Wave Propagation in Unsaturated Poroelastic Media: Boundary Integral Formulation and Three-dimensional Fundamental Solution

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Wave Propagation in Unsaturated Poroelastic Media: Boundary Integral Formulation and Three-dimensional Fundamental Solution

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Abstract   This paper aims at obtaining boundary integral formulations as well as three dimensional (3D) fundamental solutions for unsaturated soils under dynamic loadings for the first time. The boundary integral equations are derived via the use of the weighted residuals method in a way that permits an easy discretization and implementation in a Boundary Element code. Also, the associated 3D fundamental solutions for such deformable porous medium are derived in Laplace transform domain using the method of Hörmander. The derived results are verified analytically by comparison with the previously introduced corresponding fundamental solutions in elastodynamic limiting case. These solutions can be used, afterwards, in a convolution quadrature method (CQM)-based boundary element formulations in order to model the wave propagation phenomena in such media in time domain.

Keywords: Boundary element method, Boundary integral equations, Fundamental solution, Singular behavior, Unsaturated soil, Multiphase porous media, Dynamic behavior

1.   Introduction

In compacted fills or in arid climate areas where soils are submitted to wetting-drying cycles such as ground water recharge, surface runoff and evapo-transpiration, fine-grained soils are not saturated with water, and contain some air. Due to capillary effects and soil-clay adsorption, the pore water is no more positive, and is submitted to suction.

The dynamic behavior of the saturated soils has been extensively investigated [Biot (1941, 1956); Zienkiewicz and Shiomi (1984)]. In the current state of the art, it could be claimed that behavior of the saturated porous media has been well understood. Conversely, the study of the dynamic behavior of the unsaturated porous media is a relatively new area in the field of geotechnical earthquake engineering.

Wave propagation in unsaturated soils and the dynamic response of such media are of great interest in geophysics, soil and rock mechanics, and many earthquake engineering problems. However, in geomechanics, the behavior of such media including more than two phases is not consistent with the principles and concepts of classic saturated soil mechanics.

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From the mechanical point of view, an unsaturated porous medium can be represented as a three-phase (gas, liquid, and solid), or three-component (water, dry air, and solid) system in which two phases can be classified as fluids (i.e. liquid and gas). The liquid phase is considered to be pure water containing dissolved air and the gas phase is assumed to be a binary mixture of water vapor and 'dry' air.

In order to model unsaturated soil behavior, first the governing partial differential equations should be derived and solved. Because of the complexity of the governing partial differential equations, with the exception of some simple cases, their closed-form solutions are not available. Therefore the numerical methods, such as the Finite Element Method (FEM) and the Boundary Element Method (BEM), should be used for such partial differential equations.

The BEM is a very effective numerical tool for dynamic analysis of linear elastic bounded and unbounded media. The method is very attractive for wave propagation problems, because the discretization is done only on the boundary, yielding smaller meshes and systems of equations. Another advantage is that this method represents efficiently the outgoing waves through infinite domains, which is very useful when dealing with waves scattered by topographical structures. When this method is applied to problems with semi-infinite domains, there is no need to model the far field. In this method, during the formulation of boundary integral equations, the fundamental solutions for the governing partial differential equations should be derived first. Indeed, attempting to solve numerically the boundary value problems for unsaturated soils using BEM leads one to search for the associated fundamental solutions.

To the best of the authors' knowledge, no 3D fundamental solution exists in the published literature for the dynamic modeling of unsaturated soils so far, hence the development of a BEM model for dynamic behaviour of unsaturated soil is not yet possible.

The comprehensive state-of-the-art review by Gatmiri and Kamalian (2002), Gatmiri and Nguyen (2005), Gatmiri et al. (2010), Maghoul et al. (2010), Maghoul et al. (2011) provides clearly presented information on the fundamental solution applied to the soil and the porous media. For unsaturated soils, Gatmiri and Jabbari (2005 a, b) have derived the first fundamental solutions for the nonlinear governing differential equations for quasi-static poroelastic media for both two and three dimensional problems. The corresponding thermo-poro-mechanic fundamental solutions for static and quasi-static problems are, respectively, derived by Jabbari and Gatmiri (2007) (for both two and three dimensional problems) and Gatmiri et al. (2010) (for two-dimensional problems) and Maghoul et al. (2010) (for three-dimensional problems). Also, it seems that the first attempt to obtain fundamental solutions for unsaturated soils under *dynamic* loadings (for two-dimensional problems) is referred to Maghoul et al. (2011).

This paper aims at obtaining the boundary integral equation and 3D fundamental solution for unsaturated soils under dynamic loadings in order to be able to model the wave propagation phenomena in these media by BEM.

In this paper first of all, the set of fully coupled governing differential equations of a porous medium saturated by two compressible fluids (water and air) subjected to dynamic loadings is obtained. These phenomenal formulations are presented based on the

experimental observations and with respect to the poromechanics theory within the framework of the suction-based mathematical model presented by Gatmiri (1997) and Gatmiri et al. (1998).

In this model, the effect of deformations on the suction distribution in the soil skeleton and the inverse effect are included in the formulation via a suction-dependent formulation of state surfaces of void ratio and degree of saturation. The linear constitutive law is assumed. The mechanical and hydraulic properties of porous media are assumed to be suction dependent. In this formulation, the solid skeleton displacements u_i, water pressure p_w and air pressure p_a are presumed to be independent variables.

Secondly, the Boundary Integral Equation (BIE) is developed directly from those equations via the use of the weighted residuals method for the first time in a way that permits an easy discretization and implementation in a numerical code.

The associated 3D fundamental solution in Laplace transformed domain is presented by the use of the method of Hörmander (1963) for $u_i - p_w - p_a$ formulation of unsaturated porous media. As these solutions are the basis of BE formulation their singular behavior is also discussed.

In this case that the fundamental solution is known only in the frequency domain and it seems too difficult to obtain the time-dependent fundamental solution in an explicit analytical form by an inverse transformation of the frequency domain results; the convolution integral in the BIE can be numerically approximated by a new approach called "Operational Quadrature Methods" developed by Lubich (1988 a, b). In this formulation, the convolution integral is numerically approximated by a quadrature formula whose weights are determined by the Laplace transform of the fundamental solution and a linear multistep method [Maghoul (2010); Maghoul et al. (2011)].

Finally, the derived results are verified analytically by comparison with the previously introduced corresponding fundamental solutions in the elastodynamic limiting case.

2. Governing equations

Governing differential equations consist of mass conservation equations of liquid and gaseous phases, the equilibrium equation of the skeleton associated with water and air flow equations and constitutive relation. Also to have a fully coupled model of unsaturated soil, the effect of the suction change on the skeleton deformation and on the water and air permeabilities is considered. The state variables are the net total stress $(\sigma - p_a)$ and matric suction $(p_w - p_a)$. The basic assumptions considered in this paper are the following:

1. The medium consists of the superposition of three continuum media.
2. The interconnected porous space is the space through which mass exchanges of fluids occur.
3. The displacement field is defined by the displacements of the solid skeleton u (or u_i) and the displacement of the fluids relative to the solid w^α (or w_i^α). The absolute displacement of the fluids U^α (or U_i^α) is defined in such a way that the volume of fluid α displaced through unit area normal to the x_i direction is $nS_\alpha U_i^\alpha$ where n is the porosity and S_α is the degree of saturation relative to fluid α.

4. The poroelastic medium of the skeleton is isotropic and linear.
5. The solid grains are considered incompressible.
6. The infinitesimal transformation is considered. Then, the volume dilatation of the skeleton is equal to the variation of the porous connected space:

$$\frac{dn}{dt} = (1 - n)\dot{u}_{i,i} \tag{1}$$

7. Generalized Darcy's law is valid for motion of water and air.
8. Darcy flow velocity or the Eulerian relative flow vector of fluid volume (with respect to the skeleton) for fluid α can be defined through the relation

$$\dot{w}^{\alpha} = nS_{\alpha}(\dot{U}^{\alpha} - \dot{u}) \tag{2}$$

in which $\dot{U}^{\alpha}$ is the Eulerian absolute fluid velocity.

9. Void ratio and degree of saturation state surfaces are suction-dependent.

2.1 Solid skeleton

The equilibrium equation and the constitutive law for a non-isothermal isotropic and linear medium can be written as follows,

2.1.1 Equilibrium equation

$$\left(\sigma_{ij} - \delta_{ij}p_a\right)_{,j} + p_{a,i} + f_i = \rho\ddot{u}_i \tag{3}$$

where $\rho = (1 - n)\rho_s + nS_w\rho_w + nS_a\rho_a$ is the total density of the mixture.

In this equation, the relative acceleration terms of the fluids $(\ddot{w}^w, \ddot{w}^a)$ are omitted.

2.1.2 Constitutive law:

Under the assumption of small deformations, the constitutive law for the solid skeleton of an unsaturated soil, which is under suction effect, can be written as

$$\left(\sigma_{ij} - \delta_{ij}p_a\right) = \left(\lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\right) - F_{ij}^s(p_a - p_w) \tag{4}$$

where λ, μ are Lame coefficients, δ_{ij} is the Kronecker delta and F_{ij}^s is the suction modulus matrix:

$$F_{ij}^s = D_{ijkl}(D_{kl}^s)^{-1} \tag{5}$$

in which D_{kl}^s is a vector obtained from the state surface of void ratio (e) which is a function of the independent variables of $(\sigma - p_a)$ and $(p_a - p_w)$.

$$(D_{kl}^s)^{-1} = \frac{1}{1+e}\frac{\partial e}{\partial(p_a - p_w)}[1 \quad 1 \quad 0]^T \tag{6}$$

The elasticity matrix (D_{ijkl}) can be presented by using the bulk modulus and the tangent modulus

$$D_{ijkl}(\lambda, \mu) = D_{ijkl}(K_0, E_t) = D_{ijkl}(\sigma - p_a, p_a - p_w) \tag{7}$$

where E_t is tangent elastic modulus which can be evaluated as

$$E_t = E_l + E_s \tag{8}$$

E_l is the elastic modulus in absence of suction and

$$E_s = m_s(p_a - p_w) \tag{9}$$

m_s being a constant, E_s represents the effect of suction on the elastic modulus. K_0 is the bulk modulus of an open system and evaluated from the surface state of void ratio

$$K_0^{-1} = \frac{1}{1+e}\frac{\partial e}{\partial(\sigma-p_a)} \tag{10}$$

Considering the strain–deformation relation:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{11}$$

The final equation, stating the equilibrium of solid skeleton becomes

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + F^s p_{w,i} + (1 - F^s)p_{a,i} - \rho\ddot{u}_i + f_i = 0 \tag{12}$$

2.2 Mass conservation of water

The conservation law for the mass of water is written:

$$\dot{w}_{i,i}^w = -S_w\dot{\varepsilon}_{ii} + C_{ww}\dot{p}_w + C_{wa}\dot{p}_a \tag{13}$$

where $C_{wa} = -ng_1$ and $C_{ww} = (ng_1 - C_w nS_w)$ in which $C_w = d\rho_w/(\rho_w dp_w)$ is the compressibility of water and $g_1 = dS_w/d(p_a - p_w)$.

The degree of saturation in unsaturated soil S_w depends on the net stress level $(\sigma - p_a)$ and variation of suction $(p_a - p_w)$. Numerous relations have been introduced to define the degree of saturation of unsaturated soils, but the exponential form based on suction variations is one of the most common and reliable ones. The exponential form of the degree of saturation is presented here by omitting the dependency to the net stress in the original equation [Gatmiri (1997)]:

$$S_w = 1 - \left\{1 - \exp\left(\beta_w(p_a - p_w)\right)\right\} \tag{14}$$

in which β_w is constant. By assuming a negative β_w, one can see that any increase in suction results in a decrease in S_w and any decrease in suction results in the approach of S_w to one (saturated).

2.3 Mass conservation of air

With the same approach presented for the water mass conservation, the mass conservation equation of the air can be written as

$$\dot{w}_{i,i}^a = -S_a\dot{\varepsilon}_{ii} + C_{aw}\dot{p}_w + C_{aa}\dot{p}_a \tag{15}$$

where $C_{aw} = -ng_1$ and $C_{aa} = (ng_1 - C_a nS_a)$ in which $C_a = d\rho_a/(\rho_a dp_a)$ is the compressibility of air.

2.4 Flow equation for the water

Based on generalized Darcy's law for describing the balance of the forces acting on the liquid phase of the representative elementary volume, the water velocity in the unsaturated soil takes the following form:

$$-p_{w,i} = \rho_w\ddot{u}_i + \frac{\dot{w}_i^w}{k_w} - \rho_w g_i \tag{16}$$

where $k_w = a_w 10^{e\alpha_w}\left(\frac{S_w - S_{wu}}{1 - S_{wu}}\right)^{d_w}$ denotes the water permeability in an unsaturated soil in which e is the void ratio, a_w, α_w, d_w and S_{wu} are constants depending on the soil studied.

In this equation, the relative acceleration terms of the water is omitted.

2.5 Flow equation for the air

With the same approach presented for the water based on generalized Darcy's law, the air velocity in the unsaturated soil takes the following form:

$$-p_{a,i} = \rho_a \ddot{u}_i + \frac{\dot{w}_i^a}{k_a} - \rho_a g_i \qquad (17)$$

where $k_a = \frac{c_a \gamma_a}{\mu_a}\left(e(1 - S_w)\right)^{d_a}$ is the air permeability in an unsaturated soil in which μ_a is the air viscosity, e is the void ratio, c_a and d_a are constants depending on the soil studied).

2.6 Summary of the field equations

By introducing (4) into (3), (16) into (13) and (17) into (15), we have

$$(\lambda + \mu)u_{\beta,\alpha\beta} + \mu u_{\alpha,\beta\beta} + F^s p_{w,\alpha} + (1 - F^s)p_{a,\alpha} - \rho\ddot{u}_\alpha + f_\alpha = 0 \qquad (18)$$

$$-S_w\dot{u}_{\alpha,\alpha} + \rho_w k_w \ddot{u}_{\alpha,\alpha} + k_w p_{w,\alpha\alpha} + C_{ww}\dot{p}_w + C_{wa}\dot{p}_a = 0 \qquad (19)$$

$$-S_a\dot{u}_{\alpha,\alpha} + \rho_a k_a \ddot{u}_{\alpha,\alpha} + k_a p_{a,\alpha\alpha} + C_{wa}\dot{p}_w + C_{aa}\dot{p}_a = 0 \qquad (20)$$

2.7 Governing equations in the Laplace transformed domain

The Laplace transformation is used to eliminate the time variable of a partial differential equation. Therefore, by applying the Laplace transform with the assumption of zero initial conditions,

$$u_{\alpha(t=0)} = w^w_{\alpha(t=0)} = w^a_{\alpha(t=0)} \qquad (21)$$

$$p_{w(t=0)} = p_{a(t=0)} = 0 \qquad (22)$$

we can rewrite compactly the transformed coupled differential equation system into the following matrix form:

$$\widetilde{B}\begin{bmatrix}\tilde{u}_\alpha \\ \tilde{p}_w \\ \tilde{p}_a\end{bmatrix} + \begin{bmatrix}\tilde{f}_\alpha \\ 0 \\ 0\end{bmatrix} = 0 \qquad (23)$$

with the not self-adjoint operator $\widetilde{B}$:

$$\widetilde{B} = \begin{bmatrix} (\mu\Delta - \rho s^2)\delta_{\alpha\beta} + (\lambda + \mu)\partial_\alpha\partial_\beta & F^s\partial_\alpha & (1 - F^s)\partial_\alpha \\ -s\theta_1\partial_\beta & k_w\Delta + C_{ww}s & C_{wa}s \\ -s\theta_2\partial_\beta & C_{wa}s & k_a\Delta + C_{aa}s \end{bmatrix} \qquad (24)$$

where $\theta_1 = (S_w - \rho_w k_w s)$ and $\theta_2 = (S_a - \rho_a k_a s)$.

In equations (23) and (24), $\alpha, \beta = \overline{1,3}$ in three dimensional problems. Also in (24), the partial derivative $(\ )_{,\alpha}$ is denoted by ∂_α and $\Delta = \partial_{\alpha\alpha}$ is the Laplacian operator.

Based on this equation in the next section, the boundary integral equation and fundamental solutions are derived.

3. Boundary integral equation

We aim at reaching the boundary integral equations for dynamic unsaturated poroelasticity at such a level that it allows application to physical meaningful problem. The corresponding fundamental solutions will be derived in section 4. Thank to the Boundary Element Method an easy discretization and implementation can be done in a numerical code. To that end the present section is dedicated to the derivation of a set of the boundary integral equations for dynamic multiphase poro-elasticity using the weighted residuals method. In this method, the poro-elasto-dynamic integral equation is derived directly by equating the inner product of Eq. (23) and the matrix of the adjoint fundamental solutions $\widetilde{G}^*$ implying that

$$\widetilde{B}^*\widetilde{G}^* + I\delta(x - \xi) = 0 \tag{25}$$

to a null vector, i.e.

$$\int_\Omega \widetilde{B} \begin{bmatrix} \tilde{u}_\alpha \\ \tilde{p}_w \\ \tilde{p}_a \end{bmatrix} \widetilde{G}^* d\Omega = 0 \text{ with}$$

$$\widetilde{G}^* = \begin{bmatrix} \widetilde{G}^*_{\alpha\beta} & \widetilde{G}^*_{\alpha 4} & \widetilde{G}^*_{\alpha 5} \\ \widetilde{G}^*_{4\beta} & \widetilde{G}^*_{44} & \widetilde{G}^*_{45} \\ \widetilde{G}^*_{5\beta} & \widetilde{G}^*_{54} & \widetilde{G}^*_{55} \end{bmatrix} = \begin{bmatrix} \widetilde{U}^{S^*}_{\alpha\beta} & \widetilde{U}^{w^*}_\alpha & \widetilde{U}^{a^*}_\alpha \\ \widetilde{P}^{wS^*}_\beta & \tilde{P}^{wW^*} & \tilde{P}^{wA^*} \\ \widetilde{P}^{aS^*}_\beta & \tilde{P}^{aW^*} & \tilde{P}^{aA^*} \end{bmatrix} \tag{26}$$

where the integration is performed over a domain Ω with boundary Γ and vanishing body forces and sources are assumed. By this inner product, essentially, the error in satisfying the governing differential equations (23), is forced to be orthogonal to $\widetilde{G}^*$ [Schanz (2001)].

This yields, after some algebraic manipulations, the following system of integral equations in index notation as

$$\tag{27}$$

$$\int_\Gamma \left[(\lambda \tilde{u}_{k,k} - F_s(\tilde{p}_a - \tilde{p}_w) + \tilde{p}_a) n_\beta \delta_{\alpha\beta} + \mu(\tilde{u}_{\beta,\alpha} + \tilde{u}_{\alpha,\beta}) n_\beta \right] \tilde{G}^*_{\alpha j} d\Gamma$$

$$- \int_\Gamma \tilde{u}_\alpha \left[(\lambda \tilde{G}^*_{kj,j} + s\theta_1 \tilde{G}^*_{4j} + + s\theta_2 \tilde{G}^*_{5j}) n_\beta \delta_{\alpha\beta} + \mu(\tilde{G}^*_{\alpha j,\beta} + \tilde{G}^*_{\beta j,\alpha}) \right] d\Gamma$$

$$+ k_w \int_\Gamma (\tilde{p}_{w,n} \tilde{G}^*_{4j} - \tilde{p}_w \tilde{G}^*_{4j,n}) d\Gamma + k_a \int_\Gamma (\tilde{p}_{a,n} \tilde{G}^*_{5j} - \tilde{p}_a \tilde{G}^*_{5j,n}) d\Gamma$$

$$+ \int_\Omega (\tilde{u}_i \tilde{B}^*_{im}) \tilde{G}^*_{mj} d\Omega = 0$$

in which

$$\widetilde{B}^* = \begin{bmatrix} (\mu\Delta - \rho s^2)\delta_{\alpha\beta} + (\lambda + \mu)\,\partial_\alpha\,\partial_\beta & s\theta_1\,\partial_\alpha & s\theta_2\,\partial_\alpha \\ -F_s\partial_\beta & k_w\Delta + C_{ww}s & C_{wa}s \\ -(1 - F^s)\,\partial_\beta & C_{wa}s & k_a\Delta + C_{aa}s \end{bmatrix} \tag{33}$$

By substituting Eq (30) into (32) and using the property of Dirac's delta function $\delta(x - \xi)$, we reach the transformed dynamic unsaturated poroelastic boundary integral representation for the transformed internal displacements and pressures given in matrix form, i.e.,

$$\tag{34}$$

$$c(\xi)\mathbf{I}\begin{bmatrix} \tilde{u}_\alpha(\xi; s) \\ \tilde{p}_w(\xi; s) \\ \tilde{p}_a(\xi; s) \end{bmatrix}$$

$$= \int_\Gamma \begin{bmatrix} \widetilde{U}_{\alpha\beta}^{S}{}^*(x,\xi;s) & -\widetilde{P}_\alpha^{wS}{}^*(x,\xi;s) & -\widetilde{P}_\alpha^{aS}{}^*(x,\xi;s) \\ \widetilde{U}_\beta^{W}{}^*(x,\xi;s) & -\tilde{P}^{wW}{}^*(x,\xi;s) & -\tilde{P}^{aW}{}^*(x,\xi;s) \\ \widetilde{U}_\beta^{A}{}^*(x,\xi;s) & -\tilde{P}^{wA}{}^*(x,\xi;s) & -\tilde{P}^{aA}{}^*(x,\xi;s) \end{bmatrix}\begin{bmatrix} \tilde{t}_\alpha(x;s) \\ \tilde{q}_w(x;s) \\ \tilde{q}_a(x;s) \end{bmatrix}d\Gamma$$

$$- \int_\Gamma \begin{bmatrix} \widetilde{T}_{\alpha\beta}^{S}{}^*(x,\xi;s) & \widetilde{Q}_\alpha^{wS}{}^*(x,\xi;s) & \widetilde{Q}_\alpha^{aS}{}^*(x,\xi;s) \\ \widetilde{T}_\beta^{W}{}^*(x,\xi;s) & \widetilde{Q}^{wW}{}^*(x,\xi;s) & \widetilde{Q}^{aW}{}^*(x,\xi;s) \\ \widetilde{T}_\beta^{A}{}^*(x,\xi;s) & \widetilde{Q}^{wA}{}^*(x,\xi;s) & \widetilde{Q}^{aA}{}^*(x,\xi;s) \end{bmatrix}\begin{bmatrix} \tilde{u}_\alpha(x;s) \\ \tilde{p}_w(x;s) \\ \tilde{p}_a(x;s) \end{bmatrix}d\Gamma$$

where the traction vector, the normal water flux and the normal air flux are respectively

$$\tilde{t}_\alpha = \sigma_{\alpha\beta}n_\beta = \big[\big(\lambda\tilde{u}_{k,k} - F_s(\tilde{p}_a - \tilde{p}_w) + \tilde{p}_a\big)\delta_{\alpha\beta} + \mu\big(\tilde{u}_{\beta,\alpha} + \tilde{u}_{\alpha,\beta}\big)\big]n_\beta \tag{35}$$

$$\tilde{q}_w = -k_w\big(\tilde{p}_{w,n} + \rho_w s^2\tilde{u}_\alpha n_\alpha\big) \tag{36}$$

$$\tilde{q}_a = -k_a\big(\tilde{p}_{a,n} + \rho_a s^2\tilde{u}_\alpha n_\alpha\big) \tag{37}$$

The coefficient c_{ij} has a value δ_{ij} for points inside Ω and zero outside Ω. The value of c_{ij} for points on the boundary Γ is determined from the Cauchy principal value of the integrals. It is equal to $0.5\delta_{ij}$ for points on Γ where the boundary is smooth.

Also the $\widetilde{\mathbf{T}}^{S}{}^*$, $\widetilde{\mathbf{Q}}^{wS}{}^*$ and $\widetilde{\mathbf{Q}}^{aS}{}^*$ in Eq (34) can be interpreted as the adjoint terms to the traction vector $\tilde{t}_\alpha$, the water flux $\tilde{q}_w$ and the air flux $\tilde{q}_a$ as follows:

$$\widetilde{T}_{\alpha\beta}^{S}{}^* = \big[\big(\lambda\widetilde{U}_{k\beta,k}^{S}{}^* + sS_w\widetilde{P}_\beta^{wS}{}^* + sS_a\widetilde{P}_\beta^{aS}{}^*\big)\delta_{\alpha l} + \mu\big(\widetilde{U}_{\alpha\beta,l}^{S}{}^* + \widetilde{U}_{l\beta,\alpha}^{S}{}^*\big)\big]n_l \tag{38}$$

$$\widetilde{T}_\alpha^{W}{}^* = \big[\big(\lambda\widetilde{U}_{k,k}^{W}{}^* + sS_w\tilde{P}^{wW}{}^* + sS_a\tilde{P}^{aW}{}^*\big)\delta_{\alpha l} + \mu\big(\widetilde{U}_{\alpha,l}^{W}{}^* + \widetilde{U}_{l,\alpha}^{W}{}^*\big)\big]n_l \tag{39}$$

$$\widetilde{T}_\alpha^{A}{}^* = \big[\big(\lambda\widetilde{U}_{k,k}^{A}{}^* + sS_w\tilde{P}^{wA}{}^* + sS_a\tilde{P}^{aA}{}^*\big)\delta_{\alpha l} + \mu\big(\widetilde{U}_{\alpha,l}^{A}{}^* + \widetilde{U}_{l,\alpha}^{A}{}^*\big)\big]n_l \tag{40}$$

$$\widetilde{Q}_\alpha^{wS}{}^* = k_w\widetilde{P}_{\alpha,n}^{wS}{}^* \tag{41}$$

$$\widetilde{Q}^{wW}{}^* = k_w\widetilde{P}_{,n}^{wW}{}^* \tag{42}$$

$$\widetilde{Q}^{wA}{}^* = k_w\widetilde{P}_{,n}^{wA}{}^* \tag{43}$$

$$\widetilde{Q}_\alpha^{aS}{}^* = k_a\widetilde{P}_{\alpha,n}^{aS}{}^* \tag{44}$$

$$\widetilde{Q}^{aW}{}^* = k_a\widetilde{P}_{,n}^{aW}{}^* \tag{45}$$

$$\tilde{Q}^{aA^*} = k_a \widetilde{P}^{aA^*}_{,n} \tag{46}$$

Eq (34) can be compacted in index notation for the 3D case as follows

$$c(\xi)\mathbf{I}\tilde{u}_j(\xi;s) = \int_\Gamma \tilde{G}^*_{ij}(x,\xi;s)\tilde{t}_i(x;s) - \tilde{F}^*_{ij}(x,\xi;s)\tilde{u}_i(x;s) \tag{47}$$

where $\tilde{t}_i = [\tilde{t}_\alpha \quad \tilde{q}_w \quad \tilde{q}_a]^T$, $\tilde{u}_i = [\tilde{u}_\alpha \quad \tilde{p}_w \quad \tilde{p}_a]^T$ and also

$$\tilde{G}^*_{ij} = \begin{bmatrix} \tilde{U}^{S\,*}_{\alpha\beta} & -\widetilde{P}^{wS^*}_\alpha & -\widetilde{P}^{aS^*}_\alpha \\ \tilde{U}^{W^*}_\beta & -\tilde{P}^{wW^*} & -\tilde{P}^{aW^*} \\ \tilde{U}^{A^*}_\beta & -\tilde{P}^{wA^*} & -\tilde{P}^{aA^*} \end{bmatrix} \tag{48}$$

$$\tilde{F}^*_{ij} = \begin{bmatrix} \tilde{T}^{S\,*}_{\alpha\beta} & \tilde{Q}^{wS^*}_\alpha & \tilde{Q}^{aS^*}_\alpha \\ \tilde{T}^{W^*}_\beta & \tilde{Q}^{wW^*} & \tilde{Q}^{aW^*} \\ \tilde{T}^{A^*}_\beta & \tilde{Q}^{wA^*} & \tilde{Q}^{aA^*} \end{bmatrix} \tag{49}$$

with i,j varies from one to five and α,β,k varies from one to three.

The time dependent boundary integral equation for the unsaturated soil is obtained by a transformation to time domain.

$$c(\xi)\mathbf{I}u_j(\xi;t) = \int_0^t \int_\Gamma \big[G^*_{ij}(x,\xi;t-\tau)t_i(x;\tau) - F^*_{ij}(x,\xi;t-\tau)u_i(x;\tau)\big]d\Gamma \tag{50}$$

4. Fundamental solutions

The objective of this section is to derive the fundamental solutions for the unsaturated poroelastodynamic governing equation (23) in the Laplace transform domain.

These solutions are used in the time-dependent convolution quadrature-based BE formulation which needs only Laplace transformed fundamental solutions.

The physical interpretation of the fundamental solutions for the system of equations (23) is the response of the medium in the point x to, respectively, a unit point excitation $\tilde{F}_{ij} = \tilde{F}_i e_j = \delta(x-\xi)\delta_{ij}$ in the solid skeleton domain with infinite boundaries, represented by $\tilde{U}^S_{\alpha\beta}$, $\widetilde{P}^{wS}_\beta$ and $\widetilde{P}^{aS}_\beta$, as well as a unit source in the water fluid $\tilde{\gamma}_w = \delta(x-\xi)$ represented by $\tilde{U}^w_\alpha$, $\tilde{P}^{wW}$, $\tilde{P}^{aW}$ and a unit source in the air fluid $\tilde{\gamma}_a = \delta(x-\xi)$ represented by $\tilde{U}^a_\alpha$, $\tilde{P}^{wA}$, $\tilde{P}^{aA}$. Here, $\tilde{U}_\alpha, \tilde{P}^w$ and $\tilde{P}^a$ denote, respectively, the displacement of the solid skeleton in the α direction, the water and air pressures and the superscripts S, W, A designs the applied force in, respectively, solid skeleton, water and air fluids. Also, the second subscript β presents the direction of applied force in the solid skeleton.

Mathematically speaking a fundamental solution is a solution of the equation $BG + \mathbf{I}\delta(x-\xi)\delta(t-\tau) = 0$ where the matrix of fundamental solutions is denoted by G, the identity matrix by $\mathbf{I}$ and the matrix differential operator by B.

As in a 3D unsaturated poroelastic problem there are five unknowns $(u_1, u_2, u_3, p_w, p_a)$ per each observation point therefore, the dimension of the fundamental solution matrix is 5×5 per each point:

$$\widetilde{\mathbf{G}} = \begin{bmatrix} \tilde{G}_{\alpha\beta} & \tilde{G}_{\alpha4} & \tilde{G}_{\alpha5} \\ \tilde{G}_{4\beta} & \tilde{G}_{44} & \tilde{G}_{45} \\ \tilde{G}_{5\beta} & \tilde{G}_{54} & \tilde{G}_{55} \end{bmatrix} = \begin{bmatrix} \tilde{U}_{\alpha\beta}^s & \tilde{U}_{\alpha}^w & \tilde{U}_{\alpha}^a \\ \tilde{P}_{\beta}^{wS} & \tilde{P}^{wW} & \tilde{P}^{wA} \\ \tilde{P}_{\beta}^{aS} & \tilde{P}^{aW} & \tilde{P}^{aA} \end{bmatrix} \tag{51}$$

In this study, because the operator type of the governing equations is an elliptical operator the explicit 3D Laplace transform domain fundamental solution are derived by using the method of Hörmander [Hörmander (1963)]. The idea of this method is to reduce the highly complicated operator given in (24) to simple well known operators. In this method, in the Laplace transform domain, the first stage is to find the matrix of cofactors B^{co} to calculate the inverse matrix of B ($B^{-1} = B^{co}/\det B$). For the second stage, we assume that φ is a scalar solution to the equation

$$\det(B)\,I\varphi + I\delta(x-\xi) = 0 \quad \leftrightsquigarrow \quad B\,B^{co}\varphi + \delta(x-\xi) = 0 \tag{52}$$

Consequently, we get

$$G = B^{co}\varphi \tag{53}$$

From the mathematical theory of Green's formula, it is known that the fundamental solution should satisfy the adjoint operator [Stakgold 1998]. As shown in equation (24), all the operators are elliptic and not self-adjoint. Therefore, for the deduction of fundamental solutions, the adjoint operator $\widetilde{B}^*$ has to be used:

$$\widetilde{B}^* = \begin{bmatrix} (\mu\Delta - \rho s^2)\delta_{\alpha\beta} + (\lambda+\mu)\,\partial_\alpha\partial_\beta & s\theta_1\,\partial_\alpha & s\theta_2\,\partial_\alpha \\ -F^s\,\partial_\beta & k_w\Delta + C_{ww}s & C_{wa}s \\ -(1-F^s)\,\partial_\beta & C_{wa}s & k_a\Delta + C_{aa}s \end{bmatrix} \tag{54}$$

At first following Hörmander's idea (52) the determinant of the operator $\widetilde{B}^*$ are calculated:

$$det(\widetilde{B}^*) = \mu^2(\lambda+2\mu)k_w k_a(\Delta - \lambda_1^2)^2(\Delta - \lambda_2^2)(\Delta - \lambda_3^2)(\Delta - \lambda_4^2) \tag{55}$$

in which the coefficients λ_i^2 ($i = \overline{1,4}$) are the coefficients corresponding to the wave velocity propagating through the medium in a way that $\lambda_1^2 = \rho s^2/\mu$ is related to the shear wave velocity and λ_2^2, λ_3^2 and λ_4^2 correspond to the three compressional waves which are affected by the degree of saturation and the spatial distribution of fluids within the medium [Maghoul et al. (2011)]. These three roots must be determined as these which satisfy:

$$\lambda_2^2 + \lambda_3^2 + \lambda_4^2 = \frac{\rho s^2 + F_s\rho_w s^2 + \rho_a(1-F_s)s^2}{(\lambda+2\mu)} - \frac{C_{aa}s}{k_a} - \frac{C_{ww}s}{k_w} - \frac{S_w F_s s}{(\lambda+2\mu)k_w} - \frac{S_a(1-F_s)s}{(\lambda+2\mu)k_a} \tag{56}$$

$$\lambda_2^2\lambda_3^2 + \lambda_2^2\lambda_4^2 + \lambda_3^2\lambda_4^2 =$$
$$-\frac{\rho C_{aa}s^3}{(\lambda+2\mu)k_a} - \frac{\rho C_{ww}s^3}{(\lambda+2\mu)k_w} - \frac{\rho_w(F_s C_{aa}-(1-F_s)C_{wa})s^3}{(\lambda+2\mu)k_a} - \frac{\rho_a(-F_s C_{wa}+(1-F_s)C_{ww})s^3}{(\lambda+2\mu)k_w} +$$
$$\frac{(C_{ww}C_{aa}-C_{wa}^2)s^2}{k_w k_a} + \frac{S_w(F_s C_{aa}-(1-F_s)C_{wa})s^2}{(\lambda+2\mu)k_w k_a} + \frac{S_a(-F_s C_{wa}+(1-F_s)C_{ww})s^2}{(\lambda+2\mu)k_w k_a} \tag{57}$$

$$\lambda_2^2\lambda_3^2\lambda_4^2 = \frac{\rho(C_{ww}C_{aa}-C_{wa}^2)s^4}{(\lambda+2\mu)k_w k_a}$$

Secondly, by introducing the determinant, the scalar equation corresponding to (52) is given by

$$(\Delta - \lambda_1^2)(\Delta - \lambda_2^2)(\Delta - \lambda_3^2)(\Delta - \lambda_4^2)\Phi + \delta(x - \xi) = 0 \qquad (58)$$

in which Φ is an interim operator, i.e.

$$\Phi = \mu^2(\lambda + 2\mu)k_w k_a(\Delta - \lambda_1^2)\varphi \qquad (59)$$

Equation (58) can be expressed as either of four equations (60), (61), (62) and (63):

$$(\Delta - \lambda_1^2)\varphi_1 + \delta(x - \xi) = 0; \; \varphi_1 = (\Delta - \lambda_2^2)(\Delta - \lambda_3^2)(\Delta - \lambda_4^2)\Phi \qquad (60)$$

$$(\Delta - \lambda_2^2)\varphi_2 + \delta(x - \xi) = 0; \; \varphi_2 = (\Delta - \lambda_1^2)(\Delta - \lambda_3^2)(\Delta - \lambda_4^2)\Phi \qquad (61)$$

$$(\Delta - \lambda_3^2)\varphi_3 + \delta(x - \xi) = 0; \; \varphi_3 = (\Delta - \lambda_1^2)(\Delta - \lambda_2^2)(\Delta - \lambda_4^2)\Phi \qquad (62)$$

$$(\Delta - \lambda_4^2)\varphi_4 + \delta(x - \xi) = 0; \; \varphi_4 = (\Delta - \lambda_1^2)(\Delta - \lambda_2^2)(\Delta - \lambda_3^2)\Phi \qquad (63)$$

The above differential equations are of the familiar Helmholtz type. The fundamental solution of Helmholtz differential equations for an only r-dependent fully symmetric three-dimensional domain is

$$\varphi_i = \frac{exp(-\lambda_i r)}{4\pi r}, i = \overline{1,4} \qquad (64)$$

By definition of φ_1, φ_2, φ_3 and φ_4, it is deduced:

$$(65)$$

$$\Phi = \frac{1}{(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_1^2)}\left[\frac{\varphi_3 - \varphi_2}{\lambda_3^2 - \lambda_2^2} - \frac{\varphi_3 - \varphi_1}{\lambda_3^2 - \lambda_1^2} + \frac{\varphi_4 - \varphi_1}{\lambda_4^2 - \lambda_1^2} - \frac{\varphi_4 - \varphi_2}{\lambda_4^2 - \lambda_2^2}\right]$$

Replacing equation (64) into (65), one obtains

$$(66)$$

$$\varphi = \frac{1}{4\pi r}\left\{\frac{exp(-\lambda_1 r)}{(\lambda_1^2 - \lambda_3^2)(\lambda_1^2 - \lambda_4^2)(\lambda_1^2 - \lambda_2^2)} + \frac{exp(-\lambda_2 r)}{(\lambda_2^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)}\right.$$
$$\left. + \frac{exp(-\lambda_3 r)}{(\lambda_3^2 - \lambda_2^2)(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_4^2)} + \frac{exp(-\lambda_4 r)}{(\lambda_4^2 - \lambda_1^2)(\lambda_4^2 - \lambda_2^2)(\lambda_4^2 - \lambda_3^2)}\right\}$$

in which the argument $r = |x - \xi|$ denotes the distance between a load point and an observation point.

Finally, we can determine the components of fundamental solution tensor by applying the matrix of cofactors $\widetilde{B}^{*^{co}}$ to the scalar function φ which are:

− *Displacement caused by a Dirac force in the solid*

$$(67)$$

$$\widetilde{G}_{\alpha\beta}^* = \widetilde{U}_{\alpha\beta}^{S^*} = \frac{1}{4\pi\mu}\frac{-(\lambda + \mu)\Lambda^2}{\rho s^2}\left(\frac{(\lambda_i^2 - K_{ss1}^2)(\lambda_i^2 - K_{ss2}^2)}{(\lambda_i^2 - \lambda_{i+1}^2)(\lambda_i^2 - \lambda_{i+2}^2)(\lambda_i^2 - \lambda_{i+3}^2)}(R_1 + R_2\lambda_i\right.$$

$$\left. + R_3\lambda_i^2)exp(-\lambda_i r)\right) + \frac{\delta_{\alpha\beta}}{4\pi\mu r}$$

in which $\lambda_5^2 = \lambda_1^2, \lambda_6^2 = \lambda_2^2, \lambda_7^2 = \lambda_3^2$, $R_1 = \frac{(3r_{,\alpha}r_{,\beta} - \delta_{\alpha\beta})}{r^3}$, $R_2 = \frac{(3r_{,\alpha}r_{,\beta} - \delta_{\alpha\beta})}{r^2}$, $R_3 = \frac{r_{,\alpha}r_{,\beta}}{r}$,

$\Lambda^2 = \frac{\rho s^2}{(\lambda + 2\mu)}$ and $K_{ss1}^2 + K_{ss2}^2 = -\frac{S_w F_s s}{(\lambda + \mu)k_w} - \frac{S_a(1 - F_s)s}{(\lambda + \mu)k_a} - \frac{(k_w C_{aa} + k_a C_{ww})s}{k_w k_a} + \frac{\rho_w F_s s^2}{(\lambda + \mu)} +$

$$\frac{\rho_a(1-F_s)s^2}{(\lambda+\mu)}, \quad K_{ss1}^2 K_{ss2}^2 = \frac{(C_{ww}C_{aa}-C_{wa}^2)s^2}{k_w k_a} + \frac{S_w(F^s C_{aa}-C_{wa}(1-F^s))s^2}{(\lambda+\mu)k_w k_a} + \frac{S_a(-F^s C_{wa}+C_{ww}(1-F^s))s^2}{(\lambda+\mu)k_w k_a}$$
$$- \frac{\rho_w(F^s C_{aa}-C_{wa}(1-F^s))s^3}{(\lambda+\mu)k_a} - \frac{\rho_a(-F^s C_{wa}+C_{ww}(1-F^s))s^3}{(\lambda+\mu)k_w}.$$

– *Water pressure caused by a Dirac force in the solid*

$$\widetilde{G}_{4\beta}^* = \widetilde{P}_\beta^{wS^*} = \frac{-F^s}{4\pi(\lambda+2\mu)k_w} \frac{r_{,\beta}}{r^2} \frac{(1+r\lambda_i)exp(-\lambda_i r)}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)}\left(\lambda_i^2 - \frac{(C_{wa}(1-F^s)-F^s C_{aa})}{F^s k_a}s\right) \qquad (68)$$

in which $i = \overline{2,4}$ and $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$.

– *Air pressure caused by a Dirac force in the solid*

$$\widetilde{G}_{5\beta}^* = \widetilde{P}_\beta^{aS^*} = \frac{-(1-F^s)}{4\pi(\lambda+2\mu)k_a} \frac{r_{,\beta}}{r^2} \frac{(1+r\lambda_i)exp(-\lambda_i r)}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)}\left(\lambda_i^2 - \frac{(C_{wa}F^s-C_{ww}(1-F^s))}{(1-F^s)k_w}s\right) \qquad (69)$$

in which $i = \overline{2,4}$ and $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$.

– *Displacement caused by a Dirac source in the water fluid*

$$\widetilde{G}_{\alpha 4}^* = \widetilde{U}_\alpha^{W^*} = \frac{(S_w-k_w\rho_w s)s}{4\pi(\lambda+2\mu)k_w} \frac{r_{,\beta}}{r^2} \frac{(1+r\lambda_i)exp(-\lambda_i r)}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)}\left(\lambda_i^2 - K_{\alpha w}\right) \qquad (70)$$

in which $i = \overline{2,4}$, $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$ and $K_{\alpha w} = \frac{(C_{wa}(S_a-\rho_a k_a s)-C_{aa}(S_w-\rho_w k_w s))}{k_a(S_w-\rho_w k_w s)}s$.

– *Displacement caused by a Dirac source in the air fluid*

$$\widetilde{G}_{\alpha 5}^* = \widetilde{U}_\alpha^{A^*} = \frac{(S_a-k_a\rho_a s)s}{4\pi(\lambda+2\mu)k_a} \frac{r_{,\beta}}{r^2} \frac{(1+r\lambda_i)exp(-\lambda_i r)}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)}\left(\lambda_i^2 - K_{\alpha a}\right) \qquad (71)$$

in which $i = \overline{2,4}$, $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$ and $K_{\alpha a} = \frac{(C_{wa}(S_w-\rho_w k_w s)-C_{ww}(S_a-\rho_a k_a s))}{k_w(S_a-\rho_a k_a s)}s$.

– *Water pressure caused by a Dirac source in the water fluid*

$$\widetilde{G}_{44}^* = \widetilde{P}^{wW^*} = \frac{1}{4\pi k_w r} \frac{exp(-\lambda_i r)}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)}\left(\lambda_i^2 - K_w^2\right)\left(\lambda_i^2 - \Lambda_w^2\right) \qquad (72)$$

in which $i = \overline{2,4}$ and $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$ and $K_w^2 \Lambda_w^2 = \frac{-\rho C_{aa}s^3}{(\lambda+2\mu)k_a}$ and $K_w^2 + \Lambda_w^2 = \frac{-S_a(1-F^s)s}{(\lambda+2\mu)k_a} - \frac{C_{aa}s}{k_a} + \frac{\rho_a k_a(1-F^s)s^2}{(\lambda+2\mu)k_a} + \frac{\rho s^2}{(\lambda+2\mu)}$.

– *Air pressure caused by a Dirac source in the air fluid*

$$\widetilde{G}_{55}^* = \widetilde{P}^{aA^*} = \frac{1}{4\pi k_a r} \frac{exp(-\lambda_i r)}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)}\left(\lambda_i^2 - K_a^2\right)\left(\lambda_i^2 - \Lambda_a^2\right) \qquad (73)$$

in which $i = \overline{2,4}$ and $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$ and $K_a^2 \Lambda_a^2 = \frac{-\rho C_{ww}s^3}{(\lambda+2\mu)k_w}$ and $K_w^2 + \Lambda_w^2 = \frac{-S_w F^s s}{(\lambda+2\mu)k_w} - \frac{C_{ww}s}{k_w} + \frac{\rho_w k_w F^s s^2}{(\lambda+2\mu)k_w} + \frac{\rho s^2}{(\lambda+2\mu)}$.

– *Air pressure caused by a Dirac source in the water fluid*

$$\qquad (74)$$

$$\widetilde{G}_{54}^* = \widetilde{P}^{aW^*}$$

$$= \frac{s}{4\pi(\lambda+2\mu)k_w k_a r} \frac{\left(-(\lambda+2\mu)C_{wa} + (\rho_w k_w s - S_w)(1-F^s)\right)\lambda_i^2 + \rho C_{wa}s^2}{(\lambda_{i+2}^2-\lambda_i^2)(\lambda_{i+1}^2-\lambda_i^2)} exp(-\lambda_i r)$$

in which $i = \overline{2,4}$ and $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$.

(75)

$$\tilde{G}_{45}^* = \tilde{P}^{wA^*}$$

$$= \frac{s}{4\pi(\lambda + 2\mu)k_w k_a r} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_a k_a s - S_a)F^s)\lambda_i^2 + \rho C_{wa} s^2}{(\lambda_{i+2}^2 - \lambda_i^2)(\lambda_{i+1}^2 - \lambda_i^2)} exp(-\lambda_i r)$$

in which $i = \overline{2,4}$ and $\lambda_5^2 = \lambda_2^2, \lambda_6^2 = \lambda_3^2$.

In the derivation of the multiphase poroelastodynamic boundary integral equation (34) several abbreviations corresponding to an 'adjoint' traction or flux are introduced (Eqs. (38)-(46)). At first, the 'adjoint' traction solution is presented. However, for simplicity, only parts are given

$$- \quad \tilde{T}_{\alpha\beta}^{S^*} = [(\lambda \tilde{U}_{k\beta,k}^{S^*} + sS_w \tilde{P}_\beta^{wS^*} + sS_a \tilde{P}_\beta^{aS^*})\delta_{\alpha l} + \mu(\tilde{U}_{\alpha\beta,l}^{S^*} + \tilde{U}_{l\beta,\alpha}^{S^*})]n_l$$

(76)

$$(\tilde{U}_{\alpha\beta,l}^{S^*} + \tilde{U}_{l\beta,\alpha}^{S^*})n_l$$

$$= \frac{n_l}{2\pi r}\left(C_i^{SS}\left(\frac{R_5}{r^2}\left(\lambda_i + \frac{1}{r}\right) + \frac{R_6}{r}\lambda_i^2 - r_{,\alpha}r_{,\beta}r_{,l}\lambda_i^3\right) exp(-\lambda_i r)\right)$$

$$- \frac{(r_{,\alpha}n_\beta + r_{,n}\delta_{\alpha\beta})}{4\pi\mu r}\left(\lambda_1 + \frac{1}{r}\right) exp(-\lambda_1 r)$$

where $\quad R_5 = 3(r_{,\alpha}\delta_{\beta l} + r_{,\beta}\delta_{\alpha l} + r_{,l}\delta_{\alpha\beta} - 5r_{,\alpha}r_{,\beta}r_{,l}), \quad R_6 = (r_{,\alpha}\delta_{\beta l} + r_{,\beta}\delta_{\alpha l} + r_{,l}\delta_{\alpha\beta} - 6r_{,\alpha}r_{,\beta}r_{,l})$.

$$\tilde{U}_{k\beta,k}^{S^*}\delta_{\alpha l}n_l = -\frac{r_{,\beta}n_\alpha}{4\pi r}\left(C_i^{SS}\left(\lambda_i + \frac{1}{r}\right)\lambda_i^2 exp(-\lambda_i r)\right) - \frac{r_{,\beta}n_\alpha}{4\pi\mu r}\left(\lambda_1 + \frac{1}{r}\right) exp(-\lambda_1 r) \quad (77)$$

$$- \quad \tilde{T}_{\alpha}^{W^*} = [(\lambda \tilde{U}_{k,k}^{W^*} + sS_w \tilde{P}^{wW^*} + sS_a \tilde{P}^{aW^*})\delta_{\alpha l} + \mu(\tilde{U}_{\alpha,l}^{W^*} + \tilde{U}_{l,\alpha}^{W^*})]n_l$$

$$(\tilde{U}_{\alpha,l}^{W^*} + \tilde{U}_{l,\alpha}^{W^*})n_l = \frac{n_l}{2\pi r}C_{i-1}^{sW}R_{i-1}exp(-\lambda_i r) \tag{78}$$

$$\tilde{U}_{k,k}^{W^*}\delta_{\alpha l}n_l = -\frac{n_\alpha}{4\pi r}\left(C_{i-1}^{sW}\lambda_i^2 exp(-\lambda_i r)\right) \tag{79}$$

with $i = \overline{2,4}$ and $R_k = \frac{(\delta_{\alpha l} - 3r_{,\alpha}r_{,l})}{r^2} + \lambda_k\frac{(\delta_{\alpha l} - 3r_{,\alpha}r_{,l})}{r} - \lambda_k^2 r_{,\alpha}r_{,l}$

$$- \quad \tilde{T}_{\alpha}^{A^*} = [(\lambda \tilde{U}_{k,k}^{A^*} + sS_w \tilde{P}^{wA^*} + sS_a \tilde{P}^{aA^*})\delta_{\alpha l} + \mu(\tilde{U}_{\alpha,l}^{A^*} + \tilde{U}_{l,\alpha}^{A^*})]n_l$$

$$(\tilde{U}_{\alpha,l}^{A^*} + \tilde{U}_{l,\alpha}^{A^*})n_l = \frac{n_l}{2\pi r}C_{i-1}^{sA}R_{i-1}exp(-\lambda_i r) \tag{78}$$

$$\tilde{U}_{k,k}^{A^*}\delta_{\alpha l}n_l = -\frac{n_\alpha}{4\pi r}\left(C_{i-1}^{sA}\lambda_i^2 exp(-\lambda_i r)\right) \tag{79}$$

with $i = \overline{2,4}$ and $R_k = \frac{(\delta_{\alpha l} - 3r_{,\alpha}r_{,l})}{r^2} + \lambda_k\frac{(\delta_{\alpha l} - 3r_{,\alpha}r_{,l})}{r} - \lambda_k^2 r_{,\alpha}r_{,l}$

The other explicit expressions are

$$\tilde{Q}_\alpha^{wS^*} = \frac{n_\alpha}{4\pi r} C_{i-1}^{wS} R_{i-1} \, exp(-\lambda_i r) \tag{80}$$

$$\tilde{Q}^{wW^*} = -\frac{r_{,n}}{4\pi r} \left(C_{i-1}^{wW} \left(\lambda_i + \frac{1}{r} \right) exp(-\lambda_i r) \right) \tag{81}$$

$$\tilde{Q}^{wA^*} = -\frac{r_{,n}}{4\pi r} \left(C_{i-1}^{wA} \left(\lambda_i + \frac{1}{r} \right) exp(-\lambda_i r) \right) \tag{82}$$

$$\tilde{Q}_\alpha^{aS^*} = \frac{n_\alpha}{4\pi r} C_{i-1}^{aS} R_{i-1} \, exp(-\lambda_i r) \tag{83}$$

$$\tilde{Q}^{aW^*} = -\frac{r_{,n}}{4\pi r} \left(C_{i-1}^{aW} \left(\lambda_i + \frac{1}{r} \right) exp(-\lambda_i r) \right) \tag{84}$$

$$\tilde{Q}^{aA^*} = -\frac{r_{,n}}{4\pi r} \left(C_{i-1}^{aA} \left(\lambda_i + \frac{1}{r} \right) exp(-\lambda_i r) \right) \tag{85}$$

in which $i = \overline{2,4}$ and the coefficients are presented in Appendix A.

5. Singular behavior

As shown in part 3, the boundary integral equation is obtained by moving ξ to the boundary Γ. Then in order to determine the unknown boundary data, it is necessary to know the behaviour of the fundamental solutions when $r = |\xi - x|$ tends to zero, i.e. when an integration point x approaches a collocation point ξ. Simple series expansions of the fundamental solutions with respect to the variable $r = |\xi - x|$ show that the singularity of these solutions in the limit $r \to 0$ is equal to the elastostatic, poro-elastostatic or the acoustic fundamental solutions (Table 1).

The variable r in the 3D fundamental solutions is in the exponential functions. Then, as $r \to 0$, so does the argument of the modified exponential functions. Consequently, one has:

$$exp(-\lambda_k r) = \sum_{l=0}^{\infty} \frac{(-\lambda_k r)^l}{l!} = 1 - \lambda_k r + o(r^2) \tag{86}$$

Thus, by replacing Eq. (86) into the 3D solutions and after some algebraic manipulations one obtains:

$$\tilde{U}_{\alpha\beta}^{S^*} = \underbrace{\frac{1}{16\pi} \frac{1}{\mu(1-\nu)} \frac{1}{r} \left\{ \frac{x_\alpha x_\beta}{r^2} + \delta_{\alpha\beta}(3-4\nu) \right\}}_{elastostatic\ fundamental\ solution} + o(r^2) \tag{87}$$

$$\tilde{P}^{wW^*} = \underbrace{\frac{1}{4\pi k_w} \frac{1}{r}}_{poroelastostatic\ fundamental\ solution} + o(r^2) \tag{88}$$

$$\tilde{P}^{aA^*} = \frac{1}{4\pi k_a} \frac{1}{r} + o(r^2) \tag{89}$$

$$\tilde{U}_\alpha^{W^*} = \tilde{U}_\alpha^{A^*} = o(r^2) \tag{90}$$

$$\tilde{P}_\beta^{wS^*} = \tilde{P}_\beta^{aS^*} = o(r^2) \tag{91}$$

$$\tilde{P}^{wA^*} = \tilde{P}^{aW^*} = o(r^2) \tag{92}$$

Also, for adjoint fundamental solutions we have:

$$\tag{93}$$

$$\tilde{T}^S_{\alpha\beta}{}^* = \underbrace{-\frac{1}{8\pi}\frac{1}{(1-\nu)}\frac{1}{r^2}\left\{\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{\alpha\beta} + 3r_{,\alpha}r_{,\beta}\right] - (1-2\nu)\left(r_{,\alpha}n_\beta - r_{,\beta}n_\alpha\right)\right\}}_{elastostatic\ fundamental\ solution} + \mathcal{O}(r^2)$$

$$\tilde{T}^W_\alpha{}^* = \frac{s}{4\pi k_w}\left\{\frac{4\pi(S_w - k_w\rho_w s)}{(\lambda+2\mu)}r_{,n}r_{,\alpha} + k_w\rho_w sn_\alpha\right\}\frac{1}{r} + \mathcal{O}(r^2) \tag{94}$$

$$\tilde{T}^A_\alpha{}^* = \frac{s}{4\pi k_a}\left\{\frac{4\pi(S_a - k_a\rho_a s)}{(\lambda+2\mu)}r_{,n}r_{,\alpha} + k_a\rho_a sn_\alpha\right\}\frac{1}{r} + \mathcal{O}(r^2) \tag{95}$$

$$\tilde{Q}^{wW}{}^* = \tilde{Q}^{aA}{}^* = \underbrace{-\frac{r_{,n}}{4\pi r^2}}_{acoustic\ fundamental\ solution} + \mathcal{O}(r^2) \tag{96}$$

$$\tilde{Q}^{wS}_\alpha{}^* = \frac{F^S}{4\pi(\lambda+2\mu)}\frac{1}{r}\left(n_\alpha - 2r_{,n}r_{,\alpha}\right) + \mathcal{O}(r^2) \tag{97}$$

$$\tilde{Q}^{aS}_\alpha{}^* = \frac{(1-F^S)}{4\pi(\lambda+2\mu)}\frac{1}{r}\left(n_\alpha - 2r_{,n}r_{,\alpha}\right) + \mathcal{O}(r^2) \tag{98}$$

Table 1. Kind of singularity of 3D fundamental solutions

Components	Singularity
$\widetilde{U}^S_{\alpha\beta}{}^*$	weakly singular $(1/r)$
$\widetilde{U}^W_\beta{}^*$	regular $(1/r)$
$\widetilde{U}^A_\beta{}^*$	regular $(1/r)$
$\widetilde{P}^{wS}_\alpha{}^*$	regular $(1/r)$
$\widetilde{P}^{aS}_\alpha{}^*$	regular $(1/r)$
$\tilde{P}^{wW}{}^*$	weakly singular $(1/r)$
$\tilde{P}^{aA}{}^*$	weakly singular $(1/r)$
$\tilde{P}^{aW}{}^*$	regular $(1/r)$
$\tilde{P}^{wA}{}^*$	regular $(1/r)$
$\tilde{T}^S_{\alpha\beta}{}^*$	hyper singular $(1/r^2)$
$\tilde{T}^W_\beta{}^*$	weakly singular $(1/r)$
$\tilde{T}^A_\beta{}^*$	weakly singular $(1/r)$
$\tilde{Q}^{wS}_\alpha{}^*$	weakly singular $(1/r)$
$\tilde{Q}^{aS}_\alpha{}^*$	weakly singular $(1/r)$
$\tilde{Q}^{wW}{}^*$	hyper singular $(1/r^2)$
$\tilde{Q}^{aA}{}^*$	hyper singular $(1/r^2)$
$\tilde{Q}^{wA}{}^*$	regular $(1/r)$
$\tilde{Q}^{aW}{}^*$	regular $(1/r)$

6. Analytical verification of the fundamental solutions

Having derived the fundamental solution, at this stage, it is of interest to verify the validity of these solutions in somewhat more detail. One limiting case is presented here. Investigate the solution form as k_w and k_a approach infinity, ρ_w, ρ_a and F^s approach zero, to see if they would exactly take the same form as the elastodynamic fundamental solution in the Laplace transform domain.

6.1 Limiting case: Elastodynamic

Letting k_w and k_a approach infinity and ρ_w, ρ_a and F^s equal zero, the roots of the determinant equation (55) reduce to two and we will have

$$\lambda_1^2 = \frac{\rho s^2}{\mu}, \lambda_2^2 = \lambda_3^2 = 0, \lambda_4^2 = \Lambda^2 = \frac{\rho s^2}{(\lambda+2\mu)} \tag{99}$$

Then,

$$\widetilde{U}_\beta^{W^*} = \widetilde{U}_\beta^{A^*} = 0 \tag{100}$$

$$\widetilde{P}_\alpha^{wS^*} = \widetilde{P}_\alpha^{aS^*} \tag{101}$$

$$\tilde{P}^{wW^*} = \tilde{P}^{aA^*} = 0 \tag{102}$$

$$\widetilde{P}^{aW^*} = \tilde{P}^{wA^*} = 0 \tag{103}$$

$$\widetilde{U}_{\alpha\beta}^{S\,*} = \frac{1}{2\pi\rho C_2^2}\left(a\delta_{\alpha\beta} - b\frac{x_\alpha x_\beta}{r^2}\right) \tag{104}$$

in which

$$a = \left(\frac{1}{r} + \frac{C_2}{sr^2} + \frac{C_2^2}{s^2r^3}\right)exp\left(-\frac{sr}{C_2}\right) - \frac{C_2^2}{C_1^2}\left(\frac{C_1}{sr^2} + \frac{C_1^2}{s^2r^3}\right)exp\left(-\frac{sr}{C_1}\right) \tag{105}$$

$$b = \left(\frac{1}{r} + \frac{3C_2}{sr^2} + \frac{3C_2^2}{s^2r^3}\right)exp\left(-\frac{sr}{C_2}\right) - \frac{C_2^2}{C_1^2}\left(\frac{1}{r} + \frac{3C_1}{sr^2} + \frac{3C_1^2}{s^2r^3}\right)exp\left(-\frac{sr}{C_1}\right) \tag{106}$$

$$C_1^2 = \frac{(\lambda+2\mu)}{\rho}, \quad C_2^2 = \frac{\mu}{\rho} \tag{107}$$

Eqs. (100) to (104) show the fundamental singular solutions in the Laplace transform domain for a point force in 3D solid of infinite extent. This limiting case supports that the Laplace transform domain fundamental solutions of dynamic unsaturated poroelasticity for 3D cases derived in previous sections are likely to be correct.

7. Conclusion

In this paper, firstly coupled governing differential equations of a porous medium saturated by two compressible fluids (water and air) subjected to dynamic loadings are presented based on the poromechanics theory within the framework of the suction-based mathematical model presented by Gatmiri (1997) and Gatmiri et al. (1998). After that, the Boundary Integral Equation (BIE) is developed directly from those equations via the use of the weighted residuals method for the first time. Finally, the associated fundamental solution in the Laplace transformed domain is presented by the use of the method of Hörmander (1963) for 3D $u_i - p_w - p_a$ formulation of unsaturated porous media. Also, the singular behavior of the fundamental solutions is studied in order to be able to determine the unknown boundary data. It is observed that the singularity of these

solutions is equal to the elastostatic, poroelastostayic or the acoustic fundamental solutions.

The derived Laplace transform domain fundamental solutions can be directly implemented in time domain BEM in which the convolution integral is numerically approximated by a new approach so-called "Operational Quadrature Methods" developed by Lubich (1988 a, b) to model the transient behaviour of unsaturated porous media. This enables one to develop more effective numerical hybrid BE/FE methods to solve 3D nonlinear wave propagation problems in the near future.

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Appendix A   Example of appendix

$$C_1^{sS} = \frac{1}{\mu} \frac{-(\lambda+\mu)\Lambda^2}{\rho s^2} \frac{(\lambda_1^2 - K_{sS1}^2)(\lambda_1^2 - K_{sS2}^2)}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_1^2 - \lambda_4^2)} \tag{A.1}$$

$$C_2^{sS} = \frac{1}{\mu} \frac{-(\lambda+\mu)\Lambda^2}{\rho s^2} \frac{(\lambda_2^2 - K_{sS1}^2)(\lambda_2^2 - K_{sS2}^2)}{(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)} \tag{A.2}$$

$$C_3^{sS} = \frac{1}{\mu} \frac{-(\lambda+\mu)\Lambda^2}{\rho s^2} \frac{(\lambda_3^2 - K_{sS1}^2)(\lambda_3^2 - K_{sS2}^2)}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2)} \tag{A.3}$$

$$C_4^{sS} = \frac{1}{\mu} \frac{-(\lambda+\mu)\Lambda^2}{\rho s^2} \frac{(\lambda_4^2 - K_{sS1}^2)(\lambda_4^2 - K_{sS2}^2)}{(\lambda_4^2 - \lambda_1^2)(\lambda_4^2 - \lambda_2^2)(\lambda_4^2 - \lambda_3^2)} \tag{A.4}$$

$$C_1^{sW} = \frac{(S_w - k_w \rho_w s)s}{(\lambda+2\mu)k_w(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)} \left(\lambda_2^2 - \frac{(C_{wa}(S_a - \rho_a k_a s) - C_{aa}(S_w - \rho_w k_w s))}{k_a(S_w - \rho_w k_w s)} s \right) \tag{A.5}$$

$$C_2^{sW} = \frac{(S_w - k_w \rho_w s)s}{(\lambda+2\mu)k_w(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)} \left(\lambda_3^2 - \frac{(C_{wa}(S_a - \rho_a k_a s) - C_{aa}(S_w - \rho_w k_w s))}{k_a(S_w - \rho_w k_w s)} s \right) \tag{A.6}$$

$$C_3^{sW} = \frac{(S_w - k_w \rho_w s)s}{(\lambda+2\mu)k_w(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)} \left(\lambda_4^2 - \frac{(C_{wa}(S_a - \rho_a k_a s) - C_{aa}(S_w - \rho_w k_w s))}{k_a(S_w - \rho_w k_w s)} s \right) \tag{A.7}$$

$$C_1^{sA} = \frac{(S_a - k_a \rho_a s)s}{(\lambda+2\mu)k_a(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)} \left(\lambda_2^2 - \frac{(C_{wa}(S_w - \rho_w k_w s) - C_{ww}(S_a - \rho_a k_a s))}{k_w(S_a - \rho_a k_a s)} s \right) \tag{A.8}$$

$$C_2^{sA} = \frac{(S_a - k_a \rho_a s)s}{(\lambda+2\mu)k_a(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)} \left(\lambda_3^2 - \frac{(C_{wa}(S_w - \rho_w k_w s) - C_{ww}(S_a - \rho_a k_a s))}{k_w(S_a - \rho_a k_a s)} s \right) \tag{A.9}$$

$$C_3^{sA} = \frac{(S_a - k_a \rho_a s)s}{(\lambda+2\mu)k_a(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)} \left(\lambda_4^2 - \frac{(C_{wa}(S_w - \rho_w k_w s) - C_{ww}(S_a - \rho_a k_a s))}{k_w(S_a - \rho_a k_a s)} s \right) \tag{A.10}$$

$$C_1^{wS} = -\frac{F^s}{(\lambda+2\mu)(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)} \left(\lambda_2^2 - \frac{(C_{wa}(1-F^s) - F^s C_{aa})}{F^s k_a} s \right) \tag{A.11}$$

$$C_2^{wS} = -\frac{F^s}{(\lambda+2\mu)(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)} \left(\lambda_3^2 - \frac{(C_{wa}(1-F^s) - F^s C_{aa})}{F^s k_a} s \right) \tag{A.12}$$

$$C_3^{wS} = -\frac{F^s}{(\lambda+2\mu)(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)} \left(\lambda_4^2 - \frac{(C_{wa}(1-F^s) - F^s C_{aa})}{F^s k_a} s \right) \tag{A.13}$$

$$C_1^{wW} = \frac{1}{(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)} (\lambda_2^2 - K_w^2)(\lambda_2^2 - \Lambda_w^2) \tag{A.14}$$

$$C_2^{wW} = \frac{1}{(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)}(\lambda_3^2 - K_w^2)(\lambda_3^2 - \Lambda_w^2) \tag{A.15}$$

$$C_3^{wW} = \frac{1}{(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)}(\lambda_4^2 - K_w^2)(\lambda_4^2 - \Lambda_w^2) \tag{A.16}$$

$$C_1^{wA} = \frac{s}{(\lambda + 2\mu)k_a} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_a k_a s - S_a)F^s)\lambda_2^2 + \rho C_{wa} s^2}{(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)} \tag{A.17}$$

$$C_2^{wA} = \frac{s}{(\lambda + 2\mu)k_a} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_a k_a s - S_a)F^s)\lambda_3^2 + \rho C_{wa} s^2}{(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)} \tag{A.18}$$

$$C_3^{wA} = \frac{s}{(\lambda + 2\mu)k_a} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_a k_a s - S_a)F^s)\lambda_4^2 + \rho C_{wa} s^2}{(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)} \tag{A.19}$$

$$C_1^{aS} = -\frac{(1 - F^s)}{(\lambda + 2\mu)(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)}\left(\lambda_2^2 - \frac{(C_{ww}(1 - F^s) - F^s C_{wa})}{F^s k_a}s\right) \tag{A.20}$$

$$C_2^{aS} = -\frac{(1 - F^s)}{(\lambda + 2\mu)(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)}\left(\lambda_3^2 - \frac{(C_{ww}(1 - F^s) - F^s C_{wa})}{F^s k_a}s\right) \tag{A.21}$$

$$C_3^{aS} = -\frac{(1 - F^s)}{(\lambda + 2\mu)(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)}\left(\lambda_4^2 - \frac{(C_{ww}(1 - F^s) - F^s C_{wa})}{F^s k_a}s\right) \tag{A.22}$$

$$C_1^{aW} = \frac{s}{(\lambda + 2\mu)k_w} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_w k_w s - S_w)(1 - F^s))\lambda_2^2 + \rho C_{wa} s^2}{(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)} \tag{A.23}$$

$$C_2^{aW} = \frac{s}{(\lambda + 2\mu)k_w} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_w k_w s - S_w)(1 - F^s))\lambda_3^2 + \rho C_{wa} s^2}{(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)} \tag{A.24}$$

$$C_3^{aW} = \frac{s}{(\lambda + 2\mu)k_w} \frac{(-(\lambda + 2\mu)C_{wa} + (\rho_w k_w s - S_w)(1 - F^s))\lambda_4^2 + \rho C_{wa} s^2}{(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)} \tag{A.25}$$

$$C_1^{aA} = \frac{1}{(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2)}(\lambda_2^2 - K_a^2)(\lambda_2^2 - \Lambda_a^2) \tag{A.26}$$

$$C_2^{aA} = \frac{1}{(\lambda_4^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)}(\lambda_3^2 - K_a^2)(\lambda_3^2 - \Lambda_a^2) \tag{A.27}$$

$$C_3^{aA} = \frac{1}{(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)}(\lambda_4^2 - K_a^2)(\lambda_4^2 - \Lambda_a^2) \tag{A.28}$$