



**HAL**  
open science

# Nonlinear Convection in Reaction-diffusion Equations under dynamical boundary conditions

Gaëlle Pincet Mailly, Jean-François Rault

► **To cite this version:**

Gaëlle Pincet Mailly, Jean-François Rault. Nonlinear Convection in Reaction-diffusion Equations under dynamical boundary conditions. 2011. hal-00668330v1

**HAL Id: hal-00668330**

**<https://hal.science/hal-00668330v1>**

Preprint submitted on 10 Feb 2012 (v1), last revised 25 Sep 2012 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Nonlinear Convection in Reaction-diffusion Equations under dynamical boundary conditions

Gaëlle Pincet Mailly<sup>a</sup>, Jean-François Rault<sup>a,\*</sup>

<sup>a</sup>*LMPA Joseph Liouville FR 2956 CNRS, Université Lille Nord de France  
50 rue F. Buisson, B.P. 699, F-62228 Calais Cedex, France  
(+33)3 21 46 56 50*

---

## Abstract

We investigate blow-up phenomena for positive solutions of nonlinear reaction-diffusion equations including a nonlinear convection term  $\partial_t u = \Delta u - g(u) \cdot \nabla u + f(u)$  in a bounded domain of  $\mathbb{R}^N$  under the dissipative dynamical boundary conditions  $\sigma \partial_t u + \partial_\nu u = 0$ . Some conditions on  $g$  and  $f$  are discussed to state if the positive solutions blow up in finite time or not. Moreover, for certain classes of nonlinearities, an upper-bound for the blow-up time can be derived and the blow-up rate can be determined.

*Keywords:* Nonlinear parabolic problem, Dynamical boundary conditions, Lower and upper-solution, Blow-up, Global solution

*2010 MSC:* 35K55, 35B44

---

## Introduction

We consider the following nonlinear parabolic problem

$$\begin{cases} \partial_t u = \Delta u - g(u) \cdot \nabla u + f(u) & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 \geq 0 & \text{in } \overline{\Omega}, \end{cases} \quad (1)$$

where  $g : \mathbb{R} \mapsto \mathbb{R}^N$ ,  $f : \mathbb{R} \mapsto \mathbb{R}$ ,  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  with  $\mathcal{C}^2$ -boundary  $\partial\Omega$ . We denote by  $\nu : \partial\Omega \mapsto \mathbb{R}^N$  the outer unit normal vector

---

\*Corresponding author

*Email addresses:* [mailly@lmpa.univ-littoral.fr](mailto:mailly@lmpa.univ-littoral.fr) (Gaëlle Pincet Mailly),  
[jfrault@lmpa.univ-littoral.fr](mailto:jfrault@lmpa.univ-littoral.fr) (Jean-François Rault)

field, and by  $\partial_\nu$  the outer normal derivative.

These equations arise in different areas, especially in population growth, chemical reactions and heat conduction. For instance, in the case of a heat transfer in a medium  $\Omega$ , the first equation  $\partial_t u = \Delta u - g(u) \cdot \nabla u + f(u)$  is a heat equation including a nonlinear convection term  $g(u) \cdot \nabla u$  and a nonlinear source  $f$ . On the boundary  $\partial\Omega$ , if  $\sigma$  is positive, the dynamical boundary conditions describe the fact that a heat wave with the propagation speed  $\frac{1}{\sigma}$  is sent into the region into an infinitesimal layer near the boundary due to the heat flux across the boundary (see [6] and [11]).

There are various results in the literature about the theory of blow-up for semilinear parabolic equations, in particular for reaction-diffusion equations, see e.g. [9], [10], [12] and [8]. In this work, we discuss a problem involving a nonlinear convection term. Whereas a Burgers' equation has been studied in [5] in the one-dimensional case, we now consider a more general convection term and we set in a regular domain of  $\mathbb{R}^N$ . After recalling some qualitative properties in Section 1, we construct a global upper-solution for Problem (1) in Section 2 and we deduce some conditions on  $f$  and  $g$  guaranteeing global existence of the solutions (Theorem 4). In Section 3, we investigate two methods to ensure the blow-up of solutions of Problem (1). The first one is an eigenfunction method applied to the model problem

$$\begin{cases} \partial_t u = \Delta u - g(u) \cdot \nabla u + u^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega}, \end{cases} \quad (2)$$

with  $p > 1$  (Theorem 10). We also derive some upper bounds for the blow-up time. In the second method, we construct a self-similar lower-solution of Problem (1) which blows up in finite time. We prove the blow-up of solutions of the following problem

$$\begin{cases} \partial_t u = \Delta u - g(u) \cdot \nabla u + e^{pu} & \text{in } \overline{\Omega} \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega}, \end{cases} \quad (3)$$

with  $p > 0$ . Finally, in Section 4, we determine the blow-up rate of the solutions of Problem (2) in the  $L^\infty$ -norm when approaching the blow-up time (Theorem 16).

Throughout, we shall assume the dissipativity condition

$$\sigma \geq 0 \text{ on } \partial\Omega \times (0, \infty). \quad (4)$$

In order to deal with classical solutions, we always assume that the parameters in the equations of Problem (1) are smooth

$$\sigma \in \mathcal{C}^1(\partial\Omega \times (0, \infty)), \quad (5)$$

$$f \in \mathcal{C}^1(\mathbb{R}), \quad f \geq 0, \quad (6)$$

and

$$g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^N). \quad (7)$$

The initial data is continuous, non-trivial and non-negative in  $\overline{\Omega}$

$$u_0 \in \mathcal{C}(\overline{\Omega}), \quad u_0 \not\equiv 0, \quad u_0 \geq 0. \quad (8)$$

Let  $T = T(\sigma, u_0)$  denote the maximal existence time of the unique maximal classical solution of Problem (1)

$$u_\sigma \in \mathcal{C}(\overline{\Omega} \times [0, T)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0, T))$$

with the coefficient  $\sigma$  in the boundary conditions and the initial data  $u_0$ . As for the well-posedness and the local existence of the solutions of Problem (1), we refer to [2], [6] and [7]. From [6], since the convection term depends linearly on the gradient  $\nabla u$  of the solution, the maximal existence time  $T$  is the blow-up time of the solution with respect to the  $L^\infty$ -norm:

$$T = \inf \left\{ s > 0 \mid \limsup_{t \nearrow s} \sup_{\overline{\Omega}} |u(x, t)| = \infty \right\}$$

## 1. Qualitative properties

The aim of this section is to compare the solutions for different parameters  $\sigma$  and initial data  $u_0$  and to summarize some positivity results on the classical solutions of Problem (1). Let  $0 \leq \sigma_1 \leq \sigma_2$  be two coefficients satisfying condition (5),  $v_0 \leq u_0$  be two initial data fulfilling hypothesis (8) and  $w_0$  a function in  $\mathcal{C}_0(\overline{\Omega})$  with  $0 \leq w_0 \leq v_0$ . Denote by  $u_{\sigma_1}$ ,  $u_{\sigma_2}$ ,  $v$  and  $w$  the maximal solutions of the following problems

$$\begin{cases} \partial_t u_{\sigma_1} = \Delta u_{\sigma_1} - g(u_{\sigma_1}) \cdot \nabla u_{\sigma_1} + f(u_{\sigma_1}) & \text{in } \Omega \text{ for } t > 0, \\ \sigma_1 \partial_t u_{\sigma_1} + \partial_\nu u_{\sigma_1} = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u_{\sigma_1}(\cdot, 0) = u_0 & \text{in } \overline{\Omega}, \end{cases}$$

$$\begin{cases} \partial_t u_{\sigma_2} = \Delta u_{\sigma_2} - g(u_{\sigma_2}) \cdot \nabla u_{\sigma_2} + f(u_{\sigma_2}) & \text{in } \Omega \text{ for } t > 0, \\ \sigma_2 \partial_t u_{\sigma_2} + \partial_\nu u_{\sigma_2} = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u_{\sigma_2}(\cdot, 0) = u_0 & \text{in } \overline{\Omega}, \end{cases}$$

$$\begin{cases} \partial_t v = \Delta v - g(v) \cdot \nabla v + f(v) & \text{in } \Omega \text{ for } t > 0, \\ \sigma_2 \partial_t v + \partial_\nu v = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ v(\cdot, 0) = v_0 & \text{in } \overline{\Omega}, \end{cases}$$

and

$$\begin{cases} \partial_t w = \Delta w - g(w) \cdot \nabla w + f(w) & \text{in } \Omega \text{ for } t > 0, \\ w = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ w(\cdot, 0) = w_0 & \text{in } \overline{\Omega}. \end{cases}$$

Let  $T(\sigma_1, u_0)$ ,  $T(\sigma_2, u_0)$ ,  $T(\sigma_2, v_0)$  and  $T(w_0)$  be their respective maximal existence times. For the reader convenience, we recall some results stemming from the comparison principle [2].

**Theorem 1** ([4]). *Under the aforementioned hypotheses, we have*

$$T(\sigma_2, u_0) \leq T(\sigma_2, v_0) \leq T(w_0)$$

and

$$0 \leq w \leq v \leq u_{\sigma_2} \text{ in } \overline{\Omega} \times [0, T(\sigma_2, u_0)].$$

In addition, if  $u_0 \in \mathcal{C}^2(\overline{\Omega})$  with

$$\Delta u_0 - g(u_0) \cdot \nabla u_0 + f(u_0) \geq 0 \text{ in } \Omega, \quad (9)$$

we have

$$T(\sigma_1, u_0) \leq T(\sigma_2, u_0)$$

and

$$u_{\sigma_2} \leq u_{\sigma_1} \text{ in } \overline{\Omega} \times [0, T(\sigma_1, u_0)].$$

Now, using the maximum principle from [2], we extend some results obtained in [3] in the case of reaction-diffusion to our problem with convection.

**Theorem 2.** *Assume hypotheses (4) - (9). Suppose that  $f > 0$  in  $(0, \infty)$  and that  $\sigma$  does not depend on time*

$$\sigma \in \mathcal{C}^1(\partial\Omega). \quad (10)$$

Then the solution  $u$  of Problem (1) satisfies

$$\begin{aligned} u &> 0 \quad \text{in } \overline{\Omega} \times (0, T(\sigma, u_0)), \\ \partial_t u &\geq 0 \quad \text{in } \overline{\Omega} \times [0, T(\sigma, u_0)), \\ \partial_t u &> 0 \quad \text{in } \overline{\Omega} \times (0, T(\sigma, u_0)). \end{aligned}$$

Moreover, for all  $\xi \in (0, T(\sigma, u_0))$ , there exists  $d > 0$  such that

$$\partial_t u > d \quad \text{in } \overline{\Omega} \times [\xi, T(\sigma, u_0)).$$

PROOF. Let  $\tau \in (0, T(\sigma, u_0))$ . Since  $u$  is  $\mathcal{C}^{2,1}(\overline{\Omega} \times [0, \tau])$  and because  $f$  and  $g$  are smooth ((6) and (7)), we can define these constants

$$C = \sup_{\overline{\Omega} \times [0, \tau]} g(u) \quad \text{and} \quad M = \sup_{\overline{\Omega} \times [0, \tau]} g'(u) \cdot \nabla u - f'(u).$$

First, the comparison principle from [2] applied to Problem (1) implies  $u \geq 0$  in  $\overline{\Omega} \times [0, \tau]$  since  $f \geq 0$  by condition (6). Thus we obtain

$$\begin{cases} \partial_t u \geq \Delta u - g(u) \cdot \nabla u \geq \Delta u - C|\nabla u| & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega}. \end{cases}$$

The strong maximum principle from [2] implies

$$m := \min_{\overline{\Omega} \times [0, \tau]} u = \min_{\overline{\Omega}} u_0,$$

and if this minimum  $m$  is attained in  $\overline{\Omega} \times (0, \tau]$ ,  $u \equiv m$  in  $\overline{\Omega} \times [0, \tau]$ . Since  $f > 0$  in  $(0, \infty)$ , the first equation in Problem (1) leads to  $m = 0$ , and we obtain  $u_0 \equiv 0$ , a contradiction with equation (8). Hence  $u > m \geq 0$  in  $\overline{\Omega} \times (0, \tau]$ .

Then, since the coefficients in the equations of Problem (1) are sufficiently smooth, classical regularity results in [13] imply that  $u \in \mathcal{C}^{2,2}(\overline{\Omega} \times [0, \tau])$ . Thus  $y = \partial_t u \in \mathcal{C}^{2,1}(\overline{\Omega} \times [0, \tau])$  and satisfies

$$\begin{cases} \partial_t y = \Delta y - g(u) \cdot \nabla y - (g'(u) \cdot \nabla u)y + f'(u)y & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t y + \partial_\nu y = 0 & \text{on } \partial\Omega \text{ for } t > 0. \end{cases}$$

By continuity, condition (9) implies  $y(\cdot, 0) \geq 0$  in  $\overline{\Omega}$ . The comparison principle from [2] implies  $y \geq 0$  in  $\overline{\Omega} \times [0, \tau]$ . In order to apply properly the strong

maximum principle, we have to introduce  $w = ye^{Mt} \geq 0$ . By definition of  $C$  and  $M$ , we obtain

$$\begin{cases} \partial_t w \geq \Delta w - g(u) \cdot \nabla w \geq \Delta w - C|\nabla w| & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t w + \partial_\nu w \geq 0 & \text{on } \partial\Omega \text{ for } t > 0. \end{cases}$$

Again, the strong maximum principle from [2] implies

$$\tilde{m} := \min_{\overline{\Omega} \times [0, \tau]} w = \min_{\overline{\Omega}} w(\cdot, 0),$$

and if this minimum  $\tilde{m}$  is attained in  $\overline{\Omega} \times (0, \tau]$ ,  $w \equiv \tilde{m}$  in  $\overline{\Omega} \times [0, \tau]$ . In particular, if  $\tilde{m} = 0$ , we have  $\partial_t u \equiv 0$  in  $\overline{\Omega} \times [0, \tau]$ , thus  $u(\cdot, t) = u_0$  for all  $t \in [0, \tau]$ . Hence  $u$  attains its minimum in  $\overline{\Omega} \times (0, \tau]$ , which is impossible due to the first part of the proof. Thus  $w$  and  $\partial_t u$  are positive in  $\overline{\Omega} \times (0, \tau]$ . Finally, let  $\xi \in (0, \tau)$ . Because  $y$  is continuous and thanks to the previous point, there exists  $d > 0$  such that  $y(\cdot, \xi) > d$  in  $\overline{\Omega}$ . As  $y$  satisfies

$$\begin{cases} \partial_t y = \Delta y - g(u) \cdot \nabla y - \left( g'(u) \cdot \nabla u + f'(u) \right) y & \text{in } \Omega \times [\xi, \tau], \\ \sigma \partial_t y + \partial_\nu y = 0 & \text{on } \partial\Omega \times [\xi, \tau], \end{cases}$$

the weak maximum principle from [2] implies

$$\min_{\overline{\Omega} \times [0, \tau]} y = \min_{\overline{\Omega}} y(\cdot, 0).$$

Hence  $y > d$  in  $\overline{\Omega} \times [\xi, \tau]$ . Note that  $d$  depends only on  $\xi$ , not on  $\tau$ . Without this step, we only have  $y \geq \tilde{m}e^{-M\tau}$  which may vanish as  $\tau \rightarrow T(\sigma, u_0)$ .  $\square$

An important fact comes from the last statement of the previous theorem. For any positive solution  $u$  of Problem (1), the maximum principle implies that for any  $s \in (0, T(\sigma, u_0))$ , there exists  $c > 0$  such that  $u(\cdot, s) \geq c$  in  $\overline{\Omega}$ . Then, consider the solution  $\tilde{u}$  of (1) with the constant initial data  $c$  and the same  $\sigma$  in the boundary conditions. Theorem 1 implies  $\tilde{u} \leq u$ . Since  $c$  satisfies equation (9), Theorem 2 leads to  $\partial_t \tilde{u} > d > 0$ . Thus,  $\tilde{u}$  can be big enough after a long time (maybe it blows up). So does  $u$ , even if  $u_0$  does not satisfy condition (9).

## 2. Global existence

In this section, we give some conditions on the function  $g$  in the convection term, which ensure global existence of the solutions of Problem (1) for various reaction terms  $f$ . We use the comparison method from [2]. Thus, we just need to find an appropriate upper-solution of Problem (1) which does not blow up. This is our first lemma.

**Lemma 3.** *Let  $\alpha > 0$  and  $K > 0$  be two real numbers and let  $\eta \in C^1([0, \infty))$  with  $\eta' \geq \alpha^2$ . For any integer  $1 \leq j \leq N$ , the function  $U$  defined in  $\Omega \times [0, \infty)$  by*

$$U(x, t) = K \exp\left(\alpha x_j + \eta(t)\right),$$

*satisfies*

$$\begin{cases} \partial_t U \geq \Delta U - g(U) \cdot \nabla U + f(U) & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t U + \partial_\nu U \geq 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ U(\cdot, 0) > 0 & \text{in } \bar{\Omega}, \end{cases}$$

*if*

$$\alpha g_j(\omega) \geq \frac{f(\omega)}{\omega} \text{ for all } \omega \geq 0 \quad (11)$$

*and if*

$$\sigma(x, t) \geq \frac{\alpha}{\eta'(t)} \text{ for all } t > 0. \quad (12)$$

PROOF. A simple computation of the derivatives of  $U$  leads us to

$$\partial_t U - \Delta U + g(U) \cdot \nabla U = \left(\eta' - \alpha^2\right)U + \alpha g_j(U)U \text{ in } \Omega \text{ for } t > 0.$$

Since we assume  $\eta' \geq \alpha^2$ , hypothesis (11) implies

$$\partial_t U - \Delta U + g(U) \cdot \nabla U - f(U) \geq 0 \text{ in } \Omega \times (0, \infty).$$

Furthermore, on the boundary  $\partial\Omega$  for  $t > 0$ , we have

$$\begin{aligned} \sigma \partial_t U + \partial_\nu U &= \left(\sigma \eta'(t) + \alpha \nu_j(x)\right)U \\ &\geq \left(\sigma \eta'(t) - \alpha\right)U \geq 0, \end{aligned} \quad (13)$$

by hypothesis (12) since  $\nu$  is normalized. Finally,  $U(x, 0) = K \exp\left(\alpha x_j + \eta(0)\right) > 0$  in  $\bar{\Omega}$  is clear.  $\square$



**Remark 1.** In the case of the Dirichlet boundary conditions, we can use this upper-solution with the special choice  $\eta \equiv 0$  (see [14]). But for the dynamical boundary conditions, we must use a positive  $\eta$  for the time dependence because our solutions are not bounded, see Theorem 2.

Now we can state the following theorems for a nonlinear reaction term  $f$  growing as a power of  $u$  (Problem (2)), or as an exponential function (Problem (3)).

**Theorem 4.** *Let  $\sigma$  be a coefficient fulfilling conditions (4), (5) and such that there exists  $\delta > 0$  with*

$$\inf_{\partial\Omega} \sigma \geq \delta \sup_{\partial\Omega} \sigma \text{ for } t > 0 \text{ and } \left( \sup_{x \in \partial\Omega} \sigma(x, \cdot) \right)^{-1} \in L^1_{loc}(\mathbb{R}^+).$$

*Assume  $u_0$  satisfies condition (8). If there exists an integer  $1 \leq j \leq N$  such that*

$$\liminf_{\omega \rightarrow \infty} \frac{g_j(\omega)}{\omega^{p-1}} > 0, \tag{14}$$

*then the solution of Problem (2) is a global solution.*

PROOF. In view of Theorem 2, we can suppose that  $u_0$  is sufficiently big such that there exists  $C > 0$  with

$$g_j(u) \geq Cu^{p-1} \text{ in } \Omega \text{ for } t > 0.$$

For  $\eta(t) = C\delta^{-1} \int_0^t \left( \sup_{x \in \partial\Omega} \sigma(x, s) \right)^{-1} ds + C^2t$ , we have  $\eta' \geq C^2$  and Equation (12) is satisfied. Let  $K$  be a positive number such that

$$K \geq u_0(x)e^{-Cx_j - \eta(0)} \text{ for all } x \in \overline{\Omega}.$$

Then by hypotheses (5), (8) and (12), the function  $U$  defined in Lemma 3 is an upper-solution of Problem (2) since  $U(\cdot, 0) \geq u_0$  in  $\overline{\Omega}$ . Using the comparison principle from [2], the unique solution  $u$  of Problem (1) satisfies

$$0 \leq u(x, t) \leq U(x, t) \text{ for all } x \in \overline{\Omega} \text{ and } t > 0 ,$$

thus  $u$  can not blow up. □

This theorem holds in particular for a nonlinearity  $g$  in the form  $g(u) = (\alpha_1 u^{q_1}, \dots, \alpha_i u^{q_i}, \dots, \alpha_N u^{q_N})$  with at least one integer  $j$  such that  $\alpha_j > 0$  and  $q_j \geq p - 1$ . A similar result can be derived for Problem (3):

**Theorem 5.** *Under the aforementioned assumptions, the solution of Problem (3) is a global solution if the convection term  $g(u) \cdot \nabla u$  has (at least) one component  $g_j$  satisfying  $g_j(u) = \alpha_j e^{q_j u}$  with  $\alpha_j > 0$  and  $q_j > p$ .*

PROOF. Thanks to  $q_j > p$ , condition (11) is fulfilled because  $\alpha_j e^{q_j u} \geq \alpha_j e^{pu}/u$  for  $u$  sufficiently big.  $\square$

**Remark 2.** Condition (14) is optimal for Problem (2), see Theorems 4 and 10. But it can be improved in some special cases, for example, if the reaction term is  $f(u) = u \ln u$ . Lemma 3 implies that all solutions of Problem (1) are global if one component  $g_j$  of  $g$  satisfies  $g_j(u) \geq \alpha_j \ln u$ . In fact, in that case, every positive solution of (1) is global, without any assumption on the convection term  $g$ . We use this more general argument:

**Theorem 6.** *Assume that  $\sigma$  and  $u_0$  satisfy conditions (4), (5) and (8). If  $f$  is positive in  $(0, \infty)$  and fulfills*

$$\int_c^\infty \frac{1}{f(y)} dy = \infty \text{ for some } c > 0, \quad (15)$$

*then the solution of Problem (1) is global.*

PROOF. In view of the comparison principle, we suppose without loss of generality that  $c \leq \|u_0\|_\infty$ . Consider  $z$  the maximal solution of the Cauchy problem

$$\begin{cases} \dot{z} = f(z), \\ z(0) = \|u_0\|_\infty. \end{cases}$$

Its maximal existence time  $T_z$  satisfies

$$T_z = \int_{\|u_0\|_\infty}^\infty \frac{1}{f(y)} dy.$$

Condition (15) implies that  $z$  is a global solution in  $[0, \infty)$ .  $z$  is clearly an upper-solution of Problem (1) and the comparison principle from [2] yields  $0 \leq u(\cdot, t) \leq z(t)$  in  $\bar{\Omega}$  for  $t > 0$ . Thus  $u$  is a global solution.  $\square$

Condition (12) on  $\sigma$  allows us to consider fast decaying functions  $\sigma$ , but, to ensure global existence, it is essential that  $\sigma$  does not vanish on the whole  $\partial\Omega$ . Indeed let us prove the following blow-up result related to the Neumann boundary conditions, for  $\sigma \equiv 0$  on  $\partial\Omega$ .

**Theorem 7.** *Assume that  $\sigma \equiv 0$ ,  $u_0$  fulfills hypothesis (8) and  $f$  is positive in  $(0, \infty)$  such that*

$$\int_c^\infty \frac{1}{f(y)} dy < \infty \text{ for some } c > 0. \quad (16)$$

*Then every positive solution of Problem (1) blows up in finite time.*

PROOF. Let  $u$  be a non-trivial positive solution of

$$\begin{cases} \partial_t u = \Delta u - g(u) \cdot \nabla u + f(u) & \text{in } \Omega \text{ for } t > 0, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega}. \end{cases} \quad (17)$$

Using the maximum principle from [2], we have  $u(\cdot, \xi) > 0$  in  $\overline{\Omega}$  for  $\xi > 0$ . Hence, without loss of generality, we suppose  $u_0 > c$  in  $\overline{\Omega}$ . Now, consider the maximal solution  $z$  of the ODE  $\dot{z} = f(z)$  with the initial data  $z(0) = \inf\{u_0(x) / x \in \overline{\Omega}\}$ . Condition (16) implies that its maximal existence time  $T_z$  is finite:

$$T_z = \int_{z(0)}^\infty \frac{1}{f(y)} dy < \infty.$$

Since  $\nabla z = 0$ ,  $z$  is a lower solution of Problem (17). Using the comparison principle from [2], we obtain  $z(t) \leq u(\cdot, t)$  in  $\overline{\Omega}$  for  $t > 0$ . Thus,  $u$  must blow up in finite time with  $0 < T < T_z$ .  $\square$

**Remark 3.** This section illustrates the damping effect of the dissipative dynamical boundary conditions: we have shown that for nontrivial  $\sigma \geq 0$  the blow-up time of the solutions of Problem (1) can be strictly greater than the ones under the Neumann boundary conditions.

### 3. Blow-up

In this section, we investigate the blow-up in finite time for the solutions of the Problems (2) and (3). Let  $G$  be a primitive of  $g$  and suppose that there exist  $\alpha > 0$  and  $q < p$  such that

$$G(\omega) \leq \alpha\omega^q \text{ for } \omega > 0. \quad (18)$$

By applying the eigenfunction method (see [4], [9] and [12]), we obtain some conditions on the initial data  $u_0$  which guarantee the finite time blow-up and we derive some upper bounds for the blow-up times. Henceforth, we denote by  $\lambda$  the first eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$  and by  $\varphi$  an eigenfunction associated to  $\lambda$  satisfying

$$\varphi \in H_0^1(\Omega), \quad 0 < \varphi \leq 1 \text{ in } \Omega. \quad (19)$$

**Theorem 8.** *Let  $\alpha > 0$ ,  $1 < q < p$ ,  $m = p/(p - q)$  and suppose  $G$  satisfies condition (18). Assume hypotheses (4) - (8) are fulfilled. If*

$$\int_{\Omega} u_0 \varphi^m dx > (2|\Omega|^{p-1}C)^{\frac{1}{p}} \quad (20)$$

with

$$C = (p - 1)|\Omega| \left( \frac{4\lambda}{p - q} \right)^{\frac{1}{p-1}} + \left( \frac{4q}{p - q} \right)^{\frac{q}{p-q}} \alpha^m \int_{\Omega} |\nabla \varphi|^m dx ,$$

then the maximal classical solution  $u$  of Problem (2) blows up in finite time  $T$  satisfying

$$T \leq \frac{2 \int_{\Omega} u_0 \varphi^m dx}{(p - 1) \left( |\Omega|^{1-p} \left( \int_{\Omega} u_0 \varphi^m dx \right)^p - 2C \right)} =: \tilde{T}. \quad (21)$$

PROOF. Define

$$M(t) = \int_{\Omega} u \varphi^m dx.$$

Thus,

$$\dot{M}(t) = \int_{\Omega} \Delta u \varphi^m dx - \int_{\Omega} g(u) \cdot \nabla u \varphi^m dx + \int_{\Omega} u^p \varphi^m dx.$$

First, we prove that

$$\int_{\Omega} \Delta u \varphi^m dx \geq -m\lambda |\Omega|^{\frac{p-1}{p}} \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{1}{p}}. \quad (22)$$

Note that in the case where  $m \in (1, 2)$ ,  $\varphi^m \notin \mathcal{C}^2(\bar{\Omega})$ . Thus, we consider  $\int_{\Omega} (\varphi + \varepsilon)^m \Delta u dx$ , with  $\varepsilon > 0$ . We have

$$\Delta((\varphi + \varepsilon)^m) = m(\varphi + \varepsilon)^{m-1} \Delta \varphi + m(m-1)(\varphi + \varepsilon)^{m-2} |\nabla \varphi|^2 \geq -m\lambda \varphi (\varphi + \varepsilon)^{m-1}.$$

Then Green's formula used twice yields

$$\int_{\Omega} \Delta u (\varphi + \varepsilon)^m dx \geq -m\lambda \int_{\Omega} u \varphi (\varphi + \varepsilon)^{m-1} dx - \int_{\partial\Omega} \left( u \partial_{\nu} ((\varphi + \varepsilon)^m) - \partial_{\nu} u (\varphi + \varepsilon)^m \right) ds.$$

Since (19) implies

$$\int_{\partial\Omega} u \partial_{\nu} (\varphi^m) ds \leq 0 \text{ and } \int_{\partial\Omega} \partial_{\nu} u \varphi^m ds = 0 ,$$

letting  $\varepsilon \rightarrow 0$  leads to

$$\int_{\Omega} \Delta u \varphi^m dx \geq -m\lambda \int_{\Omega} u \varphi^m dx. \quad (23)$$

Since  $\varphi \leq 1$ ,  $\int_{\Omega} u \varphi^m dx \leq \int_{\Omega} u \varphi^{\frac{m}{p}} dx$  and by Hölder's inequality, (22) holds. Now, we show that

$$- \int_{\Omega} g(u) \cdot \nabla u \varphi^m dx \geq -m\alpha \left( \int_{\Omega} |\nabla \varphi|^m dx \right)^{\frac{1}{m}} \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{q}{p}}. \quad (24)$$

By Green's formula and by definition of  $G$  and  $\varphi$ , we have

$$- \int_{\Omega} g(u) \cdot \nabla u \varphi^m dx = - \int_{\Omega} \operatorname{div}(G(u)) \varphi^m dx = m \int_{\Omega} (G(u) \cdot \nabla \varphi) \varphi^{m-1} dx .$$

Equation (18) and Hölder's inequality lead to

$$\begin{aligned} \left| \int_{\Omega} (G(u) \cdot \nabla \varphi) \varphi^{m-1} dx \right| &\leq \alpha \int_{\Omega} u^q \varphi^{m-1} |\nabla \varphi| dx \\ &\leq \alpha \left( \int_{\Omega} |\nabla \varphi|^m dx \right)^{\frac{1}{m}} \left( \int_{\Omega} u^p \varphi^{\frac{(m-1)p}{q}} dx \right)^{\frac{q}{p}} . \end{aligned}$$

Then by the definition of  $m$ , (24) is satisfied.

Henceforth, introduce

$$C_1 = m\lambda |\Omega|^{\frac{p-1}{p}} \text{ and } C_2 = m\alpha \left( \int_{\Omega} |\nabla \varphi|^m dx \right)^{\frac{1}{m}} .$$

Then we obtain

$$\dot{M}(t) \geq \int_{\Omega} u^p \varphi^m dx - C_1 \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{1}{p}} - C_2 \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{q}{p}}. \quad (25)$$

Set

$$\varepsilon_1 = \frac{p^{\frac{1}{p}}}{4^{\frac{1}{p}}C_1} \text{ and } \varepsilon_2 = \frac{p^{\frac{q}{p}}}{(4q)^{\frac{q}{p}}C_2}.$$

Recall Young's inequality: for  $a > 0$  and  $\varepsilon > 0$ ,  $a = \frac{\varepsilon a}{\varepsilon} \leq \frac{\varepsilon^r a^r}{r} + \frac{1}{s\varepsilon^s}$  for  $r, s > 1$  with  $r^{-1} + s^{-1} = 1$ . It yields

$$C_1 \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{1}{p}} \leq \frac{1}{4} \int_{\Omega} u^p \varphi^m dx + \underbrace{\frac{p-1}{p\varepsilon_1^{\frac{p}{p-1}}}}_{:=C_3}.$$

In the same way, we have

$$C_2 \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{q}{p}} \leq \frac{1}{4} \int_{\Omega} u^p \varphi^m dx + C_4,$$

with

$$C_4 = \frac{1}{m\varepsilon_2^m}.$$

Then

$$\dot{M}(t) \geq \frac{1}{2} \int_{\Omega} u^p \varphi^m dx - C$$

with  $C = C_3 + C_4 > 0$ . By (19) and Hölder's inequality, we obtain that

$$\dot{M}(t) \geq \frac{1}{2} |\Omega|^{1-p} M^p - C.$$

Since  $M$  is increasing with respect to  $t$ , owing to (20) we have

$$\dot{M}(t) \geq \left( \frac{1}{2} |\Omega|^{1-p} - CM(0)^{-p} \right) M^p,$$

and we can conclude that  $u$  can not exist globally. To derive an upper bound for the blow-up time, we integrate the previous differential inequality between 0 and  $t > 0$ . We obtain

$$M(t) \geq \left( M(0)^{1-p} - (p-1) \left( \frac{1}{2} |\Omega|^{1-p} - CM(0)^{-p} \right) t \right)^{\frac{-1}{p-1}}.$$

Hence  $M$  blows up before  $\tilde{T} = M(0)^{1-p} (p-1)^{-1} \left( \frac{1}{2} |\Omega|^{1-p} - CM(0)^{-p} \right)^{-1}$ , so does  $u$ . Thus,  $T \leq \tilde{T}$ .  $\square$

With a minor change in the proof of Theorem 8, we can give another criterion on the initial data for the blow-up in finite time of the solution of Problem (2).

**Corollary 9.** *Under the aforementioned hypotheses, if*

$$\int_{\Omega} u_0 \varphi^m dx > \max \left\{ (4m\lambda)^{\frac{1}{p-1}} |\Omega|, (4C' |\Omega|^{p-1})^{\frac{1}{p}} \right\} \quad (26)$$

with

$$C' = \left( \frac{2q}{p-q} \right)^{\frac{q}{p-q}} \alpha^m \int_{\Omega} |\nabla \varphi|^m dx ,$$

then the classical maximal solution of Problem (2) blows up in finite time  $T$  satisfying

$$T \leq \frac{2 \int_{\Omega} u_0 \varphi^m dx}{(p-1) \left( |\Omega|^{1-p} \left( \int_{\Omega} u_0 \varphi^m dx \right)^p - 2m\lambda \left( \int_{\Omega} u_0 \varphi^m dx \right) - 2C' \right)}$$

PROOF. As in the proof of Theorem 8, we consider  $M(t) = \int_{\Omega} u \varphi^m dx$  satisfying Equation (25). We use (23) instead of (22)

$$\dot{M} \geq -m\lambda M + \int_{\Omega} u^p \varphi^m dx - C_2 \left( \int_{\Omega} u^p \varphi^m dx \right)^{\frac{q}{p}}$$

where  $C_2 = m\alpha \left( \int_{\Omega} |\nabla \varphi|^m dx \right)^{\frac{1}{m}}$ . Proceeding as before with  $\varepsilon'_2 = \frac{p^{\frac{q}{p}}}{(4q)^{\frac{q}{p}} C_2}$ , we obtain the differential inequality

$$\dot{M} \geq -m\lambda M + \frac{1}{2} |\Omega|^{1-p} M^p - C',$$

where  $C'$  is defined as in the assertion. Then

$$\dot{M} \geq M \left( \frac{|\Omega|^{1-p}}{4} M^{p-1} - m\lambda \right) + \frac{1}{4} |\Omega|^{1-p} M^p - C'$$

and the blow-up occurs in finite time by (26). To obtain the upper bound for the blow-up time, it suffices to integrate the ordinary differential inequality

$$\dot{M} \geq M^p \left( -m\lambda M(0)^{1-p} + \frac{1}{2} |\Omega|^{1-p} - C' M(0)^{-p} \right).$$

□

We can note that Conditions (20) and (26) on the size of the initial data are necessary only to derive an upper bound for the maximal existence time. Thanks to Theorem 2, we obtain:

**Theorem 10.** *Let  $q < p$  and suppose  $G$  satisfies*

$$\limsup_{\omega \rightarrow \infty} \frac{G(\omega)}{\omega^q} < \infty.$$

*Assume that  $\sigma$  and  $u_0$  satisfy conditions (4), (5) and (8). All the positive solutions of Problem (2) blow up in finite time.*

PROOF. Let  $u$  be a positive solution of Problem (2). Theorem 2 permits us to ensure that there exist  $t_0 > 0$  and  $C > 0$  such that  $u(\cdot, t_0)$  is big enough to satisfy Equation (20) and  $G(u) \leq Cu^q$  in  $\Omega$  for  $t > t_0$ . Thus applying Theorem 8 to  $v(x, t) = u(x, t + t_0)$ , we prove that  $v$  blows up in a finite time  $T_v$  satisfying (21). Hence,  $u$  blows up in a finite time  $T_u = t_0 + T_v$ .  $\square$

Now, we present another method to prove the blow-up of positive solutions of Problems (2) and (3). As in [16], we first construct a lower bound for the solutions of the following problem

$$\begin{cases} \partial_t u = \Delta u - \mu |\nabla u|^q + \kappa u^p & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \overline{\Omega}, \end{cases} \quad (27)$$

with  $p > q$ ,  $\mu > 0$  and  $\kappa > 0$ , which blows up in finite time.

**Lemma 11.** *Let  $t_0$  and  $\varepsilon$  be two real numbers with  $0 < t_0 < 1/\varepsilon$ . Define in  $\Omega \times [t_0, 1/\varepsilon)$  the function  $V$  by*

$$V(x, t) = \frac{1}{(1 - \varepsilon t)^k} W\left(\frac{|x|}{(1 - \varepsilon t)^m}\right),$$

*where  $k = 1/(p - 1)$ ,  $0 < m < \min\{\frac{1}{2}, \frac{p-q}{q(p-1)}\}$ ,  $W(y) = 1 + \frac{A}{2} - \frac{y^2}{2A}$  for  $y \geq 0$  with  $A > \frac{k}{m}$ , and  $\varepsilon < \frac{\kappa}{k(1+A/2)}$ . If  $1/\varepsilon - t_0$  is small enough, and if  $u_0 \geq V(\cdot, t_0)$  in  $\Omega$ , then every positive solution of Problem (27) blows up in finite time.*

PROOF. We refer to the proof of Theorem 36.2 from [14] in the case  $\kappa = 1$ .  $\square$



**Theorem 12.** *Under hypotheses (4) - (8), assume that there exists  $q > 1$  with  $q < p$  such that*

$$\limsup_{\omega \rightarrow \infty} \frac{|g(\omega)|}{\omega^{q-1}} < \infty. \quad (28)$$

*Then all the positive solutions of Problem (2) blow up in finite time.*

PROOF. Let  $u$  be a positive solution of (2). Thanks to Theorem 2, we can suppose that  $u_0$  and  $u$  are sufficiently big to verify  $u_0 \geq V(\cdot, t_0)$  in  $\bar{\Omega}$ , where  $V$  is the function defined in Lemma 11. According to condition (28), there exists  $\beta > 0$  such that  $|g(u)| \leq \beta u^{q-1}$  in  $\Omega$  for  $t > 0$ . Then, Young's inequality implies

$$|g(u) \cdot \nabla u| \leq \beta u^{q-1} |\nabla u| \leq \kappa u^p + \mu |\nabla u|^m,$$

with  $\kappa = \frac{q-1}{p}$ ,  $\mu = \frac{1}{m} \beta^m$  and  $m = \frac{p}{p-q+1}$ . By definition, we have  $m < p$ . Thus, we obtain

$$\partial_t u \geq \Delta u - \mu |\nabla u|^m + (1 - \kappa) u^p \text{ in } \Omega \text{ for } t > 0,$$

and  $u$  is an upper-solution of Problem (27), and  $u$  blows up in finite time according to Lemma 11.  $\square$

**Theorem 13.** *Assume  $\sigma$  and  $u_0$  satisfy conditions (4) - (8). If*

$$\limsup_{\omega \rightarrow \infty} \frac{|g(\omega)|}{e^{q\omega}} < \infty,$$

*then all the positive solutions of Problem (3) blow up in finite time.*

PROOF. Let  $u$  be a positive solution of (3) and define  $v = e^{\gamma u}$  with  $\gamma \in (q, p)$  and  $\gamma > 1/2$ . As in the previous proof, we suppose that  $u$  is sufficiently big such that for some  $C > 0$

$$|g(u)| \leq C e^{qu} \text{ in } \Omega \text{ for } t > 0. \quad (29)$$

Computing the derivatives of  $v$ , we obtain

$$\partial_t v = \Delta v - \frac{1}{v} |\nabla v|^2 - g(u) \cdot \nabla v + \gamma v^{\frac{p+\gamma}{\gamma}} \text{ in } \Omega \text{ for } t > 0.$$

Using condition (29), we obtain

$$\partial_t v \geq \Delta v - \frac{1}{v} |\nabla v|^2 - C v^{\frac{q}{\gamma}} |\nabla v| + \gamma v^{\frac{p+\gamma}{\gamma}} \text{ in } \Omega \text{ for } t > 0.$$

Young's inequality

$$Cv^{\frac{q}{\gamma}}|\nabla v| \leq \frac{C^2}{2}|\nabla v|^2 + \frac{1}{2}v^{\frac{2q}{\gamma}},$$

leads to

$$\partial_t v \geq \Delta v - \frac{2+C^2}{2}|\nabla v|^2 + \gamma v^{\frac{p+\gamma}{\gamma}} - \frac{1}{2}v^{\frac{2q}{\gamma}} \text{ in } \Omega \text{ for } t > 0,$$

since  $v \geq 1$ . Moreover, we have

$$\gamma v^{\frac{p+\gamma}{\gamma}} - \frac{1}{2}v^{\frac{2q}{\gamma}} \geq \left(\gamma - \frac{1}{2}\right)v^{\frac{p+\gamma}{\gamma}}$$

by definition of  $\gamma$ . Thus, in  $\Omega$  for  $t > 0$  we obtain  $\partial_t v \geq \Delta v - \mu|\nabla v|^2 + (\gamma - \frac{1}{2})v^{\frac{p+\gamma}{\gamma}}$  with  $\mu = (2 + C^2)/2$ . On the boundary  $\partial\Omega$ , we have

$$\sigma\partial_t v + \partial_\nu v = \gamma v \left( \sigma\partial_t u + \partial_\nu u \right) = 0.$$

Without loss of generality (see Theorem 2), we can suppose that  $v(\cdot, 0) \geq V(\cdot, t_0)$  in  $\bar{\Omega}$  where  $V$  is the function in Lemma 11. Hence, by the comparison principle from [2],  $v$  is an upper-solution of Problem (27) and must blow up in finite time according to Lemma 11. Thus  $u$  blows up in finite time.  $\square$

**Remark 4.** In this section, we point out the accelerating effect of the dynamical boundary conditions, in comparison with the Dirichlet boundary conditions. Indeed, we prove that, even if the initial data  $u_0$  is small, the solutions of Problem (2) blow up in finite time. But, if we replace the dynamical boundary conditions by the Dirichlet boundary conditions in the second equation of Problem (2), it is well known that the solutions are global and decay to 0 if the initial data are small enough, see for instance references [17] and [18] and the following theorem:

**Theorem 14.** *Let  $g$  be any non-negative function in  $\mathcal{C}(\mathbb{R}, \mathbb{R}^N)$  and  $u_0$  be an initial data satisfying (8) and  $u_0 = 0$  on  $\partial\Omega$ . Then for any  $p > 1 + 2/N$ , the problem*

$$\begin{cases} \partial_t u = \Delta u - g(u) \cdot \nabla u + u^p & \text{in } \bar{\Omega} \text{ for } t > 0, \\ u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \bar{\Omega}, \end{cases} \quad (30)$$

*admits global solutions if  $u_0$  is small enough.*

PROOF. We use this well known upper-solution (see [1], [15] and references therein) defined in  $\bar{\Omega} \times [0, \infty)$  by

$$U(x, t) = A(t+1)^{-\gamma} \exp\left(\frac{-\|x\|_2^2}{4(t+1)}\right)$$

where  $\gamma = \frac{1}{p-1}$  and  $A = \frac{1}{2}\left(\frac{N}{2} - \gamma\right)^\gamma$ . In  $\Omega$ , for  $t > 0$ ,  $U$  satisfies

$$\partial_t U - \Delta U + g(u) \cdot \nabla U - U^p = \left(\frac{N-2\gamma}{2(t+1)} - \sum_{i=1}^N \frac{x_i g_i(u)}{2(t+1)} - U^{p-1}\right)U.$$

Up to a translation, we can suppose that  $x_j \leq 0$  in  $\Omega$  for all  $1 \leq j \leq N$ . Thus

$$\partial_t U - \Delta U + g(u) \cdot \nabla U - U^p \geq \left(\frac{N-2\gamma}{2(t+1)} - U^{p-1}\right)U.$$

Then by definition of  $\gamma$ ,  $N - 2\gamma > 0$ , and  $U^{p-1} \leq \frac{A^{p-1}}{t+1}$ . Hence, choosing  $A$  small enough, we are led to

$$\partial_t U - \Delta U + g(u) \cdot \nabla U - U^p \geq 0 \text{ in } \Omega \times (0, \infty).$$

On the boundary,  $U \geq 0$  is clear. Finally, we just have to consider an initial data  $u_0$  with  $u_0 \leq U(\cdot, 0)$  in  $\bar{\Omega}$ , and the comparison principle implies that the solution  $u$  of (30) is bounded from above by  $U$ . Thus,  $u$  is global and goes to 0 as  $t \rightarrow \infty$ .  $\square$

#### 4. Growth Order

In this section, we are interested in the blow-up rate for Problem (2) when approaching the blow-up time  $T$ . For the convection term, we assume that

$$g(u) = (g_1(u), \dots, g_n(u)) \text{ with } g_i(u) = u^q \forall i = 1, \dots, n, \quad 1 < q \in \mathbb{R}. \quad (31)$$

First, we derive a lower blow-up estimate for  $p > q + 1$ , valid for any non-negative initial data  $u_0 \in \mathcal{C}(\bar{\Omega})$ .

**Lemma 15.** *Let  $p > q + 1$ , and assume hypotheses (4) - (8). Then the classical maximal solution  $u$  of Problem (2) satisfies*

$$\|u(\cdot, t)\|_\infty \geq (p-1)^{\frac{-1}{p-1}} (T-t)^{\frac{-1}{p-1}}$$

for  $0 < t < T$ .

PROOF. Let  $t \in [0, T)$ . Denote by  $\zeta \in \mathcal{C}^1((0, t_1))$  the maximal solution of the IVP

$$\begin{cases} \dot{\zeta} &= \zeta^p & \text{in } (0, t_1) \\ \zeta(0) &= \|u(\cdot, t)\|_\infty \end{cases}$$

with  $t_1 = \frac{1}{p-1} \|u(\cdot, t)\|_\infty^{1-p}$ . Introduce  $v \in \mathcal{C}(\overline{\Omega} \times [0, T-t)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0, T-t))$  defined by  $v(x, s) = u(x, s+t)$  for  $x \in \overline{\Omega}$  and  $s \in [0, T-t)$ . Then  $v$  is the maximal solution of the problem

$$\begin{cases} \partial_t v = \Delta v - g(v) \cdot \nabla v + v^p & \text{in } \Omega \text{ for } 0 < s < T-t, \\ \sigma \partial_t v + \partial_\nu v = 0 & \text{on } \partial\Omega \text{ for } 0 < s < T-t, \\ v(\cdot, 0) = u(\cdot, t) & \text{in } \overline{\Omega}. \end{cases}$$

The comparison principle from [2] implies that  $t_1 \leq T-t$ .  $\square$

This result remains valid for Problem (1) as soon as blow-up occurs. We just need a positive function  $f$  such that an explicit primitive of  $\frac{1}{f}$  is known.

We follow the technique developed in Theorem 2.3 in [5] for an one-dimensional Burgers' problem and inspired by Friedman & McLeod [10] to prove that the growth order of the solution of Problem (2) amounts to  $-1/(p-1)$  for  $p > 2q+1$  with  $4p(p-1)(p-1-2q) > Nq^2$ , when the time  $t$  approaches the blow-up time  $T$ .

**Theorem 16.** *Suppose conditions (4), (8), (10) and (31) are fulfilled. For*

$$p > 2q+1 \quad \text{and} \quad p(p-1)(p-2q-1) > \frac{Nq^2}{4} \quad (32)$$

*there exists a positive constant  $C$  such that the classical maximal solution  $u$  of Problem (2) satisfies*

$$\|u(\cdot, t)\|_\infty \leq \frac{C}{(T-t)^{1/p-1}} \quad \text{for } t \in [0, T).$$

PROOF. Let  $\alpha > 1$  such that

$$p(p-1)(p-2q-1) = \frac{\alpha^2 Nq^2}{4}, \quad (33)$$

and choose  $M > 1$  such that

$$M \geq \frac{Nq}{2(2q+1)}(\alpha - 1)^{\frac{2q}{2q+1-p}}.$$

First, for  $\xi \in (0, T)$ , we shall prove that there exists  $\delta > 0$  such that

$$\partial_t u \geq \delta e^{-Mt}(u^p + u^{2q+1})$$

in  $\bar{\Omega} \times [\xi, T)$ . Introduce

$$J = \partial_t u - \delta d(t)k(u)$$

with  $d(t) = e^{-Mt}$  and  $k(u) = u^p + u^{2q+1}$ . Note that classical regularity results from [13] yield  $J \in \mathcal{C}^{2,1}(\bar{\Omega} \times [\xi, T))$ . We recall that Theorem 2 implies that there exists  $c > 0$  such that  $\partial_t u \geq c > 0$  in  $\bar{\Omega} \times [\xi, T)$ . Thus, we can choose  $\delta > 0$  sufficiently small such that

$$J(\cdot, \xi) \geq 0 \quad \text{in } \bar{\Omega}.$$

$J$  fulfills the boundary condition

$$\sigma \partial_t J + \partial_\nu J = \partial_t(\sigma \partial_t u + \partial_\nu u) - \delta d k'(u)(\sigma \partial_t u + \partial_\nu u) - \sigma \delta d' k(u) = \sigma \delta M e^{-Mt} k(u) \geq 0.$$

Furthermore,  $J$  satisfies

$$\partial_t J - \Delta J + g(u) \cdot \nabla J - (pu^{p-1} - g'(u) \cdot \nabla u)J = \delta d H(u) \text{ in } \bar{\Omega} \times [\xi, T),$$

where

$$H(u) := pu^{p-1}k(u) - k'(u)u^p + k''(u)|\nabla u|^2 - \frac{d'}{d}k(u) - k(u)g'(u) \cdot \nabla u.$$

To prove that  $H(u) \geq 0$ , we shall show that

$$\begin{aligned} q\sqrt{N}u^{q-1}|\nabla u|(u^p + u^{2q+1}) &\leq M(u^p + u^{2q+1}) + (p - 2q - 1)u^{p+2q} \\ &+ (p(p-1)u^{p-2} + 2q(2q+1)u^{2q-1})|\nabla u|^2. \end{aligned} \quad (34)$$

Inequality (34) is trivial in the case where  $M \geq q\sqrt{N}u^{q-1}|\nabla u|$ . Now, suppose that  $M < q\sqrt{N}u^{q-1}|\nabla u|$ . When  $q\sqrt{N}u^{q+1} \leq 2q(2q+1)|\nabla u|$ , we have  $q\sqrt{N}u^{q-1}u^p|\nabla u| \leq p(p-1)u^{p-2}|\nabla u|^2$  and  $q\sqrt{N}u^{3q}|\nabla u| \leq 2q(2q+$

1) $u^{2q-1}|\nabla u|^2$  since  $p > 3$  then (34) follows. In the case where  $q\sqrt{N}u^{q+1} > 2q(2q+1)|\nabla u|$ , since

$$u > \left( \frac{2(2q+1)}{Nq} M \right)^{\frac{1}{2q}} = (\alpha - 1)^{\frac{1}{2q+1-p}},$$

we obtain

$$u^p + u^{2q+1} \geq \alpha u^p. \quad (35)$$

Moreover, (33) yields

$$\begin{aligned} \alpha\sqrt{N}qu^{q+1}|\nabla u| &\leq 2\sqrt{p(p-1)(p-2q-1)}u^{q+1}|\nabla u| \\ &\leq \left( \sqrt{p-2q-1}u^{q+1} - \sqrt{p(p-1)}|\nabla u| \right)^2 \\ &\quad + 2\sqrt{p(p-1)(p-2q-1)}u^{q+1}|\nabla u| \\ &\leq (p-2q-1)u^{2(q+1)} + p(p-1)|\nabla u|^2. \end{aligned}$$

Thus, multiplying by  $u^{p-2}$ , we are led to

$$\alpha\sqrt{N}qu^{q-1}|\nabla u|u^p \leq (p-2q-1)u^{p+2q} + p(p-1)u^{p-2}|\nabla u|^2$$

and by (35), the inequality (34) holds. Finally, we can conclude by the comparison principle from [2] that  $J \geq 0$  in  $\bar{\Omega} \times [\xi, T)$ , in particular,  $\partial_t u \geq \varepsilon u^p$  with  $\varepsilon > 0$ .

Now, we shall derive the upper blow-up rate estimate of  $\|u(\cdot, t)\|_\infty$  for  $t \in [\xi, T)$ . For each  $x \in \Omega$ , the integral

$$\int_t^\tau \frac{\partial_t u(x, s)}{u^p(x, s)} ds = \int_{u(x, t)}^{u(x, \tau)} \frac{1}{\eta^p} d\eta$$

converges as  $\tau \rightarrow T$ . Integrating the inequality  $\partial_t u \geq \varepsilon u^p$  leads to

$$\varepsilon(\tau - t) \leq \frac{u(x, \tau)^{1-p} - u(x, t)^{1-p}}{1-p} \leq \frac{u(x, t)^{1-p}}{p-1}.$$

Letting  $\tau \rightarrow T$  implies  $u(x, t) \leq \left( \varepsilon(p-1)(T-t) \right)^{\frac{-1}{p-1}}$  and we can conclude as in the proof of Theorem 2.3 from [5].  $\square$

The latest result can be improved thanks to Theorem 38.1 developed by Quittner and Souplet in [14] in the Dirichlet case. We extend their theorem to the case of the dynamical boundary conditions. For the reader's convenience, we precise the minor changes appearing in the proof.

**Theorem 17.** *Assume  $\sigma > 0$  on the boundary  $\partial\Omega$  and conditions (8), (10) and (31) are fulfilled, and suppose that*

$$1 < p < p_B := \begin{cases} \infty & \text{if } N = 1 \\ \frac{N(N+2)}{(N-1)^2} & \text{if } N > 1 \end{cases}$$

*Assume in addition that  $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the growth assumption*

$$|h(u, \xi)| \leq C_0(1 + |u|^{p_1} + |\xi|^{p_2}), \quad (36)$$

*for some  $1 \leq p_1 < p$  and  $1 < p_2 < 2p/(p+1)$ . Then, any nonnegative classical solution of*

$$\begin{cases} \partial_t u = \Delta u + h(u, \nabla u) + u^p & \text{in } \bar{\Omega}, 0 < t < T, \\ \sigma \partial_t u - \partial_\nu u = 0 & \text{on } \partial\Omega, 0 < t < T \end{cases} \quad (37)$$

*satisfies*

$$u(x, t) + |\nabla u(x, t)|^{\frac{2}{p+1}} \leq C(1 + t^{\frac{-1}{p-1}} + (T-t)^{\frac{-1}{p-1}}), \quad (38)$$

*for  $x \in \Omega$ ,  $0 < t < T$ , with  $C > 0$  depending on  $p, p_1, p_2, C_0, \Omega$ .*

**PROOF.** We assume that the estimate (38) fails. Then as in Theorem 38.1 from [14], there exist sequences  $u_k, T_k \in (0, \infty)$ ,  $x_k \in \Omega$  and  $t_k \in (0, T_k)$  such that  $u_k$  is a solution of (37) in  $(0, T_k)$  with:

$$M_k = u_k^{\frac{p-1}{2}}, \quad M_k(x_k, t_k) > 2k, \quad M_k(x_k, t_k) > 2k \min\{s_k, T_k - t_k\}^{\frac{-1}{2}}$$

and  $\lambda_k = 1/M_k(x_k, t_k)$ . The function  $v_k(y, s) := \lambda_k^{\frac{2}{p-1}} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s)$  defined in  $\tilde{D}_k = \left(\lambda_k^{-1}(\Omega - x_k) \cap \{|y| < k/2\}\right) \times (-k^2/4, k^2/4)$  satisfies

$$\begin{cases} \partial_s v_k = \Delta v_k + h_k + v_k^p & \text{in } \tilde{D}_k, \\ \frac{\sigma}{\lambda_k} \partial_s v_k + \partial_\nu v_k = 0 & \text{if } y \in \lambda_k^{-1}(\partial\Omega - x_k), |y| < \frac{k}{2}, |s| < \frac{k^2}{4}. \end{cases}$$

with

$$\begin{aligned} h_k(y, s) &:= \lambda_k^{\frac{2p}{p-1}} h(\lambda_k^{\frac{-2}{p-1}} v_k(y, s), \lambda_k^{\frac{-(p+1)}{p-1}} \nabla v_k(y, s)), \\ v_k^{\frac{p-1}{2}}(0) + |\nabla v_k|^{\frac{p-1}{p+1}}(0) &= 1, \end{aligned}$$

and

$$v_k^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}} \leq 2 \text{ in } \tilde{D}_k.$$

Condition (38) implies  $|h_k| \leq C\lambda_k^m$  with  $m = \min\{\frac{2(p-p_1)}{p-1}, \frac{2p-q(p+1)}{p-1}\}$ . Then, as in the proof of Souplet and Quittner, we obtain a subsequence of  $(v_k)$  converging in  $\mathcal{C}^{1+\alpha, \alpha/2}(\mathbb{R}^N \times \mathbb{R})$  with  $0 < \alpha < 1$  to a nonnegative function  $v$ . Observe that in the limit case  $\lambda_k \rightarrow 0$ , the dynamical condition  $\partial_s v_k + \frac{\lambda_k}{\sigma} \partial_\nu v_k = 0$  on the time lateral boundary of  $\tilde{D}_k$ , implies that  $v$  does not depend on time  $t$  at space infinity, that is  $v = v(y, 0) \geq 0$ . Two cases can occur. In the first case  $v$  is a solution of

$$\partial_t u = \Delta u + u^p \text{ for } x \in \mathbb{R}^N, t > 0,$$

and in the second one,  $v$  satisfies

$$\begin{cases} \partial_s v = \Delta_y v + lv^p & y \in H_c, s \in \mathbb{R}, \\ v \geq 0 & y \in \partial H_c, s \in \mathbb{R}, \end{cases}$$

with  $H_c := \{y \in \mathbb{R}^N / y_1 > -c\}$ . Since  $v^{\frac{p-1}{2}}(0) + |\nabla v|^{\frac{p-1}{2}}(0) = 1$ ,  $v$  is non-trivial and  $v$  and  $\nabla v$  are bounded. Hence, we are led to a contradiction with non-existence results (see Theorems 21.1 and 21.8 from [14]) for positive solutions of  $\partial_t u = \Delta u + u^p$  in unbounded domains.  $\square$

Thanks to this last theorem, we can improve the result obtained in Theorem 16. Indeed, we derive a blow-up rate estimate for  $p > 1 + 2q > 3$  and  $p < p_B$  where the constant  $C$  involved in the upper bound is independent on the initial data unlike in Theorem 16.

**Theorem 18.** *Under the aforementioned hypotheses, suppose  $p > 1$ ,  $q > 1$  and*

$$2q + 1 < p < p_B .$$

*Then, the maximal classical solution of Problem (2) fulfills*

$$\sup_{\bar{\Omega}} \left( u + |\nabla u|^{\frac{2}{p+1}} \right) \leq C \left( 1 + t^{\frac{-1}{p-1}} + (T-t)^{\frac{-1}{p-1}} \right)$$

*for  $t \in [0, T)$ , with a constant  $C$  such that  $0 < C = C(p, q, \Omega)$ .*

PROOF. Young's inequality yields

$$|g(u) \cdot \xi| \leq \sqrt{N} u^q |\xi| \leq \sqrt{N} \left( \frac{q}{p_1} (u^q)^{\frac{p_1}{q}} + \frac{1}{p_2} |\xi|^{p_2} \right),$$

for  $\xi \in \mathbb{R}^N$ . Equation (36) is satisfied with  $p_1 = \frac{p+2q+1}{2}$  and  $p_2 = \frac{p+2q+1}{p+1}$ , thus Theorem 17 permits to conclude.  $\square$



Since  $p > 1 + 2q > 3$ , this theorem applies when  $p_B > 3$ , i.e. for  $N = 1, 2$  and 3. In these cases, our last result completes Theorem 16 giving a blow-up estimate for  $p \in (3, p_B)$ . In particular for  $q = 1$ , Equation (32) is fulfilled for any  $p > p_B$ , then the blow-up rate for the solutions of Problem (2) is fully determined for all  $p > 3$ .

## Acknowledgment

The authors would like to thank Dr Mabel Cuesta for helpful discussions and valuable advices.

## References

- [1] C. Bandle and H. Levine, *Fujita type results for convective-like reaction diffusion equations in exterior domains*. ZAMP, **40** (1989), 665–676.
- [2] J. von Below and C. De Coster, *A Qualitative Theory for Parabolic Problems under Dynamical Boundary Conditions*. J. of Inequal. and Appl., **5** (2000), 467-486.
- [3] J. von Below and G. Pincet Mailly, *Blow Up for Reaction Diffusion Equations Under Dynamical Boundary Conditions*. Commun. in Partial Diff. Equ. **28**, (2003), 223-247.
- [4] J. von Below and G. Pincet Mailly, *Blow Up for some non linear parabolic problems with convection under dynamical boundary conditions*. Discret. Contin. Dyn. Syst. - Suppl. Vol., (2007), 1031-1041.
- [5] J. von Below, G. Pincet Mailly and J-F. Rault, *Growth order and blow up points for the parabolic burgers' equation under dynamical boundary conditions*. (Submitt.).
- [6] A. Constantin and J. Escher, *Global existence for fully parabolic boundary value problems*. NoDEA **13**, (2006), 91–118.
- [7] J. Escher, *Quasilinear parabolic systems with dynamical boundary conditions*. Commun. in Partial Diff. Equ. **18**, (1993), 1309-1364.
- [8] M. Fila, H. Ninomiya and J.L. Vázquez, *Dirichlet boundary conditions can prevent blow-up reaction-diffusion equations and systems*. Discret. Contin. Dyn. Syst. **14** (2006), 63-74.

- [9] A. Friedman: *Blow up of solutions of nonlinear parabolic equations*. In: W.M. Ni et al. (eds.) *Nonlinear Diffusion Equations and Their Equilibrium States*, pp. 301–318. Springer (1988).
- [10] A. Friedman and B. McLeod, *Blow up of positive solutions of semilinear heat equations*. *Indiana Univ. Math. J.* **34**, (1985), 425–447.
- [11] G.R. Goldstein, *Derivation and physical interpretation of general boundary conditions*. *Adv. in Diff. Equ.* **11**, (2006), 457–480.
- [12] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*. *Commun. Pure and Appl. Math.* **16**, (1963), 305–330.
- [13] O.A. Ladyženskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and quasilinear equations of parabolic type*. *Transl. of Math. Monogr.* 23, A.M.S., Providence, R.I., 1968.
- [14] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhäuser, 2007.
- [15] J-F. Rault, *The Fujita phenomenon in exterior domains under dynamical boundary conditions*. *Asymptot. Anal.* **66**, (2010), 1–8.
- [16] Ph. Souplet and F.B. Weissler, *Self-similar subsolutions and blowup for non-linear parabolic equations*. *J. Math. Anal. Appl.* **212**, (1997), 60–74.
- [17] B. Straughan, *The energy method, Stability and Nonlinear Convection*. Springer Verlag, 2004.
- [18] F.B. Weissler, *Existence and nonexistence of global solutions for a semi-linear heat equation*. *Isr. J. Math.* **38** (1981), 29–40.