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# Uniqueness of large solutions

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## Abstract

Given a nondecreasing nonlinearity  $f$ , we prove uniqueness of large solutions to the equation (1) below, in the following two cases: the domain is the ball or the domain has nonnegative mean curvature and the nonlinearity is asymptotically convex.

## 1 Introduction

In this paper, we are interested in the so-called large solutions of a certain class of partial differential equations. Let us recall what they are: given  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 1$  and  $f \in C^1(\mathbb{R})$ , a large solution is a function  $u \in C^2(\Omega)$  satisfying

$$(1) \quad \begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where the boundary condition is understood in the sense that

$$\lim_{x \rightarrow x_0, x \in \Omega} u(x) = +\infty \quad \text{for all } x_0 \in \partial\Omega$$

and where  $f$  is assumed to be positive at infinity, in the sense that

$$(2) \quad \exists a \in \mathbb{R} \quad \text{s.t.} \quad f(a) > 0 \quad \text{and} \quad f(t) \geq 0 \quad \text{for } t > a.$$

When the boundary of  $\Omega$  is smooth enough, existence of a solution of (1) is equivalent to the so-called Keller-Osserman condition :

$$(3) \quad \int^{+\infty} \frac{dt}{\sqrt{F(t)}} < +\infty, \quad \text{where } F(t) = \int_a^t f(s) \, ds.$$

For a proof of this fact, see the seminal works of J.B. Keller [6] and R. Osserman [8] for the case of monotone  $f$ , as well as [4] for the general case. From here on, we always assume that (3) holds.

Uniqueness of solutions of (1) turns out to be delicate. As one might expect, it fails in the presence of oscillations. For example, if  $f(u) = u^2 \sin^2(u)$ , the equation has infinitely many solutions (see [4]). It is also known (see e.g. the remark on p. 325 in [13]) that uniqueness fails for a nonlinearity of the form  $f(u) = u^p$ ,  $p > 1$ , if the domain is not smooth enough:

**Proposition 1.1** *Assume that  $\Omega = B \setminus \{0\}$  is the punctured unit ball of  $\mathbb{R}^N$ ,  $N \geq 2$ . Let  $p \in (1, \frac{N}{N-2})$  if  $N \geq 3$  (respectively  $p \in (1, +\infty)$  if  $N = 2$ ) and  $f(u) = u^p$ . Then, there exists infinitely many solutions of (1).*

However, one could hope that uniqueness holds under the simple assumptions that  $f$  is a nondecreasing function and that  $\Omega$  has smooth boundary. As of today, this question remains open. In [3], we proved uniqueness in the case where  $\Omega$  is a ball.

**Theorem 1.2** ([3]) *Assume that  $\Omega$  is the unit ball in  $\mathbb{R}^N$ ,  $N \geq 1$ . Assume that  $f$  is a nondecreasing function such that (2) and (3) hold. Then, there exists a unique solution of (1).*

In this paper, we give a shorter proof of this fact. Under extra convexity assumptions, we obtain the following answer for a more general class of domains.

**Theorem 1.3** *Assume that  $\partial\Omega$  is of class  $C^3$  and that its mean curvature is nonnegative. Assume that  $f$  is a nondecreasing function such that (2) and (3) hold. Assume in addition that there exists  $M \in \mathbb{R}$  such that  $\sqrt{F}$  is convex in  $(M, +\infty)$ . Then, there exists a unique solution of (1).*

**Remark 1.4** *If  $f$  is asymptotically convex, then so is  $\sqrt{F}$ .*

Let us turn to the proofs.

## 2 Proof of Theorem 1.2

**Step 1.** Reduction to the radial case.

Assume  $\Omega$  is the ball. It is well-known (see e.g. Lemma 2.4 in [3]) that the equation has a minimal and a maximal solution, each of which is radial. That is, there exist two large radial solutions  $U_1, U_2$  such that any large solution  $u$  satisfies  $U_1 \leq u \leq U_2$ . In particular, it suffices to prove that  $U_1 \geq U_2$ .

**Step 2.** Let  $u$  be a large radial solution. There exists  $r_0 \in (0, 1)$  such that in  $(r_0, 1)$ ,  $u$  is strictly increasing and

$$(4) \quad \frac{1}{2N} F(u) \leq \left( \frac{du}{dr} \right)^2 \leq 4F(u)$$

This is essentially Keller's classical argument (see [6]): let  $u$  be a large radial solution. Using (2), it follows that for  $r$  close to 1,

$$(5) \quad r^{1-N} \frac{d}{dr} \left( r^{N-1} \frac{du}{dr} \right) = \Delta u = f(u) \geq 0.$$

Since  $u$  is unbounded, there exists  $r_1$  close to 1 such that  $du/dr(r_1) > 0$ . By (5),  $du/dr > 0$  in  $[r_1, 1)$ . Integrating (5), we also have for  $r \in (r_1, 1)$ ,

$$\begin{aligned} r^{N-1} \frac{du}{dr} &= r_1^{N-1} \frac{du}{dr}(r_1) + \int_{r_1}^r s^{N-1} f(u(s)) ds \\ &\leq r_1^{N-1} \frac{du}{dr}(r_1) + f(u(r)) \frac{r^N}{N}. \end{aligned}$$

Since  $f$  is nondecreasing and satisfies the Keller-Osserman condition (3),  $\lim_{+\infty} f = +\infty$ . Using this in the above, given  $\epsilon > 0$ , we find  $r_2 \in [r_1, 1)$  such that for  $r \in (r_2, 1)$ ,

$$\frac{1}{r} \frac{du}{dr} \leq \left( \frac{1}{N} + \epsilon \right) f(u)$$

Taking  $\epsilon = \frac{1}{2(N-1)}$  and recalling that

$$\frac{d^2 u}{dr^2} + \frac{N-1}{r} \frac{du}{dr} = f(u),$$

we deduce that

$$\frac{1}{2N} f(u) \leq \frac{d^2 u}{dr^2} \leq f(u) \quad \text{in } [r_2, 1).$$

Multiplying by  $2du/dr$ , integrating and letting  $c = du/dr(r_2)^2 - F(u(r_2))$ , we obtain

$$\frac{1}{N} F(u) + c \leq \left( \frac{du}{dr} \right)^2 \leq 2F(u) + c \quad \text{for } r \in [r_2, 1)$$

and so we find  $r_0 \in [r_2, 1)$  such that (4) holds in  $[r_0, 1)$ .

**Step 3.** Change of independent variable.

Thanks to Step 2, for  $r$  close to 1, given  $i \in \{1, 2\}$ , we may perform the change of variable  $u = U_i(r)$ . Let  $r = r_i(u)$  denote the inverse mapping of  $U_i$  and  $V_i = \frac{dU_i}{dr} \circ r_i$ . By the chain rule,

$$(6) \quad V_i \frac{dV_i}{du} + \frac{N-1}{r_i} V_i = f,$$

while  $dr_i/du = 1/V_i$ , so that

$$(7) \quad 1 - r_i = \int_u^{+\infty} \frac{1}{V_i} du'.$$

**Step 4.** There exists  $u_0 > 0$  such that  $r_1 \geq r_2$  and  $V_1 \geq V_2$  in  $[u_0, +\infty)$ .

Since  $r_i$  is the inverse mapping of  $U_i$  and  $U_1 \leq U_2$ , we have  $r_1 \geq r_2$ . By (6), the function  $z = V_2 - V_1$  satisfies

$$\frac{dz}{du} + (N-1) \left\{ \frac{1}{r_2} - \frac{1}{r_1} \right\} = \left( \frac{1}{V_2} - \frac{1}{V_1} \right) f = -\frac{f}{V_1 V_2} z.$$

Since  $r_1 \geq r_2$ , we deduce that  $w$  satisfies the differential inequality

$$(8) \quad \frac{dz}{du} + az \leq 0,$$

where  $a = \frac{f}{V_1 V_2} \geq 0$  for large  $u$ . By (7), we also have

$$\int_u^{+\infty} \frac{1}{V_2} du' \geq \int_u^{+\infty} \frac{1}{V_1} du'.$$

So, there must exist  $u_0$  such that  $1/V_2(u_0) \geq 1/V_1(u_0)$  i.e.  $w(u_0) \leq 0$ . Using this together with (8), we deduce that  $z \leq 0$  in  $[u_0, +\infty)$ , as desired.

**Step 5.** The function  $w = r_1^{2N-2} V_1^2 - r_2^{2N-2} V_2^2$  is bounded.

To see this, observe first that

$$(9) \quad \frac{dw}{du} = 2(r_1^{2N-2} - r_2^{2N-2})f.$$

Hence,  $w$  is a nonnegative nondecreasing function and

$$\frac{dw}{du} \leq 4(N-1)(r_1 - r_2)f = 4(N-1) \left( \int_u^{+\infty} \left( \frac{1}{V_2} - \frac{1}{V_1} \right) du' \right) f$$

Now, if  $u_0$  is chosen so large that  $\frac{1}{2} \leq r_2$  in  $[u_0, +\infty)$ ,

$$(10) \quad \frac{1}{V_2} - \frac{1}{V_1} = \frac{V_1^2 - V_2^2}{V_1 V_2 (V_1 + V_2)} \leq \frac{2^{2N-2} w}{V_1 V_2 (V_1 + V_2)}.$$

Integrating (9) and using (4), it follows that for  $u \geq u_0$ ,

$$w(u) \leq w(u_0) + C(N) \int_{u_0}^u \left( \int_{u'}^{+\infty} \frac{w}{F^{\frac{3}{2}}} du'' \right) f du'.$$

Integrating by parts

$$w(u) \leq w(u_0) + C(N) \left( F(u) \int_u^{+\infty} \frac{w}{F^{\frac{3}{2}}} du' + \int_{u_0}^u \frac{w}{F^{\frac{1}{2}}} du' \right).$$

Thanks to the Keller-Osserman condition (3), if  $u_0$  is chosen large enough,

$$\int_{u_0}^u \frac{w}{F^{\frac{1}{2}}} du' \leq w(u) \int_{u_0}^{+\infty} \frac{1}{\sqrt{F}} \leq \frac{1}{2C(N)} w(u).$$

We have then obtained

$$(11) \quad w(u) \leq 2w(u_0) + 2C(N)F(u) \int_u^{+\infty} \frac{w}{F^{\frac{3}{2}}} du'.$$

Introduce  $G(u) = \int_u^{+\infty} \frac{w}{F^{\frac{3}{2}}} du'$ . Thanks to (4) and (3), we have  $G(+\infty) = 0$ .

In addition, letting  $c = 2C(N)$ , (11) can be rewritten as

$$-\frac{dG}{du} \leq \frac{2w(u_0)}{F^{\frac{3}{2}}} + \frac{c}{F^{\frac{1}{2}}} G.$$

That is,

$$-\frac{d}{du} \left( G \exp \left( -c \int_u^{+\infty} \frac{1}{\sqrt{F}} du' \right) \right) \leq \frac{2w(u_0)}{F^{\frac{3}{2}}} \exp \left( -c \int_u^{+\infty} \frac{1}{\sqrt{F}} du' \right) \leq \frac{2w(u_0)}{F^{\frac{3}{2}}}.$$

Integrating between  $u$  and  $+\infty$ , we then obtain, using once again (3),

$$G(u) \leq C \int_u^{+\infty} \frac{1}{F^{\frac{3}{2}}} = o\left(\frac{1}{F}\right).$$

Going back to (11), we deduce that  $w$  is bounded above.

**Step 6.** The difference  $U_2(r) - U_1(r)$  converges to 0 as  $r \rightarrow 1$ .

Given  $r$  close to 1 and  $i \in \{1, 2\}$ , let  $u_i = U_i(r)$ . Then,

$$\int_{u_1}^{+\infty} \frac{1}{V_1} du = 1 - r = \int_{u_2}^{+\infty} \frac{1}{V_2} du.$$

That is,

$$\int_{u_1}^{u_2} \frac{1}{V_1} du = \int_{u_2}^{+\infty} \left( \frac{1}{V_2} - \frac{1}{V_1} \right) du.$$

Using (10), (4), and the previous step, we deduce that

$$\int_{u_1}^{u_2} \frac{1}{\sqrt{F}} du \leq C \int_{u_2}^{+\infty} \frac{1}{F^{3/2}} du.$$

It follows that

$$0 \leq \frac{u_2 - u_1}{\sqrt{F(u_2)}} \leq \frac{C}{\sqrt{F(u_2)}} \int_{u_2}^{+\infty} \frac{1}{F} du$$

and the claim follows promptly.

**Step 7.** End of proof.

Let  $w = U_2 - U_1$ . Since  $U_2 \geq U_1$  and  $f$  is nondecreasing, we see from the previous step that

$$\begin{cases} \Delta w = f(U_2) - f(U_1) \geq 0 & \text{in } B, \\ w = 0 & \text{on } \partial B. \end{cases}$$

By the maximum principle,  $w \leq 0$  in  $B$ , as desired.

### 3 Proof of Theorem 1.3

Take a solution  $u$  to (1). Let  $a$  be the constant appearing in (2),  $M$  the constant beyond which  $\sqrt{F}$  is convex, and fix  $\tilde{M} > \max(0, a, M)$ . Fix  $\varepsilon > 0$  so small that  $u > \tilde{M}$  in  $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$ .

**Step 1.** We begin by proving that there exists a sequence of functions  $(u_N)_{N \in \mathbb{N}}$  solving

$$(12) \quad \begin{cases} \Delta u_N = f(u_N) & \text{in } \Omega_\varepsilon, \\ u_N = N & \text{on } \partial\Omega, \\ u_N = u & \text{on } \{x \in \Omega : \text{dist}(x, \partial\Omega) = \varepsilon\}, \end{cases}$$

such that

$$(13) \quad 0 \leq u_N \leq u \quad \text{in } \Omega_\varepsilon.$$

We may always assume that  $f(0) = 0$ .\* In particular,  $\underline{u} = 0$  and  $\bar{u} = u$  are respectively a sub and supersolution of (12) and they are ordered. It follows that there exists a solution  $u_N$  to (12) such that (13) holds.

A standard application of the maximum principle shows that  $u_N$  is the unique solution to (12) and that  $(u_N)$  is a nondecreasing sequence. Thanks to (13) and elliptic regularity, we may also assert that  $(u_N)$  converges in  $C_{loc}^2(\bar{\Omega}_\varepsilon \setminus \partial\Omega)$  to a function  $\tilde{u}$  solving

$$(14) \quad \begin{cases} \Delta \tilde{u} = f(\tilde{u}) & \text{in } \Omega_\varepsilon, \\ \tilde{u} = +\infty & \text{on } \partial\Omega, \\ \tilde{u} = u & \text{on } \{x \in \Omega : \text{dist}(x, \partial\Omega) = \varepsilon\}, \end{cases}$$

**Step 2.** There holds

$$(15) \quad |\nabla u_N|^2 - 2F(u_N) \leq M_N \quad \text{in } \Omega_\varepsilon,$$

where

$$(16) \quad M_N = \sup_{\text{dist}(x, \partial\Omega) = \varepsilon} [|\nabla u_N|^2 - 2F(u_N)].$$

The proof is a straightforward adaptation of an argument due to Bandle and Marcus ([1]), which uses the method of  $P$ -functions. We give the full argument here for convenience of the reader. Let

$$P_N = |\nabla u_N|^2 - 2F(u_N).$$

By a result of Payne and Stackgold ([9], see also Chapter 5 in [10]), there exists a bounded continuous vector field  $A$ , such that

$$\Delta P_N - \frac{A \cdot \nabla P_N}{|\nabla u_N|^2} \geq 0$$

at every point in  $\Omega_\varepsilon$  where  $\nabla u_N \neq 0$ . Hence,  $P_N$  attains its maximum over  $\bar{\Omega}_\varepsilon$  either on  $\partial\Omega$ , on  $\{x \in \Omega : \text{dist}(x, \partial\Omega) = \varepsilon\}$ , or at a critical point of  $u_N$ . It only remains to prove that the first case cannot occur. We claim that  $\partial P_N / \partial n \leq 0$  on  $\partial\Omega$ , where  $n$  is the outward unit normal to  $\partial\Omega$ . The boundary-point lemma then implies that  $P_N$  cannot attain its maximum on  $\partial\Omega$ . It remains to prove our claim. Observe that since  $u_N$  is constant on  $\partial\Omega$ ,  $|\nabla u_N| = \partial u_N / \partial n$  on  $\partial\Omega$ . Hence,

$$\frac{\partial P_N}{\partial n} = 2 \frac{\partial u_N}{\partial n} \frac{\partial^2 u_N}{\partial n^2} - 2f(u_N) \frac{\partial u_N}{\partial n}, \quad \text{on } \partial\Omega.$$

Furthermore, letting  $H$  denote the mean curvature of  $\partial\Omega$ ,

$$\Delta u_N = \frac{\partial^2 u_N}{\partial n^2} + (N-1)H \frac{\partial u_N}{\partial n} \quad \text{on } \partial\Omega.$$

Since  $\partial u_N / \partial n > 0$  and  $H \geq 0$ , this implies that

$$\frac{\partial^2 u_N}{\partial n^2} - f(u_N) \leq 0$$

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\*If  $f(0) \neq 0$ , work with any nondecreasing  $C^1$  function  $\tilde{f}$  such that  $\tilde{f}(0) = 0$ ,  $\tilde{f} = f$  on  $[\tilde{M}, +\infty)$ .

and consequently  $\partial P_N / \partial n \leq 0$ , as desired. We have just proved (15).

**Step 3.** The function  $\tilde{u} = \lim_{N \rightarrow +\infty} u_N$  coincides with  $u$  in  $\Omega_\varepsilon$ .

The proof of this fact bears resemblances with a trick due to L. Nirenberg given in [2]. By (13), we already have  $\tilde{u} \leq u$  in  $\Omega_\varepsilon$  and it remains to prove the reverse inequality. Thanks to (13) and elliptic regularity, there exists a constant  $M$  such that

$$2M \geq M_N,$$

where  $M_N$  is given by (16). Now let  $\tilde{F} = F + M$  and define

$$v_N = \int_{u_N}^{+\infty} \frac{dt}{\sqrt{2\tilde{F}(t)}}.$$

Then, (15) can be rewritten as

$$|\nabla v_N| \leq 1 \quad \text{in } \Omega_\varepsilon$$

from which it easily follows that

$$(17) \quad |\nabla \tilde{v}| \leq 1 \quad \text{in } \Omega_\varepsilon,$$

where we defined similarly

$$\tilde{v} = \int_{\tilde{u}}^{+\infty} \frac{dt}{\sqrt{2\tilde{F}(t)}}.$$

Let at last

$$v = \int_u^{+\infty} \frac{dt}{\sqrt{2\tilde{F}(t)}}.$$

It remains to prove that  $u \leq \tilde{u}$  i.e.  $\tilde{v} \leq v$  in  $\Omega_\varepsilon$ . Using the equations satisfied by  $u$  and  $\tilde{u}$ , we see that  $w = v - \tilde{v}$  solves

$$\begin{aligned} -\Delta w &= \frac{f}{\sqrt{2\tilde{F}}}(u) (1 - |\nabla v|^2) - \frac{f}{\sqrt{2\tilde{F}}}(\tilde{u}) (1 - |\nabla \tilde{v}|^2) \\ &= \left[ \frac{f}{\sqrt{2\tilde{F}}}(u) - \frac{f}{\sqrt{2\tilde{F}}}(\tilde{u}) \right] (1 - |\nabla \tilde{v}|^2) + \frac{f}{\sqrt{2\tilde{F}}}(u) (|\nabla \tilde{v}|^2 - |\nabla v|^2) \end{aligned}$$

Since  $\sqrt{2F}$  is convex,  $\frac{f}{\sqrt{2\tilde{F}}}$  is nondecreasing. Using this and (17), we deduce that

$$\begin{cases} -\Delta w + b(x) \cdot \nabla w \geq 0, & \text{in } \Omega_\varepsilon \\ w = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where  $b(x) = \frac{f}{\sqrt{2\tilde{F}}}(u) \nabla(v + \tilde{v})$  is locally bounded in  $\Omega$ . We may now apply the maximum principle to conclude that  $w \geq 0$  in  $\Omega$ , as desired.

**Step 4.** End of proof. The rest of the proof is similar to an argument due to Garcia-Melian [5]. We take two arbitrary solutions  $u, \bar{u}$  of our equation



(1). We let  $u_N, \bar{u}_N$  be the corresponding solutions to the approximated problem (12). In particular,  $w_N = u_N - \bar{u}_N$  solves

$$(18) \quad \begin{cases} \Delta w_N = f(u_N) - f(\bar{u}_N) & \text{in } \Omega_\varepsilon, \\ w_N = 0 & \text{on } \partial\Omega, \\ w_N = u - \bar{u} & \text{on } \{x \in \Omega : \text{dist}(x, \partial\Omega) = \varepsilon\}, \end{cases}$$

By the maximum principle,

$$w_N \leq \sup_{\text{dist}(x, \partial\Omega) = \varepsilon} (u - \bar{u}) \quad \text{in } \Omega_\varepsilon,$$

with equality at some point  $x_N$  such that  $\text{dist}(x_N, \partial\Omega) = \varepsilon$ . Extracting a sequence if necessary, we deduce that  $w = u - \bar{u}$  satisfies

$$(19) \quad w \leq \sup_{\text{dist}(x, \partial\Omega) = \varepsilon} (u - \bar{u}) \quad \text{in } \Omega_\varepsilon,$$

with equality at some point  $z$  such that  $\text{dist}(z, \partial\Omega) = \varepsilon$ . Now, we also have

$$\begin{cases} \Delta w = f(u) - f(\bar{u}) & \text{in } \Omega \setminus \Omega_\varepsilon, \\ w = u - \bar{u} & \text{on } \{x \in \Omega : \text{dist}(x, \partial\Omega) = \varepsilon\}. \end{cases}$$

By the maximum principle, we deduce that inequality (19) holds throughout  $\Omega$ , with equality at the point  $z$ . The strong maximum principle implies that  $w$  is equal to a constant  $c$ . Since  $u, \bar{u}$  solve (1), we deduce that  $f(u) = f(u + c)$ , which is possible only if  $c = 0$ .  $\square$

## 4 Proof of Proposition 1.1

We thank Laurent Véron ([11]) for the following proof. Given  $p \in (1, N/(N-2))$ ,  $k \in \mathbb{N}$  and  $\lambda > 0$ , we begin by solving

$$(20) \quad \begin{cases} -\Delta u + u^p = \lambda \delta_0 & \text{in } B, \\ u = k & \text{on } \partial B, \end{cases}$$

Since 0 is a subsolution, while a large constant multiple of the fundamental solution is a supersolution, we deduce from the method of sub and supersolution (see e.g. [7] for the appropriate statement) that there exists a solution  $u = u_k$  to (20). By the maximum principle,  $u_k$  is the unique solution to (20), and the sequence  $(u_k)$  is nondecreasing. Thanks to the Keller-Osserman estimate (see e.g. [6]), the sequence  $(u_k)$  is uniformly bounded on compact subsets of the punctured ball  $B \setminus \{0\}$ . It follows from elliptic regularity that  $u_k$  converges to a solution  $u = u_\lambda$  of

$$\begin{cases} -\Delta u + u^p = \lambda \delta_0 & \text{in } B, \\ u = +\infty & \text{on } \partial B, \end{cases}$$

By the results of [12],  $u_\lambda$  behaves like a constant multiple of the fundamental solution near the origin. In particular, each  $u_\lambda$  is a large solution in the punctured ball.

There exists yet another large solution. Simply note that for an appropriate constant  $c = c(N, p) > 0$ , the function  $u_1(x) = c|x|^{-2/(p-1)}$  solves  $\Delta u = u^p$  in  $\mathbb{R}^N \setminus \{0\}$ . Let also  $u_2$  be the unique solution to

$$\begin{cases} \Delta u = u^p & \text{in } B, \\ u = +\infty & \text{on } \partial B, \end{cases}$$

Then,  $\underline{u} = \max(u_1, u_2)$  and  $\overline{u} = u_1 + u_2$  form an ordered pair of sub and supersolution to the equation in the punctured ball. The method of sub and supersolutions implies the existence of a new large solution  $u_\infty$  which behaves like  $c|x|^{-2/(p-1)}$  near the origin, hence distinct from  $u_\lambda$ .

Finally, observe that for the nonlinearity  $f(u) = u^p$ , if  $u$  is a large solution and  $\epsilon > 0$ , then  $(1 + \epsilon)u$  is a supersolution. From this, the classification of singularities both at the origin (see [12]) and on the boundary (see e.g. [1]), and the maximum principle, it easily follows that the set of positive large solutions in the punctured ball is exactly  $\{u_\lambda\}_{\lambda \in (0, +\infty]}$ .

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## References

- [1] Catherine Bandle and Moshe Marcus, “Large” solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. Anal. Math. **58** (1992), 9–24, DOI 10.1007/BF02790355. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. MR1226934 (94c:35081)
- [2] Haïm Brezis and Shoshana Kamin, Sublinear elliptic equations in  $\mathbf{R}^n$ , Manuscripta Math. **74** (1992), no. 1, 87–106, DOI 10.1007/BF02567660. MR1141779 (93f:35062)
- [3] O. Costin and L. Dupaigne, Boundary blow-up solutions in the unit ball: asymptotics, uniqueness and symmetry, J. Differential Equations **249** (2010), no. 4, 931–964, DOI 10.1016/j.jde.2010.02.023. MR2652158 (2011e:35125)
- [4] Serge Dumont, Louis Dupaigne, Olivier Goubet, and Vicentiu Rădulescu, Back to the Keller-Osserman condition for boundary blow-up solutions, Adv. Nonlinear Stud. **7** (2007), no. 2, 271–298. MR2308040 (2008e:35062)
- [5] Jorge García-Melián, Uniqueness of positive solutions for a boundary blow-up problem, J. Math. Anal. Appl. **360** (2009), no. 2, 530–536, DOI 10.1016/j.jmaa.2009.06.077. MR2561251 (2011a:35174)
- [6] J. B. Keller, On solutions of  $\Delta u = f(u)$ , Comm. Pure Appl. Math. **10** (1957), 503–510. MR0091407 (19,964c)
- [7] Marcelo Montenegro and Augusto C. Ponce, The sub-supersolution method for weak solutions, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2429–2438, DOI 10.1090/S0002-9939-08-09231-9. MR2390510 (2010h:35160)
- [8] Robert Osserman, On the inequality  $\Delta u \geq f(u)$ , Pacific J. Math. **7** (1957), 1641–1647. MR0098239 (20 #4701)

- [9] L. E. Payne and Ivar Stakgold, “*Nonlinear problems in nuclear reactor analysis*” in *Nonlinear problems in the physical sciences and biology* (Ivar Stakgold, Daniel D. Joseph, and David H. Sattinger, eds.), Lecture Notes in Mathematics, Vol. 322, Springer-Verlag, Berlin, 1973. MR0371548 (51 #7766)
- [10] René P. Sperb, *Maximum principles and their applications*, Mathematics in Science and Engineering, vol. 157, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981. MR615561 (84a:35033)
- [11] Laurent Véron, *Personal communication*.
- [12] ———, *Solutions singulières d’équations elliptiques semilinéaires*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 18, A867–A869 (French, with English summary). MR538992 (80h:35038)
- [13] ———, *Generalized boundary value problems for nonlinear elliptic equations*, Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000), Electron. J. Differ. Equ. Conf., vol. 6, Southwest Texas State Univ., San Marcos, TX, 2001, pp. 313–342 (electronic). MR1804784 (2001j:35099)