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# Stability and resonance conditions of elementary fractional transfer functions

Rachid Malti, Xavier Moreau, Firas Khemane, and Alain Oustaloup

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## Abstract

Elementary fractional transfer functions are studied in this paper. Some basic properties of elementary transfer functions of the first kind are recalled. Then, two main results are presented regarding elementary fractional transfer functions of the second kind, written in a canonical form and characterized by a commensurate order, a pseudo-damping factor, and a natural frequency. First, stability conditions are established in terms of the pseudo-damping factor and the commensurate order, as a corollary to Matignon's stability theorem. They extend the previous result into conditions that are simpler to check. Then, resonance conditions are established numerically in terms of the commensurate order and the pseudo-damping factor and give interesting information on the frequency behavior of fractional systems. It is shown that elementary transfer functions of the second kind might have up to two resonant frequencies. Moreover, three abaci are given allowing to determine the pseudo-damping factor and the commensurate order for respectively a desired normalized gain at each resonance, a desired phase at each resonance, and a desired normalized first or second resonant frequency.

*Key words:* Fractional calculus, elementary transfer function, second order, stability condition, resonance condition.

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## 1 Introduction

Although the fractional calculus remained for a long time purely a mathematical concept, the last two decades witnessed considerable development in the use of fractional operators in various fields. In electrochemistry for instance, diffusion of charges in acid batteries is governed by Randles models (Sabatier *et al.*, 2006) that involve Warburg impedance with an integrator of order 0.5. Electrochemical diffusion, investigated in semi-infinite planar, spherical and cylindrical media by (Oldham and Spanier, 1974), showed to have a tight relation with derivatives of order 0.5. In thermal diffusion in a semi-infinite homogeneous medium, (Battaglia *et al.*, 2001) have shown that the exact solution for the heat equation links thermal flux to a half order derivative of the surface temperature on which the flux is applied. Fractional system modeling and identification is treated in

detail in (Malti *et al.*, 2008a).

Stability of fractional systems was treated in different contexts (linear, non linear, commensurate, non commensurate, time-variant, time invariant, delayed, non delayed, analytical, numerical) by different authors as presented in the state of the art by (Sabatier *et al.*, 2010b) with more than 20 references on the subject. Fractional orthogonal bases functions were synthesized in (Aoun *et al.*, 2007; Malti *et al.*, 2004; Malti *et al.*, 2005; Aoun, 2005; Akçay, 2008). Initial conditions problem, more difficult to tackle in the fractional case, was treated in (Lorenzo and Hartley, 2008; Sabatier *et al.*, 2010a) and their own references. Time-domain simulation of fractional systems was discussed in (Oustaloup, 1995; Chen and Moore, 2002; Aoun *et al.*, 2004; Tavazoei and Haeri, 2010) and all references therein. As shown by this quick and non exhaustive bibliography, fractional systems are gaining more and more interest in the scientific and industrial communities.

The main objectives of this paper are to study stability and resonance of elementary fractional transfer functions. First, some basic properties of elementary transfer functions of the first kind are recalled in section 2. Then, the main contributions of this paper are presented

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in section 3. Stability conditions of elementary transfer functions of the second kind are first established in terms of the pseudo-damping factor, as a corollary to Matignon's stability theorem. Then, resonance conditions are determined numerically. All combinations of the pseudo-damping factor and the commensurate order, allowing to obtain a resonant transfer function, are given. Moreover, three abaci are provided for each resonant frequency allowing to obtain the pseudo-damping factor and the commensurate order for respectively a desired normalized gain at resonance, a desired phase at resonance, and a desired normalized resonant frequency.

### Fractional transfer functions

Commensurate fractional systems can be represented as rational transfer function in  $s^\nu$ :

$$H(s) = \frac{T(s^\nu)}{R(s^\nu)} = \frac{\sum_{j=0}^M b_j s^{\nu j}}{1 + \sum_{i=1}^N a_i s^{\nu i}}, \quad (1)$$

where the commensurate order  $\nu > 0$ , is a strictly positive real number,  $(a_i, b_j) \in \mathbb{R}^2$ ,  $M$ , and  $N$  are respectively numerator and denominator degrees, with  $N > M$  for strictly proper systems.

Bounded input bounded output stability of fractional systems is considered in this paper in the sense defined by (Matignon, 1998). The most well known stability criterion for commensurate fractional systems is Matignon's (1998) theorem [1], who proved the stability of commensurate fractional systems with commensurate orders between 0 and 1. It allows to check system stability straightforwardly through the location of its  $s^\nu$ -poles instead of computing the  $s$ -poles. Matignon's theorem is extended to commensurate orders,  $\nu$ , between 1 and 2 in (Moze *et al.*, 2005) and to any commensurate order in the PhD thesis (Aoun, 2005). In the former reference, the authors define a new commensurate order  $\nu' = \frac{\nu}{2}$  and apply the initial Matignon's theorem on system  $s^{\nu'}$ -poles. The latter reference presents a more general result as the instability of fractional systems having commensurate orders greater than 2 is also proven. Hence, the extension by (Aoun, 2005) is presented below in theorem 1.1 together with its proof. Theorem 1.1 is used later for determining the stability of elementary transfer functions.

**Theorem 1.1 ((Matignon, 1998) extended)** *A commensurate transfer function with a commensurate order  $\nu$ , as in (1), with  $T$  and  $R$  two coprime polynomials, is stable iff*

$$0 < \nu < 2 \quad (2)$$

and

$$\forall p \in \mathbb{C} \text{ such that } R(p) = 0, \quad |\arg(p)| > \nu \frac{\pi}{2}. \quad (3)$$

**Proof:** The extended stability theorem, for  $\nu > 0$ , is proven by contradiction. It will be proven below that  $H(s)$  is unstable iff

$$\nu \geq 2 \quad (4)$$

or

$$\exists p \in \mathbb{C} \text{ such that } R(p) = 0, \quad |\arg(p)| \leq \nu \frac{\pi}{2}. \quad (5)$$

Let  $\{p_k\}_{k=1,2,\dots,N}$  denote the set of all  $s^\nu$ -poles of  $H(s)$  such that  $R(p) = 0$ .

The multivalued function  $s \mapsto s^\nu$  becomes an analytic function in the complement of its branch cut line of the complex plane. The branch cut line is chosen to be along the negative real axis including the branching points 0 and  $\infty$ . Hence, all arguments of  $s$  are restrained to  $(-\pi, \pi)$  and all the  $s$ -poles of  $H(s)$  are given by:

$$s_{k,\ell} = |p_k|^{\frac{1}{\nu}} e^{j\left(\frac{\arg(p_k) + 2\ell\pi}{\nu}\right)}, \quad \text{for } k = 1, \dots, N \quad (6)$$

and  $\forall \ell \in \mathbb{Z}$  such that  $-\pi < \frac{\arg(p_k) + 2\ell\pi}{\nu} < \pi$ . Additionally, according to (Bonnet and Partington, 2000),  $H(s)$  is unstable iff it has at least one  $s$ -pole in the right-half complex plane, hence with arguments satisfying:

$$-\frac{\pi}{2} \leq \frac{\arg(p_k) + 2\ell\pi}{\nu} \leq \frac{\pi}{2}. \quad (7)$$

Consequently,  $H(s)$  is unstable iff there exists an  $\ell \in \mathbb{Z}$  satisfying (7) or:

$$-\nu \frac{\pi}{2} - \arg(p_k) \leq 2\ell\pi \leq \nu \frac{\pi}{2} - \arg(p_k). \quad (8)$$

For simplicity and without loss of generality, all arguments of  $p_k$  are restrained to their principal determination in  $[-\pi, \pi)$ . According to  $\nu$ , two cases are distinguished: (i) if  $\nu \geq 2$ , then  $\ell = 0$  always satisfies the inequalities in (8). Hence,  $H(s)$  has at least  $N$  unstable  $s$ -poles,  $s_{k,0}$  with  $k = 1, 2, \dots, N$ , obtained for  $\ell = 0$ . (ii) if  $1 < \nu < 2$ , then the only possible value of  $\ell$  which might satisfy (8) is  $\ell = 0$  iff 0 is in the interval defined by (8), i.e. provided  $|\arg(p_k)| \leq \nu \frac{\pi}{2}$ . This completes the proof of instability of  $H(s)$  iff conditions (4) or (5) are satisfied and by contradiction the proof of stability of  $H(s)$  iff conditions (2) and (3) are satisfied.  $\square$

When plotting the frequency response of  $H(s)$ , the factorized form is generally preferred as it shows the contribution of every elementary transfer function of the first

kind (equivalent to first order transfer function when  $\nu = 1$ )

$$F_1(s) = \left( \left( \frac{s}{\omega_n} \right)^\nu + 1 \right)^{\pm 1} \quad (9)$$

and every elementary transfer function of the second kind (equivalent to second order transfer function when  $\nu = 1$ )

$$F_2(s) = \left( \left( \frac{s}{\omega_n} \right)^{2\nu} + 2\zeta \left( \frac{s}{\omega_n} \right)^\nu + 1 \right)^{\pm 1} \quad (10)$$

as they appear in terms of every natural frequency  $\omega_n$  and the  $\zeta$  parameter, referred to as a *pseudo-damping factor* ( $\zeta$  is a damping factor only if  $\nu = 1$ ). When two complex conjugate  $s^\nu$ -poles are present in the commensurate fractional transfer function (1), the representation using elementary transfer functions of the second kind (10) must be used as the transfer functions of the first kind contain only real-valued parameters.

(Hartley and Lorenzo, 1998) studied the time response of elementary transfer functions of the first kind. Moreover, (Sabatier *et al.*, 2003) examined the time-domain performances (maximum overshoot and settling time) of elementary transfer functions of the first and of the second kind in terms of the differentiation order  $\nu$  and, for the transfer function of the second kind, the pseudo-damping factor  $\zeta$  (denoted  $\cos(\theta)$  in (Sabatier *et al.*, 2003)).

## 2 Stability and resonance of elementary transfer functions of the first kind

Some basic properties of elementary transfer functions of the first kind

$$F_1(s) = \frac{1}{\left( \frac{s}{\omega_n} \right)^\nu + 1}, \quad (11)$$

with  $\omega_n > 0$ , are recalled in this section. Since  $\omega_n > 0$ ,  $F_1(s)$  is stable iff (2) is satisfied.

Regarding resonance conditions, define  $\Omega = \frac{\omega}{\omega_n}$  as the normalized frequency. Hence, the gain in dB of  $F_1(j\Omega\omega_n) = |\mathcal{F}_1(j\Omega)|_{\text{dB}}$  is given by:

$$|\mathcal{F}_1(j\Omega)|_{\text{dB}} = -10 \log \left( 1 + 2 \cos \left( \nu \frac{\pi}{2} \right) \Omega^\nu + \Omega^{2\nu} \right). \quad (12)$$

In case  $F_1(s)$  is resonant,  $|\mathcal{F}_1(j\Omega)|_{\text{dB}}$  has at least a maximum at a positive normalized frequency. Hence,

$\frac{d|\mathcal{F}_1(j\Omega)|_{\text{dB}}}{d\Omega} = 0$  has a real and strictly positive solution corresponding to that maximum at:

$$\Omega_r^\nu = -\cos \left( \nu \frac{\pi}{2} \right), \quad (13)$$

provided the following resonance condition is satisfied:

$$1 < \nu < 2. \quad (14)$$

The gain and the phase at resonance are respectively obtained by substituting (13) in (12) and in (11):

$$|\mathcal{F}_1(j\Omega_r)|_{\text{dB}} = -20 \log \left( \sin \left( \nu \frac{\pi}{2} \right) \right), \quad (15)$$

$$\arg(\mathcal{F}_1(j\Omega_r)) = (1 - \nu) \frac{\pi}{2}. \quad (16)$$

## 3 Stability and resonance of fractional transfer functions of the second kind

Two important results are presented in this section. First, stability conditions of elementary transfer functions of the second kind are established in terms of the pseudo-damping factor and the commensurate order, as a corollary to Matignon stability theorem. Then, resonance conditions are established numerically.

The canonical form of an elementary transfer function of the second kind is given by:

$$F_2(s) = \frac{1}{1 + 2\zeta \left( \frac{s}{\omega_n} \right)^\nu + \left( \frac{s}{\omega_n} \right)^{2\nu}}, \quad (17)$$

with  $\zeta \in \mathbb{R}$ ,  $\omega_n > 0$ , and  $\nu > 0$ .

The properties of rational second order transfer function, (17) with  $\nu = 1$ , are well known: (i) it is stable if the damping factor  $\zeta > 0$ , (ii) it is resonant if  $0 < \zeta < \frac{\sqrt{2}}{2}$ , (iii) it has two complex conjugate poles if  $0 < \zeta < 1$ , (iv) it has a real double pole if  $\zeta = 1$ , (v) it is overdamped if  $\zeta > 1$ . How do these properties generalize to elementary transfer functions of the second kind, with  $\nu \neq 1$ ? This is the main concern of the paper.

### 3.1 Stability of fractional transfer functions of the second kind

Stability of fractional transfer functions of the second kind is presented here as a corollary to theorem 1.1.

**Corollary 3.1** *The transfer function (17) is stable iff:*

$$0 < \nu < 2 \quad \text{and} \quad \zeta > -\cos \left( \nu \frac{\pi}{2} \right). \quad (18)$$

**Proof:** the key idea to prove corollary 3.1 is to compute both  $s^\nu$ -poles of (17), and then to evaluate their arguments depending on  $\zeta$  and  $\nu$ , and finally to find conditions for which these arguments satisfy the inequality (3) of theorem 1.1. A complete proof is presented in (Malti *et al.*, 2008b).  $\square$

This corollary is in accordance with the rational case, as it is well known that a second order transfer function, as in (17) with  $\nu = 1$ , is stable iff  $\zeta > 0$ .

### 3.2 Resonance of fractional transfer functions of the second kind

Define, as previously,  $\Omega = \frac{\omega}{\omega_n}$  as the normalized frequency. Hence, the gain in dB of  $F_2(j\Omega\omega_n) = |\mathcal{F}_2(j\Omega)|_{\text{dB}}$  is given by:

$$|\mathcal{F}_2(j\Omega)|_{\text{dB}} = -10 \log \left[ \Omega^{4\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^{3\nu} + 2(2\zeta^2 + \cos(\nu\pi)) \Omega^{2\nu} + 4\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^\nu + 1 \right]. \quad (19)$$

In case  $F_2(s)$  is resonant,  $|\mathcal{F}_2(j\Omega)|_{\text{dB}}$  has at least a maximum at a positive normalized frequency. Hence,

$$\frac{d|\mathcal{F}_2(j\Omega)|_{\text{dB}}}{d\Omega} = 0 \Rightarrow \Omega^{3\nu} + 3\zeta \cos\left(\nu\frac{\pi}{2}\right) \Omega^{2\nu} + (2\zeta^2 + \cos(\nu\pi)) \Omega^\nu + \zeta \cos\left(\nu\frac{\pi}{2}\right) = 0 \quad (20)$$

has at least one real and strictly positive solution corresponding to that maximum. One can check easily that for rational systems, with  $\nu = 1$ , (20) reduces to  $\Omega^3 + (2\zeta^2 - 1)\Omega = 0$ , which strictly positive solution is given, as expected, by  $\Omega_r = \sqrt{1 - 2\zeta^2}$ , provided that the following known condition is satisfied  $\zeta < \frac{\sqrt{2}}{2}$ .

The third order equation in  $\Omega^\nu$  (20), can have positive real-valued, negative real-valued or complex-valued solutions. The number of resonant frequencies of the studied system, zero one or two, depends on the number of strictly positive real-valued solutions corresponding to maxima of  $|\mathcal{F}_2(j\Omega)|$ . Care must be taken, because some of the strictly positive solutions correspond to minima of  $|\mathcal{F}_2(j\Omega)|$ , especially when a double resonance is present, then the gain has a minimum between the two maxima (see example 2).

Solving (20) analytically is not an easy task. A numerical solution is obtained for various combinations of  $\nu$  and  $\zeta$  and plotted in Fig. 1. The gray region numbered 0 represents combinations of  $\nu$  and  $\zeta$  which produce non resonant systems. The yellow and the green regions numbered 1 and 2 represent combinations of  $\nu$  and  $\zeta$  which produce resonant systems with respectively one and two

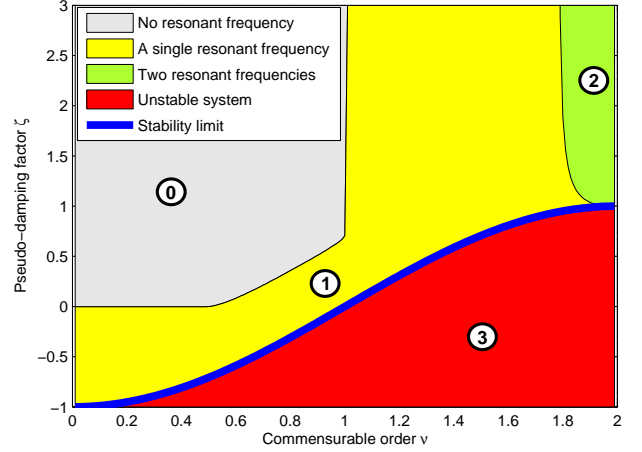


Fig. 1. Stability and resonance regions of the fractional system (17) in the  $\zeta$  versus  $\nu$  plane

resonant frequencies. The red region numbered 3 represents combinations of  $\nu$  and  $\zeta$  which produce unstable systems. As shown in Fig. 1, when the fractional transfer function is resonant, it may have one or two resonant frequencies depending on the combinations of  $\nu$  and  $\zeta$ . Indeed, when  $\zeta > 1$ , the fractional transfer function of the second kind may be decomposed into two fractional transfer functions of the first kind having two modes. Either both are resonant or only one depending on  $\zeta$  and  $\nu$ . Hence, the main properties of fractional transfer functions of the second kind, deduced from Fig. 1, are summarized below.

- if  $0 < \nu \leq 0.5$  and  $-\cos\nu\frac{\pi}{2} < \zeta < 0 \Rightarrow$  the stable transfer function  $F_2(s)$  in (17) is always resonant,
- if  $0 < \nu \leq 0.5$  and  $0 \leq \zeta \Rightarrow$  the stable transfer function  $F_2(s)$  is never resonant (for  $\zeta = 0$  it can straightforwardly be proven that (20) has no strictly positive solution),
- if  $0.5 < \nu \leq 1$  and  $-\cos\nu\frac{\pi}{2} < \zeta < \zeta_0$  where  $\zeta_0$  is computed numerically and plotted in Fig. 1 as the upper contour of the resonance yellow region numbered 1 in the interval  $\zeta \in (0.5, 1] \Rightarrow$  the stable transfer function  $F_2(s)$  is resonant. In the particular case of rational systems, with  $\nu = 1$ ,  $\zeta_0 = \frac{\sqrt{2}}{2}$ ,
- if  $0.5 < \nu \leq 1$  and  $\zeta_0 \leq \zeta \Rightarrow$  the stable transfer function  $F_2(s)$  is never resonant,
- if  $1 < \nu < 2$  and  $-\cos\nu\frac{\pi}{2} < \zeta \Rightarrow$  the stable transfer function  $F_2(s)$  is always resonant,
- if  $\nu_1 < \nu < 2$  and  $\zeta_1 < \zeta$  where  $\nu_1$  and  $\zeta_1$  are computed numerically and plotted as the lower-left contours of the upper-right green region numbered 2 of Fig. 1  $\Rightarrow$  the stable transfer function  $F_2(s)$  has two resonant frequencies.

Three abaci are given for each normalized frequency allowing to determine the pseudo-damping factor and the commensurate order: (i) for a desired normalized first

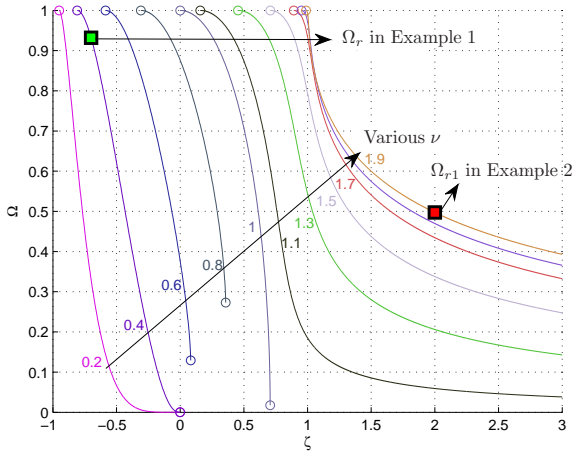


Fig. 2. Abacus – First normalized resonant frequency versus  $\zeta$ , for different values of  $\nu$

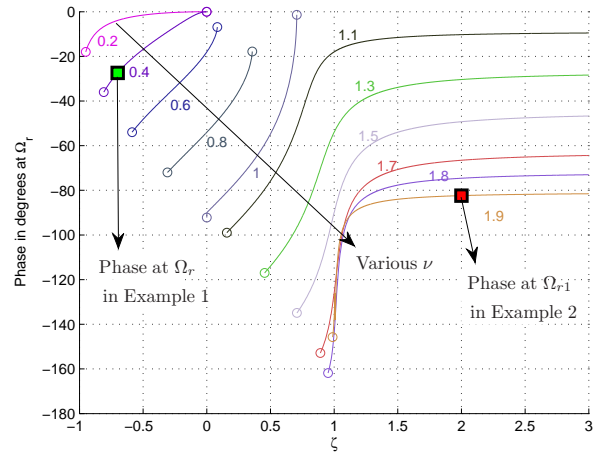


Fig. 4. Abacus – Phase at the first normalized resonant frequency versus  $\zeta$ , for different values of  $\nu$

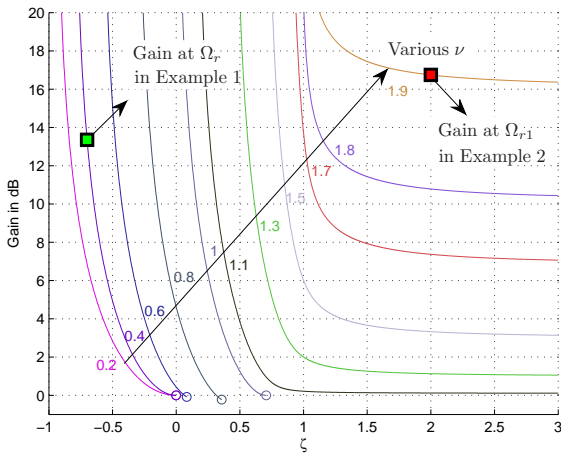


Fig. 3. Abacus – Gain at the first normalized resonant frequency versus  $\zeta$ , for different values of  $\nu$

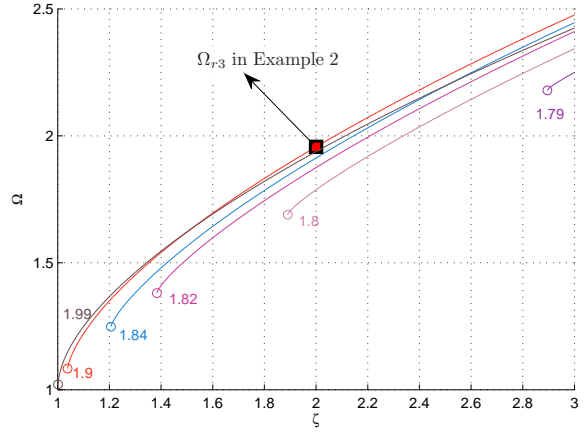


Fig. 5. Abacus – Second normalized resonant frequency versus  $\zeta$ , for different values of  $\nu$

resonant frequency in Fig. 2, (ii) for a desired normalized gain at the first resonant frequency in Fig. 3, (iii) for a desired phase at the first resonant frequency in Fig. 4, (iv) for a desired normalized second resonant frequency, when it exists, in Fig. 5, (v) for a desired normalized gain at the second resonant frequency, when it exists, in Fig. 6, (vi) for a desired phase at the second resonant frequency, when it exists, in Fig. 7.

### 3.3 Numerical examples

The first example shows how to use the abaci to set up a resonant transfer function of the second kind with a prescribed gain at resonance. The second example points out a behavior particular to fractional systems of the second kind that is the presence of two resonant frequencies.

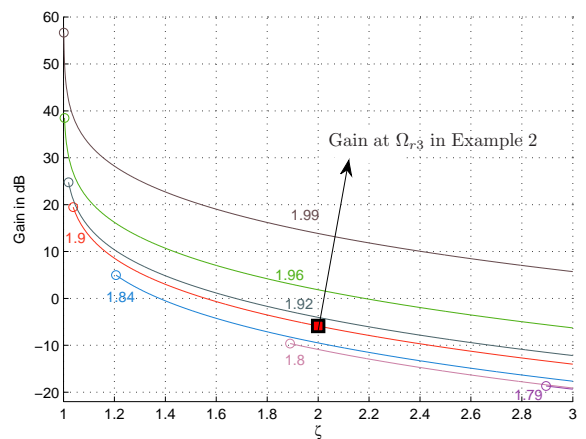


Fig. 6. Abacus – Gain at the second normalized resonant frequency versus  $\zeta$ , for different values of  $\nu$

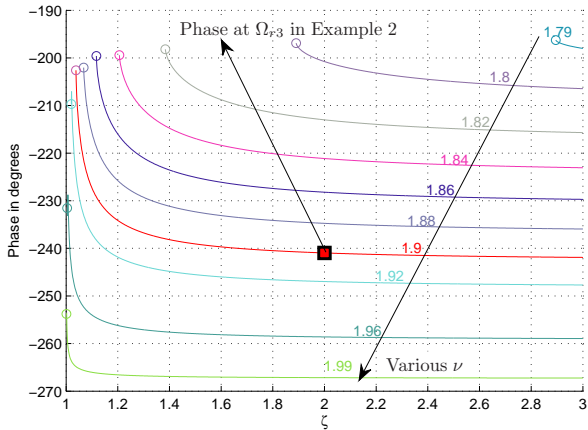


Fig. 7. Abacus – Phase at the second normalized resonant frequency versus  $\zeta$ , for different values of  $\nu$

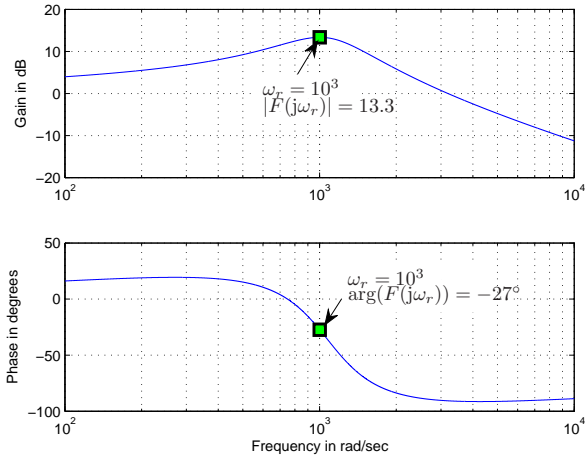


Fig. 8. Bode diagram of  $F(s)$  with  $\zeta = -0.7$  and  $\nu = 0.4$

**Example 1** Assume one wants to get a resonant transfer function with a commensurate order of  $\nu = 0.4$ , a normalized gain at resonance of  $\left| \frac{F(j\omega_r)}{F(j0)} \right|_{\text{dB}} = 13.3\text{dB}$ , a resonant frequency at  $\omega_r = 10^3$  rad/Sec, and a steady state gain of 0dB. Then from the abacus of Fig. 3, one deduces that the pseudo-damping factor must be set to  $\zeta = -0.7$ . Next, from the abacus of Fig. 2, one deduces that the normalized frequency equals  $\Omega_r = \frac{\omega_r}{\omega_n} = 0.93$  rad/Sec. Hence, the natural frequency,  $\omega_n$ , must be set to 1075 rad/Sec. The Bode diagram of the obtained transfer function:

$$F(s) = \frac{1}{1 - 2 \times 0.7 \left(\frac{s}{1075}\right)^{0.4} + \left(\frac{s}{1075^2}\right)^{0.8}}, \quad (21)$$

plotted in Fig. 8, confirms that  $F(s)$  complies with the specifications of the example. Moreover, using the abacus of Fig. 4 one can deduce that the phase at resonance equals  $-27^\circ$ . In this case, one can easily check that the

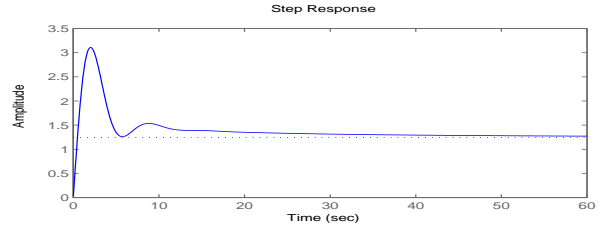


Fig. 9. Step response of  $F(s)$  with  $\zeta = -0.7$  and  $\nu = 0.4$

equation (20) reduces to:

$$\Omega^{1.2} - 1.70\Omega^{0.8} + 1.30\Omega^{0.4} - 0.57 = 0, \quad (22)$$

and that the only strictly positive real-valued solution is at  $\Omega_r = 0.93$ . The step response, plotted in Fig. 9, is underdamped as expected due to the resonant frequency<sup>1</sup>.

**Example 2** Consider now the combination  $\nu = 1.9$ , and  $\zeta = 2$ , located in the green region numbered 2 of Fig. 1. The transfer function (17) presents a double resonance. In this case, equation (20) reduces to:

$$\Omega^{5.7} - 5.93\Omega^{3.8} + 8.95\Omega^{1.9} - 1.98 = 0, \quad (23)$$

and has three real-valued solutions  $\Omega_{r1} = 0.50$ ,  $\Omega_{r2} = 1.47$ ,  $\Omega_{r3} = 1.96$ . As shown in Fig. 10,  $\Omega_{r1}$  corresponds to a resonance,  $\Omega_{r2}$  to a minimum, and  $\Omega_{r3}$  to a second resonance. When a system presents two resonant frequencies, it always has a minimum between these two maxima. The resonant frequencies  $\Omega_{r1}$  and  $\Omega_{r3}$  are pointed out in the abaci of Fig. 2 and Fig. 5. The amplitudes at the two resonant frequencies equal 16.7 and  $-5.9$  dB as shown in the abaci of Fig. 3 and Fig. 6. The phases at the two resonant frequencies equal  $-86^\circ$  and  $-240^\circ$ , in accordance with the abaci of Fig. 4 and Fig. 7. Moreover, the step response, plotted in Fig. 11, is underdamped as expected due to the resonant frequencies.

## 4 Conclusion

Elementary fractional transfer functions are studied in this paper. First, stability and resonance conditions of elementary transfer functions of the first kind are recalled. Then, the main results of the paper are presented. A corollary to Matignon's theorem is established for the stability of elementary transfer functions of the second

<sup>1</sup> For time-domain simulations, the fractional transfer functions are approximated by rational transfer functions in the frequency band  $[0.01, 100]$  with 10 cells using (Oustaloup, 1995) method, then standard continuous-time simulation algorithms are applied on the rational approximation.

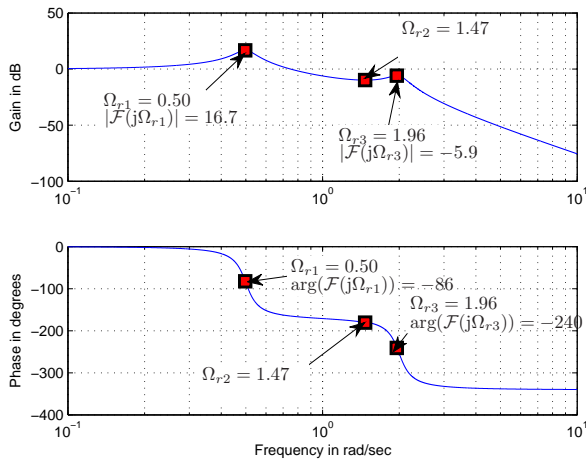


Fig. 10. Bode diagram for  $\zeta = 2$  and  $\nu = 1.9$

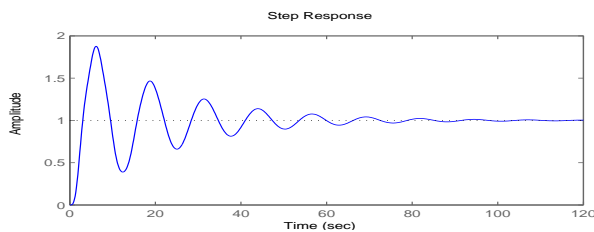


Fig. 11. Step response for  $\zeta = 2$  and  $\nu = 1.9$

kind in terms of the pseudo-damping factor and the commensurate order. It extends the previous stability results to a condition simpler to check. Moreover, resonance conditions are established numerically in terms of the pseudo-damping factor and the commensurate order and plotted in Fig. 1, which shows that some combinations of  $\zeta$  and  $\nu$  yield two resonant frequencies. Furthermore for each resonant frequency, three abaci are given allowing to determine the pseudo-damping factor and the commensurate order for respectively a desired normalized gain at resonance, a desired phase at resonance, and a desired normalized resonant frequency. Further insights related to elementary transfer functions and the computation of their  $\mathcal{H}_2$ -norm are presented in (Malti *et al.*, 2011).

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