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A PONCELET THEOREM FOR LINES

JEAN VALLÈS

Abstract. Our aim is to prove a Poncelet type theorem for a line configuration on the complex projective plane $\mathbb{P}^2$. More precisely, we say that a polygon with $2n$ sides joining $2n$ vertices $A_1, A_2, \cdots, A_{2n}$ is well inscribed in a configuration $\mathcal{L}_n$ of n lines if each line of the configuration contains exactly two points among $A_1, A_2, \cdots, A_{2n}$. Then we prove :

Theorem *Let $\mathcal{L}_n$ be a configuration of n lines and D a smooth conic in $\mathbb{P}^2$. If it exists one polygon with $2n$ sides well inscribed in $\mathcal{L}_n$ and circumscribed around D then there are infinitely many such polygons. In particular a general point in $\mathcal{L}_n$ is a vertex of such a polygon.*

This result was probably known by Poncelet or Darboux but we did not find a similar statement in their publications. Anyway, even if it was, we would like to propose an elementary proof based on Frégier's involution. We begin by recalling some facts about these involutions. Then we explore the following question : When does the product of involutions correspond to an involution? We give a partial answer in proposition 2.6. This question leads also to Pascal theorem, to its dual version proved by Brianchon, and to its generalization proved by Möbius (see [1], thm.1 and [6], page 219). In the last section, using the Frégier's involutions and the projective duality we prove the main theorem quoted above.

February 10, 2012

1. Introduction

Let $\mathcal{L}_n$ be a configuration of n lines $L_1, \cdots, L_n$ in the complex projective plane $\mathbb{P}^2$ and D be a smooth conic in the same plane. We assume that $\mathcal{L}_n \cap D$ consists in $2n$ distinct points. From a point A_1 on L_1 (not being on the other lines neither on D) we draw a tangent line to D. This line cuts L_2 in a point A_2. In the same way we define successively

$$A_3 \in L_3, \cdots, A_n \in L_n, A_{n+1} \in L_{n-1}, \cdots, A_{2n} \in L_2.$$

The second tangent line to D from A_{2n} meets L_1 in A_{2n+1}. Now two situations can occur : $A_{2n+1} = A_1$ or $A_{2n+1} \neq A_1$. This second case is clearly the general case.

If $A_{2n+1} = A_1$ the polygon $(A_1A_2) \cup \cdots \cup (A_{2n}A_{2n+1})$ is simultaneously inscribed in $\mathcal{L}_n$ and circumscribed around D. Our aim is to prove that in this case, there are infinitely many polygons with $2n$ sides simultaneously inscribed in $\mathcal{L}_n$ and circumscribed around D (see theorem 3.1). It means that the existence of such polygons does not depend on the initial point but only on D and $\mathcal{L}_n$.

On the contrary, if $A_{2n+1} \neq A_1$ the polygon will never close after $2n$ steps for any initial point on $\mathcal{L}_n$.

This kind of result is called a porism and it is of the same nature than Steiner porism (see [2], thm.7.3) for circles tangent to two given circles or Poncelet porism for two conics (see [5], [3] or [8]).

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To prove this porism we *dualize* the situation and we consider the dual polygon inscribed in D. To be more explicit, let us assume that $A_{2n+1} = A_1$. Then the situation can be dualized in the following way. Any line L_i in the configuration is the polar line of a point c_i (its pole). Any tangent line (A_iA_{i+1}) is the polar line of a point $x_{i,i+1} \in D$ for $1 \leq i \leq 2n$. For $1 \leq j \leq n$, the lines $(x_{j-1,j}x_{j,j+1})$ and $(x_{j+n-1,j+n}x_{j+n,j+n+1})$ meet in c_i. By this way we obtain an inscribed polygon $(x_{1,2}x_{2,3}) \cup \cdots \cup (x_{2n,2n+1}x_{1,2})$ with $2n$ sides passing through n fixed points $c_1, \cdots, c_n$ (this inscribed polygon corresponds to the choice of a $2n$-cycle among the permutations of $2n$ points).

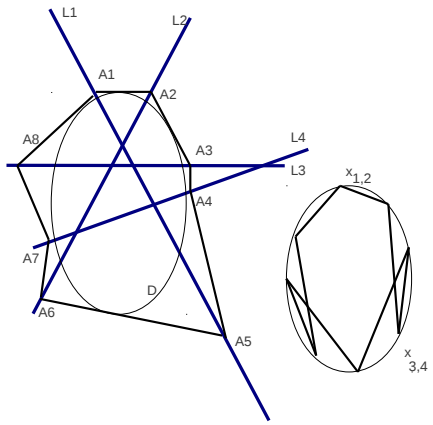

FIGURE 1. Octagon inscribed in D and its dual inscribed in 4 lines and circumscribed around D.

This inscribed polygon leads to study Frégier's involutions that are particular automorphisms of D. Indeed we verify that the porism (seetheorem 3.1) is true when the product of the n Frégier's involutions giving the incribed polygon is also a Frégier's involution.

So first of all we recall the definition and the main properties of these automorphisms. Then we study the product of involutions. More precisely we wonder when a product of n involutions is still an involution.
For $n = 2$, it is very well known (see proposition 2.3).
For $n = 3$, we prove that the product is an involution if and only if the centers are aligned (see proposition 2.4). It is an other way to set out the so-called Pascal theorem[1].
For $n \geq 4$, we propose a new proof of a generalization of Pascal theorem due to Möbius (see theorem 2.7). We prove also that the product of $2n+1$ involutions is an involution when their centers are aligned (see proposition 2.6).

The projective duality give us the dual versions of all these results, like Brianchon theorem for instance (see theorem 2.8 and proposition 2.9).

[1]Another way but probably not a new way. According to its simplicity this argument is certainly already written somewhere.

In the last section we prove our Poncelet type theorem for lines (see theorem 3.1) and we conclude by an explicit computation in the case of two lines simultaneously inscribed and circumscribed around a smooth conic.

2. Frégier's involutions

The group $\mathrm{PGL}(2,\mathbb{C})$ acts on $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ in the usual way.

Definition 2.1. An element $g \in \mathrm{PGL}(2,\mathbb{C})$ which is not the identity I on $\mathbb{P}^1$ is called an involution if $g^2 = I$.

Considering $\mathrm{GL}(2,\mathbb{C})$ as the space of 2×2 invertible matrices it is clear that any $g \in \mathrm{PGL}(2,\mathbb{C})$ has at most two fixed points (g has three fixed points if and only if $g = I$). Moreover, when g is an involution it is easy to verify that it has exactly two fixed points and that it is determined by the data of these fixed points.

The Veronese embedding $\mathbb{P}^1 \overset{v_2}{\hookrightarrow} \mathbb{P}^2$ induces an embedding for groups $\mathrm{PGL}(2,\mathbb{C}) \subset \mathrm{PGL}(3,\mathbb{C})$. Let $g \in \mathrm{PGL}(2,\mathbb{C})$ be an involution on $\mathbb{P}^1$. The corresponding transformation g in $\mathbb{P}^2$ has two fixed points on the smooth conic $D = v_2(\mathbb{P}^1)$: the images of the fixed points on $\mathbb{P}^1$.
Let us consider now the intersection point $x \notin D$ of the two tangent lines in the fixed points of g. A general line through x cuts D in two points. The map exchanging these points is an involution on D (i.e. on $\mathbb{P}^1$). Its fixed points are the intersection points of D with the polar line of x. Such an involution is called *Frégier involution* on D with center x.

Since an involution is determined by its fixed points, this involution is g. We have verified that:

Proposition 2.2. *Any involution on $D \subset \mathbb{P}^2$ is a Frégier's involution.*

2.1. **Product of two and three involutions.** Rigorously speaking we should not write product but composition of involutions. Anyway, from now, since for matrices it becomes a product we will write product in any case. Moreover we will denote uv the product (i.e the composition) of two involutions u and v and u^n the product (i.e the composition) of u with itself n times. The following proposition is classical and easy to prove.

Proposition 2.3. *Let u and v be two involutions with distincts fixed points. Then, uv is involutive if and only if the two fixed points of u and the two fixed points of v form an harmonic divison of D.*

For three involutions we recognize Pascal theorem.

Proposition 2.4. *Let u, v, and w be three involutions with distincts fixed points and let x_u, x_v and x_w be their respective centers. Then,*

$$uvw \text{ is involutive } \Leftrightarrow x_u, x_v \text{ and } x_w \text{ are aligned.}$$

Proof. [2] Assume that the three centers are aligned on a line L. The line L is not tangent to D because the three involutions do not have a common fixed point. Then let $\{x, y\} = L \cap D$. We verify that $uvw(x) = y$ and $uvw(x) = y$. The automorphism uvw has at least one fixed point $z \in D$. Now the three points x, y and z are fixed points for $(uvw)^2$. It means that uvw is an involution on D.

[2]This proposition, its proof and its corollary appear in a book project with Giorgio Ottaviani (see [7]).

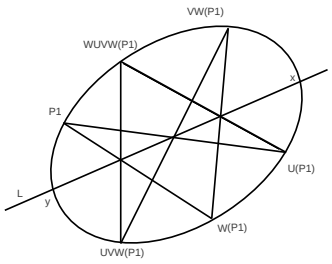

FIGURE 2. Three involutions with aligned centers.

Conversely, assume that uvw is involutive. Let $x \in D$ such that $v(x) = w(x) \neq x$. Call L the line joining x to $v(x)$, i.e passing trough x_v and x_w. From $v(x) = w(x) \neq x$ and the assumption $uvw = wvu$, we find four fixed points of wv: $x, v(x), u(x), uv(x)$. The automorphism wv has at most two distinct fixed points. The first two are distinct. Consider the third one $u(x)$. If $u(x) = v(x)$ we have $u(x) = v(x) = w(x)$ and we have finished, hence $u(x) = x$. Consider the fourth one $uv(x)$. If $uv(x) = x$ we have again $u(x) = v(x) = w(x)$ and we have finished, hence $uv(x) = v(x)$. It follows that $x, v(x)$ are fixed points of both u and uvw, that is a contradiction since an involution is uniquely determined by its fixed points. □

Corollary 2.5 (Pascal's theorem)**.** *Let $p_1, p_2, p_3, q_3, q_2, q_1$ be six (ordered) points on a smooth conic D. Let $x_{ij}, i < j$ the intersection point of the two lines joining p_i to q_j and p_j to q_i. Then the three points x_{12}, x_{13} and x_{23} are aligned.*

Proof. We denote by u the involution defined by x_{13}, v the one defined by x_{23} and w the last one defined by x_{12}. Then by following lines we verify that

$$(uvw)(p_1) = q_1, (uvw)(q_1) = p_1.$$

Let z be a fixed point of uvw. Then z, p_1, q_1 are fixed points of $(uvw)^2$. Since an element of $\mathrm{PGL}(2, \mathbb{C})$ that has more than three fixed points is the identity, we have proved that uvw is an involution. The result now follows from proposition 2.4. □

Remark. The center of the product of three involutions u_1, u_2 and u_3 with aligned centers (on a line L) belongs also to L. Indeed let us define $v = u_1u_2u_3$; since the centers are aligned v is also an ivolution. Then $u_1u_2u_3v = I$. It implies $u_1 = u_2u_3v$ i.e that the product of u_2, u_3 and v is an involution. According to theorem 2.4 their centers are aligned.

2.2. **Product of $n \geq 4$ involutions and Möbius theorem.** We show that the product of an odd number of involutions with aligned centers is still an involution.

Remark. We cannot expect an equivalence as it was in proposition 2.4 but only an implication. Indeed let us give a product of five involutions with three centers on a line L and two centers on another line D that is also an involution. So, let us consider a line L and three points x_1, x_2, x_3 on it. We associate three involutions u_1, u_2, u_3 to these centers. The product $w = u_1u_2u_3$ is also an involution by proposition 2.4 . Moreover,

according to remark 2.1 its center x belongs to L. Now we introduce another line D passing through x, and two others points x_4, x_5 on D. Let us call u_4, u_5 the associated involutions. Then

$$u_1u_2u_3u_4u_5 = wu_3u_4,$$

and wu_3u_4 is an involution because their centers x, x_4, x_5 are aligned.

Proposition 2.6. *Let $u_1, \cdots, u_{2n+1}$ be involutions on $D \subset \mathbb{P}^2$ with respective centers $c_1, \cdots, c_{2n+1}$. If $c_1, \cdots, c_{2n+1}$ are aligned then $\prod_{i=1}^{2n+1} u_i$ is an involution.*

Proof. Let $v = \prod_{i=1}^{2n+1} u_i$. We recall that the fixed points of v are also fixed points of v^2. The automorphism v possesses at least one fixed point x. Let L be the line of centers and $L \cap D = \{y, z\}$. These two points are exchanged by v and then are not fixed points of v. The points x, y, z are fixed points of v^2, then $v^2 = I$. □

Remark. The proposition is not valid for an even number of involutions. Indeed since two points are always aligned it is clearly not valid for two involutions. Consider now three involutions with aligned centers u_1, u_2, u_3. Call w their product. Let u_4 be another involution with its center on the previous line of centers. Then the product wu_4 is an involution if and only if the four fixed points form an harmonic division. But if we move the center of u_4 on L the cross-ratio is changing. So for the general point on L the product will not be involutive.

In general when the product of involutions is an involution we are not able to say something pertinent about the position of their centers. But, when the product of n involutions is still an involution and at least $n-1$ centers are aligned, Möbius proved that all the centers are aligned (see [6], page 219). We prove again this theorem in the terminology of Frégier's involution. The formulation below is the one given in Adler's article (see [1], thm. 1).

Theorem 2.7 (Möbius theorem)**.** *Let $x_1, y_1, \cdots, x_n, y_n$ be points on a smooth conic. Consider the intersection points $a_j = (x_jx_{j+1}) \cap (y_jy_{j+1})$, $j = 1, \cdots, n-1$ and*

$$a_n = \begin{cases} (x_ny_1) \cap (y_nx_1) & \text{if} \quad n = 2m+1 \\ (x_nx_1) \cap (y_ny_1) & \text{if} \quad n = 2m. \end{cases}$$

If all of these points except possibly one are collinear then the same is true for the remaining point.

Proof. As we have seen before (in proposition 2.4) it is true for three involutions since the product of three involutions is an involution if and only if the centers are aligned. Moreover as we said in remark 2.1 the center of the product is also aligned with the three others.

We need only to prove the result for four involutions. Indeed let us verify that we can reduce the general case ($n \geq 4$) to three or four involutions. Consider $n > 4$ involutions. Any group of three terms among u_{i-1}, u_i, u_{i+1} for $i = 2, \cdots, n-2$ gives a new involution with a center on the same line. So when n is odd it remains three involutions after reductions $uu_{n-1}u_n$. Since the product is an involution the centers are aligned. But the centers of u and u_{n-1} are already on the line L. Then the center or u_n is also on L. When n is even it remains four involutions after reduction $uu_{n-2}u_{n-1}u_n$ with the first three centers on L.

So let us consider the case $u_1u_2u_3u_4 = v$ with v an involution. We have $u_1v = u_2u_3u_4$. The center x_4 belongs to (x_2x_3) if and only if $u_2u_3u_4$ is an involution, i.e. if and only if u_1v is an involution. To prove it let us show that $u_1v = vu_1$. Since x_1, x_2, x_3 are aligned the product $u_1u_2u_3$ is involutive, so $u_1u_2u_3 = u_3u_2u_1$. Then

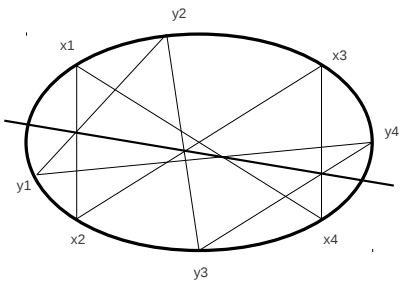

FIGURE 3. Four aligned centers.

$u_2u_3 = u_1u_3u_2u_1$ and $u_1v = (u_2u_3)u_4 = (u_1u_3u_2u_1)u_4$. Since $u_3u_2u_1 = u_1u_2u_3$ we obtain

$$u_1v = (u_1u_3u_2u_1)u_4 = u_1(u_1u_2u_3)u_4 = u_1v.$$

□

2.3. **Projective duality and involutions.** All the results obtained above can be *dualized* by considering polar lines of points and poles of lines with respect to the smooth conic D. By this way any inscribed polygon into D induces a circumscribed polygon (with the same number of sides) around D. Even if Möbius theorem was certainly *dualized* by Möbius himself, we write one more time this dual version below.

Theorem 2.8 (dual Möbius)**.** *Let $L_1 \cup \cdots \cup L_{2n}$ be a polygon tangent circumscribed to the smooth conic D. If the diagonals joining $L_i \cap L_{i+1}$ to $L_{i+n} \cap L_{i+1+n}$ for $i = 1, \cdots, n-1$ are concurrent lines then the same is true for the remaining diagonal joining $L_{2n} \cap L_1$ to $L_n \cap L_{n+1}$.*

Remark. For $n = 3$ it is Brianchon theorem.

Proposition 2.6 implies that one can construct an inscribed polygon in D when n aligned points $c_1, \cdots, c_n$ (not on D) are given. Indeed let x be a point on D and let us take successively its images by the involutions u_i associated to the centers c_i:

$$x, u_1(x), (u_2u_1)(x), \cdots, y = (u_{2n+1} \cdots u_1)(x).$$

Then let us take successively the images of y by the involutions u_i:

$$u_1(y), (u_2u_1)(y), \cdots, (u_{2n+1} \cdots u_1)(y).$$

Since the product is involutive the process stops and we have:

$$(u_{2n+1} \cdots u_1)(y) = (u_{2n+1} \cdots u_1)^2(x) = x.$$

In other words, from a general point on D we can draw by this method an inscribed polygon with $4n+2$ sides. Dualizing this statement, we verify the following proposition:

Proposition 2.9 (dual version of proposition 2.6)**.** *Let us consider $2n+1$ concurrent lines L_i meeting a smooth conic D in $4n+2$ distinct points. Then take any point P_1 on L_1 and draw a tangent to D from this point. This tangent cuts L_2 in one point P_2. Let us draw successively $P_i \in L_i$ for $1 \leq i \leq 2n+1$ and $P_{2n+1+j} \in L_j$ for $1 \leq j \leq 2n+1$. Then the line $(P_1 P_{4n+2})$ is tangent to $D^\vee$.*

3. A Poncelet theorem for lines

The following theorem is not a consequence nor of the well known Poncelet closure theorem (except when the configuration consists in two lines) neither of Darboux theorem (the configuration is not a Poncelet curve associated to D and described in [9]). We say that a polygon with $2n$ sides joining $2n$ vertices is well inscribed in a configuration $\mathcal{L}_n$ of n lines when each line of the configuration contains exactly two vertices.

Theorem 3.1. *Let $\mathcal{L}_n$ be a configuration of n lines and D a smooth conic in $\mathbb{P}^2$. If it exists a polygon with $2n$ sides well inscribed into $\mathcal{L}_n$ and circumscribed around D then there are infinitely many such polygons. In particular a general point in $\mathcal{L}_n$ is a vertex of such a polygon.*

Proof. The given polygon of $2n$ sides well inscribed into $\mathcal{L}_n$ and circumscribed to D corresponds by duality (polarity) to an inscribed polygon into D. It gives $2n$ points on D linked by $2n$ lines that are the polar lines of the considered $2n$ vertices in $\mathcal{L}_n$. These $2n$ lines meet two by two in n points $L_1^\vee, \cdots, L_n^\vee$ (poles of the n lines of the configuration).

Let us show that the product $v = (u_n \cdots u_1)$ of the n involutions $u_1, \cdots, u_n$ with respective centers $L_1^\vee, \cdots, L_n^\vee$ is involutive. Let x_1 be an intersection point of $L_1 \cap D$ and $x_2 = v(x_1)$. Following the sides of this inscribed polygon, we have $v^2(x_1) = x_2$. We have, in the same way, $v^2(x_2) = x_2$. Since the inscribed polygon has $2n$ vertices and not only n, these two fixed points of the automorphism v^2 do not coïncide; indeed they are exchanged by v. Let x be a fixed point of the product v. This point x is also a fixed point of v^2. Since x_1 and x_2 are exchanged by v they do not coïncide with x. It implies that v^2 has three fixed points, i.e. $v^2 = I$.

Then, a polygon constructed from a general point $p \in D$ by joining the vertices

$$\{p, u_1(p), (u_2 u_1)(p), \cdots, (u_n \cdots u_1)(p), \cdots, (u_{n-1} \cdots u_1 u_n \cdots u_1)(p)\}$$

is inscribed in D and corresponds by duality to an inscribed polygon in $\mathcal{L}_n$ circumscribed around D. □

3.1. **Poncelet theorem for singular conics.** Assume that $u_{2i} = u$ and $u_{2i+1} = v$ and let x and y be their respective centers. We can consider the two polar lines $L_x = x^\vee$ and $L_y = y^\vee$. Then theorem 3.1 (with $u_{2i} = u$ and $u_{2i+1} = v$) implies that $(uv)^n = I$ if and only if there exists a polygon with $2n$ sides inscribed in $L_x \cup L_y$ and circumscribed around D. In that case since an union of two lines is a conic it is a consequence of Poncelet theorem (see [10], thm.2.2).

This situation can be described in an elementary way. Since $\mathrm{PGL}(2, \mathbb{C})$ acts transitively on triplets of points in $\mathbb{P}^1$ we can choose three among the four fixed points of u and v. Then we will obtain a good matrix description. Let us introduce first a family of polynomials[3] on the affine line:

$P_0(x) = 1, P_1(x) = x$ and for $n \geq 2$, $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$.

We can give now a simple characterization for an union of two lines to be Poncelet associated to a smooth conic.

[3]Quite similar than Fibonacci polynomials.

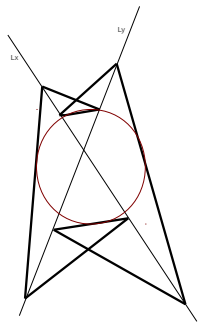

FIGURE 4. Octagon inscribed in $L_x \cup L_y$.

Proposition 3.2. *Let* $\{1, -1\}$ *be the fixed points of* u *and* $\{0, 2/x\}$ *be the fixed points of* v*. Then,*

$$(uv)^n = I \Leftrightarrow P_{n-1}(x) = 0 \text{ and } P_{n-2}(x) \neq 0.$$

Proof. Two representatives matrices of u and v are

$$M_u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M_v = \begin{pmatrix} 1 & 0 \\ x & -1 \end{pmatrix}.$$

The product is given by the matrix

$$(M_u M_v)^n = \begin{pmatrix} xP_{n-1}(x) - P_{n-2}(x) & -P_{n-1}(x) \\ P_{n-1}(x) & -P_{n-2}(x) \end{pmatrix}$$

and this matrix is a multiple of the matrix identity if and only if

$$P_{n-1}(x) = 0.$$

□

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