_0(z, Q(z)) + O(|z|^2), \quad z \neq 0, \quad z \to 0 \]

2.6.3. Behaviour of local CR orbits. Follow the construction of $E(z, \varphi(z))$; compare with $E_0(z, Q(z))$. We know the integral manifold, the orbit of $E_0(z, Q(z))$; deduce an evaluation of the integral manifold $\mathcal{K}$ of $E(z, \varphi(z))$.

**Lemma 4.** Under the above hypotheses, the local orbit $\Sigma$ corresponding to $\Sigma_0$ has two connected components in the neighborhood of 0.

*Proof.* Using the real coordinates, as for Lemma 3, $\Sigma' \cap \{y_1 = 0\}$. Locally, the connected components are obtained for $y_1 > 0$ and $y_1 < 0$ respectively, from formula (1).

We will improperly call $\Sigma' = \Sigma$ a singular CR orbit and a singular leaf of the foliation.

We intend to prove: 1) $\mathcal{K}$ does not cross the singular leaf through 0;
2) the only separatrix is the singular leaf through 0.

From the orbit $\mathcal{K}_0$, construct the differential equation defining it, and using (7), construct the differential equation defining $\mathcal{K}$.

In $\mathbb{C}^3$, we use the notations: $x = x_1, y = y_1, u = x_2, v = y_2$; it suffices to consider the particular case: $Q = 3x^2 - y^2 + u^2 + v^2$. On $S_0$, the orbit $\mathcal{K}_0$ issued from the point $(c, 0, 0, 0)$ is defined by: $3x^2 - y^2 - u^2 + v^2 = 3c^2$, i.e., for $x \geq 0$, $x = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{1/2} = A(y, u, v)$; the local coordinates on the orbit are $(y, u, v)$. $\mathcal{K}_0$ satisfies the differential equation: $dx = dA$. From (9), the orbit $\mathcal{K}$, issued from $(c, 0, 0, 0)$, satisfies $dx = dA + \Psi$ with $\Psi(y, u, v; c) = O(|z|^3)$; hence $\Psi = d\Phi$, then $x = A + \Phi$, with $\Phi = O(|z|^3)$.

More explicitly, $\mathcal{K}$ is defined by:

$$x = x_{\mathcal{K},c} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2 + 3c^2)^{1/2} + \Phi(y, u, v; c), \quad \Phi(y, u, v; c) = O(|z|^3)$$

The cone $\Sigma'_0$ whose equation is: $Q = 0$ is a separatrix for the orbits $\mathcal{K}_0$. The corresponding object $\Sigma' = \{\varphi(z) = 0\}$ for $S$ has the singular point 0 and for $x > 0, y > 0, u > 0, v > 0$ is defined by the differential equation $dx = d(A + \Phi)$, with $c = 0$, i.e. the local equation of $\Sigma'$ is

$$x = x_{\mathcal{K},0} = \frac{1}{\sqrt{3}}(y^2 - u^2 - v^2)^{1/2} + \Phi(y, u, v; 0), \quad \Phi(y, u, v; 0) = O(|z|^3)$$

For given $(y, u, v)$, $x_{\mathcal{K},c} - x_{\mathcal{K},0} = x_{\mathcal{K},0} - x_{\mathcal{K},0} + \Phi(y, u, v; c) - \Phi(y, u, v; 0)$. But $x_{\mathcal{K},0} - x_{\mathcal{K},0} = O(1)$ and $\Phi(y, u, v; c) - \Phi(y, u, v; 0) = O(|z|^3)$.

As a consequence, for $x > 0, y > 0, u > 0, v > 0$, locally, $\Sigma'$ is a separatrix for the orbits $\mathcal{K}$, and the only one. Same result for $x < 0$.

2.6.4. What has been done from the hyperboloid $H_+ = \{Q = -1\}$ can be repeated from the hyperboloid $H_+ = \{Q = 1\}$.

As at the beginning of the section 2.6.2, we consider

(i) the hyperboloid $H_+ \{Q = 1\}$ and the projection:

$$\pi_+: \mathbb{C}^3 \setminus \{z = 0\} \to H_+, \quad (z, w) \mapsto \frac{z}{(Q(z))^{1/2}},$$

(ii) for every $z \in H_+$, a real orthonormal basis $e_1(z), \ldots, e_6(z)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ such that

$$e_1(z), e_2(z) \in H_+; \quad e_3(z) \in T_z H_+,$$

where $HH_+$ is the complex tangent bundle to $H_+$.

2.6.5.

**Lemma 5.** Given $\varphi$, there exists $R > 0$ such that, in $B(0, R) \cap \{x > 0, y > 0, u > 0, v > 0\} \subset \mathbb{C}^2$, the CR orbits $\mathcal{K}$ have $\Sigma'$ as unique separatrix.
such that $(U_p)$ is compact, nowhere minimal at all its CR points, which has a unique $1$-hyperbolic flat complex point $q$ of points $U_p$ leaves in $S$. Let $q \in S$ be such that for $|z| < R$, $x_{K,c} - x_{K,0} > 0$.

2.7. CR–orbits near a subvariety containing a special $1$-hyperbolic point.

2.7.1. In the section 2.7, we will impose conditions on $S$ and give a local property in the neighborhood of a compact $(2n-3)$-subvariety of $S$.

Assume that $S \subset \mathbb{C}^n$ $(n \geq 3)$, is a locally closed $(2n-2)$-submanifold, nowhere minimal at all its CR points, which has a unique $1$-hyperbolic flat complex point $p$, and such that:

(i) $\Sigma$ being the orbit whose closure $\Sigma'$ contains $p$, then $\Sigma'$ is compact.

(ii) $\Sigma$ has two connected components $\sigma_1, \sigma_2$, whose closures are homeomorphic to spheres of dimension $2n-3$.

Then, there exists a neighborhood $V$ of $\Sigma'$ such that $V \setminus \Sigma'$ is foliated by compact real $(2n-3)$-dimensional CR orbits whose equation, in a neighborhood of $p$ is (3), and, the $w(u=x_n)$-axis being assumed to be vertical, each orbit is diffeomorphic to

- the sphere $S^{2n-3}$ above $\Sigma'$,
- the union of two spheres $S^{2n-3}$ under $\Sigma'$,

and there exists a smooth function $\nu$, having the CR orbits as the level surfaces.

Proof. From subsection 2.7.1 and the following remark:

When $x_{\sigma_n}$ tends to 0, the orbits tends to $\Sigma'$, and because of the geometry of the orbits near $p$, they are diffeomorphic to a sphere above $\Sigma'$, and to the union of two spheres under $\Sigma'$. The existence of $\nu$ is proved as in Proposition 1, namely, consider a smooth curve $\gamma : [0, \varepsilon) \to S$ such that
\[ \gamma(0) = q, \text{ where } q \text{ is a point of } \Sigma \text{ close to } p, \text{ and } \gamma \text{ is a diffeomorphism onto its image } \Gamma = \gamma([0, \varepsilon]). \] Let \( \nu = \gamma^{-1} \) on the image of \( \gamma \), then, close enough to \( q \), every CR orbit cuts \( \Gamma \) at a unique point \( q(t) \), \( t \in [0, \varepsilon] \). Hence there is a unique extension of \( \nu \) from \( \gamma([0, \varepsilon]) \) to \( V \setminus p \) where \( V \) is a neighborhood of \( \Sigma' \) having CR orbits as its level surfaces. \( \nu \) being smooth away from \( p \), it is smooth on the orbit \( \Sigma \) and, if we set \( \nu(p) = \nu(q) = 0, \nu \) is smooth on a neighborhood of \( \Sigma \cup \{p\} = \Sigma' \).

\[ \square \]

2.8. Geometry of the complex points of \( S \). The results of section 2.8 are particular cases of theorems of H-F Lai [Lai72], that I learnt from F. Forstneric in July 2011.

In [BK91] E. Bedford & W. Klingenberg cite the following theorem of E. Bishop [Bis65][section 4, p.15]: On a 2-sphere embedded in \( \mathbb{C}^2 \), the difference between the numbers of elliptic points and of hyperbolic points is the Euler-Poincaré characteristic, i.e. 2. For the proof, Bishop uses a theorem of ([CS51], section 4).

We extend the result for \( n \geq 3 \) and give proofs which are essentially the same than in the general case of [Lai72, Lai74] but simpler.

2.8.1. Let \( S \) be a smooth compact connected oriented submanifold of dimension \( 2n - 2 \). Let \( G \) be the manifold of the oriented real linear \((2n-2)\)-subspaces of \( \mathbb{C}^n \). The submanifold \( S \) of \( \mathbb{C}^n \) has a given orientation which defines an orientation \( o(p) \) of the tangent space to \( S \) at any point \( p \in S \). By mapping each point of \( S \) into its oriented tangent space, we get a smooth Gauss map

\[ t : S \to G \]

Denote \(-t(p)\) the tangent space to \( S \) at \( p \) with opposite orientation \(-o(p)\).

2.8.2. Properties of \( G \). (a) \( \dim G = 2(2n - 2) \).

Proof. \( G \) is a two-fold covering of the Grassmannian \( M_{m,k} \), of the linear \( k \)-subspaces of \( \mathbb{R}^m \) [Ste99][Part, section 7.9], for \( m = 2n \) and \( k = 2n - 2 \); they have the same dimension. We have:

\[ M_{m,k} \cong O_m/O_k \times O_{m-k} \]

But \( \dim O_k = \frac{1}{2}k(k-1) \), hence \( \dim M_{m,k} = \frac{1}{2}(m(m-1) - k(k-1) - (m-k)(m-k-1)) = k(m-k) \).

(b) \( G \) has the complex structure of a smooth quadric of complex dimension \((2n - 2)\) of \( \mathbb{CP}^{2n-1} \) [Pol08].

(c) There exists a canonical isomorphism \( h: G \to \mathbb{CP}^{n-1} \times \mathbb{CP}^{n-1} \).

(d) Homology of \( G \) (cf [Pol08]): Let \( S_1, S_2 \) be generators of \( H_{2n-2}(G, \mathbb{Z}) \); we assume that \( S_1 \) and \( S_2 \) are fundamental cycles of complex projective subspaces of complex dimension \((n-1)\) of the complex quadric \( G \). We also denote \( S_1, S_2 \) the ordered two factors \( \mathbb{CP}^{n-1} \), so that \( h: G \to S_1 \times S_2 \).
2.8.3.

**Proposition 7.** For \( n \geq 2 \), in general, \( S \) has isolated complex points.

*Proof.* Let \( \pi \in G \) be a complex hyperplane of \( \mathbb{C}^n \) whose orientation is induced by its complex structure; the set of such \( \pi \) is \( H = G_{n-1,n}^c = \mathbb{C}P^{n-1*} \subset G \), as real submanifold. If \( p \) is a complex point of \( S \), then \( t(p) \in H \) or \(-t(p) \in H \). The set of complex points of \( S \) is the inverse image by \( t \) of the intersections \( t(S) \cap H \) and \(-t(S) \cap H \) in \( G \). Since \( \dim t(S) = 2n - 2 \), \( \dim H = 2(n - 1) \), \( \dim G = 2(2n - 2) \), the intersection is 0-dimensional, in general. \( \square \)

2.8.4. Denoting also \( S \), the fundamental cycle of the submanifold \( S \) and \( t \), the homomorphism defined by \( t \), we have:

\[
t_t(S) \sim u_1S_1 + u_2S_2
\]

where \( \sim \) means homologous to.

2.8.5.

**Lemma 8** (proved for \( n = 2 \) in [CS51]). With the above notations, we have:

\[
u_1 = u_2; \quad u_1 + u_2 = \chi(S), \quad \text{Euler-Poincaré characteristic of } S.
\]

The proof for \( n = 2 \) works for any \( n \geq 3 \), namely:

Let \( G' \) be the manifold of the oriented real linear 2-subspaces of \( \mathbb{C}^n \). Let \( \alpha : G \to G' \) map each oriented \( 2(n-1) \)-subspace \( R \) onto its normal 2-subspace \( R' \) oriented so that \( R, R' \) determine the orientation of \( \mathbb{C}^n \). \( \alpha \) is a canonical isomorphism. Let \( n : S \to G' \) the map defined by taking oriented normal planes; then: \( n = \alpha t \) and \( t = \alpha^{-1} n \), hence the mapping \( hah^{-1} : S_1 \times S_2 \to S_1 \times S_2 \). Let \( (x, y) \) be a point of \( S_1 \times S_2 \), then \( (\dagger) \) \( hah^{-1}(x, y) = (x, -y) \).

Over \( G \), there is a bundle \( V \) of spheres obtained by considering as fiber over a real oriented linear \( (2n - 2) \)-subspace of \( \mathbb{C}^n \) through 0 the unit sphere \( S^{2n-3} \) of this subspace. Let \( \Omega \) be the characteristic class of \( V \), and let \( \Omega_t, \Omega_n \) denote the characteristic classes of the tangent and normal bundles of \( S \). Then \( t^* \Omega = \Omega_t, n^* \Omega = \Omega_n \).

\( V \) is the Stiefel manifold of ordered pairs of orthogonal unit vectors through in \( \mathbb{R}^{2n} \cong \mathbb{C}^n \). Let \( f : V \to G \) the projection.

From the Gysin sequence, we see that the kernel of \( f^* : H^{2n-2}(G) \to H^{2n-2}(V) \) is generated by \( \Omega \). To find the kernel of \( f^* \), we determine the morphism \( f_* : H_{2n-2}(V) \to H_{2n-2}(G) \). A generating \( 2n - 2 \)-cycle of in \( V \) is \( S^2 \times x \) where \( S^2 \cong \mathbb{C}P^{n-1} \) and \( e \) is a point. Let \( z \) be any point of \( S^2 \), then from \( (\dagger) \), we have

\[
hf(z, e) = (z, -z)
\]

Therefore, we see that \( f_*(S^2 \times e) = S_1 - S_2 \). Then, the kernel of \( f^* \) is \( \mathbb{Z} \)-generated by \( S_1^* + S_2^* \).
With convenient orientation for the fibre of the bundle $V$, we get: \( \Omega = S^1 + S^*_2 \). For convenient orientation of $S$, we get \( \Omega_t, S = \chi_S = \text{Euler characteristic of } S \). We have

\[
\Omega = t^*(S^1 + S^*_2) = t^*S^1 + t^*S^*_2
\]

\[
\Omega_n = n^*(S^1 + S^*_2) = t^*\alpha^*(S^1 + S^*_2) = t^*(S^1 - S^*_2) = t^*S^1 - t^*S^*_2
\]

Since \( \Omega_n = 0 \), we get:

\[
(t^*S^1)_n = (t^*S^*_2)_n = \frac{1}{2} \chi_S
\]

2.8.6. Local intersection numbers of $H$ and $t(S)$ when all complex points are flat and special. $H$ is a complex linear $(n - 1)$-subspace of $G$, then is homologous to one of the $S_j$, $j = 1, 2$, say $S_2$ when $G$ has its structure of complex quadric. The intersection number of $H$ and $S_1$ is 1 and the intersection number of $H$ and $S_2$ is 0. So, the intersection number of $H$ and $u_1S_1 + u_2S_2$ is $u_1$.

In the neighborhood of a complex point 0, $S$ is defined by equation (1), with $w = z_n$ and

(1') \[Q(z) = \sum_{j=1}^{n-1} \mu_j(z_j\bar{z}_j + \lambda_j R e \ z_j^2), \quad \mu_j > 0, \lambda_j \geq 0\]

Let $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, with real $x_l$. Let $e_l$ the unit vector of the $x_l$ axis, $l = 1, \ldots, 2n$.

For simplicity assume $n = 3$: \[Q(z) = \mu_1(z_1\bar{z}_1 + \lambda_1 R e \ z_1^2) + \mu_2(z_2\bar{z}_2 + \lambda_2 R e \ z_2^2), \text{ with } \mu_1 = \mu_2 = 1\]

Then, up to higher order terms, $S$ is defined by:

\[z_1 = x_1 + ix_2; \quad z_2 = x_3 + ix_4; \quad z_3 = (1 + \lambda_1)x_4^2 + (1 - \lambda_1)x_2^2 + (1 + \lambda_2)x_3^2 + (1 - \lambda_2)x_5^2\]

In the neighborhood of 0, the tangent space to $S$ is defined by the four independent vectors

\[\nu_1 = e_1 + 2(1 + \lambda_1)x_1 e_5; \quad \nu_2 = e_2 + 2(1 - \lambda_1)x_2 e_5; \quad \nu_3 = e_3 + 2(1 + \lambda_2)x_3 e_5; \quad \nu_4 = e_4 + 2(1 - \lambda_2)x_4 e_5\]

Then, if 0 is special elliptic or special $k$-hyperbolic with $k$ even, the tangent plane at 0 has the same orientation; if 0 is special elliptic or special $k$-hyperbolic with $k$ odd the tangent space has opposite orientation.

2.8.7.

**Proposition 9** (known for $n = 2$ [Bis65], here for $n \geq 3$). Let $S$ be a smooth, oriented, compact, 2-codimensional, real submanifold of $\mathbb{C}^n$ whose all complex points are flat and special elliptic or special 1-hyperbolic. Then, on $S$, \( \frac{1}{2} \) (special elliptic points) - \( \frac{1}{2} \) (special 1-hyperbolic points = $\chi(S)$. If $S$ is a sphere, this number is 2.
Proof. Let \( p \in S \) be a complex point and \( \pi \) be the tangent hyperplane to \( S \) at \( \pi \). Assume that

\[ (**) \text{ the orientation of } S \text{ induces, on } \pi, \text{ the orientation given by its complex structure,} \]

then \( \pi \in H \).

If \( p \) is elliptic, the intersection number of \( H \) and \( t(S) \) is 1; if \( p \) is 1-hyperbolic, the intersection number of \( H \) and \( t(S) \) is -1 at \( p \).

From the beginning of section 2.8.6, the sum of the intersection numbers of \( H \) and \( t(S) \) at complex points \( p \) satisfying (**) is \( u_1 \). Reversing the condition (**), and using Lemma 8, we get the Proposition. \( \square \)

3. Particular cases: horned sphere; elementary models and their gluing

3.1. We recall the following Harvey-Lawson theorem with real parameter to be used later.

3.1.1. Let \( E \cong \mathbb{R} \times \mathbb{C}^{n-1} \), and \( k: \mathbb{R} \times \mathbb{C}^{n-1} \rightarrow \mathbb{R} \) be the projection. Let \( N \subset E \) be a compact, (oriented) CR subvariety of \( \mathbb{C}^{n+1} \) of real dimension \( 2n - 2 \) and CR dimension \( n - 2 \), \( (n \geq 3) \), of class \( C^\infty \), with negligible singularities (i.e. there exists a closed subset \( \tau \subset N \) of \( (2n - 2) \)-dimensional Hausdorff measure 0 such that \( N \setminus \tau \) is a CR submanifold). Let \( \tau' \) be the set of all points \( z \in N \) such that either \( z \in \tau \) or \( z \in N \setminus \tau \) and \( N \) is not transversal to the complex hyperplane \( k^{-1}(k(z)) \) at \( z \). Assume that \( N \), as a current of integration, is \( d \)-closed and satisfies:

\[ (H) \text{ there exists a closed subset } L \subset \mathbb{R}_{x_1} \text{ with } H^1(L) = 0 \text{ such that for every } x \in k(N) \setminus L, \text{ the fiber } k^{-1}(x) \cap N \text{ is connected and does not intersect } \tau'. \]

3.1.2. Theorem 10 ([DTZ10] (see also [DTZ05])). Let \( N \) satisfy (H) with \( L \) chosen accordingly. Then, there exists, in \( E' = E \setminus k^{-1}(L) \), a unique \( C^\infty \) Levi-flat \( (2n - 1) \)-subvariety \( M \) with negligible singularities in \( E' \setminus N \), foliated by complex \( (n-1) \)-subvarieties, with the properties that \( M \) simply (or trivially) extends to \( E' \) as a \( (2n - 1) \)-current (still denoted \( M \)) such that \( dM = N \) in \( E' \). The leaves are the sections by the hyperplanes \( E_{x_1^0}, x_1^0 \in k(N) \setminus L \), and are the solutions of the “Harvey-Lawson problem” for finding a holomorphic subvariety in \( E_{x_1^0} \cong \mathbb{C}^n \) with prescribed boundary \( N \cap E_{x_1^0} \).

3.1.3. Remark 11. Theorem 10 is valid in the space \( E \cap \{ \alpha_1 < x_1 < \alpha_2 \} \), with the corresponding condition (H). Moreover, since \( N \) is compact, for convenient coordinate \( x_1 \), we can assume \( x_1 \in [0,1] \).
3.2. To solve the boundary problem by Levi-flat hypersurfaces, $S$ has to satisfy necessary and sufficient local conditions. A way to prove that these conditions can occur is to construct an example for which the solution is obvious.

3.3. Sphere with one special 1-hyperbolic point (sphere with two horns): Example.

3.3.1. In $\mathbb{C}^3$, let $(z_j)$, $j = 1, 2, 3$, be the complex coordinates and $z_j = x_j + iy_j$. In $\mathbb{R}^5 \cong \mathbb{C}^3$, consider the 4-dimensional subvariety (with negligible singularities) $S$ defined by:

\begin{align*}
y_3 &= 0 \\
0 &\leq x_3 \leq 1; \\
&\quad x_3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 - 1) + (1 - x_3)(x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2) = 0 \\
-1 &\leq x_3 \leq 0; \\
&\quad x_3 = x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2
\end{align*}

The singular set of $S$ is the 3-dimensional section $x_3 = 0$ along which the tangent space is not everywhere (uniquely) defined. $S$ being in the real hyperplane $\{y_3 = 0\}$, the complex tangent spaces to $S$ are $\{x_3 = x^0\}$ for convenient $x^0$.

3.3.2. The tangent space to the hypersurface $f(x_1, y_1, x_2, y_2, x_3) = 0$ in $\mathbb{R}^5$ is

\[X_1f'_x + Y_1f'_y + X_2f'_x + Y_2f'_y + X_3f'_x = 0,\]

Then, the tangent space to $S$ in the hyperplane $\{y_3 = 0\}$ is:

\[2x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)]X_1 + 2y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)]Y_1 + 2x_2[x_3 + (1 - x_3)(2x_2^2 + 1)]X_2 + 2y_2[x_3 + (1 - x_3)(2y_2^2 + 1)]Y_2 + [(x_1^2 + y_1^2 + x_2^2 + y_2^2 + 3x_3^2 - 1) - (x_1^4 + y_1^4 + x_2^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2)]X_3 = 0,\]

for $x_3 \leq 0$,

\[4(x_1^2 + 2)x_1X_1 + 4(y_1^2 - 1)y_1Y_1 + 2(2x_2^2 + 1)x_2X_2 + 2(2y_2^2 + 1)y_2Y_2 - X_3 = 0.\]

3.3.3. The complex points of $S$ are defined by the vanishing of the coefficients of $X_j$, $j=1,2,3,4$ in the equation of the tangent spaces for $0 \leq x_3 \leq 1$,

\begin{align*}
x_1[x_3 + 2(1 - x_3)(x_1^2 + 2)] &= 0, \\
y_1[x_3 + 2(1 - x_3)(y_1^2 - 1)] &= 0, \\
x_2[x_3 + (1 - x_3)(2x_2^2 + 1)] &= 0, \\
y_2[x_3 + (1 - x_3)(2y_2^2 + 1)] &= 0.
\end{align*}

We have the solutions
$h$:  $x_j = 0, y_j = 0, (j = 1, 2), x_3 = 0$;
\(e_3: x_j = 0, y_j = 0, (j = 1, 2), x_3 = 1.\)
for \(x_3 \leq 0,\)
\[
(x^2_3 + 2)x_1 = 0,
\]
\[
(y^2_3 - 1)y_1 = 0,
\]
\[
(2x^2_3 + 1)x_2 = 0,
\]
\[
(2y^2_3 + 1)y_2 = 0.
\]
We have the solutions
\(h: x_j = 0, y_j = 0, (j = 1, 2), x_3 = 0;\)
\(e_1, e_2: x_1 = 0, y_1 = \pm 1, x_2 = 0, y_2 = 0, x_3 = -1.\)
Remark that the tangent space to \(S\) at \(h\) is well defined. Moreover, the set \(S\) will be smoothed along its section by the hyperplane \(\{x_3 = 0\}\) by a small deformation leaving \(h\) unchanged. In the following \(S\) will denote this smooth submanifold.

### 3.3.4.

**Lemma 12.** The points \(e_1, e_2, e_3\) are special elliptic; the point \(h\) is special \(\{1\}\)-hyperbolic.

**Proof.** Point \(e_3\): Let \(x_3' = 1 - x_3\), then the equation of \(S\) in the neighborhood of \(e_3\) is:
\[
(1 - x_3')(x^2_1 + y^2_1 + x^2_2 + y^2_2 + x^2_3 - 2x'_3) - x'_3(x^4_1 + y^4_1 + x^2_2 + y^2_2 + 4x^2_1 - 2y^2_1 + x^2_2 + y^2_2) = 0, \quad i.e.
\]
\[
2x'_3 = x^2_1 + y^2_1 + x^2_2 + y^2_2) + O(|z|^3), \quad \text{or} \quad w = z \bar{z} + O(|z|^3)
\]
then \(e_3\) is special elliptic.
Points \(e_1, e_2\): Let \(y_1' = y_1 \pm 1, x_3' = x_3 + 1,\) then the equation of \(S\) in the neighborhood of \(e_1, e_2\) is:
\[
x'_3 - 1 = x^4_1 + (y_1' \mp 1)^4 + x^2_2 + y^2_2 + 4x^2_1 - 2(y_1' \mp 1)^2 + x^2_3 + y^2_3
\]
\[
= x^4_1 + y_1'^4 + 4y_1'^2 + 6y_1'^2 + 4y_1' + 1 + x^2_2 + y^2_2 + 4x^2_1 - 2(y_1' \mp 1)^2 + x^2_3 + y^2_3,
\]
then
\[
x'_3 = x^4_1 + 4y_1'^4 + x^2_2 + y^2_2 + O(|z|^3), \quad \text{or} \quad w = z_1 \bar{z}_1 + z_2 \bar{z}_2,
\]
then \(e_1, e_2\) are special elliptic.
Point \(h\): The equation of \(S\) in the neighborhood of \(h\) is:
for \(x_3 \geq 0,\)
\[
x_3(x^2_1 + y^2_1 + x^2_2 + y^2_2 + x^2_3 - 1)
\]
\[
+ (1 - x_3)(x^4_1 + y^4_1 + x^4_2 + y^4_2 + 4x^2_1 - 2y^2_1 + x^2_2 + y^2_2) = 0
\]
for \(x_3 \leq 0,\)
\[
x_3 = x^4_1 + y^4_1 + x^4_2 + y^4_2 + 4x^2_1 - 2y^2_1 + x^2_2 + y^2_2, \quad i.e.
\]
\[
x_3 = 4x^4_1 - 2y^2_1 + x^2_2 + y^2_2 + O(|z|^3), \quad \text{in both cases, up to the third order terms, i.e.:} \quad w = z_1 \bar{z}_1 + z_2 \bar{z}_2 + 3Re z^2_1,
\]
then \(h\) is special \(\{1\}\)-hyperbolic. \(\square\)
3.3.5. **Section** \( \Sigma' = S \cap \{x_3 = 0\} \). Up to a small smooth deformation, its equation is:

\[
x_1^4 + y_1^4 + y_2^4 + 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0, \text{ in } \{x_3 = 0\}.
\]

The tangent cone to \( \Sigma' \) at 0 is: \( 4x_1^2 - 2y_1^2 + x_2^2 + y_2^2 = 0 \).

Locally, the section of \( S \) by the coordinate 3-space

\[
x_1, y_1, x_3 \text{ is: } x_3 = 4x_1^2 - 2y_1^2 + O(|z|^3)
\]

\[
x_2, y_2, x_3 \text{ is: } x_3 = x_2^2 + y_2^2 + O(|z|^3)
\]

3.3.1'. **Shape** of \( \Sigma' = S \cap \{x_3 = 0\} \) in the neighborhood of the origin 0 of \( \mathbb{C}^3 \).

**Lemma 13.** Under the above hypotheses and notations,

(i) \( \Sigma = \Sigma' \setminus 0 \) has two connected components \( \sigma_1, \sigma_2 \).

(ii) The closures of the three connected components of \( S \setminus \Sigma' \) are submanifolds with boundaries and corners.

**Proof.** (i) The only singular point of \( \Sigma' \) is 0. We work in the ball \( B(0, A) \) of \( \mathbb{C}^2(x_1, y_1, x_2, y_2) \) for small \( A \) and in the 3-space \( \pi_\lambda = \{y_2 = \lambda x_2\}, \lambda \in \mathbb{R} \). For \( \lambda \) fixed, \( \pi_\lambda \cong \mathbb{R}^3(x_1, y_1, x_2) \), and \( \Sigma' \cap \pi_\lambda \) is the cone of equation \( 4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0 \) with vertex 0 and basis in the plane \( x_2 = x_2^0 \) the hyperboloid \( H_\lambda \) of equation \( 4x_1^2 - 2y_1^2 + (1 + \lambda^2)x_2^2 + O(|z|^3) = 0 \); the curves \( H_\lambda \) have no common point outside 0. So, when \( \lambda \) varies, the surfaces \( \Sigma' \cap \pi_\lambda \) are disjoint outside 0. The set \( \Sigma' \cap \{y_1 = 0\} = \{0\} \), the origin of \( \mathbb{C}^3 \); from above: \( \sigma_1 = \Sigma \cap \{y_1 > 0\}; \sigma_2 = \Sigma \cap \{y_1 < 0\} \).

(ii) The three connected components of \( S \setminus \Sigma' \) are the components which contain, respectively \( e_1, e_2, e_3 \) and whose boundaries are \( \mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_1 \cup \mathfrak{f}_2 \); these boundaries have corners as shown in the first part of the proof. \( \square \)

The connected component of \( \mathbb{C}^2 \times \mathbb{R} \setminus S \) containing the point \((0, 0, 0, 0, 1/2)\) is the Levi-flat solution, the complex leaves being the sections by the hyperplanes \( x_3 = x_3^0, -1 < x_3^0 < 1 \).

The sections by the hyperplanes \( x_3 = x_3^0 \) are diffeomorphic to a 3-sphere for \( 0 < x_3^0 < 1 \) and to the union of two disjoint 3-spheres for \(-1 < x_3^0 < 0 \), as can be shown intersecting \( S \) by lines through the origin in the hyperplane \( x_3 = x_3^0 \); \( \Sigma' \) is homeomorphic to the union of two 3-spheres with a common point.

The connected component of \( \mathbb{C}^2 \times \mathbb{R} \setminus S \) containing the point \((0, 0, 0, 0, 1/2)\) is the Levi-flat solution, the complex leaves being the sections by the hyperplanes \( x_3 = x_3^0, -1 < x_3^0 < 1 \).

The sections by the hyperplanes \( x_3 = x_3^0 \) are diffeomorphic to a 3-sphere for \( 0 < x_3^0 < 1 \) and to the union of two disjoint 3-spheres for \(-1 < x_3^0 < 0 \), as can be shown intersecting \( S \) by lines through the origin in the hyperplane \( x_3 = x_3^0 \); \( \Sigma' \) is homeomorphic to the union of two 3-spheres with a common point.

3.4. **Sphere with one special 1-hyperbolic point (sphere with two horns)**. The example of section 3.3 shows that the necessary conditions of
section 2 can be realised. Moreover, from Proposition 2.8.7, the hypothesis on the number of complex points is meaningful.

3.4.1. Proposition 14. [cf [Dol08]/Proposition 2.6.1] Let $S \subset \mathbb{C}^n$ be a compact connected real 2-codimensional manifold such that the following holds:

(i) $S$ is a topological sphere; $S$ is nonminimal at every CR point;

(ii) every complex point of $S$ is flat; there exist three special elliptic points $e_j, j = 1, 2, 3$ and one special 1-hyperbolic point $h$;

(iii) $S$ does not contain complex manifolds of dimension $(n - 2)$;

(iv) the singular CR orbit $\Sigma'$ through $h$ on $S$ is compact and $\Sigma' \setminus \{h\}$ has two connected components $\sigma_1$ and $\sigma_2$ whose closures are homeomorphic to spheres of dimension $2n - 3$;

(v) the closures $S_1, S_2, S_3$ of the three connected components $S_1', S_2', S_3'$ of $S \setminus \Sigma'$ are submanifolds with (singular) boundary.

Then each $S_j \setminus \{e_j \cup \Sigma'\}, j = 1, 2, 3$ carries a foliation $F_j$ of class $C^\infty$ with 1-codimensional CR orbits as compact leaves.

Proof. From conditions (i) and (ii), $S$ satisfying the hypotheses of Proposition 1, near any elliptic flat point $e_j$, and of Proposition 6 near $\Sigma'$, all CR orbits being diffeomorphic to the sphere $S^{2n-3}$. The assumption (iii) guarantees that all CR orbits in $S$ must be of real dimension $2n - 3$. Hence, by removing small connected open saturated neighborhoods of all special elliptic points, and of $\Sigma'$, we obtain, from $S \setminus \Sigma'$, three compact manifolds $S_j''$, $j = 1, 2, 3$, with boundary and with the foliation $F_j$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbb{R}$ is 0, near $e_j$. It is easy to show that this foliation is transversely oriented. □

3.4.2. Recall the Thurston’s Stability Theorem ([CaC], Theorem 6.2.1).

Proposition 15. Let $(M, F)$ be a compact, connected, transversely-orientable, foliated manifold with boundary or corners, of codimension 1, of class $C^1$. If there is a compact leaf $L$ with $H^1(L, \mathbb{R}) = 0$, then every leaf is homeomorphic to $L$ and $M$ is homeomorphic to $L \times [0, 1]$, foliated as a product.

Then, from the above theorem, $S_j''$ is homeomorphic to $S^{2n-3} \times [0, 1]$ with CR orbits being of the form $S^{2n-3} \times \{x\}$ for $x \in [0, 1]$. Then the full manifold $S_j$ is homeomorphic to a half-sphere supported by $S^{2n-2}$ and $F_j$ extends to $S_j$; $S_3$ having its boundary pinched at the point $h$.

□

3.4.3. Theorem 16. Let $S \subset \mathbb{C}^n$, $n \geq 3$, be a compact connected smooth real 2-codimensional submanifold satisfying the conditions (i) to (v) of Proposition 15. Then there exists a Levi-flat $(2n - 1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{S}$ (in the sense of currents) such that the natural projection $\pi :$
\[ \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \] restricts to a bijection which is a CR diffeomorphism between \( \tilde{S} \) and \( S \) outside the complex points of \( S \).

**Proof.** By Proposition 1, for every \( e_j \), a continuous function \( \nu'_j \), \( C^\infty \) outside \( e_j \), can be constructed in a neighborhood \( U_j \) of \( e_j \), \( j = 1, 2, 3 \), and by Proposition 6, we have an analogous result in a neighborhood of \( \Sigma' \). Furthermore, from Proposition 15, a smooth function \( \nu''_j \) whose level sets are the leaves of \( \mathcal{F}_j \) can be obtained globally on \( S'_j \setminus \{e_j \cup \Sigma'\} \). With the functions \( \nu'_j \) and \( \nu''_j \), and analogous functions near \( \Sigma' \), then using a partition of unity, we obtain a global smooth function \( \nu_j : S_j \rightarrow \mathbb{R} \) without critical points away from the complex points \( e_j \) and from \( \Sigma' \).

Let \( \sigma_1 \), resp. \( \sigma_2 \) be the two connected, relatively compact components of \( \Sigma \setminus \{h\} \), according to condition (iv); \( \overline{\sigma}_1 \), resp. \( \overline{\sigma}_2 \) are the boundary of \( S_1 \), resp. \( S_2 \), and \( \overline{\sigma}_1 \cup \overline{\sigma}_2 \) the boundary of \( S_3 \). We can assume that the three functions \( \nu_j \) are finite valued and get the same values on \( \sigma_1 \) and \( \sigma_2 \). Hence a function \( \nu : S \rightarrow \mathbb{R} \).

The submanifold \( S \) being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set \( \tilde{S} = N = \text{gr} \nu = \{(\nu(z), z) : z \in S\} \). Let \( S_\nu = \{e_1, e_2, e_3, \sigma_1 \cup \sigma_2 \} \).

\[ \lambda : S \rightarrow \tilde{S} \quad (z \mapsto \nu((z), z)) \] is bicontinuous; \( \lambda|_{S \setminus S_\nu} \) is a diffeomorphism; moreover \( \lambda \) is a CR map. Choose an orientation on \( S \). Then \( N \) is an (oriented) CR subvariety with the negligible set of singularities \( \tau = \lambda(S_\nu) \).

At every point of \( S \setminus S_\nu \), \( d_x \nu \neq 0 \), then condition (H) (section 3.1.1) is satisfied at every point of \( N \setminus \tau \).

Then all the assumptions of Theorem 10 being satisfied by \( N = \tilde{S} \), in a particular case, we conclude that \( N \) is the boundary of a Levi-flat \( (2n-2) \)-variety (with negligible singularities) \( \tilde{M} \) in \( \mathbb{R} \times \mathbb{C}^n \).

Taking \( \pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) to be the standard projection, we obtain the conclusion. \( \square \)

### 3.5. Generalizations: elementary models and their gluing.

#### 3.5.1. The examples and the proofs of the theorems when \( S \) is homeomorphic to a sphere (sections 3.4) suggest the following definitions.

#### 3.5.2. Definitions. Let \( T' \) be a smooth, locally closed (i.e. closed in an open set), connected submanifold of \( \mathbb{C}^n \), \( n \geq 3 \). We assume that \( T' \) has the following properties:

1. \( T' \) is relatively compact, non necessarily compact, and of codimension 2.
2. \( T' \) is nonminimal at every CR point.
3. \( T' \) does not contain complex manifold of dimension \( (n-2) \).
4. \( T' \) has exactly 2 complex points which are flat and either special elliptic or special 1-hyperbolic.
(v) If \( p \in T' \) is special 1-hyperbolic, the singular orbit \( \Sigma' \) through \( p \) is compact, \( \Sigma' \setminus p \) has two connected components \( \sigma_1, \sigma_2 \), whose closures are homeomorphic to spheres of dimension \( 2n - 3 \).

(vi) If \( p \in T' \) is special 1-hyperbolic, in the neighborhood of \( p \), with convenient coordinates, the equation of \( T' \), up to third order terms is

\[
z_n = \sum_{j=1}^{n-1} (z_j \bar{z}_j + \lambda_j \Re z_j^2); \quad \lambda_1 > 1; \quad 0 \leq \lambda_j < 1 \quad \text{for} \quad j \neq 1
\]

or in real coordinates \( x_j, y_j \) with \( z_j = x_j + iy_j \),

\[
x_n = ((\lambda_1 + 1)x_1^2 - (\lambda_1 - 1)y_1^2) + \sum_{j=2}^{n-1} ((1 + \lambda_j)x_j^2 + (1 - \lambda_j)y_j^2) + O(|z|^3)
\]

(vii) the closures, in \( T', T_1, T_2, T_3 \) of the three connected components \( T'_1, T'_2, T'_3 \) of \( T' \setminus \Sigma' \) are submanifolds with (singular) boundary. Let \( T''_j, j = 1, 2, 3 \) be neighborhoods of the \( T'_j \) in \( T' \).

up- and down- 1-hyperbolic points. Let \( \tau \) be the \((2n-2)\)-submanifold with (singular) boundary contained into \( T' \) such that either \( \tau_1 \) (resp. \( \tau_2 \)) is the boundary of \( \tau \) near \( p \), or \( \Sigma' \) is the boundary of \( \tau \) near \( p \). In the first case, we say that \( p \) is 1-up, (resp. 2-up), in the second that \( p \) is down. If \( T' \) is contained in a small enough neighborhood of \( \Sigma' \) in \( \mathbb{C}^n \), such a \( T' \) will be called a local elementary model, more precisely it defines a germ of elementary model around \( \Sigma \).

The union \( T \) of \( T_1, T_2, T_3 \) and of the germ of elementary model around the singular orbit at every special 1-hyperbolic point is called an elementary model. \( T \) behaves as a locally closed submanifold still denoted \( T \).

3.5.3. Examples of elementary models. We will say that \( T \) is a elementary model of type:

(a) if it has: two elliptic points;
(b) if it has: one special elliptic point and one down-\{1\}-hyperbolic point;
(c1) if it has: one special elliptic point and one 1-up-\{1\}-hyperbolic point;
(c2) if it has: one special elliptic point and one 2-up-\{1\}-hyperbolic point;
(d1) if it has: two special 1-up-\{1\}-hyperbolic points;
(d2) if it has: two special 2-up-\{1\}-hyperbolic points;
(e) if it has: two special down-\{1\}-hyperbolic points;

Other configurations are easily imagined.

The prescribed boundary of a Levi-flat hypersurface of \( \mathbb{C}^n \) in [DTZ05] and [DTZ10], whose complex points are flat and elliptic, is an elementary model of type (a).

3.5.4. Properties of elementary models. For instance, \( T \) is 1-up and has one special elliptic point, we solve the boundary problem as in \( S_1 \) in the proof of Theorem 16.
Proposition 17. Let $T$ be a local elementary model. Then, $T$ carries a foliation $\mathcal{F}$ of class $C^\infty$ with 1-codimensional CR orbits as compact leaves.

Proof. From the definition at the end of section 3.5.2 and Proposition 6. □

3.5.5.

Theorem 18. Let $T$ be the elementary model there exists an open neighborhood $T''$ in $T'$ carrying a smooth function $\nu : T'' \to \mathbb{R}$ whose level sets are the leaves of a smooth foliation.

Proof. By removing small connected open saturated neighborhoods of every special elliptic point, and of $\Sigma'$, the singular orbit through every special 1-hyperbolic point $p$, we obtain, from $S \setminus \Sigma'$, three compact manifolds $S_j''$, $j = 1, 2, 3$, with boundary,

(a) $S_1$ and $S_2$ containing one special elliptic point $e$ or one special 1-hyperbolic point with the foliations $\mathcal{F}_1, \mathcal{F}_2$, from Propositions 1 and 17,

(b) $S_3''$ with the foliation $\mathcal{F}_3$ of codimension 1 given by its CR orbits whose first cohomology group with values in $\mathbb{R}$ is 0, near $e$, or $p$. It is easy to show that this later foliation is transversely oriented.

From the Thurston’s Stability Theorem (see section 3.4.2), $S_3''$ is homeomorphic to $S_2 \times \{x\}$ for $x \in [0, 1]$; hence smooth functions $\nu_1, \nu_2, \nu_3$, whose level sets are the leaves of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively, and using a partition of unity the desired function $\nu$ on $T$.

□

3.6.

Theorem 19. Let $T$ be an elementary model. Then there exists a Levi-flat $(2n - 1)$-subvariety $\tilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\tilde{T}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a bijection which is a CR diffeomorphism between $\tilde{T}$ and $T$ outside the complex points of $T$.

Proof. The submanifold $T$ being, locally, a boundary of a Levi-flat hypersurface, is orientable. We now set $\tilde{T} = N = \text{gr} \nu = \{(\nu(z), z) : z \in S\} \subset E \cong \mathbb{R} \times \mathbb{C}^{n-1}$. Let $T_s$ be the union of the flat complex points of $T$.

$\lambda : T \to \tilde{T}$ ($z \mapsto (\nu((z), z))$) is bicontinuous; $\lambda|_{T \setminus T_s}$ is a diffeomorphism; moreover $\lambda$ is a CR map. Choose an orientation on $T$. Then $N$ is an (oriented) CR subvariety with the negligible set of singularities $\tau = \lambda(T_s)$.

Using Remark 11, at every point of $T \setminus T_s$, $d_{z_1} \nu \neq 0$, we see that condition (H) (section 3.1.1) is satisfied at every point of $N \setminus \tau$.

Then all the assumptions of Theorem 10 being satisfied by $N = \tilde{T}$, in a particular case, we conclude that $N$ is the boundary of a Levi-flat $(2n - 2)$-variety (with negligible singularities) $\tilde{M}$ in $\mathbb{R} \times \mathbb{C}^n$.

Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we obtain the conclusion. □
3.7. Gluing of elementary models.

3.7.1. The gluing happens between two compatible elementary models along boundaries, for instance down and 1-up. Remark that the gluing can only be made at special 1-hyperbolic points. More precisely, it can be defined as follows.

The assumed properties of the submanifold $S$ in section 2 in $\mathbb{C}^n$ have a meaning in any complex analytic manifold $X$ of complex dimension $n \geq 3$, and are kept under any holomorphic isomorphism.

We will define a submanifold $S'$ of $X$ obtained by gluing of elementary models by induction on the number $m$ of models. An elementary model $T$ in $X$ is the image of an elementary model $T_0$ in $\mathbb{C}^n$ by an analytic isomorphism of a neighborhood of $T_0$ in $\mathbb{C}^n$ into $X$.

3.7.2. Let $S'$ be a closed smooth real submanifold of $X$ of dimension $2n - 2$ which is non minimal at every CR point. Assume that $S'$ is obtained by gluing of $m$ elementary models.

a) $S'$ has a finite number of flat complex points, some special elliptic and the others special 1-hyperbolic;

b) for every special 1-hyperbolic $p'$, there exists a CR-isomorphism $h$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ from a neighborhood of $p$ in $T'$ onto a neighborhood of $p'$ in $S'$.

c) for every CR-orbit $\Sigma_p'$ whose closure contains a special 1-hyperbolic point $p'$, there exists a CR-isomorphism $h$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ from a neighborhood of $\Sigma_p = \Sigma_p' \setminus p$ in $T'$ onto a neighborhood $V$ of $\Sigma_{p'}$ in $S'$.

Every special 1-hyperbolic point of $S'$ which belongs to only one elementary model in $S'$ will be called free.

We will define the gluing of one more elementary model to $S'$.

3.7.3. Gluing an elementary model $T$ of type $(d_1)$ to a free down-1-hyperbolic point of $S'$. Let $h_1$ be a CR-isomorphism from a neighborhood $V_1$ of $\sigma_1$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of $\sigma_1$ in $S'$. Let $k_1$ be a CR-isomorphism from a neighborhood $T''_1$ of $T'_1$ into $X$ such that $k_1|V_1 = h_1$.

3.7.4. Theorem 20. The compact manifold or the manifold with singular boundary $S'$, obtained by the gluing of a finite number of elementary models, is the boundary of a Levi-flat hypersurface of $X$ in the sense of currents.

Proof. From Theorem 19 and the definition of gluing. □
3.8. Examples of gluing. Denoting the gluing of the two models of type 
\((d_1)\) and \((d_2)\) to a free down-1-hyperbolic point of \(S'\) by: \(\rightarrow (d_1) - (d_2)\), and 
the converse by: \((d_1) - (d_2) \rightarrow\), and, also, analogous configurations in the 
same way, we get:

- torus: \((b) \rightarrow (d_1) - (d_2) \rightarrow (b)\); the Euler-Poincaré characteristic of a 
torus is \(\chi(T^k) = 0\): 2 special elliptic and 2 special 1-hyperbolic points.

4. Case of graphs

(see [DTZ09] for the case of elliptic points only, and dropping the property 
of the function solution to be Lipschitz).

4.1. We want to add the following hypothesis: \(S\) is embedded into the 
boundary of a strictly pseudoconvex domain of \(\mathbb{C}^n, n \geq 3\), and more pre-
cisely, let \((z, w)\) be the coordinates in \(\mathbb{C}^{n-1} \times \mathbb{C}\), with \(z = (z_1, \ldots, z_{n-1})\), \(w = u + iv = z_n\); let \(\Omega\) be a strictly pseudoconvex domain of \(\mathbb{C}^{n-1} \times \mathbb{R}_u\) (i.e. the 
second fundamental form of the boundary \(b\Omega\) of \(\Omega\) is everywhere positive 
definite); let \(S\) be the graph \(gr(g)\) of a smooth function \(g : b\Omega \to \mathbb{R}_v\). notice 
that \(b\Omega \times \mathbb{R}_v\) contains \(S\) and is strictly pseudoconvex.

Assume that \(S\) is a horned sphere (section 3.4), satisfying the hypotheses 
of Theorem 16. Denote by \(p_j, j = i, \ldots, 4\) the complex points of \(S\). Our 
aim is to prove

4.2. Theorem 21. Let \(S\) be the graph of a smooth function \(g : b\Omega \to \mathbb{R}_v\). Let \(Q = 
(q_1, \ldots, q_4) \in b\Omega\) be the projections of the complex points \(P = (p_1, \ldots, p_4)\) 
of \(S\), respectively. Then, there exists a continuous function \(f : \overline{\Omega} \to \mathbb{R}_v\) 
which is smooth on \(\overline{\Omega} \setminus Q\) and such that \(f|_{b\Omega} = g\), and \(M_0 = \text{graph}(f) \setminus S\) is 
a smooth Levi flat hypersurface of \(\mathbb{C}^n\). Moreover, each complex leaf of \(M_0\) 
is the graph of a holomorphic function \(\phi : \Omega' \to \mathbb{C}\) where \(\Omega' \subset \mathbb{C}^{n-1}\) is a 
domain with smooth boundary (that depends on the leaf) and \(\phi\) is smooth 
on \(\Omega'\).

The natural candidate to be the graph \(M\) of \(f\) is \(\pi(M)\) where \(\pi\) and \(M\) 
are as in Theorem 16. We prove that this is the case proceeding in several 
steps.

4.3. Behaviour near \(S\).

4.3.1. Assume that \(D\) is a strictly pseudoconvex domain and that \(S \subset bD\).

Recall ([HL75][Theorem 10.4]): Let \(D\) be a strictly pseudoconvex domain 
of \(\mathbb{C}^n, n \geq 3\) with boundary \(bD\), \(\Sigma \subset bD\) be a compact connected maximally 
complex smooth \((2d - 1)\)-submanifold with \(d \geq 2\). Then, \(\Sigma\) is the boundary of a uniquely determined relatively compact subset \(V \subset \overline{D}\) such that \(\overline{\Omega} \setminus \Sigma\) 
is a complex analytic subset of \(D\) with finitely many singularities of pure
dimension \( \leq d - 1 \), and near \( \Sigma \), \( \overline{V} \) is a \( d \)-dimensional complex manifold with boundary.

\( V \) is said to be the solution of the boundary problem for \( \Sigma \).

4.3.2.

**Lemma 22** ([DTZ09]). Let \( \Sigma_1, \Sigma_2 \) be compact connected maximally complex \((2d-1)\)-submanifolds of \( bD \). Let \( V_1, V_2 \) be the corresponding solutions of the boundary problem. If \( d \geq 2 \), \( 2d \geq n + 1 \) and \( \Sigma_1 \cap \Sigma_2 = \emptyset \), then \( V_1 \cap V_2 = \emptyset \).

Let \( \Sigma \) be a CR orbit of the foliation of \( S \setminus P \). Then \( \Sigma \) is a compact maximally complex \((2n-3)\)-dimensional real submanifold of \( \mathbb{C}^n \) contained in \( bD \). Let \( V = V_\Sigma \) be the solution of the boundary problem corresponding to \( \Sigma \). From Theorem 16, \( V = \pi(V) \), where \( V = (M \setminus S) \cap (\mathbb{C}^n \times \{x\}) \) for suitable \( x \in (0,1) \), the projection on the \( x \)-axis being finite, we can always assume that it lies into \((0,1)\). Moreover \( \pi|_V \) is a biholomorphism \( V \cong V \) and \( M \setminus S \subset D \).

Let \( \Sigma_1, \Sigma_2 \) be two distinct orbits of the foliation of \( S \setminus P \), and \( \overline{V}_1, \overline{V}_2 \) the corresponding leaves, then, from Lemma 22, \( \overline{V}_1 \cap \overline{V}_2 = \emptyset \).

4.3.3. Assume that \( S \) satisfies the full hypotheses of Theorem 21.

Set \( m_1 = \min_S g \), \( m_2 = \max_S g \) and \( r \gg 0 \) such that

\[
D = \Omega \times [m_1, m_2] \subset \subset B(r) \cap (\Omega \times i\mathbb{R}_v)
\]

where \( B(r) \) is the ball \( \{|(z,w)| < r\} \).

4.3.4.

**Lemma 23.** Let \( p \in S \) be a CR point. Then, near \( p \), \( M \) is the graph of a function \( \phi \) on a domain \( U \subset \mathbb{C}^{n-1}_z \times \mathbb{R}_u \) which is smooth up to the boundary of \( U \).

**Proof.** Near \( p \), each CR orbit \( \Sigma \) is smooth and can be represented as the graph of a CR function over a strictly pseudoconvex hypersurface and \( V_\Sigma \) as the graph of the local holomorphic extension of this function. From Hopf lemma, \( V \) is transversal to the strictly pseudoconvex hypersurface \( d\Omega \times i\mathbb{R}_v \) near \( p \). Hence the family of the \( V_\Sigma \), near \( p \), forms a smooth real hypersurface with boundary on \( S \) that is the graph of a smooth function \( \phi \) from a relative open neighborhood \( U \) of \( p \) on \( \Omega \) into \( \mathbb{R}_v \). Finally, Lemma 22 guarantees that this family does not intersect any other leaf \( V \) from \( M \). \( \square \)

4.3.5.

**Corollary 24.** If \( p \in S \) is a CR point, each complex leaf \( V \) of \( M \), near \( p \), is the graph of a holomorphic function on a domain \( \Omega_V \subset \mathbb{C}^{n-1}_z \), which is smooth up to the boundary of \( \Omega_V \).

4.4. Solution as a graph of a continuous function.
4.4.1. Recall results of Shcherbina [She93] from:

(a) the Main Theorem:
Let $G$ be a bounded strictly convex domain in $\mathbb{C}_z \times \mathbb{R}_u$ ($z \in \mathbb{C}$) and $\varphi : bG \to \mathbb{R}_w$ a continuous function. Then the following properties hold, where $\Gamma = \text{gr}$, and $\hat{\Gamma}(\varphi)$ means polynomial hull of $\Gamma(\varphi)$:

- (a) the set $\hat{\Gamma}(\varphi) \setminus \Gamma(\varphi)$ is the union of a disjoint family of complex discs $\{D_\alpha\}$;
- (a ii) for each $\alpha$, there is a simply connected domain $\Omega_\alpha \subset \mathbb{C}_z$ and a holomorphic function $w = f_\alpha$, defined on $\Omega_\alpha$, such that $D_\alpha$ is the graph of $f_\alpha$.
- (a iii) For each $f_\alpha$, there exists an extension $f_\alpha^* \in C(\overline{\Omega}_\alpha)$ and $bD_\alpha = \{(z, w) \in b\Omega_\alpha \times \mathbb{C}_w : w = f_\alpha^*(z)\}$.

(b)

Lemma 25. Let $\{G_n\}_{n=0}^\infty$, $G_n \subset \mathbb{C}_z \times \mathbb{R}_u$, be a sequence of bounded strictly convex domains such that $G_n \to G_0$. Let $\{\varphi_n\}_{n=0}^\infty$, $\varphi_n : \partial G_n \to \mathbb{R}_v$ be a sequence of continuous functions such that $\hat{\Gamma}(\varphi_n) \to \hat{\Gamma}(\varphi_0)$ in the Hausdorff metric. Then, if $\Phi_n$ is the continuous function : $G_n \to \mathbb{R}_v$ such that $\hat{\Gamma}(\varphi) = \Gamma(\Phi)$, we have $\Gamma(\Phi_n) \to \Gamma(\Phi_0)$ in the Hausdorff metric.

(c)

Lemma 26. Let $\mathcal{U}$ be a smooth connected surface which is properly embedded into some convex domain $G \subset \mathbb{C}_z \times \mathbb{R}_u$. Suppose that near each point of this surface, it can be defined locally by the equation $u = u(z)$. Then the surface $\mathcal{U}$ can be represented globally as a graph of some function $u = U(z)$, defined on some domain $\Omega \subset \mathbb{C}_z$.

4.4.2.

Proposition 27. $M$ is the graph of a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$.

Proof. We will intersect the graph $S$ with a convenient affine subspace of real dimension 4 to go back to the situation of Shcherbina.

Fix $a \in (\mathbb{C}_n^{n-1} \setminus \emptyset)$ and, for a given point $(\zeta, \xi) \in \Omega$, with $\zeta \in \mathbb{C}_n^{n-1}$ and $\xi \in \mathbb{R}_w$, let $H_{(\zeta, \xi)} \subset \mathbb{C}_n^{n-1} \times \{\xi\}$ be the complex line through $(\zeta, \xi)$ in the direction $(a, 0)$. Set:

$L_{(\zeta, \xi)} = H_{(\zeta, \xi)} + \mathbb{R}_u(0, 1)$, $\Omega_{(\zeta, \xi)} = L_{(\zeta, \xi)} \cap \Omega$, $S_{(\zeta, \xi)} = (H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap S$

Then $S_{(\zeta, \xi)}$ is contained in the strictly convex cylinder

$(H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap (b\Omega \times i\mathbb{R}_v)$

and is the graph of $g_{(\Omega_{(\zeta, \xi)})}$. From (a ii), the polynomial hull of $S_{(\zeta, \xi)}$ is a continuous graph over $\overline{\Omega}_{(\zeta, \xi)}$. Consider $M = \pi(M)$ and set

$M_{(\zeta, \xi)} = (H_{(\zeta, \xi)} + \mathbb{C}_w(0, 1)) \cap M$. 

It follows that $M_{\zeta,\xi}$ is contained in the polynomial hull $\hat{S}_{\zeta,\xi}$. From $(a_{iii})$, $\hat{S}_{\zeta,\xi}$ is a graph over $\overline{M}_{\zeta,\xi}$ foliated by analytic discs, so $M_{\zeta,\xi}$ is a graph over a subset $U$ of $\overline{M}_{\zeta,\xi}$.

Every analytic disc $\Delta$ of $\hat{S}_{\zeta,\xi}$ had its boundary on $S_{\zeta,\xi}$. Since all the the complex points of $S$ are isolated, $b\Delta$ contains a CR point $p$ of $S$; from Lemma 23, near $p$, $M_{\zeta,\xi}$ is a graph over $\overline{M}_{\zeta,\xi}$. Near $p$, $\Delta$ is contained in $M_{\zeta,\xi}$, then in a closed complex analytic leaf $V_S$ of $M$; so $\Delta \subset V_S \subset M$; but $\Delta \subset H_{\zeta,\xi} + \mathbb{C}^w(0,1)$; then: $\Delta \subset M_{\zeta,\xi}$. Consequently, near $p$, $M_{\zeta,\xi} = \hat{S}_{\zeta,\xi}$.

It follows that $M$ is the graph of a function $f : \overline{M} \to \mathbb{R}_w$.

One proves, using (b), that $f$ is continuous on $\Omega$, whence on $\overline{M} \setminus Q$, by Lemma 23. Then continuity at every $q_j$ is proved using the Kontinuit"atsatz on the domain of holomorphy $\Omega \times i\mathbb{R}_w$. $\square$

4.5. Regularity. The property: $M \setminus P = (p_1, \ldots, p_4)$ is a smooth manifold with boundary results from:

4.5.1.

Lemma 28. Let $U$ be a domain of $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$, $n \geq 2$, $f : U \to \mathbb{R}_v$ a continuous function. Let $A \subset \text{graph}(f)$ be a germ of complex analytic set of codimension 1. Then $A$ is a germ of complex manifold which is a graph of over $\mathbb{C}^{n-1}_z$.

Proof. Assume that $A$ is a germ at 0. Let $g \in \mathcal{O}$, $h \neq 0$ such that $A = \{h = 0\}$. For $\varepsilon << 1$, let $D_\varepsilon$ be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$, then $A \cap D_\varepsilon = \{0\}$, i.e. $A$ is $w$-regular.

Let $\pi : \mathbb{C}^n_{z,w} \to \mathbb{C}^{n-1}_z$ be the projection. The local structure theorem for analytic sets gives:

for some neighborhood $U$ of 0 in $\mathbb{C}^{n-1}_z$, there exists an analytic hypersurface $\Delta \subset U$ such that: $A_\Delta = A \cup ((U \setminus \Delta) \times D_\varepsilon)$ is a manifold;

$\pi/A_\Delta \to U \setminus \Delta$ is a $d(\in \mathbb{N})$-sheeted covering.

It is easy to show that the covering $\pi : A_\Delta \to U \setminus \Delta$ is trivial.

Then we may define $d$ holomorphic functionos $\tau_1, \ldots, \tau_d : U \setminus \Delta \to \mathbb{C}$ such that $A_\Delta$ is the union of the graphs of the $\tau_j$. By the Riemann extension theorem, the functions $\tau_j$ extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$.

Suppose that $\tau_j \neq \tau_k$, for $j \neq k$, then for some disc $D \subset U$ centered at 0, we have $\tau_j|D \neq \tau_k|D$, then $(\tau_j - \tau_k)|D$ vanishes only at 0. But, from the hypothesis, in restriction to $D$, $\{Re(\tau_j - \tau_k) = 0\} \subset \{\tau_j - \tau_k = 0\}|D = \{0\}$, impossible. $\square$

4.6.

Proof of the Theorem 21. Consider the foliation of $S \setminus P$ given by the level sets of the smooth function $\nu : S \to [0,1]$ (sections 2.3 and 2.7) and set $L_t = \{\nu = t\}$ for $t \in (0,1)$. Let $V_t \subset \overline{M} \times i\mathbb{R}_w \subset \mathbb{C}^n$ be the complex leaf of $M$ bounded by $L_t$. 
By Proposition 27, $M$ is the graph of a continuous function over $\Omega$, and, by Lemma 28, each leaf $V_t$ is a complex smooth hypersurface and $\pi|_{V_t}$ is a submersion.

Since $\Omega$ is strictly convex, as in Shcherbina (see 4.4.1, c)), $\pi|_{V_t}$ is 1-1, then, by Corollary 24, $\pi$ sends $V_t$ onto a domain $\Omega_t \subset \mathbb{C}^{n-1}$ with smooth boundary. Let

$$\pi_u : (\mathbb{C}^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_u$$

$$\pi_v : (\mathbb{C}^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_v$$

then $\pi_u|_{L_t} = a_t \pi|_{L_t}$ and $\pi_v|_{L_t} = b_t \pi|_{L_t}$ where $a_t, b_t$ are smooth functions on $b\Omega_t$. Moreover $b\Omega_t, a_t, b_t$ depend smoothly on $t$.

If $(z_t, w_t) \in M$, then $w_t$ varies on $V_t$, so $w_t$ is the holomorphic extension of $a_t + ib_t$ to $\Omega_t$. In particular $u_t$ and $v_t$ are smooth in $(z, t)$, from the Bochner-Martinelli formula.

$$\frac{\partial u_t}{\partial t}$$
is harmonic on $\Omega_t$ for each $t$ and has a smooth extension on $b\Omega_t$.

From Lemma 23 and Corollary 24, $\frac{\partial u_t}{\partial t}$ does not vanish on $b\Omega_t$. Since the CR orbits $L_t$ are connected from Proposition 14, $b\Omega_t$ is also connected, hence $\frac{\partial u_t}{\partial t}$ has constant sign on $b\Omega_t$. Then, by the maximum principle, also $\frac{\partial u_t}{\partial t}$ on $\Omega_t$ and, in particular does not vanish. This implies that $M \setminus S$ is the graph of a smooth function over $\Omega$ which smoothly extends to $\overline{\Omega} \setminus Q$.

From Proposition 27, $M$ is the graph of a continuous function over $\Omega$. □

4.7. Elementary smooth models.

4.7.1. Definition. An elementary smooth model in $\mathbb{C}^n$ is an elementary model in the sense of section 3.5.2 and satisfying the further condition which makes sense from Theorem 21:

(G) Let $(z, w)$ be the coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$, with $z = (z_1, \ldots, z_{n-1}), w = u + iv = z_n$, let $\Omega$ be a strictly pseudoconvex domain of $\mathbb{C}^{n-1} \times \mathbb{R}_u$; assume that $T'$ is the graph of a smooth function $g : b\Omega \to \mathbb{R}_v$.

4.7.2. Theorem 29. Let $T$ be an elementary smooth model. Then, there exists a continuous function $f : \overline{\Omega} \to \mathbb{R}_v$, which is smooth on $\overline{\Omega} \setminus Q$ and such that $f|_{b\Omega} = g$, and $M_0 = \text{graph}(f) \setminus S$ is a smooth Levi flat hypersurface of $\mathbb{C}^n$; in particular, $S$ is the boundary of the hypersurface $M = \text{graph}(f)$

Proof. similar to the proof of Theorem 21. □

4.7.3. Gluing of elementary smooth models. In an open set of $\mathbb{C}^n$, a coordinate system $(z, w)$ of $\mathbb{C}^{n-1} \times \mathbb{R}_u$ defines an $(n - 1, 1)$-frame.

To define the gluing of elementary models (section 3.7) we considered a CR-isomorphism from an open set of $\mathbb{C}^n$ induced by a holomorphic isomorphism of the ambient space $\mathbb{C}^n$ onto a an open set of $\mathbb{C}^n$. To define the
gluing of elementary smooth models, we have to consider a holomorphic
isomorphism of the ambient space $\mathbb{C}^n$ onto an open set of $\mathbb{C}^n$ sending an
$(n-1,1)$-frame of $\mathbb{C}^{n-1}_x \times \mathbb{R}_{\alpha}$ onto an $(n-1,1)$-frame of $\mathbb{C}^{n-1}_x' \times \mathbb{R}_{\alpha'}$.

As in section 3.7.1, we will define a submanifold $S'$ of $X$ obtained by gluing
of elementary smooth models by induction on the number $m$ of models. An
elementary smooth model $T$ in $X$ is the image of an elementary smooth
model $T_0$ of $\mathbb{C}^n$ by an analytic isomorphism of a neighborhood of $T_0$ in $\mathbb{C}^n$
into $X$.

\textit{Gluing an elementary smooth model $T$ of type $(d_1)$ to a free down-1-hyperbolic
point of $S'$}.

Every elementary smooth model is contained in a cylinder $b\Omega \times \mathbb{R}_\alpha$ determined by $\Omega$ and an $(n-1,1)$-frame. Two sets $\Omega$ are compatible if either
they coincide or one is part of the other.

The announced gluing is defined in the following way: there exists a CR-
isomorphism $h_1$ from a neighborhood $V_1$ of $\sigma'_1$ induced by a holomorphic
isomorphism of the ambient space $\mathbb{C}^n$ onto a neighborhood of $\sigma_1$ in $S'$.
Let $k_1$ be a CR-isomorphism from a neighborhood $T^*_{1}$ of $T'_1$ into $X$ such
that $k_1|V_1 = h_1$, and there exists a common $(n-1,1)$-frame on which the
Corresponding sets $\Omega$ are compatible. The existence of such a situation is
possible as the example of the horned (almost everywhere) smooth sphere
shows (Theorem 21.).

Remark that the gluing implies that the obtained submanifold $S'$ is $C^0$
and smooth except at the complex points.

Other gluing are obtained in a similar way. Hence:

\textbf{Theorem 30.} The manifold $S'$ obtained by gluing of elementary smooth
models is of class $C^0$, and smooth except at the complex points.

\textbf{Corollary 31.} The manifold $S'$ is the boundary of manifold $M$ of class $C^\infty$
whose interior is a Levi-flat smooth hypersurface.

\textbf{References}

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