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# Numerical controllability of the wave equation through a primal method and Carleman estimates

NICOLAE CÎNDEA\*, ENRIQUE FERNÁNDEZ-CARA† and ARNAUD MÜNCH\*

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## Abstract

This paper deals with the numerical computation of boundary null controls for the 1D wave equation with a potential. The goal is to compute an approximation of controls that drive the solution from a prescribed initial state to zero at a large enough controllability time. We do not use in this work duality arguments but explore instead a direct approach in the framework of global Carleman estimates. More precisely, we consider the control that minimizes over the class of admissible null controls a functional involving weighted integrals of the state and of the control. The optimality conditions show that both the optimal control and the associated state are expressed in terms of a new variable, the solution of a fourth-order elliptic problem defined in the space-time domain. We first prove that, for some specific weights determined by the global Carleman inequalities for the wave equation, this problem is well-posed. Then, in the framework of the finite element method, we introduce a family of finite-dimensional approximate control problems and we prove a strong convergence result. Numerical experiments confirm the analysis. We complete our study with several comments.

**Keywords:** one-dimensional wave equation, null controllability, finite element methods, Carleman inequalities.

**Mathematics Subject Classification (2010)-** 35L10, 65M12, 93B40.

## 1 Introduction. The null controllability problem

We are concerned in this work with the null controllability for the 1D wave equation with a potential. The state equation is the following:

$$\begin{cases} y_{tt} - (a(x)y_x)_x + b(x,t)y = 0, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = 0, \quad y(1,t) = v(t), & t \in (0,T) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in (0,1). \end{cases} \quad (1)$$

Here,  $T > 0$ ,  $a \in C^1([0,1])$  with  $a(x) \geq a_0 > 0$  in  $[0,1]$ ,  $b \in L^\infty((0,1) \times (0,T))$ ,  $y_0 \in L^2(0,1)$ ,  $y_1 \in H^{-1}(0,1)$ ;  $v = v(t)$  is the *control* (a function in  $L^2(0,T)$ ) and  $y = y(x,t)$  is the associated state.

In the sequel, for any  $\tau > 0$  we denote by  $Q_\tau$  and  $\Sigma_\tau$  the sets  $(0,1) \times (0,\tau)$  and  $\{0,1\} \times (0,\tau)$  respectively. We will also use the following notation :

$$Ly := y_{tt} - (a(x)y_x)_x + b(x,t)y. \quad (2)$$

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For any  $(y_0, y_1) \in \mathbf{V} := L^2(0, 1) \times H^{-1}(0, 1)$  and any  $v \in L^2(0, T)$ , it is well known that there exists exactly one solution  $y$  to (1), with the following regularity

$$y \in C([0, T], L^2(0, 1)) \cap C^1([0, T], H^{-1}(0, 1)) \quad (3)$$

(see for instance [18]).

Thus, for any final time  $T > 0$ , the null controllability problem for (1) at time  $T$  is the following: for each  $(y_0, y_1) \in \mathbf{V}$ , find  $v \in L^2(0, T)$  such that the associated solution to (1) satisfies

$$y(x, T) = y_t(x, T) = 0, \quad x \in (0, 1). \quad (4)$$

It is well known that (1) is null-controllable at any *large* time  $T > T^*$  for some  $T^*$  that depends on  $a$  (for instance, see [18] for  $a \equiv 1$  leading to  $T^* = 2$  and see [26] for a general situation). As a consequence of the *Hilbert Uniqueness Method* of J.-L. Lions [18], it is also known that the null controllability of (1) is equivalent to an observability inequality for the associated adjoint problem.

The goal of this paper is to design and analyze a numerical method allowing to solve the previous null controllability problem.

So far, the approximation of the minimal  $L^2$ -norm control — the so-called HUM control — has focused most of the attention. The earlier contribution is due to Glowinski and Lions in [14] (see also [16] for an update), who made use of duality arguments. Duality allows to transform the original constrained minimization problem in an unconstrained and *a priori* easier minimization (dual) problem. However, as observed in [14], depending on the approximation method that is used, this approach can lead to some numerical difficulties.

More precisely, it is easily seen that the HUM control is given by  $v(t) = \phi_x(1, t)$ , where  $\phi$  solves the backwards wave system

$$\begin{cases} L\phi = 0 & \text{in } Q_T \\ \phi = 0 & \text{on } \Sigma_T \\ (\phi(\cdot, T), \phi_t(\cdot, T)) = (\phi_0, \phi_1) & \text{in } (0, 1) \end{cases} \quad (5)$$

and  $(\phi_0, \phi_1)$  minimizes the strictly convex and coercive functional

$$\mathcal{I}(\phi_0, \phi_1) = \frac{1}{2} \|\phi_x(1, \cdot)\|_{L^2(0, T)}^2 + \int_0^1 y_0(x) \phi_t(x, 0) dx - \langle y_1, \phi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad (6)$$

over  $\mathbf{H} = H_0^1(0, 1) \times L^2(0, 1)$ . Here and henceforth  $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$  denotes the duality product for  $H^{-1}(0, 1)$  and  $H_0^1(0, 1)$ .

The coercivity of  $\mathcal{I}$  over  $\mathbf{H}$  is a consequence of the *observability inequality*

$$\|\phi_0\|_{H_0^1(0, 1)}^2 + \|\phi_1\|_{L^2(0, 1)}^2 \leq C \|\phi_x(1, \cdot)\|_{L^2(0, T)}^2 \quad \forall (\phi_0, \phi_1) \in \mathbf{H}, \quad (7)$$

that holds for some constant  $C = C(T)$ . This inequality has been derived in [18] using the *multipliers method*.

At the numerical level, for standard approximation schemes (based on finite difference or finite element methods), the discrete version of (7) may not hold uniformly with respect to the discretization parameter, say  $h$ . In other words, the constant  $C = C(h)$  may blow up as  $h$  goes to zero. Consequently, in such cases the functional  $\mathcal{I}_h$  (the discrete version of  $\mathcal{I}$ ) fails to be coercive uniformly with respect to  $h$  and the sequence  $\{v_h\}_{h>0}$  does not converge to  $v$  as  $h \rightarrow 0$ , but diverges exponentially. These pathologies, by now well-known and understood, are due to the spurious discrete high frequencies generated by the finite dimensional approximation; we refer to [29] for a review on that topic; see [19] for detailed examples of that behavior observed with finite difference methods.

Several remedies based on more elaborated approximations have been proposed and analyzed in the last decade. Let us mention the use of mixed finite elements [5], additional viscosity terms which have the effect to restore the uniform property [1, 19] and also filtering technics [7]. Also, notice that some error estimates have been obtained recently, see [6, 7].

In this paper, following the recent work [9] devoted to the heat equation, we consider a more general norm. Specifically, we consider the following extremal problem:

$$\begin{cases} \text{Minimize } J(y, v) = \frac{1}{2} \iint_{Q_T} \rho^2 |y|^2 dx dt + \frac{1}{2} \int_0^T \rho_0^2 |v|^2 dt \\ \text{Subject to } (y, v) \in \mathcal{C}(y_0, y_1, T) \end{cases} \quad (8)$$

where  $\mathcal{C}(y_0, y_1, T)$  denotes the linear manifold

$$\mathcal{C}(y_0, y_1, T) = \{ (y, v) : v \in L^2(0, T), y \text{ solves (1) and satisfies (4)} \}.$$

Here, we assume that the weights  $\rho$  and  $\rho_0$  are strictly positive and continuous in  $Q_T$  and  $(0, T)$ , respectively.

As for the  $L^2$ -norm situation (for which we simply have  $\rho \equiv 0$  and  $\rho_0 \equiv 1$ ), we may also apply duality arguments in order to find a solution to (8), introducing the unconstrained problem

$$\begin{cases} \text{Minimize } J^*(\mu, \phi_0, \phi_1) = \frac{1}{2} \iint_{Q_T} \rho^{-2} |\mu|^2 dx dt + \frac{1}{2} \int_0^T \rho_0^{-2} |\phi_x(1, t)|^2 dt \\ \quad + \int_0^1 y_0(x) \phi_t(x, 0) dx - \langle y_1, \phi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \\ \text{Subject to } (\mu, \phi_0, \phi_1) \in L^2(Q_T) \times \mathbf{H}, \end{cases} \quad (9)$$

where  $\phi$  solves the nonhomogeneous backwards problem

$$\begin{cases} L\phi = \mu & \text{in } Q_T \\ \phi = 0 & \text{on } \Sigma_T \\ (\phi(\cdot, T), \phi_t(\cdot, T)) = (\phi_0, \phi_1) & \text{in } (0, 1). \end{cases}$$

Clearly,  $J^*$  is the conjugate function of  $J$  in the sense of Fenchel and Rockafellar [8, 23] and, if  $\rho_0 \in L^\infty(Q_T)$  (that is,  $\rho_0^{-2}$  is positively bounded from below),  $J^*$  is coercive in  $L^2(Q_T) \times \mathbf{H}$  thanks to (7). Therefore, if  $(\hat{\mu}, \hat{\phi}_0, \hat{\phi}_1)$  denotes the minimizer of  $J^*$ , the corresponding optimal pair for  $J$  is given by

$$v = -a(1)\rho_0^{-2}\phi_x(1, \cdot) \text{ in } (0, T) \text{ and } y = -\rho^{-2}\mu \text{ in } Q_T.$$

At the discrete level, we may suspect that such coercivity may not hold uniformly with respect to the discretization parameters, leading to the pathologies and the lack of convergence we have just mentioned.

On the other hand, the explicit occurrence of the state variable  $y$  in the cost  $J$  allows to avoid the use of duality. Following [9] and based on an idea due to Fursikov and Imanuvilov [11] that allows to get general controllability results, we may use directly a primal method that provides an optimal pair  $(y, v) \in \mathcal{C}(y_0, y_1, T)$ . More precisely, under some conditions on the coefficient  $a$ , one is led to a fourth-order *elliptic* problem that is well-posed for an appropriate choice of the weights  $\rho$  and  $\rho_0$ , deduced from an appropriate global Carleman estimate (an updated version of the inequalities established in [2]).

This is the approach that we present and analyze in this work. As will be shown, it ensures numerical convergence results for standard approximation methods.

This paper is organized as follows.

In Section 2, adapting the arguments and results in [9], we show that the solution to (8) can be expressed in terms of the unique solution  $p$  of the variational problem (23) (see Proposition 2.2) in the Hilbert space  $P$ , defined as the completion of  $P_0$  with respect to the inner product (17). The well-posedness is deduced from the application of Riesz's Theorem: a suitable global Carleman inequality (see Theorem 2.1) ensures the continuity of the linear form in (23) for  $T$  large enough when  $\rho$  and  $\rho_0$  are given by (19).

In Section 3, we perform the numerical analysis of the variational problem (23) in the finite element theory framework. Thus, we replace  $P$  by the finite element space  $P_h$  of  $C^1(\overline{Q_T})$  functions defined by (32) and we show that the unique solution  $\hat{p}_h \in P_h$  of the finite dimensional problem (37) converges (strongly) for the  $P$ -norm to  $p$  as  $h \rightarrow 0^+$ . Section 4 contains some numerical experiments that illustrate and confirm the convergence of the sequence  $\{\hat{p}_h\}_{h>0}$ .

Finally, we present some additional comments in Section 5 and we provide some details of the proof of Theorem 2.1 in an Appendix.

## 2 A variational approach to the null controllability problem

With the notation introduced in Section 1, the following result holds.

**PROPOSITION 2.1** *Let us assume that  $\rho$  and  $\rho_0$  are positive and satisfy  $\rho \in C^0(Q_T)$  and  $\rho_0 \in C^0(0, T)$ . For any  $(y_0, y_1) \in \mathbf{V}$  and any  $T > 0$  large enough, there exists exactly one solution to the extremal problem (8).*

The proof is simple. Indeed, for  $T \geq T^*$ , the controllability holds and  $C(y_0, y_1, T)$  is non-empty. Furthermore, it is a closed convex set of  $L^2(Q_T) \times L^2(0, T)$ . On the other hand,  $(y, v) \mapsto J(y, v)$  is strictly convex, proper and lower-semicontinuous in  $L^2(Q_T) \times L^2(0, T)$  and

$$J(y, v) \rightarrow +\infty \quad \text{as} \quad \|(y, v)\|_{L^2(Q_T) \times L^2(\Sigma_T)} \rightarrow +\infty.$$

Hence, the extremal problem (8) certainly possesses a unique solution.

In this paper, it will be convenient to assume that the coefficient  $a$  belongs to the family

$$\begin{aligned} \mathcal{A}(x_0, a_0) = \{ a \in C^1([0, 1]) : a(x) \geq a_0 > 0, \\ - \min_{[0,1]} \left( a(x) + (x - x_0)a_x(x) \right) < \min_{[0,1]} \left( a(x) + \frac{1}{2}(x - x_0)a_x(x) \right) \}, \end{aligned} \quad (10)$$

where  $x_0 < 0$  and  $a_0$  is a positive constant.

It is easy to check that the constant function  $a(x) \equiv a_0$  belongs to  $\mathcal{A}(x_0, a_0)$ . Similarly, any non-decreasing smooth function  $\geq a_0$  belongs to  $\mathcal{A}(x_0, a_0)$ . Roughly speaking,  $a \in \mathcal{A}(x_0, a_0)$  means that  $a$  is sufficiently smooth, strictly positive and not too decreasing in  $[0, 1]$ .

Under the assumption (10), there exists “good” weight functions  $\rho$  and  $\rho_0$  which provide a very suitable solution to the original null controllability problem. They can be deduced from global Carleman inequalities.

The argument is the following. First, let us introduce a constant  $\beta$ , with

$$- \min_{[0,1]} \left( a(x) + (x - x_0)a_x(x) \right) < \beta < \min_{[0,1]} \left( a(x) + \frac{1}{2}(x - x_0)a_x(x) \right) \quad (11)$$

and let us consider the function

$$\phi(x, t) := |x - x_0|^2 - \beta t^2 + M_0, \quad (12)$$

where  $M_0$  is such that

$$\phi(x, t) \geq 1 \quad \forall (x, t) \in (0, 1) \times (-T, T), \quad (13)$$

i.e.  $M_0 \geq 1 - |x_0|^2 + \beta T^2$ . Then, for any  $\lambda > 0$  we set

$$\varphi(x, t) := e^{\lambda \phi(x, t)}. \quad (14)$$

The Carleman estimates for the wave equation are given in the following result:

**THEOREM 2.1** *Let us assume that  $x_0 < 0$ ,  $a_0 > 0$  and  $a \in \mathcal{A}(x_0, a_0)$ . Let  $\beta$  and  $\varphi$  be given respectively by (11) and (14). Moreover, let us assume that*

$$T > \frac{1}{\beta} \max_{[0,1]} a(x)^{1/2} (x - x_0). \quad (15)$$

*Then there exist positive constants  $s_0$  and  $M$ , only depending on  $x_0$ ,  $a_0$ ,  $\|a\|_{C^1([0,1])}$ ,  $\|b\|_{L^\infty(Q_T)}$  and  $T$ , such that, for all  $s > s_0$ , one has*

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt \end{aligned} \quad (16)$$

for any  $w \in L^2(-T, T; H_0^1(0, 1))$  satisfying  $Lw \in L^2((0, 1) \times (-T, T))$  and  $w_x(1, \cdot) \in L^2(-T, T)$ .

There exists an important literature related to (global) Carleman estimates for the wave equation. Almost all references deal with the particular case  $a \equiv 1$ ; we refer to [2, 3, 13, 27]. The case where  $a$  is non-constant is less studied; we refer to [12].

The proof follows closely the ideas used in the proofs of [3, Theorems 2.1 and 2.5] to obtain a global Carleman estimate for the wave equation when  $a \equiv 1$ . The parts of the proof which become different for a non-constant  $a$  are detailed in the Appendix of this paper.

Now, let us consider the linear space

$$P_0 = \{q \in C^\infty(\overline{Q_T}) : q = 0 \text{ on } \Sigma_T\}.$$

The bilinear form

$$(p, q)_P := \iint_{Q_T} \rho^{-2} Lp Lq dx dt + \int_0^T \rho_0^{-2} a(1)^2 p_x(1, t) q_x(1, t) dt \quad (17)$$

is a scalar product in  $P_0$ . Indeed, recall that, for  $T$  large enough, the unique continuation property for the wave equation holds. Accordingly, if  $q \in P_0$ ,  $Lq = 0$  in  $Q_T$  and  $q_x = 0$  on  $\{1\} \times (0, T)$ , then  $q \equiv 0$ . This shows that  $(\cdot, \cdot)_P$  is certainly a scalar product in  $P_0$ .

Let  $P$  be the completion of  $P_0$  with respect to this scalar product. Then  $P$  is a Hilbert space for  $(\cdot, \cdot)_P$  and we can deduce from Theorem 2.1 the following result, that indicates which are the appropriate weights  $\rho$  and  $\rho_0$  for our controllability problem:

**LEMMA 2.1** *Let  $x_0$ ,  $a_0$  and  $a$  be as in Theorem 2.1 and let us assume that*

$$T > \frac{2}{\beta} \max_{[0,1]} a(x)^{1/2} (x - x_0), \text{ with } \beta \text{ satisfying (11)}. \quad (18)$$

*Let us introduce the weights  $\rho$  and  $\rho_0$ , respectively given by*

$$\rho(x, t) := e^{-s\varphi(x, 2t-T)}, \quad \rho_0(t) := \rho(1, t), \quad (19)$$

where  $s > s_0$  and let us consider the corresponding Hilbert space  $P$ . There exists a constant  $C_0 > 0$ , only depending on  $x_0$ ,  $a_0$ ,  $\|a\|_{C^1([0,1])}$ ,  $\|b\|_{L^\infty(Q_T)}$  and  $T$ , such that

$$\|p(\cdot, 0)\|_{H_0^1(0,1)}^2 + \|p_t(\cdot, 0)\|_{L^2(0,1)}^2 \leq C_0 (p, p)_P \quad \forall p \in P. \quad (20)$$

PROOF: For every  $p \in P$ , we denote by  $\bar{p} \in L^2((0, 1) \times (-T, T))$  the function defined by

$$\bar{p}(x, t) = p\left(x, \frac{t+T}{2}\right).$$

It is easy to see that  $\bar{p} \in L^2(-T, T; H_0^1(\Omega))$ ,  $L\bar{p} \in L^2((0, 1) \times (-T, T))$  and  $\bar{p}_x(1, \cdot) \in L^2(-T, T)$ , so that we can apply Theorem 2.1 to  $\bar{p}$ . Accordingly, we have

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|\bar{p}_t|^2 + |\bar{p}_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |\bar{p}|^2 dx dt \\ & \leq C \int_{-T}^T \int_0^1 e^{2s\varphi} |L\bar{p}|^2 dx dt + Cs \int_{-T}^T e^{2s\varphi(1,t)} |\bar{p}_x(1, t)|^2 dt \end{aligned} \quad (21)$$

where  $C$  depends on  $x_0$ ,  $a_0$ ,  $\|a\|_{C^1([0,1])}$ ,  $\|b\|_{L^\infty(Q_T)}$  and  $T$ .

Replacing  $\bar{p}$  by its definition in (21) and changing the variable  $t$  by  $t' = 2t - T$  we obtain the following for any  $T$  satisfying (18):

$$\begin{aligned} & s \iint_{Q_T} \rho^{-2} (|p_t|^2 + |p_x|^2) dx dt + s^3 \iint_{Q_T} \rho^{-2} |p|^2 dx dt \\ & \leq C \iint_{Q_T} \rho^{-2} |Lp|^2 dx dt + Cs \int_0^T \rho_0^{-2} |p_x(1, t)|^2 dt, \end{aligned}$$

where  $C$  is replaced with a slightly different constant. Finally, combining the arguments in Corollary 2.8 and Remark 2.7 in [3], we obtain the estimate (20).  $\square$

**Remark 1** The estimate (20) must be viewed as an *observability inequality*. As expected, it holds if and only if  $T$  is large enough. Notice that, when  $a(x) \equiv 1$ , the assumption (18) reads

$$T > 2(1 - x_0).$$

This confirms that, in this case, whenever  $T > 2$ , (20) holds (it suffices to choose  $x_0$  appropriately and apply Lemma 2.1; see [18]).  $\square$

The previous results allow us to find a very useful characterization of the optimal pair  $(y, v)$  for  $J$ :

**PROPOSITION 2.2** *Let the assumptions of Lemma 2.1 be verified and let  $\rho$  and  $\rho_0$  be given by (19). Let  $(y, v) \in \mathcal{C}(y_0, y_1, T)$  be the solution to (8). Then there exists  $p \in P$  such that*

$$y = -\rho^{-2} Lp, \quad v = -(a(x)\rho_0^{-2} p_x)|_{x=1}. \quad (22)$$

Moreover,  $p$  is the unique solution of the following variational equality:

$$\begin{cases} \iint_{Q_T} \rho^{-2} Lp Lq dx dt + \int_0^T \rho_0^{-2} a^2(1) p_x(1, t) q_x(1, t) dt \\ = \int_0^1 y_0(x) q_t(x, 0) dx - \langle y^1, q(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad \forall q \in P; \quad p \in P. \end{cases} \quad (23)$$

Here and in the sequel, we use the following duality pairing:

$$\langle y^1, q(\cdot, 0) \rangle_{H^{-1}, H_0^1} = \int_0^1 \frac{\partial}{\partial x} ((-\Delta)^{-1} y_1)(x) q_x(x, 0) dx,$$

where  $-\Delta$  is the Dirichlet Laplacian in  $(0, 1)$ .

PROOF: From the definition of the scalar product in  $P$ , we see that  $p$  solves (23) if and only if

$$(p, q)_P = l(q) := \int_0^1 y_0(x) q_t(x, 0) dx - \langle y^1, q(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad \forall q \in P; \quad p \in P.$$

In view of Lemma 2.1 and *Riesz's Representation Theorem*, problem (23) possesses exactly one solution in  $P$ .

Let us now introduce  $y$  and  $v$  according to (22) and let us check that  $(y, v)$  solves (8). First, notice that  $y \in L^2(Q_T)$  and  $v \in L^2(0, T)$ . Then, by replacing  $y$  and  $v$  in (23), we obtain the following:

$$\iint_{Q_T} y Lq dx dt + \int_0^T a(1)v(t)q_x(1, t) dt = l(q) \quad \forall q \in P. \quad (24)$$

Hence,  $(y, v)$  is the solution of the controlled wave system (1) in the transposition sense. Since  $y \in L^2(Q_T)$  and  $v \in L^2(0, T)$  the pair  $(y, v)$  belongs to  $\mathcal{C}(y_0, y_1, T)$ .

It remains to check that  $(y, v)$  minimizes the cost function  $J$  in (8). But this is easy. Indeed, for any  $(z, w) \in \mathcal{C}(y_0, y_1, T)$  such that  $J(z, w) < +\infty$ , in view of (24), one has:

$$\begin{aligned} J(z, w) &\geq J(y, v) + \iint_{Q_T} \rho^2 y (z - y) dx dt + \int_0^T \rho_0^2 v (w - v) dt \\ &= J(y, v) - \iint_{Q_T} Lp(z - y) dx dt + \int_0^T \rho_0^2 v (w - v) dt = J(y, v). \end{aligned}$$

□

**Remark 2** From (22) and (23), we see that the function  $p$  furnished by Proposition 2.2 solves, at least in the distributional sense, the following differential problem, that is of the fourth-order in time and space:

$$\begin{cases} L(\rho^{-2}Lp) = 0, & (x, t) \in Q_T \\ p(0, t) = 0, \quad (\rho^{-2}Lp)(0, t) = 0, & t \in (0, T) \\ p(1, t) = 0, \quad (\rho^{-2}Lp + a\rho_0^{-2}p_x)(1, t) = 0, & t \in (0, T) \\ (\rho^{-2}Lp)(x, 0) = y_0(x), \quad (\rho^{-2}Lp)(x, T) = 0, & x \in (0, 1) \\ (\rho^{-2}Lp)_t(x, 0) = y_1(x), \quad (\rho^{-2}Lp)_t(x, T) = 0, & x \in (0, 1). \end{cases} \quad (25)$$

Notice that the “boundary” conditions at  $t = 0$  and  $t = T$  are of the Neumann kind. □

**Remark 3** The weights  $\rho^{-1}$  and  $\rho_0^{-1}$  behave exponentially with respect to  $s$ . For instance, we have

$$\rho(x, t)^{-1} = \exp \left( s e^{\lambda(|x-x_0|^2 - \beta(2t-T)^2 + M_0)} \right).$$

Since the parameter  $s$  has to be taken large enough, greater than  $s_0 > 0$  (in order that the Carleman inequality (16) ensures the well-posedness of (23)), the weights  $\rho^{-2}$  and  $\rho_0^{-2}$  may lead in practice to numerical overflow. One may overcome this situation by introducing a suitable change of variable.

More precisely, let us introduce the variable  $z = \rho p$  and the Hilbert space  $M = \rho P$ , so that the formulation (23) becomes :

$$\begin{cases} \iint_{Q_T} \rho^{-2} L(\rho z) L(\rho \bar{z}) dx dt + \int_0^T \rho_0^{-2} a^2(1) (\rho z)_x(1, t) (\rho \bar{z})_x(1, t) dt \\ = \int_0^1 y_0(x) (\rho \bar{z})_t(x, 0) dx - \langle y_1, (\rho \bar{z})(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad \forall \bar{z} \in M; \quad z \in M. \end{cases} \quad (26)$$

The well-posedness of this formulation is a consequence of the well-posedness of (23). Then, after some computations, the following is found:



$$\begin{aligned}\rho L(\rho^{-1}z) &= \rho^{-1} \left( (\rho z)_t - (a(\rho z)_x)_x + b\rho z \right) \\ &= (\rho^{-1}\rho_t)z + z_t - a_x((\rho^{-1}\rho_x)z + z_x) - a(2\rho^{-1}\rho_x z_x + \rho^{-1}\rho_{xx}z + z_{xx}) + bz\end{aligned}$$

with

$$\rho^{-1}\rho_x = -s\varphi_x(x, 2t - T), \quad \rho^{-1}\rho_t = -2s\varphi_t(x, 2t - T), \quad \rho^{-1}\rho_{xx} = -s\varphi_{xx} + (s\varphi_x)^2.$$

Similarly,

$$(\rho_0^{-1}(\rho z)_x)(1, t) = -s\varphi_x(1, 2t - T)z(1, t) + z_x(1, t).$$

Consequently, in the bilinear part of (26) (equivalent to (23)), there is no (positive) exponential, but only polynomial, function of  $s$ . In the right hand side (the linear part), the change of variable introduces negative exponentials in  $s$ . A similar trick has been used in [9] in the context of the heat equation, for which blowing up (exponential) functions as  $t \rightarrow T^-$  appear.  $\square$

**Remark 4** The estimate (20) can be proven for a weight  $\rho_0$  which blows up at  $t = 0$  and  $t = T$ . For this purpose, we consider a function  $\theta_\delta \in C^2([0, T])$  with  $\theta_\delta(0) = \theta_\delta(1) = 0$  and  $\theta_\delta(x) = 1$  for every  $x \in (\delta, T - \delta)$ . Then, introducing again  $\bar{p}(x, t) := \theta_\delta(t)p(x, (t + T)/2)$ , it is not difficult to see that the proofs of Lemma 2.1 and Theorem 2.1 can be adapted to obtain (20) with

$$\rho(x, t) = e^{-s\varphi(x, 2t - T)}, \quad \rho_0(t) = \theta_\delta(t)^{-1}\rho(1, t).$$

Thanks to the properties of  $\theta_\delta$ , the control  $v$  defined by

$$v = -\theta_\delta^2 \rho_0^{-2} a(x) p_x \Big|_{x=1}$$

vanishes at  $t = 0$  and also at  $t = T$ , a property which is very natural in the boundary controllability context. In the sequel, we will use this modified weight  $\rho_0$ , assuming in addition, for numerical purposes, the following behavior near  $t = 0$  and  $t = T$ :

$$\lim_{t \rightarrow 0^+} \frac{\theta_\delta(t)}{\sqrt{t}} = \mathcal{O}(1), \quad \lim_{t \rightarrow T^-} \frac{\theta_\delta(t)}{\sqrt{T - t}} = \mathcal{O}(1). \quad (27)$$

$\square$

### 3 Numerical analysis of the variational approach

We now highlight that the variational formulation (23) allows to obtain a sequence of approximations  $\{v_h\}$  that converge strongly towards the null control  $v$  furnished by the solution to (8).

#### 3.1 Finite dimensional approximation

Let us introduce the bilinear form  $m(\cdot, \cdot)$  with

$$m(p, \bar{p}) := \iint_{Q_T} \rho^{-2} Lp L\bar{p} dx dt + \int_0^T a(1)^2 \rho_0^{-2} p_x \bar{p}_x dt$$

and the linear form  $l$ , with

$$\langle l, \bar{p} \rangle := \int_0^1 y_0(x) \bar{p}_t(x, 0) dx - \langle y^1, \bar{p}(\cdot, 0) \rangle_{H^{-1}, H_0^1}.$$

Then (23) reads as follows :

$$m(p, \bar{p}) = \langle l, \bar{p} \rangle, \quad \forall \bar{p} \in P; \quad p \in P. \quad (28)$$

For any dimensional space  $P_h \subset P$ , we can introduce the following *approximate* problem:

$$m(p_h, \bar{p}_h) = \langle l, \bar{p}_h \rangle, \quad \forall \bar{p}_h \in P_h; \quad p_h \in P_h. \quad (29)$$

Obviously, (29) is well-posed. Furthermore, we have the classical result:

LEMMA 3.1 *Let  $p \in P$  be the unique solution to (28) and let  $p_h \in P_h$  be the unique solution to (29). We have*

$$\|p - p_h\|_P \leq \inf_{\bar{p}_h \in P_h} \|p - \bar{p}_h\|_P.$$

PROOF: We write that

$$\|p_h - p\|_P^2 = m(p_h - p, p_h - p) = m(p_h - p, p_h - \bar{p}_h) + m(p_h - p, \bar{p}_h - p).$$

The first term vanishes for all  $\bar{p}_h \in P_h$ . The second one is bounded by  $\|p_h - p\|_P \|\bar{p}_h - p\|_P$ . So, we get

$$\|p - p_h\|_P \leq \|p - \bar{p}_h\|_P \quad \forall \bar{p}_h \in P_h$$

and the result follows.  $\square$

As usual, this result can be used to prove the convergence of  $p_h$  towards  $p$  as  $h \rightarrow 0$  when the spaces  $P_h$  are chosen appropriately.

More precisely, assume that  $\mathcal{H} \subset \mathbf{R}^d$  is a *net* (i.e. a generalized sequence) that converges to zero and let  $P_h$  be as above for each  $h \in \mathcal{H}$ . Let us introduce the interpolation operators  $\Pi_h : P_0 \rightarrow P_h$  and let us assume that the finite dimensional spaces  $P_h$  are chosen such that

$$\|p - \Pi_h p\|_P \rightarrow 0 \quad \text{as } h \in \mathcal{H}, \quad h \rightarrow 0, \quad \forall p \in P_0. \quad (30)$$

We then have:

PROPOSITION 3.1 *Let  $p \in P$  be the solution to (28) and let  $p_h \in P_h$  be the solution to (29) for each  $h \in \mathcal{H}$ . Then*

$$\|p - p_h\|_P \rightarrow 0 \quad \text{as } h \in \mathcal{H}, \quad h \rightarrow 0. \quad (31)$$

PROOF: Let us choose  $\epsilon > 0$ . From the density of  $P_0$  in  $P$ , there exists  $p_\epsilon \in P_0$  such that  $\|p - p_\epsilon\|_P \leq \epsilon$ . Therefore, from Lemma 3.1, we find that

$$\begin{aligned} \|p - p_h\|_P &\leq \|p - \Pi_h p_\epsilon\|_P \\ &\leq \|p - p_\epsilon\|_P + \|p_\epsilon - \Pi_h p_\epsilon\|_P \\ &\leq \epsilon + \|p_\epsilon - \Pi_h p_\epsilon\|_P. \end{aligned}$$

From (30),  $\|p_\epsilon - \Pi_h p_\epsilon\|_P$  goes to zero as  $h \in \mathcal{H}, h \rightarrow 0$  and the result follows.  $\square$

### 3.2 The finite dimensional space $P_h$

The spaces  $P_h$  have to be chosen so that  $\rho^{-1} L p_h$  belongs to  $L^2(Q_T)$  for any  $p_h \in P_h$ . This means that  $p_h$  must possess second-order derivatives in  $L^2_{loc}(Q_T)$ . Therefore, a conformal approximation based on a standard quadrangulation of  $Q_T$  requires spaces of functions that must be  $C^1$  in both variables  $x$  and  $t$ .

For large integers  $N_x$  and  $N_t$ , we set  $\Delta x = 1/N_x$ ,  $\Delta t = T/N_t$  and  $h = (\Delta x, \Delta t)$ . We introduce the associated uniform quadrangulations  $\mathcal{Q}_h$ , with  $Q_T = \bigcup_{K \in \mathcal{Q}_h} K$  and we assume that  $\{\mathcal{Q}_h\}_{h>0}$  is a regular family. Then, we introduce the space  $P_h$  as follows:

$$P_h = \{z_h \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{Q}_h, z_h = 0 \text{ on } \Sigma_T\}. \quad (32)$$

Here,  $\mathbb{P}(K)$  denotes the following space of polynomial functions in  $x$  and  $t$ :

$$\mathbb{P}(K) = (\mathbb{P}_{3,x} \otimes \mathbb{P}_{3,t})(K), \quad (33)$$

where  $\mathbb{P}_{\ell,\xi}$  is by definition the space of polynomial functions of order  $\ell$  in the variable  $\xi$ .

Obviously,  $P_h$  is a finite dimensional subspace of  $P$ .

According to the specific geometry of  $Q_T$ , we shall analyze the situation for a uniform quadrangulation  $\mathcal{Q}_h$ . In that case,  $\mathbb{P}(K)$  coincides with  $\mathbb{Q}_3(K)$  for each element  $K \in \mathcal{Q}_h$  of the form

$$K = K_{kl} = (x_k, x_{k+1}) \times (t_l, t_{l+1}),$$

where

$$x_{k+1} = x_k + \Delta x, \quad t_{l+1} = t_l + \Delta t, \quad \text{for } k = 1, \dots, N_x, \quad l = 1, \dots, N_t.$$

Any function  $z_h \in \mathbb{P}(K_{kl})$  is uniquely determined by the real numbers

$$z_h(x_{k+m}, t_{l+n}), \quad (z_h)_x(x_{k+m}, t_{l+n}), \quad (z_h)_t(x_{k+m}, t_{l+n}) \quad \text{and} \quad (z_h)_{xt}(x_{k+m}, t_{l+n}),$$

with  $m, n = 0, 1$ .

More precisely, for any  $k$  and  $l$ , let us introduce the functions  $L_{ik}$  and  $\mathcal{L}_{jl}$ , with

$$\begin{cases} L_{0k}(x) := \frac{(\Delta x + 2x - 2x_k)(\Delta x - x + x_k)^2}{(\Delta x)^3}, & L_{1k}(x) := \frac{(x - x_k)^2(-2x + 2x_k + 3\Delta x)}{(\Delta x)^3}, \\ L_{2k}(x) := \frac{(x - x_k)(\Delta x - x + x_k)^2}{(\Delta x)^2}, & L_{3k}(x) := \frac{-(x - x_k)^2(\Delta x - x + x_k)}{(\Delta x)^2} \end{cases}$$

and

$$\begin{cases} \mathcal{L}_{0l}(t) := \frac{(\Delta t + 2t - 2t_l)(\Delta t - t + t_l)^2}{(\Delta t)^3}, & \mathcal{L}_{1l}(t) := \frac{(t - t_l)^2(-2t + 2t_l + 3\Delta t)}{(\Delta t)^3}, \\ \mathcal{L}_{2l}(t) := \frac{(t - t_l)(\Delta t - t + t_l)^2}{(\Delta t)^2}, & \mathcal{L}_{3l}(t) := \frac{-(t - t_l)^2(\Delta t - t + t_l)}{(\Delta t)^2}. \end{cases}$$

Then the following result is not difficult to prove:

LEMMA 3.2 *Let  $u \in P_0$  be given and let us define the function  $\Pi_h u$  as follows: on each  $K_{kl} = (x_k, x_k + \Delta x) \times (t_l, t_l + \Delta t)$ , we set*

$$\begin{aligned} \Pi_h u(x, t) := & \sum_{i,j=0}^1 L_{ik}(x) \mathcal{L}_{jl}(t) u(x_{i+k}, t_{j+l}) + \sum_{i,j=0}^1 L_{i+2,k}(x) \mathcal{L}_{jl}(t) u_x(x_{i+k}, t_{j+l}) \\ & + \sum_{i,j=0}^1 L_{ik}(x) \mathcal{L}_{j+2,l}(t) u_t(x_{i+k}, t_{j+l}) + \sum_{i,j=0}^1 L_{i+2,k}(x) \mathcal{L}_{j+2,l}(t) u_{xt}(x_{i+k}, t_{j+l}). \end{aligned}$$

Then  $\Pi_h u$  is the unique function in  $P_h$  that satisfies the following for all  $k = 1, \dots, N_x$  and  $l = 1, \dots, N_t$ :

$$\begin{aligned} \Pi_h u(x_k, t_l) &= u(x_k, t_l), & (\Pi_h u(x_k, t_l))_x &= u_x(x_k, t_l), \\ (\Pi_h u(x_k, t_l))_t &= u_t(x_k, t_l), & (\Pi_h u(x_k, t_l))_{xt} &= u_{xt}(x_k, t_l). \end{aligned}$$

The linear mapping  $\Pi_h : P_0 \mapsto P_h$  is by definition the interpolation operator associated to  $P_h$ .

In the next section, we will use the following result:

LEMMA 3.3 *For any  $u \in P_0$  and any  $(x, t) \in K_{kl}$ ,  $k \in \{1, N_x\}$  and  $l \in \{1, N_t\}$ , we have*

$$\begin{aligned} u - \Pi_h u &= \sum_{i,j=0}^1 \left( m_{ij} u_x(x_{i+k}, t_{j+l}) + n_{ij} u_t(x_{i+k}, t_{j+l}) + p_{ij} u_{tx}(x_{i+k}, t_{j+l}) \right) \\ &\quad + \sum_{i,j=0}^1 L_{ik} \mathcal{L}_{jl} \mathcal{R}[u; x_{i+k}, t_{j+l}], \end{aligned} \quad (34)$$

where the  $m_{ij}$ ,  $n_{ij}$  and  $p_{ij}$  are given by

$$\begin{cases} m_{ij}(x, t) := \left( L_{ik}(x)(x - x_i) - L_{i+2,k}(x) \right) \mathcal{L}_j(t), \\ n_{ij}(x, t) := L_{ik}(x) \left( \mathcal{L}_j(t)(t - t_j) - \mathcal{L}_{j+2}(t) \right), \\ p_{ij}(x, t) := L_{ik}(x) \mathcal{L}_{jl}(t)(x - x_i)(t - t_j) - L_{i+2}(x) \mathcal{L}_{j+2}(t) \end{cases}$$

and

$$\begin{aligned} \mathcal{R}[u; x_{i+k}, t_{j+l}](x, t) &:= \int_{t_{j+l}}^t (t-s) u_{tt}(x_{i+k}, s) ds \\ &\quad + (x - x_{i+k}) \int_{t_{j+l}}^t (t-s) u_{xtt}(x_{i+k}, s) ds \\ &\quad + \int_{x_{i+k}}^x (x-s) u_{xx}(s, t) ds. \end{aligned}$$

The proof is very simple. Indeed, (34) is a consequence of the following Taylor expansion with integral remainder :

$$\begin{aligned} u(x, t) &= u(x_i, t_j) + (t - t_j) u_t(x_i, t_j) + \int_{t_j}^t (t-s) u_{tt}(x_i, s) ds \\ &\quad + (x - x_i) \left( u_x(x_i, t_j) + (t - t_j) u_{xt}(x_i, t_j) + \int_{t_j}^t (t-s) u_{xtt}(x_i, s) ds \right) \\ &\quad + \int_{x_i}^x (x-s) u_{xx}(s, t) ds \end{aligned}$$

and the fact that  $\sum_{i,j=0}^1 L_{ik}(x) \mathcal{L}_{jl}(t) \equiv 1$ . □

### 3.3 An estimate of $\|p - \Pi_h p\|_P$ and some consequences

We now prove that (30) holds for  $P_h$  given by (32).

Thus, let us fix  $p \in P_0$  and let us first check that

$$\iint_{Q_T} \rho^{-2} |L(p - \Pi_h(p))|^2 dx dt \rightarrow 0 \quad \text{as } h = (\Delta x, \Delta t) \rightarrow (0, 0). \quad (35)$$

For each  $K_{kl} \in \mathcal{Q}_h$  (simply denoted by  $K$  in the sequel), we write :

$$\begin{aligned} \iint_K \rho^{-2} |L(p - \Pi_h p)|^2 dx dt &\leq \|\rho^{-2}\|_{L^\infty(K)} \iint_K |L(p - \Pi_h p)|^2 dx dt \\ &\leq 3 \|\rho^{-2}\|_{L^\infty(K)} \left( \iint_K |(p - \Pi_h p)_{tt}|^2 dx dt + \iint_K |(a(x)(p - \Pi_h p)_x)_x|^2 dx dt \right. \\ &\quad \left. + \|b\|_{L^\infty(K)}^2 \iint_K |p - \Pi_h p|^2 dx dt \right). \end{aligned} \quad (36)$$

Using Lemma 3.3, we have:

$$\begin{aligned}
& \iint_K |p - \Pi_h p|^2 dx dt \\
&= \iint_K \left| \sum_{i,j} (m_{ij} p_x(x_i, t_j) + n_{ij} p_t(x_i, t_j) + p_{ij} p_{tx}(x_i, t_j) + L_i \mathcal{L}_j \mathcal{R}[p; x_i, t_j]) \right|^2 dx dt \\
&\leq 16 \|p_x\|_{L^\infty(K)}^2 \sum_{i,j} \iint_K |m_{ij}|^2 dx dt + 16 \|p_t\|_{L^\infty(K)}^2 \sum_{i,j} \iint_K |n_{ij}|^2 dx dt \\
&\quad + 16 \|p_{tx}\|_{L^\infty(K)}^2 \sum_{i,j} \iint_K |p_{ij}|^2 dx dt + 16 \sum_{i,j} \iint_K |L_i \mathcal{L}_j \mathcal{R}[p; x_i, t_j]|^2 dx dt,
\end{aligned}$$

where we have omitted the indices  $k$  and  $l$ .

Moreover,

$$\begin{aligned}
|\mathcal{R}[p; x_{i+k}, t_{j+l}]|^2 &\leq |t - t_j|^3 \|p_{tt}(x_i, \cdot)\|_{L^2(t_l, t_{l+1})}^2 + |x - x_i|^2 |t - t_j|^3 \|p_{xtt}(x_i, \cdot)\|_{L^2(t_l, t_{l+1})}^2 \\
&\quad + |x - x_i|^3 \|p_{xx}(\cdot, t)\|_{L^2(x_k, x_{k+1})}^2.
\end{aligned}$$

Consequently, we get:

$$\begin{aligned}
& \sum_{i,j} \iint_K |L_i \mathcal{L}_j \mathcal{R}[p; x_{i+k}, t_{j+l}]|^2 dx dt \\
&\leq \sup_{x \in (x_k, x_{k+1})} \|p_{tt}(x, \cdot)\|_{L^2(t_l, t_{l+1})}^2 \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j|^3 dx dt \\
&\quad + \sup_{x \in (x_k, x_{k+1})} \|p_{xtt}(x, \cdot)\|_{L^2(t_l, t_{l+1})}^2 \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j|^3 |x - x_i|^2 dx dt \\
&\quad + \sup_{t \in (t_l, t_{l+1})} \|p_{xx}(\cdot, t)\|_{L^2(x_k, x_{k+1})}^2 \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |x - x_i|^3 dx dt.
\end{aligned}$$

After some tedious computations, one finds that

$$\begin{aligned}
\sum_{i,j} \iint_K |m_{ij}|^2 dx dt &= \frac{104}{11025} (\Delta x)^3 \Delta t, \quad \sum_{i,j} \iint_K |n_{ij}|^2 dx dt = \frac{104}{11025} \Delta x (\Delta t)^3, \\
\sum_{i,j} \iint_K |p_{ij}|^2 dx dt &= \frac{353}{198450} (\Delta x)^3 (\Delta t)^3
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j|^3 dx dt &= \frac{143}{7350} \Delta x (\Delta t)^4, \\
\sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |x - x_i|^3 dx dt &= \frac{143}{7350} (\Delta x)^4 \Delta t, \\
\sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |x - x_i|^2 |t - t_j|^3 dx dt &= \frac{209}{132300} (\Delta x)^3 (\Delta t)^4.
\end{aligned}$$

This leads to the following estimate for any  $K = K_{kl} \in \mathcal{Q}_h$ :

$$\begin{aligned}
\iint_K |p - \Pi_h p|^2 dx dt &\leq \frac{1664}{11025} (\Delta x)^3 \Delta t \|p_x\|_{L^\infty(K)}^2 \\
&\quad + \frac{1664}{11025} \Delta x (\Delta t)^3 \|p_t\|_{L^\infty(K)}^2 \\
&\quad + \frac{2824}{99225} (\Delta x)^3 (\Delta t)^3 \|p_{tx}\|_{L^\infty(K)}^2 \\
&\quad + \frac{1144}{3675} (\Delta x)^4 \Delta t \sup_{x \in (x_k, x_{k+1})} \|p_{tt}(\cdot, t)\|_{L^2(t_l, t_{l+1})}^2 \\
&\quad + \frac{1144}{3675} \Delta x (\Delta t)^4 \sup_{t \in (t_l, t_{l+1})} \|p_{xx}(\cdot, t)\|_{L^2(x_k, x_{k+1})}^2 \\
&\quad + \frac{836}{33075} (\Delta x)^3 (\Delta t)^3 \sup_{x \in (x_k, x_{k+1})} \|p_{xtt}(\cdot, t)\|_{L^2(t_l, t_{l+1})}^2.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\iint_{Q_T} |p - \Pi_h p|^2 dx dt &\leq K_1 T \|p_x\|_{L^\infty(Q_T)}^2 (\Delta x)^2 \\
&\quad + K_1 T \|p_t\|_{L^\infty(Q_T)}^2 (\Delta t)^2 \\
&\quad + K_2 T \|p_{tx}\|_{L^\infty(Q_T)}^2 (\Delta x)^2 (\Delta t)^2 \\
&\quad + K_3 \|p_{tt}(\cdot, t)\|_{L^2(0, T; L^\infty(0, 1))}^2 (\Delta x)^3 \\
&\quad + K_3 \|p_{xx}(\cdot, t)\|_{L^\infty(0, T; L^2(0, 1))}^2 (\Delta t)^3 \\
&\quad + K_4 \|p_{xtt}(\cdot, t)\|_{L^2(0, T; L^\infty(0, 1))}^2 (\Delta x)^2 (\Delta t)^2
\end{aligned}$$

for some positive constants  $K_i$ . Hence, for any  $p \in P_0$  one has

$$\iint_{Q_T} |p - \Pi_h p|^2 dx dt \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Proceeding as above, we show that the other terms in (36) also converge to 0 as  $h = (\Delta x, \Delta t) \rightarrow (0, 0)$ . Hence, (35) holds.

On the other hand, a similar argument yields

$$\int_0^T \rho_0^{-2} a(1)^2 |(p - \Pi_h p)_x|^2 dx dt \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

This proves that (30) holds.

We can now use Proposition 3.1 and deduce convergence results for the approximate control and state variables:

**PROPOSITION 3.2** *Let  $p_h \in P_h$  be the unique solution to (29), where  $P_h$  is given by (32)–(33). Let us set*

$$y_h := \rho^{-2} L p_h, \quad v_h := -\rho_0^{-2} a(x) p_{h,x} \Big|_{x=1}.$$

*Then one has*

$$\|y - y_h\|_{L^2(Q_T)} \rightarrow 0 \quad \text{and} \quad \|v - v_h\|_{L^2(0, T)} \rightarrow 0,$$

*where  $(y, v)$  is the solution to (8).* □

### 3.4 A second approximate problem

For simplicity, we will assume in this section that  $y_1 \in C^0([0, 1])$ .

In order to take into account the numerical approximation of the weights and the data that we necessarily have to perform in practice, we also consider a second approximate problem. It is the following:

$$m_h(\hat{p}_h, \bar{p}_h) = \langle l_h, \bar{p}_h \rangle \quad \forall \bar{p}_h \in P_h; \quad \hat{p}_h \in P_h, \quad (37)$$

where the bilinear form  $m_h(\cdot, \cdot)$  is given by

$$m_h(p_h, \bar{p}_h) := \iint_{Q_T} \pi_h(\rho^{-2}) L p_h L \bar{p}_h dx dt + \int_0^T a(1)^2 \pi_{\Delta x}(\rho_0^{-2}) p_h \bar{p}_h dt$$

and the linear form  $l_h$  is given by

$$\langle l_h, \bar{p} \rangle := \int_0^1 (\pi_{\Delta x} y_0)(x) \bar{p}_t(x, 0) dx - \langle \pi_{\Delta x} y_1, \bar{p}(\cdot, 0) \rangle_{H^{-1}, H_0^1}.$$

Here, for any function  $f \in C^0(\bar{Q}_T)$ ,  $\pi_h(f)$  denotes the piecewise linear function which coincides with  $f$  at all vertices of  $\mathcal{Q}_h$ . Similar (self-explanatory) meanings can be assigned to  $\pi_{\Delta x}(z)$  and  $\pi_{\Delta t}(w)$  when  $z \in C^0([0, 1])$  and  $w \in C^0([0, T])$ , respectively.

Since the weight  $\rho^{-2}$  is strictly positive in  $Q_T$  and bounded (actually  $\rho^{-2} \geq 1$ ), we easily see that the ratio  $\pi_h(\rho^{-2})/\rho^{-2}$  is bounded uniformly with respect to  $h$  (for  $|h|$  small enough). The same holds for the vanishing weight  $\theta_\delta^2 \rho(1, \cdot)^{-2}$  under the assumptions (27).

As a consequence, it is not difficult to prove that (37) is well-posed. Moreover, we have:

LEMMA 3.4 *Let  $p_h$  and  $\hat{p}_h$  be the solutions to (29) and (37), respectively. Then,*

$$\begin{aligned} \|\hat{p}_h - p_h\|_P &\leq \max\left(\left\|\frac{\pi_h(\rho^{-2})}{\rho^{-2}} - 1\right\|_{L^\infty(Q_T)}, \left\|\frac{\pi_{\Delta t}(\rho_0^{-2})}{\rho_0^{-2}} - 1\right\|_{L^\infty(0, T)}\right) \|\hat{p}_h\|_P \\ &\quad + C_1 \|\pi_{\Delta x}(y_0) - y_0\|_{L^2} + C_2 \|\pi_{\Delta x}(y_1) - y_1\|_{H^{-1}}, \end{aligned} \quad (38)$$

where  $C_1$  and  $C_2$  are positive constants independent of  $h$ .

PROOF: Since  $p_h$  and  $\hat{p}_h$  respectively solve (29) and (37), one has:

$$\begin{aligned} \|\hat{p}_h - p_h\|_P^2 &= m(\hat{p}_h - p_h, \hat{p}_h - p_h) \\ &= m(\hat{p}_h, \hat{p}_h - p_h) - m_h(\hat{p}_h, \hat{p}_h - p_h) + \langle l_h, \hat{p}_h - p_h \rangle - \langle l, \hat{p}_h - p_h \rangle \\ &= \iint_{Q_T} (\rho^{-2} - \pi_h(\rho^{-2})) L \hat{p}_h L (\hat{p}_h - p_h) dx dt + \int_0^T (\rho_0^{-2} - \pi_{\Delta t}(\rho_0^{-2})) a(1)^2 \hat{p}_{x,h} (\hat{p}_{x,h} - p_{x,h}) dt \\ &\quad + \langle l_h, \hat{p}_h - p_h \rangle - \langle l, \hat{p}_h - p_h \rangle \\ &= \iint_{Q_T} \left(1 - \frac{\pi_h(\rho^{-2})}{\rho^{-2}}\right) (\rho^{-1} L \hat{p}_h) (\rho^{-1} L (\hat{p}_h - p_h)) dx dt \\ &\quad + \int_0^T \left(1 - \frac{\pi_{\Delta t} \rho_0^{-2}}{\rho_0^{-2}}\right) a(1)^2 (\rho_0^{-1} \hat{p}_h) \rho_0^{-1} (\hat{p}_{h,x} - p_{h,x}) dt \\ &\quad + \int_\Omega (\pi_{\Delta x}(y_0) - y_0)(x) (\hat{p}_{t,h} - p_{t,h})(x, 0) dx - \langle \pi_{\Delta x}(y_1) - y_1, (\hat{p}_h - p_h)(x, 0) \rangle_{H^{-1}, H_0^1}. \end{aligned}$$

In view of the definitions of the bilinear forms  $m(\cdot, \cdot)$  and  $m_h(\cdot, \cdot)$ , we easily find (38).  $\square$

Taking into account that (30) holds and

$$\max\left(\left\|\frac{\pi_h(\rho^{-2})}{\rho^{-2}} - 1\right\|_{L^\infty(Q_T)}, \left\|\frac{\pi_{\Delta t}(\rho_0^{-2})}{\rho_0^{-2}} - 1\right\|_{L^\infty(0, T)}\right) \rightarrow 0,$$

we find that, as  $h$  goes to zero, the unique solution to (37), converges in  $P$  to the unique solution to (28):

$$\|p - \hat{p}_h\|_P \leq \|p - p_h\|_P + \|p_h - \hat{p}_h\|_P \rightarrow 0.$$

An obvious consequence is the following:

**PROPOSITION 3.3** *Let  $\hat{p}_h \in P_h$  be the unique solution to (37), where  $P_h$  is given by (32)–(33). Let us set*

$$\hat{y}_h := \pi_h(\rho^{-2})L\hat{p}_h, \quad \hat{v}_h := -\pi_{\Delta x}(\rho_0^{-2})a(x)\hat{p}_{h,x}|_{x=1}. \quad (39)$$

Then one has

$$\|y - \hat{y}_h\|_{L^2(Q_T)} \rightarrow 0 \text{ and } \|v - \hat{v}_h\|_{L^2(0,T)} \rightarrow 0,$$

where  $(y, v)$  is the solution to (8).  $\square$

## 4 Numerical experiments

We now present some numerical experiments concerning the solution of (37), which can in fact viewed as a linear system involving a band sparse, definite positive, symmetric matrix of order  $4N_x N_t$ . We will denote by  $\mathcal{M}_h$  this matrix. If  $\{\hat{p}_h\}$  stands for the corresponding vector solution of size  $4N_x N_t$ , we may write  $(p_h, \bar{p}_h)_{P_h} = (\mathcal{M}_h\{\hat{p}_h\}, \{\bar{p}_h\})$ .

We will use an exact integration method in order to compute the components of  $\mathcal{M}_h$  and the (direct) Cholesky method with reordering to solve the linear system.

After the computation of  $\hat{p}_h$ , the control  $\hat{v}_h$  is given by (39). We highlight, that by the definition of the space  $P_h$ , the derivative with respect to  $x$  of  $\hat{p}_h$  is a degree of freedom of  $\{\hat{p}_h\}$ ; hence the computation of  $\hat{v}_h$  does not require any additional calculus.

The corresponding controlled state  $\hat{y}_h$  may be obtained using the pointwise first equality (39) or, equivalently, by solving (24). However, in order to check the action of the control function  $\hat{v}_h$  properly, we compute the approximation  $\hat{y}_h$  by solving (1) using a  $C^1$  finite element method in space and the standard centered scheme in time (of second order).

More precisely, let us introduce the finite dimensional spaces

$$Z_h = \{z_h \in C^1([0, 1]) : z_h|_{[x_i, x_i + \Delta x]} \in \mathbb{P}_{3,x} \quad \forall i = 1, \dots, N_x\}$$

and  $Z_{0h} = \{z_h \in Z_h : z_h(0) = z_h(1) = 0\}$ . Then, a suitable approximation  $\hat{y}_h$  of the controlled state  $y$  is defined in the following standard way:

- At time  $t = 0$ ,  $\hat{y}_h$  is given by  $y_h(\cdot, 0) = P_{Z_h}(y_0)$ , the projection of  $y_0$  on  $Z_h$ ;
- At time  $t_1 = \Delta t$ ,  $\hat{y}_h(\cdot, t_1) \in Z_h$  is given by the solution to

$$\begin{cases} 2 \int_0^1 \frac{(\hat{y}_h(x, t_1) - \hat{y}_h(x, t_0) - \Delta t y_1(x))}{(\Delta t)^2} \phi dx \\ + \int_0^1 [a(x)\hat{y}_{h,x}(x, t_0)\phi_x + b(x, t_0)\hat{y}_h(x, t_0)\phi] dx = 0 \\ \forall \phi \in Z_{h0}; \quad \hat{y}_h(0, t_1) \in Z_h, \quad \hat{y}_h(0, t_1) = 0, \quad \hat{y}_h(1, t_1) = \hat{v}_h(t_1). \end{cases} \quad (40)$$

- At time  $t = t_n$ ,  $n = 2, \dots, N_t$ ,  $\hat{y}_h(\cdot, t_n)$  solves the following linear problem:

$$\begin{cases} 2 \int_0^1 \frac{(\hat{y}_h(x, t_n) - 2\hat{y}_h(x, t_{n-1}) + \hat{y}_h(\cdot, t_{n-2}))}{(\Delta t)^2} \phi dx \\ + \int_0^1 [a(x)\hat{y}_{h,x}(x, t_{n-1})\phi_x + b(x, t_n)\hat{y}_h(x, t_{n-1})\phi] dx = 0 \\ \forall \phi \in Z_{h0}; \quad \hat{y}_h(0, t_n) \in Z_h, \quad \hat{y}_h(0, t_n) = 0, \quad \hat{y}_h(1, t_n) = \hat{v}_h(t_n). \end{cases} \quad (41)$$



This requires a preliminary projection of  $\hat{v}_h$  on a grid on  $(0, T)$  fine enough in order to fulfill the underlying CFL condition. To this end, we use the following interpolation formula: for any  $p_h \in P_h$  and any  $\theta \in [0, 1]$ , we have:

$$\begin{aligned} p_{h,x}(1, t_j + \theta\Delta t) &= (2\theta + 1)(\theta - 1)^2 p_{h,x}(1, t_j) + \Delta t \theta(1 - \theta)^2 p_{h,x,t}(1, t_j) \\ &\quad + \theta^2(3 - 2\theta) p_{h,x}(1, t_{j+1}) + \Delta t \theta^2(\theta - 1) p_{h,x,t}(1, t_{j+1}) \end{aligned} \quad (42)$$

for all  $t \in [t_j, t_{j+1}]$ .

We will consider a constant coefficient  $a(x) \equiv a_0 = 1$  and a constant potential  $b(x, t) \equiv 1$  in  $Q_T$ . We will take  $T = 2.2$ ,  $x_0 = -1/20$ ,  $\beta = 0.99$  and  $M_0 = 1 - x_0^2 + \beta T^2$ , so that (18) holds. Finally, concerning the parameters  $\lambda$  and  $s$  (which appear in (21)), we will take  $\lambda = 0.1$  and  $s = 1$ .

#### 4.1 Estimating the Carleman constant

Before prescribing the initial data, let us check that the finite dimensional analog of the constant  $C_0$  in (20) is uniformly bounded with respect to  $h$  when (18) is satisfied.

In the space  $P_h$ , the approximate version of (20) is

$$(A_h\{p_h\}, \{p_h\}) \leq C_{0h}(\mathcal{M}_h\{p_h\}, \{p_h\}) \quad \forall \{p_h\} \in P_h,$$

where  $A_h$  is the square matrix of order  $4N_t N_x$  defined by the identities

$$(A_h\{p_h\}, \{q_h\}) := \int_0^1 (p_{h,x}(x, 0) q_{h,x}(x, 0) + p_{h,t}(x, 0) q_{h,t}(x, 0)) dx.$$

Therefore,  $C_{0h}$  is the solution of a generalized eigenvalue problem:

$$C_{0h} = \max\{\lambda : A_h\{p_h\} = \lambda \mathcal{M}_h\{p_h\} \quad \forall p_h \in P_h\}. \quad (43)$$

We can easily solve (43) by the power iteration algorithm. Table 1 collects the values of  $C_{0h}$  for various  $h = (\Delta x, \Delta t)$  for  $T = 2.2$  and  $T = 1.5$ , with  $\Delta t = \Delta x$ . As expected  $C_{0h}$  is bounded in the first case only. This computation also suggests that, for  $T = 2.2$ , the value  $s = 1$  is large enough to ensure the Carleman estimate. The same results are obtained for  $\Delta t \neq \Delta x$ .

$\Delta x, \Delta t$	1/10	1/20	1/40	1/80
$T = 2.2$	$6.60 \times 10^{-2}$	$7.61 \times 10^{-2}$	$8.56 \times 10^{-2}$	$9.05 \times 10^{-2}$
$T = 1.5$	0.565	2.672	17.02	289.29

Table 1: The constant  $C_{0h}$  with respect to  $h$ .

#### 4.2 Smooth initial data and constant speed of propagation

We now solve (23) with  $a \equiv 1$  and smooth initial data. For simplicity, we also take a constant potential  $b \equiv 1$ .

For  $(y_0, y_1) = (\sin(\pi x), 0)$ , Table 2 collects relevant numerical values with respect to  $h = (\Delta x, \Delta t)$ . We have taken  $\Delta t = \Delta x$  for simplicity but, in this finite element framework, any other choice is possible. In particular, we have reported the condition number  $\kappa(\mathcal{M}_h)$  of the matrix  $\mathcal{M}_h$ , defined by

$$\kappa(\mathcal{M}_h) = \|\|\mathcal{M}_h\|\|_2 \|\|\mathcal{M}_h^{-1}\|\|_2$$

(the norm  $\|\|\mathcal{M}_h\|\|_2$  stands for the largest singular value of  $\mathcal{M}_h$ ). We observe that this number behaves polynomially with respect to  $h$ .

$\Delta x, \Delta t$	1/10	1/20	1/40	1/80
$\kappa(\mathcal{M}_h)$	$3.06 \times 10^8$	$1.57 \times 10^{10}$	$6.10 \times 10^{11}$	$2.47 \times 10^{13}$
$\ \hat{p}_h\ _{P_h}$	$1.541 \times 10^{-1}$	$1.548 \times 10^{-1}$	$1.550 \times 10^{-1}$	$1.550 \times 10^{-1}$
$\ \hat{p}_h - p\ _{P_h}$	$4.46 \times 10^{-2}$	$1.45 \times 10^{-2}$	$4.01 \times 10^{-3}$	$8.38 \times 10^{-4}$
$\ \hat{v}_h\ _{L^2(0,T)}$	$5.421 \times 10^{-1}$	$5.431 \times 10^{-1}$	$5.434 \times 10^{-1}$	$5.434 \times 10^{-3}$
$\ \hat{v}_h - v\ _{L^2(0,T)}$	$2.39 \times 10^{-2}$	$8.12 \times 10^{-3}$	$2.48 \times 10^{-3}$	$9.57 \times 10^{-4}$
$\ \hat{y}_h(\cdot, T)\ _{L^2(0,1)}$	$1.80 \times 10^{-2}$	$8.18 \times 10^{-3}$	$1.64 \times 10^{-3}$	$5.85 \times 10^{-4}$
$\ \hat{y}_{t,h}(\cdot, T)\ _{H^{-1}(0,1)}$	$3.06 \times 10^{-2}$	$8.25 \times 10^{-3}$	$3.59 \times 10^{-3}$	$1.93 \times 10^{-3}$

Table 2:  $(y_0(x), y_1(x)) \equiv (\sin(\pi x), 0)$ ,  $a := 1, b := 1 - T = 2.2$ .

Table 2 clearly exhibits the convergence of the variables  $\hat{p}_h$  and  $\hat{v}_h$  as  $h$  goes to zero. Assuming that  $h = (1/160, 1/160)$  provides a reference solution, we have also reported in Table 2 the estimates  $\|p - \hat{p}_h\|_P$  and  $\|v - \hat{v}_h\|_P$ . We observe then that

$$\|p - \hat{p}_h\|_P = \mathcal{O}(h^{1.91}), \quad \|v - \hat{v}_h\|_{L^2(0,T)} = \mathcal{O}(h^{1.56}).$$

The corresponding state  $\hat{y}_h$  is computed from the main equation (1), as explained above, taking  $\Delta t = \Delta x/4$ . That is, we use (42) with  $\theta = 0, 1/4, 1/2$  and  $3/4$  on each interval  $[t_j, t_{j+1}]$ . We observe the following behavior with respect to  $h$ :

$$\|\hat{y}_h\|_{L^2(0,1)} = \mathcal{O}(h^{1.71}), \quad \|\hat{y}_{t,h}\|_{H^{-1}(0,T)} = \mathcal{O}(h^{1.31}),$$

which shows that the control  $\hat{v}_h$  given by the second equality in (39) is a good approximation of a null control for (1).

Figure 2-Left displays the function  $\hat{p}_h \in P_h$  (the unique solution of (37)) for  $h = (1/80, 1/80)$ . Figure 2-Right displays the associated control  $\hat{v}_h$ . As a consequence of the introduction of the function  $\theta_\delta$  in the weight, we see that  $\hat{v}_h$  vanishes at times  $t = 0$  and  $t = T$ . Finally, Figure 3 displays the corresponding controlled state  $\hat{y}_h$ .

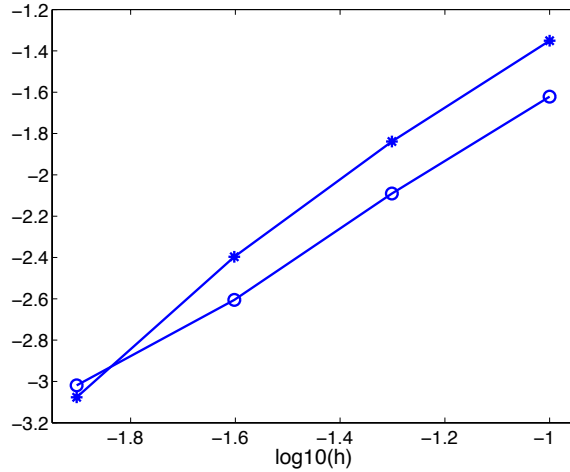
Figure 1:  $\log_{10} \|p - \hat{p}_h\|_P$  (\*) and  $\log_{10} \|v - \hat{v}_h\|_{L^2(0,T)}$  (o) vs.  $\log_{10}(h)$ .

Table 3 and Figures 4 and 5 provide the results for  $y_0(x) \equiv e^{-500(x-0.2)^2}$ , the other data being unchanged. We still observe the convergence of the variables  $\hat{p}_h$ ,  $\hat{v}_h$  and  $\hat{y}_h$ , with a lower rate. This is due in part to the shape of the initial condition  $y_0$ . Precisely, we get  $\|p - \hat{p}_h\|_{P_h} = \mathcal{O}(h^{1.74})$ ,  $\|\hat{v}_h - v\|_{L^2(0,T)} = \mathcal{O}(h^{0.68})$ ,  $\|\hat{y}_h\|_{L^2(0,1)} = \mathcal{O}(h^{1.35})$  and  $\|\hat{y}_{t,h}\|_{H^{-1}(0,T)} = \mathcal{O}(h^{1.11})$ .

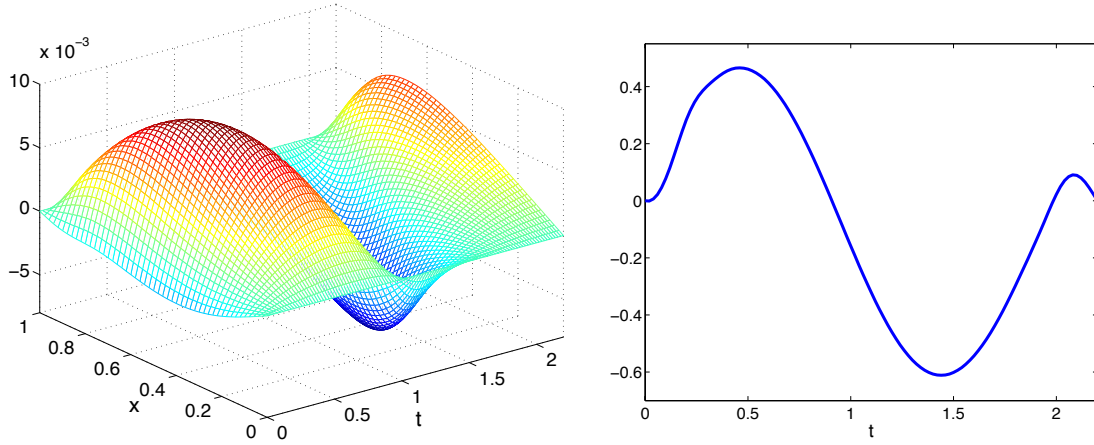


Figure 2:  $y_0(x) \equiv (\sin(\pi x))$  and  $a := 1$  - The solution  $\hat{p}_h$  over  $Q_T$  (**Left**) and the corresponding variable  $\hat{v}_h$  on  $(0, T)$  (**Right**) -  $h = (1/80, 1/80)$ .

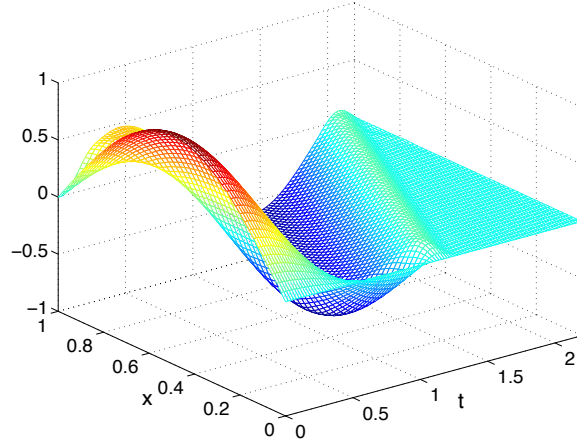


Figure 3:  $y_0(x) \equiv (\sin(\pi x))$  and  $a := 1$  - The solution  $\hat{y}_h$  over  $Q_T$  -  $h = (1/80, 1/80)$ .

$\Delta x, \Delta t$	1/10	1/20	1/40	1/80	1/160
$\ \hat{p}_h\ _{P_h}$	$4.38 \times 10^{-2}$	$3.95 \times 10^{-2}$	$4.20 \times 10^{-2}$	$4.31 \times 10^{-2}$	$4.33 \times 10^{-2}$
$\ \hat{p}_h - p\ _{P_h}$	$1.80 \times 10^{-1}$	$6.30 \times 10^{-2}$	$1.66 \times 10^{-2}$	$2.78 \times 10^{-3}$	-
$\ \hat{v}_h\ _{L^2(0,T)}$	$1.48 \times 10^{-1}$	$1.33 \times 10^{-1}$	$1.53 \times 10^{-1}$	$1.64 \times 10^{-1}$	$1.67 \times 10^{-1}$
$\ \hat{v}_h - v\ _{L^2(0,T)}$	$9.81 \times 10^{-2}$	$6.28 \times 10^{-2}$	$3.80 \times 10^{-2}$	$1.11 \times 10^{-2}$	-
$\ \hat{y}_h(\cdot, T)\ _{L^2(0,1)}$	$1.09 \times 10^{-1}$	$7.67 \times 10^{-2}$	$3.70 \times 10^{-2}$	$1.11 \times 10^{-2}$	$1.87 \times 10^{-3}$
$\ \hat{y}_{t,h}(\cdot, T)\ _{H^{-1}(0,1)}$	$1.36 \times 10^{-1}$	$8.82 \times 10^{-2}$	$5.16 \times 10^{-2}$	$1.76 \times 10^{-2}$	$2.82 \times 10^{-3}$

Table 3:  $y_0(x) \equiv e^{-500(x-0.2)^2}$  and  $a := 1, b := 1 - T = 2.2$ .

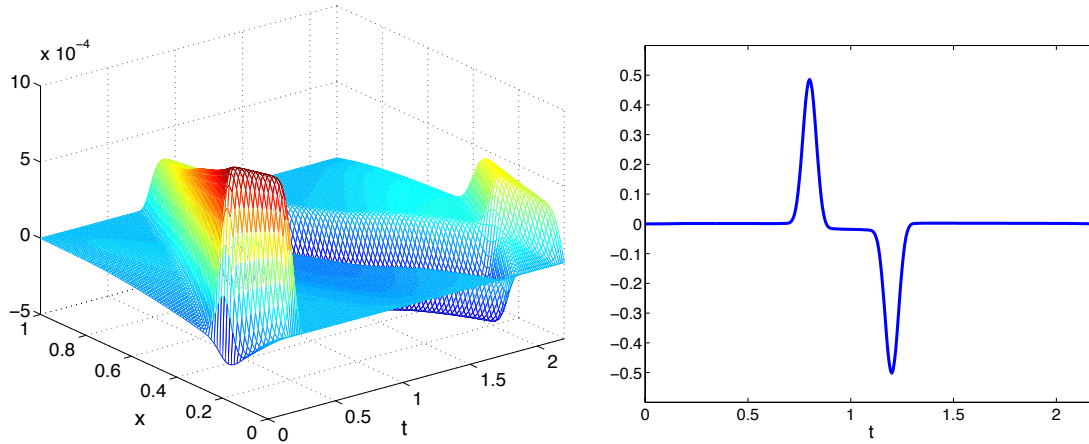


Figure 4:  $y_0(x) \equiv e^{-500(x-0.2)^2}$  and  $a := 1$  - The solution  $\hat{p}_h$  over  $Q_T$  (**Left**) and the corresponding variable  $\hat{v}_h$  on  $(0, T)$  (**Right**) -  $h = (1/80, 1/80)$ .

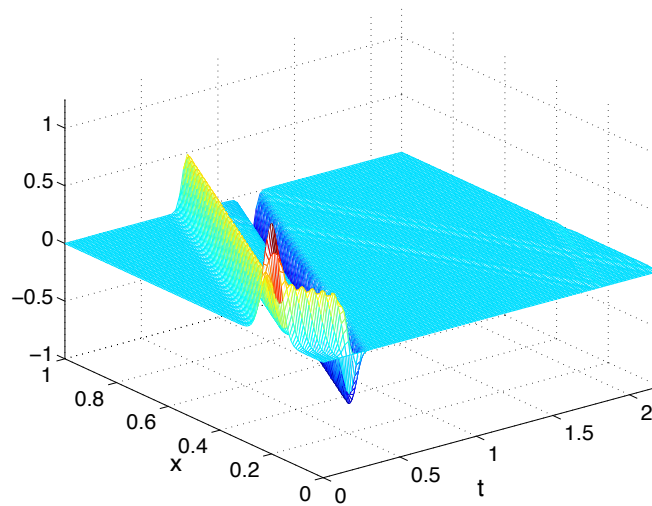


Figure 5:  $y_0(x) \equiv e^{-500(x-0.2)^2}$  and  $a := 1$  -The solution  $\hat{y}_h$  over  $Q_T$  -  $h = (1/80, 1/80)$ .

### 4.3 Initial data $(y_0, y_1) \in H_0^1(0, 1) \times L_2(0, 1)$ and constant speed of propagation

Let us enhance that our approach, in agreement with the theoretical results, also provides convergent results for irregular initial data. We take a continuous but not differentiable initial state  $y_0$  and a piecewise constant initial speed  $y_1$ :

$$y_0(x) \equiv x 1_{[0,1/2]}(x) + (1-x) 1_{[1/2,1]}(x), \quad y_1(x) \equiv 10 \times 1_{[1/5,1/2]}(x). \quad (44)$$

The other data are unchanged, except  $b$ , that is taken equal to zero.

Observe that these functions remain compatible with the  $C^1$  finite element used to approximate  $p$ , since  $y_0$  and  $y_1$  only appear in the right hand side of the variational formulation and  $\pi_{\Delta x} y_0$  and  $\pi_{\Delta x} y_1$  make sense; see (37). The unique difference is that, once  $\hat{p}_h$  and  $\hat{v}_h$  are known,  $\hat{y}_h$  must be computed from (40)–(41) using a  $C^0$  (and not  $C^1$ ) spatial finite element method.

Recall however that these initial data typically generate pathological numerical behavior when a dual approach is used.

Some numerical results are given in Table 4 and Figures 6 and 7. As before, we observe the convergence of the variable  $\hat{p}_h$  and therefore  $\hat{v}_h$  and  $\hat{y}_h$  as  $h \rightarrow 0$ . We see that  $\|\hat{p}_h - p\|_{P_h} = \mathcal{O}(h^{1.48})$  and  $\|\hat{v}_h - v\|_{L^2(0,1)} = \mathcal{O}(h^{1.23})$ . In particular, we do not observe oscillations for the control or the functions  $\hat{p}_h$  and  $\hat{p}_{h,t}$  at the initial time.

We also mention that similar convergent results can be obtained for discontinuous  $y_0 \in L^2(0, 1)$ .

$\Delta x, \Delta t$	1/10	1/20	1/40	1/80	1/160
$\ \hat{p}_h\ _{P_h}$	$3.16 \times 10^{-1}$	$2.89 \times 10^{-2}$	$2.73 \times 10^{-2}$	$2.65 \times 10^{-2}$	$2.61 \times 10^{-1}$
$\ \hat{p}_h - p\ _{P_h}$	$1.12 \times 10^{-1}$	$4.62 \times 10^{-2}$	$1.70 \times 10^{-2}$	$5.12 \times 10^{-3}$	-
$\ \hat{v}_h\ _{L^2(0,T)}$	1.23	1.11	1.05	1.02	1.004
$\ \hat{v}_h - v\ _{L^2(0,T)}$	$2.52 \times 10^{-1}$	$1.25 \times 10^{-1}$	$5.57 \times 10^{-2}$	$1.90 \times 10^{-2}$	-
$\ \hat{y}_h(\cdot, T)\ _{L^2(0,1)}$	$1.09 \times 10^{-1}$	$5.40 \times 10^{-2}$	$2.20 \times 10^{-2}$	$1.09 \times 10^{-2}$	$6.20 \times 10^{-3}$
$\ \hat{y}_{t,h}(\cdot, T)\ _{H^{-1}(0,1)}$	$7.25 \times 10^{-2}$	$4.62 \times 10^{-2}$	$2.85 \times 10^{-2}$	$5.12 \times 10^{-3}$	$6.75 \times 10^{-3}$

Table 4:  $(y_0, y_1)$  given by (44) and  $a := 1 - T = 2.2$ .

### 4.4 Non constant smooth speed of propagation

Finally, let us consider a non constant function  $a \in C^1([0, 1])$  (we refer to [15] for the dual approach in this case). We take

$$a(x) = \begin{cases} 1, & x \in [0, 0.45], \\ \in [1., 5.] \quad (a'(x) > 0), & x \in (0.45, 0.55), \\ 5, & x \in [0.55, 1], \end{cases} \quad (45)$$

so that condition (18) is equivalent to  $T > 2(1 + 1/20)\sqrt{5} \approx 4.69$  (taking again  $x_0 = -1/20$ ). In order to reduce the computational cost, we take as before  $T = 2.2$  and we still observe that the constant  $C_{0h}$  in (43) is uniformly bounded. This clearly suggests that the estimate (18) is not optimal.

We take again  $(y_0(x), y_1(x)) \equiv (e^{-500(x-0.2)^2}, 0)$  and  $b \equiv 0$ . Table 5 illustrates the convergence of the approximations with respect to  $h$ . Figures 8 and 9 depict for  $h = (1/80, 1/80)$  the functions  $\hat{p}_h$ ,  $\hat{v}_h$  and  $\hat{y}_h$ . In particular, in Figure (9), we may observe the diffraction of the wave when crossing the transitional zone  $(0.45, 0.55)$ .

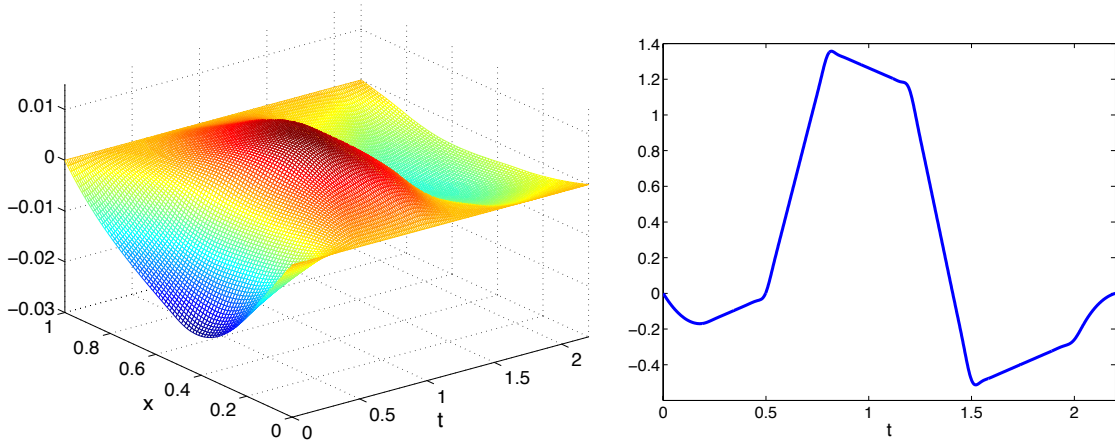


Figure 6:  $(y_0, y_1)$  given by (44) and  $a := 1$ - The solution  $\hat{p}_h$  over  $Q_T$  (**Left**) and the corresponding variable  $\hat{v}_h$  on  $(0, T)$  (**Right**) -  $h = (1/80, 1/80)$ .

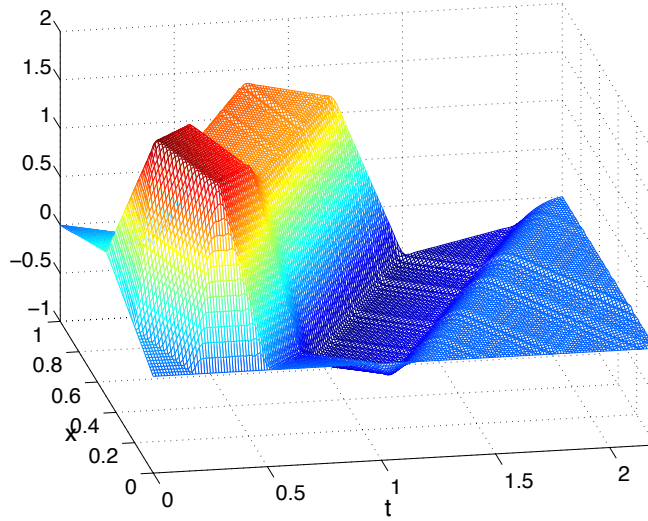


Figure 7:  $(y_0, y_1)$  given by (44) and  $a := 1$  -The solution  $\hat{y}_h$  over  $Q_T$  -  $h = (1/80, 1/80)$ .

$\Delta x, \Delta t$	1/10	1/20	1/40	1/80	1/160
$\ \hat{p}_h\ _{P_h}$	$3.87 \times 10^{-2}$	$3.44 \times 10^{-2}$	$3.75 \times 10^{-2}$	$3.85 \times 10^{-2}$	$3.86 \times 10^{-2}$
$\ \hat{p}_h - p\ _{P_h}$	$1.25 \times 10^{-1}$	$5.75 \times 10^{-2}$	$2.64 \times 10^{-2}$	$1.01 \times 10^{-2}$	-
$\ \hat{v}_h\ _{L^2(0,T)}$	$7.74 \times 10^{-2}$	$6.53 \times 10^{-2}$	$9.16 \times 10^{-2}$	$1.01 \times 10^{-1}$	$1.03 \times 10^{-1}$
$\ \hat{v}_h - v\ _{L^2(0,T)}$	$5.07 \times 10^{-1}$	$4.17 \times 10^{-2}$	$2.03 \times 10^{-2}$	$4.86 \times 10^{-3}$	-
$\ \hat{y}_h(\cdot, T)\ _{L^2(0,1)}$	$1.09 \times 10^{-1}$	$7.89 \times 10^{-2}$	$1.81 \times 10^{-2}$	$1.16 \times 10^{-2}$	$1.71 \times 10^{-3}$
$\ \hat{y}_{t,h}(\cdot, T)\ _{H^{-1}(0,1)}$	$1.01 \times 10^{-1}$	$8.39 \times 10^{-2}$	$4.81 \times 10^{-2}$	$7.52 \times 10^{-3}$	$1.55 \times 10^{-3}$

Table 5:  $y_0(x) \equiv e^{-500(x-0.2)^2}$  and  $a$  given by (45) -  $T = 2.2$ .

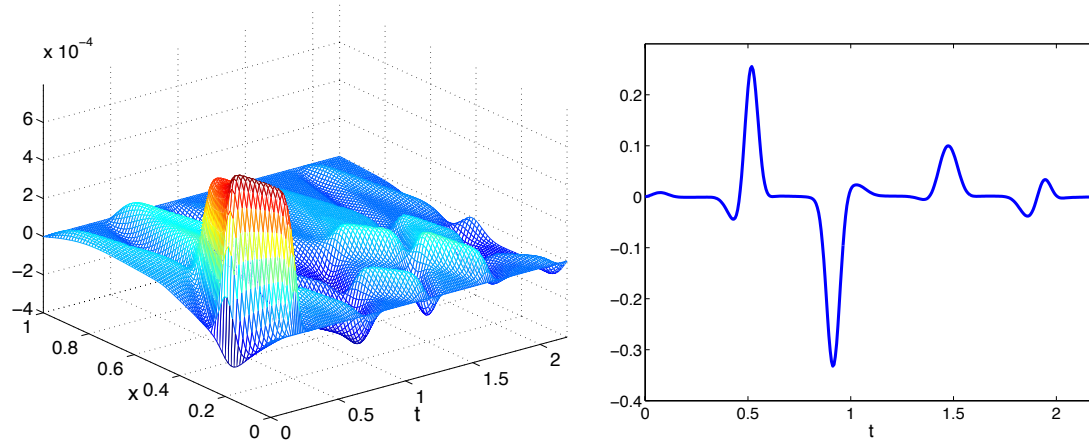


Figure 8:  $y_0(x) \equiv e^{-500(x-0.2)^2}$  and  $a$  given by (45) - The solution  $\hat{p}_h$  over  $Q_T$  (**Left**) and the corresponding variable  $\hat{v}_h$  on  $(0, T)$  (**Right**) -  $h = (1/80, 1/80)$ .

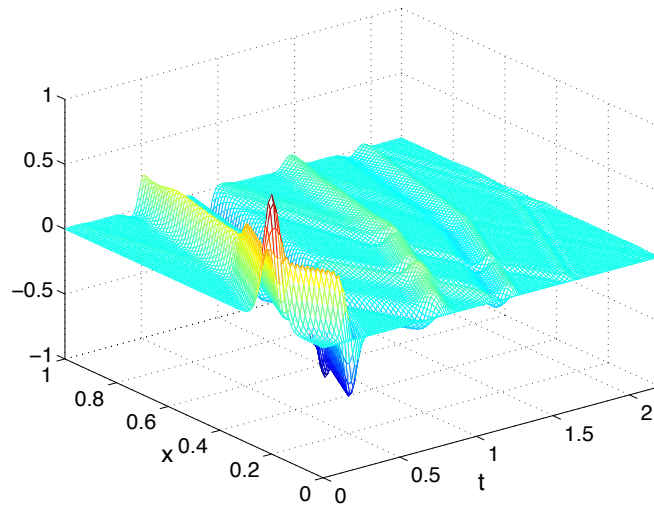


Figure 9:  $y_0(x) \equiv e^{-500(x-0.2)^2}$  and  $a$  given by (45) - The solution  $\hat{y}_h$  over  $Q_T$  -  $h = (1/80, 1/80)$ .

## 5 Further comments and concluding remarks

### 5.1 Primal versus dual approach (I): analogies

The solution to the variational formulation (23) is also the unique minimizer of the functional  $I$ , with

$$I(p) := \frac{1}{2} \iint_{Q_T} \rho^{-2} |Lp|^2 dx dt + \frac{1}{2} \int_0^T \rho_0^{-2} a(1)^2 |p_x(1, \cdot)|^2 dt - \int_0^1 y_0(x) p_t(x, 0) dx + \langle y_1, p(\cdot, 0) \rangle_{H^{-1}, H_0^1}. \quad (46)$$

This is similar to the conjugate functional  $J^*$  in (9).

Obviously,  $J^*(\mu, \phi_0, \phi_1) = I(-\phi)$  for all  $(\mu, \phi_0, \phi_1) \in L^2(Q_T) \times \mathbf{H}$ . Remark that this analogy implies, in the case  $\rho, \rho_0 \in L^\infty(Q_T)$  (given by (19)), that the space  $P$  coincides with  $C^0([0, T]; H_0^1(0, 1)) \cap C^1([0, T]; L^2(0, 1))$ . The fact that

$$a(x)p_x|_{x=1} \in L^2(0, T)$$

is a consequence of the well known *hidden regularity property*; conversely, if we only have  $\rho_0 \in L^\infty(Q_{T-\delta})$  for any  $\delta > 0$ , the extremal problem (9) is not well-posed in  $L^2(Q_T) \times \mathbf{H}$  but in a larger space (namely, the completion of  $L^2(Q_T) \times \mathbf{H}$  for the norm in  $P$ ; see Remark 4).

Therefore, the extremal problems (9) and (46) are connected to each other having (8) as starting point. The primal problem (23) belongs to the framework of elliptic variational problems in two dimensions and is well tailored for a resolution with finite elements. The dual problem (9) is of hyperbolic nature: the time variable is kept explicitly and time integration is required.

Note that we may also derive the optimality conditions for  $J^*$  (as we did in Section 2 for  $J$ ): this leads, at least formally, to the problem (23).

We also mention [21] where a (different) variational approach is introduced.

### 5.2 Primal versus dual approaches (II): discrete properties

The variational approach used here leads to satisfactory convergence results, in particular the strong convergence of the approximate controls  $\hat{v}_h$  towards a null control of the wave equation. This relies in a fundamental way on the fact that  $P_h$  is a finite dimensional subspace of  $P$ . Indeed, this allows to write directly the Carleman estimate in  $P_h$  and get that the function  $I$  (given by (46)) is uniformly coercive with respect to the discretization parameter  $h$ .

On the other hand, notice that no wave equation has to be solve in order to compute the approximations  $\hat{v}_h$ . For each  $h$ , once  $\hat{v}_h$  is known, we must solve the wave equation, in a post-treatment process, to compute the corresponding state  $\hat{y}_h$  (recall that, actually, this may be avoided by using directly the third optimality condition  $y = -\rho^{-2}Lp$ ).

This is in contrast with the dual approach. Indeed, the minimization of  $J^*$  by an iterative process requires the resolution of wave equations, through a decoupled space and time discretization. As recalled in the introduction, this may exhibit numerical pathologies (the occurrence of spurious high frequency solutions) and, therefore, needs some specific numerical approximations and techniques. We mention the work [4], where the authors prove, in a close context and within a dual approach, a weaker uniform semi-discrete Carleman estimate with an additional term in the right hand side, necessary to absorb these possibly spurious high frequencies (see [4], Theorem 2.3).

It is important to note that the computed functions  $\hat{v}_h$  are not *a priori* null controls for discrete systems (associated to the wave equation (1)), but simply approximations of the control  $v$  furnished by the solution to (8).

Let us also observe that the (primal) approach in this paper is straightforward to implement. In practice, the resolution is reduced to solve a linear system, with a banded sparse, symmetric



and definite positive matrix, for which efficient direct  $LU$  type solvers are known and available. Furthermore, we may want to adapt (and refine locally) the mesh of  $Q_T$  in order to improve convergence and such adaptation is much simpler than in the dual approach, where  $t$  is "conserved" as a time variable.

### 5.3 Mixed formulation and $C^0$ -approximation

The approach can be extended to the higher dimensional case of the wave equation in a bounded set  $\Omega \subset \mathbb{R}^N$ , with  $N \geq 2$ . However, the use of  $C^1$ -finite element is a bit more involved. Arguing as in [9], we may avoid this difficulty by introducing a mixed formulation equivalent to (23).

The idea is to keep explicit the variable  $y$  in the formulation and to introduce a Lagrange multiplier, associated to the constraint  $\rho^2 y + Lp = 0$  (see (22)). We obtain the following mixed formulation: find  $(y, p, \lambda) \in Z \times P \times Z$  such that

$$\left\{ \begin{array}{l} \iint_{Q_T} \rho^2 y \bar{y} \, dx \, dt + \int_0^T \rho_0^{-2} a(1)^2 p_x(1, t) \bar{p}_x(1, t) \, dt + \iint_{Q_T} \lambda (\rho^2 \bar{y} + L\bar{p}) \, dx \, dt \\ = \int_0^1 y_0(x) \bar{p}_t(x, 0) \, dx - \langle y_1, \bar{p}(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad \forall (\bar{y}, \bar{p}) \in Z \times P, \\ \iint_{Q_T} \bar{\lambda} (\rho^2 y + Lp) \, dx \, dt = 0 \quad \forall \bar{\lambda} \in Z, \end{array} \right. \quad (47)$$

where

$$Z = L^2(\rho^2; Q_T) := \{ z \in L_{loc}^1(Q_T) : \iint_{Q_T} \rho^2 |z|^2 \, dx \, dt < +\infty \}.$$

Taking advantage of the global estimate (16), we may show, through an appropriate inf-sup condition, that (47) is well-posed in  $Z \times P \times Z$ . Moreover, the approximation of this formulation may be addressed using  $C^0$ -finite element, which is very convenient. The approximation is non-conformal. More precisely, the variable  $p$  is now sought in a space  $R_h$  of  $C^0$ -functions that is not included in  $P$ .

At the discrete level, (47) reduces the controllability problem to the inversion of a square, banded and symmetric matrix. Moreover, as before, no wave equation has to be solves, whence the numerical pathology described above is not expected. However, since the underlying approximation is not conformal (this is the price to pay to avoid  $C^1$  finite elements), a careful (and *a priori* not straightforward) choice for  $R_h$  has to be done in order to guarantee a uniform discrete inf-sup condition. The analysis of this point, as well as the use of stabilized finite elements, will be detailed in a future work.

### 5.4 Extensions

The approach presented here can be extended and adapted to other equations and systems. What is needed is, essentially, an appropriate Carleman estimate.

In particular, we can adapt the previous ideas and results to the inner controllability case, i.e. the null controlability of the wave equation with distributed controls acting on a (small) sub-domain  $\omega$  of  $(0, 1)$ . Furthermore, using finite element tools, we can also get results in the case where the sub-domain  $\omega$  varies in time, that is non-cylindrical control domains  $q_T$  of the form

$$q_T = \{ (x, t) \in Q_T : g_1(t) < x < g_2(t), \quad t \in (0, T) \},$$

where  $g_1$  and  $g_2$  are smooths functions on  $[0, T]$ , with  $0 \leq g_1 < g_2 \leq 1$ . This opens the possibility to optimize numerically the domain  $q_T$ , as was done in a cylindrical situations in [20] (see also [22]).

Let us finally mention that many non-linear situations can be considered through a suitable linearization and iterative process. We refer to [10] for some ideas in a similar parabolic situation.

## A Appendix: On the proof of Theorem 2.1

We first prove a global Carleman estimate for functions  $w$  satisfying vanishing initial and final conditions. In what follows,  $L$  stands for the operator given in (2) with  $b \equiv 0$ . It is easy to check that, if the estimate (16) holds for in this particular case, then the same estimate holds for any potential  $b \in L^\infty((0, 1) \times (-T, T))$ .

**THEOREM A.1** *With the notation of Section 2, let  $x_0 < 0$  be a fixed point, let  $\psi$  and  $\varphi$  be the weight functions defined by (12)–(14) and let  $a \in \mathcal{A}(x_0, a_0)$ . Then there exist positive constants  $s_0$  and  $M$ , only depending on  $x_0, a_0, \|a\|_{C^1([0,1])}, \|b\|_{L^\infty(Q_T)}$  and  $T$  such that, for all  $s > s_0$  one has*

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|v_t|^2 + |v_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lv|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |v_x(1, t)|^2 dt. \end{aligned} \quad (48)$$

for any  $v \in L^2(-T, T; H_0^1(0, 1))$  satisfying  $Lv \in L^2((0, 1) \times (-T, T))$ ,  $v_x(1, \cdot) \in L^2(-T, T)$  and

$$v(\cdot, \pm T) = v_t(\cdot, \pm T) = 0.$$

The proof of this result follows step-by-step the proof of Theorem 2.1 in [3]. However, since the argument provides conditions on the set of admissible  $a$  and, to our knowledge, these conditions have not been stated in this form before, we provide here the detailed proof.

**PROOF :** Let us introduce  $w = e^{s\varphi}v$  and let us set

$$Pw := e^{s\varphi}L(e^{-s\varphi}w) = e^{s\varphi}((e^{-s\varphi}w)_{tt} - (a(e^{-s\varphi}w)_x)_x).$$

After some computations, we find that  $Pw = P_1w + P_2w + Rw$ , with

$$\begin{aligned} P_1w &= w_{tt} - (aw_x)_x + s^2\lambda^2\varphi^2w (|\psi_t|^2 - a|\psi_x|^2) \\ P_2w &= (\alpha - 1)s\lambda\varphi w (\psi_{tt} - (a\psi_x)_x) - s\lambda^2\varphi w (|\psi_t|^2 - a|\psi_x|^2) - 2s\lambda\varphi (\psi_t w_t - a\psi_x w_x) \\ R w &= -\alpha s\lambda\varphi w (\psi_{tt} - (a\psi_x)_x), \end{aligned}$$

where the parameter  $\alpha$  will be chosen below.

Recall that

$$\psi(x, t) \equiv |x - x_0|^2 - \beta t^2 + M_0, \quad \varphi(x, t) \equiv e^{\lambda\psi(x, t)}$$

and

$$\psi(x, t) \geq 1 \quad \forall (x, t) \in (0, 1) \times (-T, T).$$

In this proof, we will denote by  $M$  a generic positive constant that can depend on  $x_0, a_0, \|a\|_{C^1([0,1])}, \|b\|_{L^\infty(Q_T)}$  and  $T$ .

By density, it will suffice to establish (48) under the additional assumption  $a \in C^3([0, 1])$ . As in the constant case  $a \equiv 1$ , the first part of the proof is devoted to estimate from below the integral

$$I = \int_{-T}^T \int_\Omega (P_1w)(P_2w) dx dt = \sum_{i,j=1}^3 I_{ij}. \quad (49)$$

By integrating by parts in time and/or space, we can compute the integrals  $I_{ij}$  in (49). We obtain:

$$\begin{aligned}
I_{11} &= (\alpha - 1)s\lambda \int_{-T}^T \int_0^1 w_{tt} \varphi w (\psi_{tt} - (a\psi_x)_x) dx dt \\
&= (1 - \alpha)s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad - \frac{(1 - \alpha)}{2} s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 \psi_{tt} (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad - \frac{(1 - \alpha)}{2} s\lambda^3 \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_t|^2 (\psi_{tt} - (a\psi_x)_x) dx dt,
\end{aligned}$$

$$\begin{aligned}
I_{12} &= -s\lambda^2 \int_{-T}^T \int_0^1 w_{tt} \varphi w (|\psi_t|^2 - a|\psi_x|^2) dx dt \\
&= s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w_t|^2 (|\psi_t|^2 - a|\psi_x|^2) dx dt - s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_{tt}|^2 dx dt \\
&\quad - \frac{3s\lambda^3}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_t|^2 \psi_{tt} dx dt + \frac{s\lambda^3}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 a |\psi_x|^2 \psi_{tt} dx dt \\
&\quad - \frac{s\lambda^4}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_t|^2 (|\psi_t|^2 - a|\psi_x|^2) dx dt
\end{aligned}$$

and

$$\begin{aligned}
I_{13} &= -2s\lambda \int_{-T}^T \int_0^1 w_{tt} \varphi (\psi_t w_t - a\psi_x w_x) dx dt \\
&= s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 \psi_{tt} dx dt + s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w_t|^2 |\psi_t|^2 dx dt \\
&\quad + s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (a\psi_x)_x dx dt + s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w_t|^2 a |\psi_x|^2 dx dt \\
&\quad - 2s\lambda^2 \int_{-T}^T \int_0^1 \varphi a \psi_x \psi_t w_x w_t dx dt.
\end{aligned}$$

Also,

$$\begin{aligned}
I_{21} &= (1 - \alpha)s\lambda \int_{-T}^T \int_0^1 (aw_x)_x \varphi w (\psi_{tt} - (a\psi_x)_x) dx dt \\
&= -(1 - \alpha)s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad + \frac{(1 - \alpha)}{2} s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 (a\psi_x)_x (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad + \frac{(1 - \alpha)}{2} s\lambda^3 \int_{-T}^T \int_0^1 \varphi a |w|^2 |\psi_x|^2 (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad - (1 - \alpha)s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 a \psi_x (a\psi_x)_{xx} dx dt \\
&\quad - \frac{(1 - \alpha)}{2} s\lambda \int_{-T}^T \int_0^1 \varphi |w|^2 (a_x (a\psi_x)_{xx} + a (a\psi_x)_{xxx}) dx dt,
\end{aligned}$$

$$\begin{aligned}
I_{22} &= s\lambda^2 \int_{-T}^T \int_0^1 (aw_x)_x \varphi w (|\psi_t|^2 - a|\psi_x|^2) dx dt \\
&= -s\lambda^2 \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (|\psi_t|^2 - a|\psi_x|^2) dx dt \\
&\quad - \frac{s\lambda^2}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 ( (|a_x|^2 + aa_{xx}) |\psi_x|^2 + 4aa_x \psi_x \psi_{xx} + 2a(a\psi_x)_x \psi_{xx} ) dx dt \\
&\quad + \frac{s\lambda^3}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 (a\psi_x)_x (|\psi_t|^2 - a|\psi_x|^2) dx dt \\
&\quad + \frac{s\lambda^4}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 a |\psi_x|^2 (|\psi_t|^2 - a|\psi_x|^2) dx dt \\
&\quad - s\lambda^3 \int_{-T}^T \int_{\Omega} \varphi |w|^2 a \psi_x (a_x |\psi_x|^2 + 2a\psi_x \psi_{xx}) dx dt
\end{aligned}$$

and

$$\begin{aligned}
I_{23} &= 2s\lambda \int_{-T}^T \int_0^1 (aw_x)_x \varphi (\psi_t w_t - a\psi_x w_x) dx dt \\
&= s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\psi_{tt} + a\psi_{xx}) dx dt \\
&\quad + s\lambda^2 \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (|\psi_t|^2 + a|\psi_x|^2) dx dt - 2s\lambda^2 \int_{-T}^T \int_0^1 \varphi a \psi_x \psi_t w_x w_t dx dt \\
&\quad - s\lambda \int_{-T}^T [a(1)^2 |w_x(1, t)|^2 \varphi(1, t) \psi_x(1, t) - a(0)^2 |w_x(0, t)|^2 \varphi(0, t) \psi_x(0, t)] dt.
\end{aligned}$$

Finally,

$$\begin{aligned}
I_{31} &= (\alpha - 1) s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt, \\
I_{32} &= -s^3 \lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt
\end{aligned}$$

and

$$\begin{aligned}
I_{33} &= -2s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 w (|\psi_t|^2 - a|\psi_x|^2) (\psi_t w_t - a\psi_x w_x) dx dt \\
&= s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad + 2s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 \psi_{tt} + aa_x \psi_x |\psi_x|^2 + a^2 |\psi_x|^2 \psi_{xx}) dx dt \\
&\quad + 3s^3 \lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt.
\end{aligned}$$

Gathering together all terms  $I_{ij}$  for  $i, j \in \{1, 2, 3\}$ , we obtain

$$\begin{aligned}
I &= \int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt \\
&= s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (2\psi_{tt} - \alpha(\psi_{tt} - (a\psi_x)_x)) dx dt \\
&+ s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\alpha(\psi_{tt} - (a\psi_x)_x) + 2(a\psi_x)_x - a_x \psi_x) dx dt \\
&+ 2s\lambda^2 \int_{-T}^T \int_0^1 \varphi (|w_t|^2 |\psi_t|^2 - 2a\psi_x \psi_t w_x w_t + a^2 |w_x|^2 |\psi_x|^2) dx dt \\
&+ 2s^3 \lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt \\
&+ s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (2|\psi_t|^2 \psi_{tt} + a a_x \psi_x |\psi_x|^2 + 2a^2 |\psi_x|^2 \psi_{xx}) dx dt \\
&+ \alpha s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt \\
&- s\lambda \int_{-T}^T (a(1)^2 |w_x(1, t)|^2 \varphi(1, t) \psi_x(1, t) - a(0)^2 |w_x(0, t)|^2 \varphi(0, t) \psi_x(0, t)) dx dt \\
&+ X_0,
\end{aligned}$$

where  $X_0$  is the sum of all “lower order terms”:

$$|X_0| \leq Ms\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt.$$

Let us analyze the high order terms arising in the previous expression of  $I$ . First, remark that

$$s\lambda^2 \int_{-T}^T \int_0^1 \varphi (|w_t|^2 |\psi_t|^2 - 2a\psi_x \psi_t w_x w_t + a^2 |w_x|^2 |\psi_x|^2) dx dt \geq 0. \quad (50)$$

Secondly, notice that, under the assumption  $a \in \mathcal{A}(x_0, a_0)$ , if  $\beta$  satisfies (11), we can choose  $\alpha$  in such a way that the terms of order  $s\lambda$  are positive. Indeed, we have in this case

$$-a(x) - (x - x_0)a_x(x) < \beta < a(x) + \frac{1}{2}(x - x_0)a_x(x) \quad \forall x \in [0, 1],$$

whence

$$\frac{2\beta}{\beta + a(x) + (x - x_0)a_x(x)} < \frac{2a(x) + (x - x_0)a_x(x)}{\beta + a(x) + (x - x_0)a_x(x)} \quad \forall x \in [0, 1].$$

Let  $\alpha$  satisfy

$$\sup_{[0,1]} \left( \frac{2\beta}{\beta + a(x) + (x - x_0)a_x(x)} \right) < \alpha < \inf_{[0,1]} \left( \frac{2a(x) + (x - x_0)a_x(x)}{\beta + a(x) + (x - x_0)a_x(x)} \right).$$

Then, an explicit computation of the derivatives of  $\psi$  shows that

$$2\psi_{tt} - \alpha(\psi_{tt} - (a\psi_x)_x) > 0 \quad \text{and} \quad \alpha(\psi_{tt} - (a\psi_x)_x) + 2(a\psi_x)_x - a_x \psi_x > 0 \quad \text{in} \quad [0, 1] \times [-T, T]$$

and, consequently,

$$\begin{aligned}
&s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (2\psi_{tt} - \alpha(\psi_{tt} - (a\psi_x)_x)) dx dt \\
&+ s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\alpha(\psi_{tt} - (a\psi_x)_x) + 2(a\psi_x)_x - a_x \psi_x) dx dt \\
&\geq Ms\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 dx dt + Ms\lambda \int_{-T}^T \int_0^1 \varphi |w_x|^2 dx dt.
\end{aligned}$$

The remaining terms in  $I$  can be written in the form

$$\begin{aligned}
& 2s^3\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt \\
& + s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (2|\psi_t|^2 \psi_{tt} + aa_x \psi_x |\psi_x|^2 + 2a^2 |\psi_x|^2 \psi_{xx}) dx dt \\
& + \alpha s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt \\
& = s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 F_\lambda(x, Y(x, t)) dx dt,
\end{aligned}$$

where  $Y := |\psi_t|^2 - a|\psi_x|^2$  and

$$\left\{ \begin{aligned}
F_\lambda(x, Y) &:= 2\lambda Y^2 + (2\psi_{tt} + \alpha(\psi_{tt} - (a(x)\psi_x)_x)) Y \\
&\quad + a(x)|\psi_x|^2(2\psi_{tt} + a_x(x)\psi_x + 2a(x)\psi_{xx}) \\
&= 2\lambda Y^2 + (4\beta + \alpha(2\beta + a(x) + (x - x_0)a_x(x))) Y \\
&\quad + 8a(x)(x - x_0)^2(-2\beta + 2a(x) + (x - x_0)a_x(x)).
\end{aligned} \right.$$

Since  $F_\lambda$  is polynomial of the second degree in  $Y$ , one has

$$\begin{aligned}
F_\lambda(x, Y) &\geq 8a(x)(x - x_0)^2(-2\beta + 2a(x) + (x - x_0)a_x(x)) \\
&\quad - \frac{1}{8\lambda} [4\beta + \alpha(2\beta + a(x) + (x - x_0)a_x(x))]^2
\end{aligned}$$

for all  $x \in [0, 1]$  and  $Y \in \mathbf{R}$ . Therefore, if  $\beta$  satisfies (11), for  $\lambda$  large enough (depending on  $x_0$  and  $\|a\|_{C^1([0,1])}$ ), we obtain:

$$s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 F_\lambda(X) dx dt \geq Ms^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt \quad (51)$$

Putting together the estimates (50)–(51), the following is found:

$$\begin{aligned}
& \int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt \geq Ms\lambda \int_{-T}^T \int_0^1 \varphi (|w_t|^2 + |w_x|^2) dx dt \\
& + Ms^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt \\
& - Ms\lambda \int_{-T}^T |w_x(1, t)|^2 dx dt - Ms\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt.
\end{aligned} \quad (52)$$

On the other hand, recalling the definition of  $P$ ,  $P_1$ ,  $P_2$  and  $R$ , we observe that

$$\int_{-T}^T \int_0^1 (|P_1 w|^2 + |P_2 w|^2) dx dt + 2 \int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt = \int_{-T}^T \int_0^1 |Pw - Rw|^2 dx dt$$

It is not difficult to see that there exists  $M$  such that

$$\int_{-T}^T \int_0^1 |Pw - Rw|^2 dx dt \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms^2\lambda^2 \int_{-T}^T \int_0^1 \varphi^2 |w|^2 dx dt.$$

In particular, we have

$$\int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms^2\lambda^2 \int_{-T}^T \int_0^1 \varphi^2 |w|^2 dx dt \quad (53)$$

and combining (52) and (52) we obtain:

$$\begin{aligned} & s\lambda \int_{-T}^T \int_0^1 \varphi (|w_t|^2 + |w_x|^2) dx dt + s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms\lambda \int_{-T}^T |w_x(1, t)|^2 dt \\ & \quad + Ms\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt + Ms^2 \lambda^2 \int_{-T}^T \int_0^1 \varphi^2 |w|^2 dx dt. \end{aligned}$$

Obviously, the last two terms in the right hand side can be absorbed by the second term in the left for  $s$  large enough. Therefore, there exists  $s_0 > 0$ , only depending on  $x_0$ ,  $a_0$ ,  $\|a\|_{C^1([0,1])}$ ,  $\|b\|_{L^\infty(Q_T)}$  and  $T$ , such that, for all  $s > s_0$ , one has:

$$\begin{aligned} & s\lambda \int_{-T}^T \int_0^1 \varphi (|w_t|^2 + |w_x|^2) dx dt + s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms\lambda \int_{-T}^T |w_x(1, t)|^2 dt. \end{aligned} \quad (54)$$

Since  $w = ve^{s\varphi}$  and  $Pw = e^{s\varphi}Lv$ , we can easily rewrite (54) in the form (48).

This ends the proof.  $\square$

In the remaining part of the Appendix, we will use the Carleman estimate (48) to prove Theorem 2.1.

Thus, let us assume that (15) holds,  $w \in L^2(-T, T; H_0^1(0, 1))$ ,  $Lw \in L^2((0, 1) \times (-T, T))$  and  $w_x(1, \cdot) \in L^2(-T, T)$ . Thanks to (15), there exists  $\eta \in (0, T)$  and  $\varepsilon > 1$  such that

$$(1 - \varepsilon)(T - \eta)\beta \geq \max_{[0,1]} a(x)^{1/2}(x - x_0).$$

Moreover, simple computations show that, for every  $t \in (-T, -T + \eta) \cup (T - \eta, T)$ , the function  $\psi(\cdot, t)$  satisfies:

$$\begin{cases} (1 - \varepsilon) \min_{[0,1]} |\psi_t(x, t)| \geq \max_{[0,1]} a(x)^{1/2} |\psi_x(x, t)| \\ \max_{[0,1]} \psi(x, t) < \min_{[0,1]} \psi(x, 0). \end{cases} \quad (55)$$

Let  $\chi \in C_c^\infty(\mathbb{R})$  a cut-off function such that  $0 \leq \chi \leq 1$  and

$$\chi(t) = \begin{cases} 1, & \text{if } |t| \leq T - \eta \\ 0, & \text{if } |t| \geq T \end{cases}$$

Then we can apply Theorem A.1 to the function  $\tilde{w} := \chi w$ , whence the following Carleman estimate holds

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|\tilde{w}_t|^2 + |\tilde{w}_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |\tilde{w}|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |L\tilde{w}|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |\tilde{w}_x(1, t)|^2 dt. \end{aligned}$$

Since  $L\tilde{w} = \chi Lw + \chi_{tt}w + 2\chi_t w_t$ , we deduce from (55) that

$$\begin{aligned} & s \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt \\ & \quad + M \int_{-T}^{-T+\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w|^2) dx dt + M \int_{T-\eta}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w|^2) dx dt. \end{aligned} \quad (56)$$

Let us denote by  $E_s = E_s(t)$  the energy associated to the operator  $L$ , that is,

$$E_s(t) := \frac{1}{2} \int_0^1 e^{2s\varphi} (|w_t|^2 + a|w_x|^2) dx. \quad (57)$$

Then, the argument employed in the proof of Theorem 2.5 in [3] (using the modified energy given by (57)) can be used to deduce (16) from (56).

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