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# Martin representation and Relative Fatou Theorem for fractional Laplacian with a gradient perturbation

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## Abstract

Let  $L = \Delta^{\alpha/2} + b \cdot \nabla$  with  $\alpha \in (1, 2)$ . We prove the Martin representation and the Relative Fatou Theorem for non-negative singular  $L$ -harmonic functions on  $\mathcal{C}^{1,1}$  bounded open sets.<sup>1</sup>

## 1 Introduction and Preliminaries

Analysis of harmonic functions related to fractional powers  $\Delta^{\alpha/2}$  of the Laplace operator is an important topic, intensely developed in recent years,

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also for perturbations of the operator  $\Delta^{\alpha/2}$ , see, e.g., [2, 5, 18, 10, 13, 23, 32, 7] and references therein.

From the probabilistic point of view, stable stochastic processes with gradient perturbations on  $\mathbb{R}^d$ ,  $d \geq 2$ , i.e. with the infinitesimal generator

$$L = \Delta^{\alpha/2} + b \cdot \nabla, \quad (1)$$

where  $\alpha \in (0, 2)$ , constitute an important class of jump processes, intensely studied in recent years. Their most celebrated case are the Ornstein-Uhlenbeck stable processes with  $b(x) = \lambda x$ ,  $\lambda \in \mathbb{R}$ . They have important physical and financial applications and form a part of Lévy-driven Ornstein-Uhlenbeck processes, cf. [12, 34].

The motivations of this paper were to:

- (i) establish the theory of the Martin representation for singular  $L$ -harmonic non-negative functions
- (ii) study boundary limit properties of  $L$ -harmonic functions and to obtain a Relative Fatou Theorem for them
- (iii) develop the theory of Hardy spaces of  $L$ -harmonic functions.

The topics (i) and (ii) are addressed in this article and the subject (iii) in a forthcoming paper.

All these topics are fundamental for the knowledge of  $L$ -harmonic functions. The topics (i) and (ii) are well developed for fractional Laplacians. The Martin representation was established in this case in [6, 10, 15, 33]. The Relative Fatou Theorem was proved for  $\alpha$ -harmonic functions on Lipschitz sets in [31], see also the survey [7, Chapter 3] and [25]. Also some important variants of stable processes such as relativistic and truncated stable processes were studied from the point of view of topics (i) and (ii), see [7, Section 3.4], [26] and [28]. Nevertheless, the methods of these extensions do not apply to the operator  $L$  of the form (1). Let us notice that all our results are also true for Ornstein-Uhlenbeck stable processes.

On the other hand, fine properties of harmonic functions were widely studied in the case of diffusion operators, see [35, 36, 3, 38, 17, 24, 1, 4] for the results on the classical Laplacian  $\Delta = \sum_{i=1}^d \partial^2 / \partial x_i^2$ , and [37, 30] for the Laplacian with various perturbations.

Let us mention that the methods of this article give also interesting new results for operators different from  $L$ . In the case of Laplacians with gradient perturbations, i.e.  $\alpha = 2$ , we get new perturbation formulas for the Green function and the Martin and Poisson kernels (Section 3.4).

The potential theory of stable stochastic processes with gradient perturbations was started in the Ornstein–Uhlenbeck case by T. Jakubowski [22, 23]. Next, in the general context of gradient perturbations and  $\alpha > 1$ , with a function  $b$  from the Kato class  $\mathcal{K}_d^{\alpha-1}$  it was developed by K. Bogdan and T. Jakubowski [8, 9]. This work is a natural continuation of the research presented in [9].

In particular, we send the reader to [9] for the definitions of the fractional Laplacian,  $\mathcal{C}^{1,1}$  sets, Green functions and Poisson kernels, with respect to both operators  $\Delta^{\alpha/2}$  and  $L$ . The definitions of  $\alpha$ -harmonic, regular  $\alpha$ -harmonic and singular  $\alpha$ -harmonic functions can be found e.g. in the monography [7, page 61] and are analogous for  $L$ -harmonic functions.

Throughout this paper, like in [9], we suppose  $1 < \alpha < 2$ , unless stated otherwise. We consider an open set  $D$  of class  $\mathcal{C}^{1,1}$  and a vector field  $b \in \mathcal{K}_d^{\alpha-1}$  on  $\mathbb{R}^d$  i.e.

$$\limsup_{\epsilon \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-z| < \epsilon} |b(z)| |x-z|^{\alpha-1-d} dz = 0.$$

The potential theory objects related to the operator  $L$  defined in (1) will be denoted with a tilde  $\sim$ , while those related to the operator  $\Delta^{\alpha/2}$  will be denoted without it. In particular  $\tilde{G}_D$  is the Green function of  $D$  for  $L$  and  $G_D$  is the Green function of  $D$  for  $\Delta^{\alpha/2}$ . We fix throughout this paper a point  $x_0 \in D$  and define the Martin kernel of  $D$  for  $\Delta^{\alpha/2}$  by

$$M_D(x, Q) = \lim_{y \rightarrow Q} \frac{G_D(x, y)}{G_D(x_0, y)}, \quad x \in D, Q \in \partial D.$$

The  $L$ -Martin kernel is defined by

$$\tilde{M}_D(x, Q) = \lim_{y \rightarrow Q} \frac{\tilde{G}_D(x, y)}{\tilde{G}_D(x_0, y)}, \quad x \in D, Q \in \partial D$$

and we show in Section 3 its existence.

The starting point of the research contained in this paper are the following mutual estimates of Green functions and Poisson kernels for  $L$  and  $\Delta^{\alpha/2}$  (see [9, Theorem 1 and (72)]).

**Comparability Theorem.** *There exists a constant  $C = C(\alpha, b, D)$  such that for all  $x, y \in D$  and  $z \in (\overline{D})^c$ ,*

$$C^{-1}G_D(x, y) \leq \tilde{G}_D(x, y) \leq CG_D(x, y), \tag{2}$$

$$C^{-1}P_D(x, z) \leq \tilde{P}_D(x, z) \leq CP_D(x, z). \tag{3}$$

One of the main elements of the proof of (2) is the following perturbation formula, that will be also very useful in our present work (see [9, Lemma 12]).

**Perturbation formula for Green functions.** *Let  $x, y \in \mathbb{R}^d, x \neq y$ . We have*

$$\tilde{G}_D(x, y) = G_D(x, y) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla_z G_D(z, y) dz. \quad (4)$$

We start our paper by proving in Section 2.1 a generalization of the Comparability Theorem: according to Lemma 1, the constant  $C$  in the estimates (2) may be chosen the same for sets  $D_r$  sufficiently close to  $D$ . The same phenomenon holds also for the Poisson kernels  $\tilde{P}_D(x, y)$  and  $P_D(x, y)$ . In Section 2.2 we prove a uniform integrability result, that will be needed in proving the main results of the paper, contained in Sections 3 and 4.

In Section 3 we develop the Martin theory of  $L$ -harmonic functions. We prove the existence of the  $L$ -Martin kernel which is  $L$ -harmonic (Theorems 8 and 12). Next we obtain the Martin representation of singular  $L$ -harmonic non-negative functions on  $D$ , see Theorem 13.

The formula (4) allows us to prove very useful perturbation formulas for Martin kernels (14), Poisson kernels (17) and singular  $\alpha$ -harmonic functions (28). In Section 3.4, (4) and (14) are proved in the diffusion case  $\alpha = 2$ . Also a perturbation formula (33) for the  $L$ -Poisson kernel is derived.

Section 4 is devoted to an important fine boundary property of singular  $L$ -harmonic functions: the Relative Fatou Theorem (Theorem 23). We provide a proof of this theorem based on the perturbation formula for singular  $\alpha$ -harmonic functions (28).

## 2 Preparatory results

In this section we prove some results, interesting independently, that will be useful in proving the main results of the paper, coming in the next sections.

### 2.1 Uniform comparability of Green functions and Poisson kernels

In what follows,  $\mathbb{R}^d$  denotes the Euclidean space of dimension  $d \geq 2$ ,  $dy$  stands for the Lebesgue measure on  $\mathbb{R}^d$ . Without further mention we will only consider Borelian sets, measures and functions in  $\mathbb{R}^d$ . By  $x \cdot y$  we denote the Euclidean scalar product of  $x, y \in \mathbb{R}^d$ . Writing  $f \approx g$  we mean that there is a constant  $C > 0$  such that  $C^{-1}g \leq f \leq Cg$ . As usual,  $a \wedge b = \min(a, b)$  and

$a \vee b = \max(a, b)$ . We let  $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ . For  $U \subset \mathbb{R}^d$  we denote

$$\delta_U(x) = \text{dist}(x, U^c),$$

the distance to the complement of  $U$ .

*In what follows  $D$  is a bounded  $\mathcal{C}^{1,1}$  open set.*

For  $r \geq 0$  define

$$D_r = \{x \in D : \delta_D(x) > r\}.$$

When  $r$  is sufficiently small, then  $D_r$  is also a  $\mathcal{C}^{1,1}$  open set, see [32, Lemma 5], and one may show that the localization radius of  $D_r$  varies continuously with respect to  $r$ .

In the sequel we will often use the estimates of the Green function ([29], [14], see also [21]) of a  $\mathcal{C}^{1,1}$  open set

$$G_D(y, z) \approx |y - z|^{\alpha-d} \left( \frac{\delta_D(y)^{\alpha/2} \delta_D(z)^{\alpha/2}}{|y - z|^\alpha} \wedge 1 \right), \quad (5)$$

and of the Martin kernel ([15])

$$M_D(x, Q) \approx \frac{\delta_D(x)^{\alpha/2}}{|x - Q|^d}. \quad (6)$$

Moreover, in the stable case, the estimates (5) are uniform when we consider the sets  $D_r$  sufficiently close to  $D$ , i.e. there exist constants  $c, \epsilon_0 > 0$  depending only on  $D$  and  $\alpha$  such that for all  $r \in [0, \epsilon_0]$  and  $x, y \in D_r$  we have

$$\begin{aligned} c^{-1} |y - z|^{\alpha-d} \left( \frac{\delta_{D_r}(y)^{\alpha/2} \delta_{D_r}(z)^{\alpha/2}}{|y - z|^\alpha} \wedge 1 \right) &\leq G_{D_r}(y, z) \\ &\leq c |y - z|^{\alpha-d} \left( \frac{\delta_{D_r}(y)^{\alpha/2} \delta_{D_r}(z)^{\alpha/2}}{|y - z|^\alpha} \wedge 1 \right), \end{aligned} \quad (7)$$

see [21, Theorem 21] and [32, Lemma 5]. We will now show analogous uniformity of constants for the fractional Laplacian with a gradient perturbation.

**Lemma 1.** *(i) There exist constants  $c, \epsilon_0 > 0$  depending only on  $D$  and  $\alpha$  such that for all  $r \in [0, \epsilon_0]$  and  $x, y \in D_r$  we have*

$$c^{-1} G_{D_r}(x, y) \leq \tilde{G}_{D_r}(x, y) \leq c G_{D_r}(x, y).$$

*(ii) There exist constants  $C, \epsilon_0 > 0$  depending only on  $D$  and  $\alpha$  such that for all  $r \in [0, \epsilon_0]$ ,  $x \in D_r$  and  $y \in D_r^c$  we have*

$$C^{-1} P_{D_r}(x, y) \leq \tilde{P}_{D_r}(x, y) \leq C P_{D_r}(x, y).$$

*Proof.* In order to show (i) we follow the proof of the Theorem 1 in [9]. We analyse below its crucial points.

1. *Comparison of Green functions  $\tilde{G}_S(x, y)$  and  $G_S(x, y)$  for "small" sets  $S$ ,* [9, Lemma 13], based on estimates from [9, Lemma 11]. Thanks to property (7), we see that the comparison of Green functions  $\tilde{G}_{S_r}(x, y)$  and  $G_{S_r}(x, y)$  for small sets  $S$  holds with a common constant  $c$ , when  $r \in [0, \epsilon_0]$ .

2. *Harnack inequalities for  $L$  and the Boundary Harnack Principle,* [9, Lemmas 15, 16]. Thanks to 1., we get them uniformly with respect to  $r \in [0, \epsilon_0]$ .

3. Now the proof of (i) for any  $\mathcal{C}^{1,1}$  open set  $D$  is the same as in Section 5 of [9].

The part (ii) is implied by (i), applying the Ikeda-Watanabe formula for the Poisson kernel  $\tilde{P}_D$ , see [9, Lemma 6 and (39)]. Recall that the Lévy system for the process  $\tilde{X}_t$  is given by the Lévy measure of the  $\alpha$ -stable process  $X_t$ .  $\square$

An immediate consequence of Lemma 1 and [21, Theorem 22] is the following estimate of the Poisson kernel of  $L$  for  $D_r$ .

**Corollary 2.** *There exist positive constants  $C, \epsilon_0$  depending only on  $D$ ,  $\alpha$  and  $b$  such that for all  $r \in [0, \epsilon_0]$ ,  $x \in D_r$  and  $y \in D_r^c$  we have*

$$\frac{C^{-1}\delta_{D_r}^{\alpha/2}(x)}{\delta_{D_r}^{\alpha/2}(y)(1+\delta_{D_r}(y))^{\alpha/2}|x-y|^d} \leq \tilde{P}_{D_r}(x, y) \leq \frac{C\delta_{D_r}^{\alpha/2}(x)}{\delta_{D_r}^{\alpha/2}(y)(1+\delta_{D_r}(y))^{\alpha/2}|x-y|^d}.$$

## 2.2 Derivatives of the Poisson kernel for $\Delta^{\alpha/2}$

In this section we prove useful gradient estimates for the Poisson kernel of  $\Delta^{\alpha/2}$  for  $D$ ,  $0 < \alpha < 2$ .

Consider a ball  $B = B(\xi_0, r) \subset \bar{B} \subset D$  and let  $P_B$  be the Poisson kernel of  $B$

$$P_B(x, y) = C_\alpha^d \left[ \frac{r^2 - |x - \xi_0|^2}{|y - \xi_0|^2 - r^2} \right]^{\alpha/2} |x - y|^{-d}, \quad x \in B, y \in \bar{B}^c \quad (8)$$

and equal to 0 elsewhere. By [11, Lemma 3.1],

$$|\nabla_x P_B(x, y)| \leq (d + \alpha) \frac{P_B(x, y)}{r - |x - \xi_0|}, \quad x \in B, y \in (\bar{B})^c. \quad (9)$$

We will now show analogous estimate for  $\mathcal{C}^{1,1}$  bounded open sets.

**Lemma 3.** Suppose  $0 < \alpha < 2$ . Let  $D$  be a bounded  $\mathcal{C}^{1,1}$  open set in  $\mathbb{R}^d$  and let  $P_D(x, y)$  be the Poisson kernel of  $\Delta^{\alpha/2}$  for  $D$ . Then we have

$$|\nabla_x P_D(x, y)| \leq (\alpha + d) \frac{P_D(x, y)}{\delta_D(x)}, \quad x \in D, y \in (\overline{D})^c. \quad (10)$$

*Proof.* For  $x \in D$  denote  $B_x = B(x, \delta_D(x))$ . In view of [10, (29)] we have

$$P_D(x, y) = P_{B_x}(x, y) + \int_{B_x^c} P_{B_x}(x, z) P_D(z, y) dz.$$

By (9) and bounded convergence we have

$$\begin{aligned} |\nabla_x P_D(x, y)| &\leq |\nabla_x P_{B_x}(x, y)| + \left| \nabla_x \int_{B_x^c} P_{B_x}(x, z) P_D(z, y) dz \right| \\ &\leq (d + \alpha) \frac{P_{B_x}(x, y)}{\delta_D(x)} + \int_{B_x^c} |\nabla_x P_{B_x}(x, z)| P_D(z, y) dz \leq (d + \alpha) \frac{P_D(x, y)}{\delta_D(x)}. \end{aligned}$$

□

From (9) and the dominated convergence theorem it follows, that if  $f$  is  $\alpha$ -harmonic in  $D$  then

$$\frac{\partial}{\partial x_i} f(x) = \int_{B^c} \frac{\partial}{\partial x_i} P_B(x, y) f(y) dy, \quad i = 1, \dots, d. \quad (11)$$

The estimate (9) and (11) gives ([11, Lemma 3.2])

**Lemma 4.** Let  $U$  be an arbitrary open set in  $\mathbb{R}^d$  and let  $\alpha \in (0, 2)$ . For every non-negative function  $u$  on  $\mathbb{R}^d$  which is  $\alpha$ -harmonic in  $U$ , we have

$$|\nabla u(x)| \leq d \frac{u(x)}{\delta_U(x)}, \quad x \in U. \quad (12)$$

Since  $G_U(\cdot, y)$  is  $\alpha$ -harmonic in  $U \setminus \{y\}$ , for every  $y \in U$  we obtain

$$|\nabla_x G_U(x, y)| \leq d \frac{G_U(x, y)}{\delta_U(x) \wedge |x - y|}, \quad x, y \in U, \quad x \neq y. \quad (13)$$



## 2.3 A uniform integrability result

One of the important results of [9] is

**Lemma 5.**  $G_D(y, w)/[\delta(w) \wedge |y - w|]$  is uniformly in  $y$  integrable against  $|b(w)|dw$ .

In the next lemma we will show a similar property for the family of functions  $G_{D_{2^{-n}}}(x, w)M_D(w, Q)\delta_{D_{2^{-n}}}(w)^{-1}$ .

**Lemma 6.** Let  $x \in D$  be fixed. There exists  $N = N(D, x) \in \mathbb{N}$  such that the functions

$$G_{D_{2^{-n}}}(x, w)M_D(w, Q)\delta_{D_{2^{-n}}}(w)^{-1}$$

are uniformly in  $Q \in \partial D$  and  $n > N$  integrable against  $|b(w)|dw$ .

*Proof.* In view of the properties of  $D$  and of the estimates of  $G_{D_{2^{-n}}}(x, w)$  and  $M_D(w, Q)$ , we can choose  $N = N(D, x) \in \mathbb{N}$  sufficiently large, such that for all  $n > N$  we have

$$\begin{aligned} \frac{G_{D_{2^{-n}}}(x, w)M_D(w, Q)}{\delta_{D_{2^{-n}}}(w)} &\leq \frac{c\mathbf{1}_{D_{2^{-n}}}(w)\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d|w - x|^{d-\alpha}} \\ &\leq \tilde{c}\mathbf{1}_{D_{2^{-n}}}(w) \left( |w - x|^{\alpha-d} + \frac{\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d} \right), \end{aligned}$$

where  $c$  and  $\tilde{c}$  depends only on  $D, \alpha$  and  $x$ . The first term in the parentheses is integrable against  $|b(w)|dw$  independently of  $Q, n$ , so we only need to consider the second one. For  $w \in D_{2^{-N}}$  and  $Q \in \partial D$  we have

$$\frac{\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d} \leq \text{diam}(D)^{\alpha/2} 2^{(N+1)(d+1-\alpha/2)},$$

and for  $w \in D_{2^{-n}} \setminus D_{2^{-N}}, Q \in \partial D$  we get

$$\begin{aligned} \frac{\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d} &= \frac{(\delta_{D_{2^{-n}}}(w) + 2^{-n})^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d} \\ &\leq 2^{\alpha/2} \left( |w - Q|^{\alpha-d-1} + \frac{2^{-n\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d} \right). \end{aligned}$$

Since  $\mathbf{1}_{D_{2^{-n}}}(w)|w - Q|^{\alpha-d-1}$  is uniformly in  $Q, n$  integrable against  $|b(w)|dw$ , we can restrict our attention to the function

$$H_n(w, Q) = \frac{2^{-n\alpha/2}\mathbf{1}_{D_{2^{-n}}}(w)}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w - Q|^d}$$

Let  $R > N$ ,  $R \in \mathbb{N}$ . For  $k, m, n \in \mathbb{N}$ ,  $k \geq R$ ,  $m \geq N$ ,  $n > N$  we define

$$W_{k,m}^n(Q, R) = \left\{ w \in D_{2^{-n}} : \frac{1}{2^{k+1}} < \delta_{D_{2^{-n}}}(w) \leq \frac{1}{2^k}, \quad \frac{1}{2^{m+1}} < |w - Q| \leq \frac{1}{2^m} \right\},$$

$$W_k^n(Q, R) = \left\{ w \in D_{2^{-n}} : \frac{1}{2^{k+1}} < \delta_{D_{2^{-n}}}(w) \leq \frac{1}{2^k}, \quad |w - Q| > \frac{1}{2^N} \right\}.$$

We note that  $W_{k,m}^n(Q, R) = \emptyset$  for  $k < m$  or  $m \geq n$ .  $W_{k,m}^n(Q, R)$  can be covered by  $c_1(2^{k-m})^{d-1}$  balls of radii  $2^{-k}$ , where  $c_1 = c_1(D)$ . For  $r > 0$  denote

$$K_r = \sup_{z \in \mathbb{R}^d} \int_{B(z, r)} |b(w)| |z - w|^{\alpha-d-1} dw.$$

Then  $K_r \rightarrow 0$  as  $r \downarrow 0$ . We have

$$\begin{aligned} & \int_{W_{k,m}^n(Q, R)} H_n(w, Q) |b(w)| dw \\ & \leq (2^{k+1})^{1-\alpha/2} (2^{m+1})^d 2^{-n\alpha/2} c_1 (2^{k-m})^{d-1} \sup_{z \in D} \int_{B(z, 2^{-k})} |b(w)| dw \\ & \leq (2^{k+1})^{1-\alpha/2} (2^{m+1})^d 2^{-n\alpha/2} c_1 (2^{k-m})^{d-1} (2^k)^{\alpha-d-1} K_{2^{-k}} \\ & \leq c_2 K_{2^{-R}} (2^k)^{\alpha/2-1} 2^m 2^{-n\alpha/2}, \end{aligned}$$

where  $c_2 = c_2(D, b, \alpha)$ . Furthermore,  $W_k^n(Q, R)$  can be covered by  $c_3(2^k)^{d-1}$  balls of radii  $2^{-k}$ , where  $c_3 = c_3(D)$ , and thus

$$\begin{aligned} & \int_{W_k^n(Q, R)} H_n(w, Q) |b(w)| dw \\ & \leq (2^{k+1})^{1-\alpha/2} 2^{Nd} 2^{-n\alpha/2} c_3 (2^k)^{d-1} \sup_{z \in D} \int_{B(z, 2^{-k})} |b(w)| dw \\ & \leq (2^{k+1})^{1-\alpha/2} 2^{Nd} 2^{-n\alpha/2} c_3 (2^k)^{d-1} (2^k)^{\alpha-d-1} K_{2^{-k}} \\ & \leq c_4 K_{2^{-R}} (2^k)^{\alpha/2-1} 2^{-n\alpha/2}, \end{aligned}$$

where  $c_4 = c_4(D, b, \alpha)$ . Let  $A_R^n = \{w \in D_{2^{-n}} : \delta_{D_{2^{-n}}}(w) \leq 2^{-R}\}$ . Then

$$A_R^n = \sum_{k=R}^{\infty} W_k^n(Q, R) + \sum_{m=N}^{n-1} \sum_{k=R \vee m}^{\infty} W_{k,m}^n(Q, R),$$

and we obtain

$$\begin{aligned}
& \int_{A_R^n} H_n(w, Q) |b(w)| dw \\
& \leq c_4 K_{2^{-R}} 2^{-n\alpha/2} \sum_{k=R}^{\infty} (2^k)^{\alpha/2-1} + \sum_{m=N}^{n-1} \sum_{k=R \vee m}^{\infty} c_2 K_{2^{-R}} (2^k)^{\alpha/2-1} 2^m 2^{-n\alpha/2} \\
& \leq c_5 K_{2^{-R}} \left( 2^{-n\alpha/2} (2^R)^{\alpha/2-1} + \sum_{m=N}^{n-1} (2^{n-m})^{-\alpha/2} \right) \leq c_6 K_{2^{-R}},
\end{aligned}$$

where  $c_6 = c_6(D, b, \alpha)$ . For  $w \in D_{2^{-n}} \setminus A_R^n$  we have

$$H_n(w, Q) < 2^{-n\alpha/2} (2^R)^{1-\alpha/2} \left( \frac{1}{2^R} + \frac{1}{2^n} \right)^{-d} < 4^{dR},$$

so  $B_R^n(Q) := \{w : H_n(w, Q) > 4^{dR}\} \subset A_R^n$  for all  $Q \in \partial D$  and  $n > N$ . Therefore

$$\lim_{R \rightarrow \infty} \sup_{Q \in \partial D, n > N} \int_{B_R^n(Q)} H_n(w, Q) |b(w)| dw \leq \lim_{R \rightarrow \infty} c_6 K_{2^{-R}} = 0.$$

□

### 3 Martin kernel and Martin representation

In this section we will discuss first the existence and the properties of the Martin kernel of  $L$  for a  $\mathcal{C}^{1,1}$  bounded open set  $D$ . Next we will investigate the Martin representation for non-negative singular  $L$ -harmonic functions on  $D$ .

#### 3.1 Existence and Perturbation formula for the $L$ -Martin kernel

In order to prove the existence of the  $L$ -Martin kernel, we will need the following property of the Green function for  $\Delta^{\alpha/2}$ .

**Lemma 7.** *For all  $x \in D$  and  $Q \in \partial D$  we have*

$$\lim_{y \rightarrow Q} \frac{\nabla_x G_D(x, y)}{G_D(x, y)} = \nabla_x M_D(x, Q).$$

*Proof.* Let  $z \in D, Q \in \partial D$  and choose  $r > 0$  such that  $\overline{B(z, r)} \subset D$  and  $B(z, r) \cap B(Q, r) = \emptyset$ . Since  $G_D(\cdot, y)$  is  $\alpha$ -harmonic in  $B(z, r)$  for  $y \in B(Q, r) \cap D$ , by (11), we have

$$\begin{aligned} \frac{\nabla_x G_D(x, y)}{G_D(x_0, y)} &= \nabla_x \int_{B(z, r)^c} P_{B(z, r)}(x, w) \frac{G_D(w, y)}{G_D(x_0, y)} dw \\ &= \int_{B(z, r)^c} \nabla_x P_{B(z, r)}(x, w) \frac{G_D(w, y)}{G_D(x_0, y)} dw, \quad x \in B(z, r). \end{aligned}$$

Furthermore, by (9) and (5),

$$|\nabla_x P_{B(z, r)}(x, w)| \frac{G_D(w, y)}{G_D(x_0, y)} \leq C \frac{P_{B(z, r)}(x, w)}{r - |x - z|} \frac{G_D(w, y)}{\delta_D(y)^{\alpha/2}}.$$

We now use the estimate [9, (25)] and by considering the cases  $\delta_D(w) > |w - y|$  and  $\delta_D(w) \leq |w - y|$  we get  $\frac{G_D(w, y)}{\delta_D(y)^{\alpha/2}} \leq C|w - y|^{\alpha/2 - d}$ . Hence the last term is uniformly in  $y \in B(Q, r/2) \cap D$  integrable against  $dw$ , and thus

$$\begin{aligned} &\lim_{y \rightarrow Q} \int_{B(z, r)^c} \nabla_x P_{B(z, r)}(x, w) \frac{G_D(w, y)}{G_D(x_0, y)} dw \\ &= \int_{B(z, r)^c} \nabla_x P_{B(z, r)}(x, w) M_D(w, Q) dw = \nabla_x M_D(x, Q). \end{aligned}$$

The last equality follows from (11) and the  $\alpha$ -harmonicity of the Martin kernel.  $\square$

Thanks Lemma 7 we obtain the main result of this subsection.

**Theorem 8.** *Let  $x \in D$  and  $Q \in \partial D$ . Let  $M_D(x, Q)$  be the Martin kernel of  $\Delta^{\alpha/2}$  for  $D$ . Denote*

$$l_D(x, Q) = M_D(x, Q) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla_z M_D(z, Q) dz$$

*The function  $l_D(x, Q)$  is well defined for  $x \in D$  and  $Q \in \partial D$  and  $l_D(x, Q) > 0$ . Moreover the following limit exists and equals:*

$$\lim_{y \rightarrow Q} \frac{\tilde{G}_D(x, y)}{\tilde{G}_D(x_0, y)} = \frac{l_D(x, Q)}{l_D(x_0, Q)}.$$

*Thus the Martin kernel of  $L = \Delta^{\alpha/2} + b \cdot \nabla$  for  $D$  exists and equals*

$$\tilde{M}_D(x, Q) = \frac{1}{l_D(x_0, Q)} \left[ M_D(x, Q) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla_z M_D(z, Q) dz \right]. \quad (14)$$

*Proof.* We divide the perturbation formula (4) for the Green function  $\tilde{G}_D(x, y)$  by  $G_D(x_0, y)$  and let  $y \rightarrow Q$ .

The exchange of  $\lim_{y \rightarrow Q}$  and  $\int_D$  is justified by Lemma 11 of [9], see the formula (49) in its proof. Note that by the Boundary Harnack Principle,  $G_D(x_0, y) \approx G_D(x, y)$  when  $y \in B(Q, \epsilon_0)$ , a sufficiently small ball around  $Q$ . We also use the estimates (5), (13) and (2).

The exchange of  $\lim_{y \rightarrow Q}$  and  $\nabla_z$  is justified by Lemma 7. Finally

$$\lim_{y \rightarrow Q} \frac{\tilde{G}_D(x, y)}{G_D(x_0, y)} = M_D(x, Q) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla M_D(z, Q) = l_D(x, Q).$$

The strict positivity of the function  $l_D(x, Q)$  follows from (2), which implies that there exists  $a > 0$  such that

$$l_D(x, Q) \geq a M_D(x, Q) > 0. \quad (15)$$

Now we consider the quotient

$$\frac{\tilde{G}_D(x, y)}{\tilde{G}_D(x_0, y)} = \frac{\tilde{G}_D(x, y)}{G_D(x_0, y)} \frac{G_D(x_0, y)}{\tilde{G}_D(x_0, y)} \rightarrow \frac{l_D(x, Q)}{l_D(x_0, Q)},$$

when  $y \rightarrow Q$ . □

Directly from the definition of  $\tilde{M}_D(x, Q)$  and (2) we obtain the following corollary.

**Corollary 9.** *There is a constant  $c$  such that for all  $x \in D$  and  $Q \in \partial D$ ,*

$$c^{-1} M_D(x, Q) \leq \tilde{M}_D(x, Q) \leq c M_D(x, Q). \quad (16)$$

### 3.2 Properties of the $L$ -Martin kernel

We will now study further properties of the Martin kernel of  $L$  for  $D$ . We start with the following useful formulas.

**Lemma 10.** *Consider a  $\mathcal{C}^{1,1}$  open set  $U \subset \bar{U} \subset D$ .*

(i) **(Perturbation formula for the Poisson kernel)** *For all  $x \in U, z \in (\bar{U})^c$*

$$\tilde{P}_U(x, z) = P_U(x, z) + \int_U \tilde{G}_U(x, w) b(w) \cdot \nabla P_U(w, z) dw. \quad (17)$$

(ii) *Let  $Q \in \partial D$ . We have the following expression for the  $L$ -Poisson integral of the Martin kernel of  $\Delta^{\alpha/2}$ :*

$$\begin{aligned} \tilde{P}_U M_D(x, Q) &:= \int_{U^c} \tilde{P}_U(x, y) M_D(y, Q) dy \\ &= M_D(x, Q) + \int_U \tilde{G}_U(x, z) b(z) \cdot \nabla M_D(z, Q) dz, \quad x \in U. \end{aligned} \quad (18)$$

*Proof.* In the following we apply the Ikeda-Watanabe formula for the Poisson kernels  $\tilde{P}_U$  and  $P_U$ . By (4) and Fubini's theorem, for any  $x \in U$  and  $z \in U^c$ ,

$$\begin{aligned}\tilde{P}_U(x, z) &= \int_U \mathcal{A}_{d,\alpha} \frac{\tilde{G}_U(x, y)}{|z - y|^{d+\alpha}} dy \\ &= \int_U \frac{\mathcal{A}_{d,\alpha}}{|z - y|^{d+\alpha}} \left[ G_U(x, y) + \int_U \tilde{G}_U(x, w) b(w) \cdot \nabla G_U(w, y) dw \right] dy \\ &= P_U(x, z) + \int_U \tilde{G}_U(x, w) b(w) \cdot \nabla P_U(w, z) dw.\end{aligned}$$

For the necessary exchanges of order of integration and derivation in the last formula, we apply (13), (2), Lemma 5 and bounded convergence theorem. In order to prove (ii), we use (i) and insert the formula (17) in  $\int_{U^c} \tilde{P}_U(x, y) M_D(y, Q) dy$ . We obtain

$$\int_{U^c} \tilde{P}_U(x, y) M_D(y, Q) dy = M_D(x, Q) + \int_U \tilde{G}_U(x, z) b(z) \cdot \nabla M_D(z, Q) dz$$

In the last equality the use of Fubini theorem and the exchange of  $\int$  and  $\nabla$  are justified by (10), (6), Lemma 5 and bounded convergence.  $\square$

**Lemma 11.** *The Martin kernel  $\tilde{M}_D(\cdot, \cdot)$  is jointly continuous on  $D \times \partial D$ .*

*Proof.* By Theorem 8 and the continuity of  $M_D(\cdot, \cdot)$ , it suffices to show the joint continuity on  $D \times \partial D$  of the function

$$f(x, Q) = \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla M_D(z, Q) dz.$$

Let  $z \in D$ . By (11) and the  $\alpha$ -harmonicity of  $M_D(\cdot, Q)$ , for  $r > 0$  sufficiently small, we have

$$\begin{aligned}\nabla M_D(z, Q) &= \nabla \int_{B(z, r)^c} P_{B(z, r)}(z, w) M_D(w, Q) dy \\ &= \int_{B(z, r)^c} \nabla P_{B(z, r)}(z, w) M_D(w, Q) dw.\end{aligned}$$

From (9) and (6) it follows, that  $\nabla P_{B(z, r)}(z, w) M_D(w, Q)$  is uniformly in  $Q$  integrable against  $dw$ . This implies that  $\nabla M_D(z, \cdot)$  is continuous on  $\partial D$  for every  $z \in D$ . Let now  $x \in D$  and choose  $r > 0$  such that  $\overline{B(x, r)} \subset D$ . By (2), (5), (12) and (6), for all  $y \in B(x, r)$ ,  $z \in D$  and  $Q \in \partial D$ , we have

$$\tilde{G}_D(y, z) |\nabla M_D(z, Q)| \leq \frac{C \delta_D(z)^{\alpha-1}}{|y - z|^{d-\alpha} |z - Q|^d} \leq \frac{C}{|y - z|^{d-\alpha} |z - Q|^{d+1-\alpha}}. \quad (19)$$

Hence,  $\tilde{G}_D(y, z)|\nabla M_D(z, Q)|$  is uniformly in  $y \in B(x, r)$  and  $Q \in \partial D$  integrable against  $|b(z)|dz$ , which gives the continuity of  $f(\cdot, \cdot)$ .  $\square$

We will now use Lemma 10 to show  $L$ -harmonicity of  $\tilde{M}(x, Q)$ .

**Theorem 12.** *For every  $Q \in \partial D$  the Martin kernel  $\tilde{M}(x, Q)$  is a singular  $L$ -harmonic function of  $x$  on  $D$ .*

*Proof.* First consider a  $\mathcal{C}^{1,1}$  open set  $U = D_r$ . We note that

$$\tilde{G}_D(x, w) = \tilde{G}_U(x, w) + \int_{U^c} \tilde{P}_U(x, z) \tilde{G}_D(z, w) dz. \quad (20)$$

By (18), (17), (20) and Fubini's theorem

$$\begin{aligned} \tilde{P}_U l_D(x, Q) &= \int_{U^c} \tilde{P}_U(x, z) l_D(z, Q) dz \\ &= \int_{U^c} \tilde{P}_U(x, z) M_D(z, Q) dz \\ &\quad + \int_{U^c} \tilde{P}_U(x, z) \int_D \tilde{G}_D(z, w) b(w) \cdot \nabla M_D(w, Q) dw dz \\ &= M_D(x, Q) + \int_{U^c} \int_U \tilde{G}_U(x, w) b(w) \cdot \nabla P_U(w, z) dw M_D(z, Q) dz \\ &\quad + \int_D \left[ \int_{U^c} \tilde{P}_U(x, z) \tilde{G}_D(z, w) dz \right] b(w) \cdot \nabla M_D(w, Q) dw \\ &= M_D(x, Q) + \int_U \tilde{G}_U(x, w) b(w) \cdot \nabla M_D(w, Q) dw \\ &\quad + \int_D [\tilde{G}_D(x, w) - \tilde{G}_U(x, w)] b(w) \cdot \nabla M_D(w, Q) dw \\ &= M_D(x, Q) + \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla M_D(w, Q) dw = l_D(x, Q). \end{aligned}$$

Thus the function  $l_D(x, Q)$  is regular  $L$ -harmonic on each set  $U = D_r$  for  $r$  sufficiently small. By the strong Markov property, it has the mean value property on each open set  $U \subset \bar{U} \subset D$ .  $\square$

### 3.3 $L$ -Martin representation

The objective of this section is to prove the following Martin representation theorem for non-negative singular  $L$ -harmonic functions on  $D$ .

**Theorem 13.** *For every non-negative finite measure  $\nu$  on  $\partial D$  the function  $u$  given by*

$$u(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\nu(Q), \quad (21)$$

*is singular  $L$ -harmonic on  $D$ . Conversely, if  $u$  is non-negative singular  $L$ -harmonic on  $D$ , then there exists a unique non-negative finite measure  $\nu$  on  $\partial D$  verifying (21).*

*Proof.* The  $L$ -harmonicity of the Martin integral (21) and the uniqueness of the representation follow from Theorem 12, Lemma 11, (6), (16) and Fubini theorem, in the same way as in the case of the Martin representation for  $\alpha$ -harmonic functions in [6, proof of Theorem 1]. We will now focus on the existence part. By  $L$ -harmonicity of  $u$  and by (17) we have for each  $n$

$$\begin{aligned} u(x) &= \int_{D_{1/n}^c} \tilde{P}_{D_{1/n}}(x, y) u(y) dy = \\ &= \int_{D_{1/n}^c} u(y) \left[ P_{D_{1/n}}(x, y) + \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x, w) b(w) \cdot \nabla P_{D_{1/n}}(w, y) dw \right] dy. \end{aligned}$$

Denote

$$u_n^*(x) = \int_{D_{1/n}^c} P_{D_{1/n}}(x, y) u(y) dy.$$

By (10), [9, (72)] and Lemma 5 we have

$$\begin{aligned} &\int_{D_{1/n}^c} \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x, w) |b(w)| |\nabla P_{D_{1/n}}(w, y)| u(y) dw dy \\ &\leq C \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x, w) |b(w)| \frac{u(w)}{\delta_{D_{1/n}}(w)} dw < \infty, \end{aligned}$$

where  $C = C(\alpha, b, D_{1/n}) > 0$ . Hence, by Fubini theorem

$$u(x) = u_n^*(x) + \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x, w) b(w) \cdot \int_{D_{1/n}^c} \nabla P_{D_{1/n}}(w, y) u(y) dy dw.$$

The function  $u_n^*$  is  $\alpha$ -harmonic on  $D_{1/n}$ , so it is differentiable. In order to justify the exchange of  $\int$  and  $\nabla$  in the last integral we fix  $w \in D_{1/n}$ . Then by (10) and [9, (72)], for  $\varepsilon > 0$  sufficiently small and all  $w' \in B(w, \varepsilon)$  and  $y \in D_{1/n}^c$  we have

$$|\nabla_{w'} P_{D_{1/n}}(w', y) u(y)| \leq C \frac{u(y)}{\delta_{D_{1/n}}(y)^{\alpha/2}},$$



where  $C = C(\alpha, b, D_{1/n}, \varepsilon) > 0$ . Since the last term is integrable on  $D_{1/n}^c$ , by the dominated convergence we obtain

$$u(x) = u_n^*(x) + \int_D \tilde{G}_{D_{1/n}}(x, w) b(w) \cdot \nabla u_n^*(w) dw. \quad (22)$$

We now study the sequence  $u_n^*(x)$  in the same way as K. Bogdan [6] in the proof of the existence part of the  $\Delta^{\alpha/2}$ -Martin representation, with the difference that in our case the function  $u$  under the integral defining  $u_n^*$  is not  $\alpha$ -harmonic.

Like in [6, (2.27)] we have

$$u_n^*(x) = \int_{D_{1/n}^c} P_{D_{1/n}}(x, y) u(y) dy = \int_{D_{1/n}^c} \int_{D_{1/n}} u(y) \mathcal{A}_{d, \alpha} \frac{G_{D_{1/n}}(x, \xi)}{|\xi - y|^{d+\alpha}} d\xi dy$$

Set  $\mu_n(d\xi) = \mathcal{A}_{d, \alpha} G_{D_{1/n}}(x_0, \xi) \int_{D_{1/n}} \frac{u(y)}{|\xi - y|^{d+\alpha}} dy d\xi$ . Lemma 1 implies that

$$\mu_n(\mathbb{R}^d) = \int_{D_{1/n}^c} P_{D_{1/n}}(x_0, y) u(y) dy \leq C \int_{D_{1/n}^c} \tilde{P}_{D_{1/n}}(x_0, y) u(y) dy = cu(x_0) < \infty$$

(recall that if  $u$  was  $\alpha$ -harmonic, then  $\mu_n(\mathbb{R}^d) = u(x_0)$ ). We obtain

$$u_n^*(x) = \int_{D_{1/n}} \frac{G_{D_{1/n}}(x, \xi)}{G_{D_{1/n}}(x_0, \xi)} \mu_n(d\xi).$$

The only other property of the function  $u$  intervening in the proof of the existence part of the  $\Delta^{\alpha/2}$ -Martin representation in [6] is

$$\lim_n \int_{D_{1/n}^c} u(y) dy = 0$$

and it also holds in our case: the  $L$ -harmonic function  $u$  is integrable on  $D_{1/n}^c$  for every  $n$ . The sequence  $(\mu_n)$  of simultaneously bounded finite measures with support contained in  $\bar{D}$  is tight. We choose a subsequence  $\mu_{n_k}$  converging to a finite (perhaps zero) measure  $\mu$ . This choice is common for all  $x$ . Without loss of generality, we may suppose that  $(n_k)$  is a subsequence of  $(2^{-n})$ . The limit measure  $\mu$  satisfies

$$\text{supp}(\mu) \subset \partial D.$$

Exactly as in the proof of the existence part of the  $\Delta^{\alpha/2}$ -Martin representation in [6], we deduce that for all  $x \in D$  the limit

$$\lim_k u_{n_k}^*(x) = u^*(x)$$

exists and

$$u^*(x) = \int_{\partial D} M_D(x, Q) d\mu(Q). \quad (23)$$

Furthermore, in view of (11), for  $x \in D_{1/n}$  and  $r > 0$  sufficiently small we have

$$\nabla u_n^*(x) = \nabla \int_{B(x,r)^c} P_{B(x,r)}(x, y) u_n^*(y) dy = \int_{B(x,r)^c} \nabla P_{B(x,r)}(x, y) u_n^*(y) dy.$$

By Lemma 1 and (9) we have

$$|\nabla P_{B(x,r)}(x, y) u_n^*(y)| \leq C \frac{P_{B(x,r)}(x, y)}{r - |x|} u(y),$$

and by the dominated convergence we get  $\nabla u_{n_k}^*(x) \rightarrow \nabla u^*(x)$  as  $k \rightarrow \infty$ . We also have  $\tilde{G}_{D_{1/n}}(x, w) \nearrow \tilde{G}_D(x, w)$ . In order to justify the passage with the limit under the integral sign in (22) with  $n_k$  instead of  $n$  we observe that the functions  $\tilde{G}_{D_{1/n_k}}(x, w) b(w) \cdot \nabla u_{n_k}^*(w)$  are uniformly integrable on  $D$ . Clearly, by Lemma 1 we have  $c^{-1} u_n^*(w) \leq u(w) \leq c u_n^*(w)$ , where  $c$  does not depend on  $n$ , thus  $u_n^*(w) \leq c u^*(w)$ . By the gradient estimates we get

$$\tilde{G}_{D_{1/n}}(x, w) |b(w)| |\nabla u_n^*(w)| \leq \tilde{G}_{D_{1/n}}(x, w) |b(w)| \frac{u^*(w)}{\delta_{D_{1/n}}(w)},$$

and the uniform integrability follows from (23), Lemma 1 and Lemma 6. Therefore

$$u(x) = u^*(x) + \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla u^*(w) dw, \quad (24)$$

which, using (23), becomes

$$\begin{aligned} u(x) &= \\ &= \int_{\partial D} M_D(x, Q) d\mu(Q) + \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla \int_{\partial D} M_D(w, Q) d\mu(Q) dw. \end{aligned} \quad (25)$$

By the gradient estimates and dominated convergence we also get

$$\nabla \int_{\partial D} M_D(w, Q) d\mu(Q) = \int_{\partial D} \nabla M_D(w, Q) d\mu(Q), \quad w \in D.$$

Define a measure  $\nu$  on  $\partial D$  by  $\nu(dQ) = l_D(x_0, Q) d\mu(Q)$ . As the function  $Q \rightarrow l_D(x_0, Q)$  is continuous positive, the measure  $\nu$  is finite positive on  $\partial D$ . Using

Fubini theorem in (25) and the perturbation formula for  $\tilde{M}_D$  from Theorem 8, we obtain

$$u(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\nu(Q).$$

□

**Remark 1.** We point out that the proof of Theorem 13 is based on the perturbation formula. In fact, the methods used in [6] in order to prove the Martin representation theorem for singular  $\alpha$ -harmonic functions can not be applied in the present case because the Green function  $\tilde{G}_D(x, y)$  is not  $L$ -harmonic on  $D \setminus \{x\}$  as a function of  $y$ .

**Corollary 14. (Perturbation formula for singular  $L$ -harmonic functions)** Let  $v(x) \geq 0$  be a singular  $L$ -harmonic function on  $D$  with the Martin representation

$$v(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\nu(Q), \quad x \in D. \quad (26)$$

Define a singular  $\alpha$ -harmonic function  $v^*$  on  $D$  by

$$v^*(x) = \int_{\partial D} M_D(x, Q) \frac{d\nu(Q)}{l(x_0, Q)}, \quad x \in D \quad (27)$$

Then the following formula holds

$$v(x) = v^*(x) + \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla v^*(w) dw. \quad (28)$$

*Proof.* Observe that by (15) there exists  $\delta > 0$  such that

$$l_D(x_0, Q) > \delta > 0$$

for all  $Q \in \partial D$ . Thus the measure  $d\mu(Q) = \frac{d\nu(Q)}{l_D(x_0, Q)}$  is finite and the function  $v^*$  is well defined. By the unicity of the Martin representation and the formula (23), the function  $v^*$  defined by (27) is the same as the function  $v^*$  defined by a limit procedure and associated to  $v$  in the proof of the Theorem 13. Hence the formula (24) holds for  $v$  and  $v^*$ . It is equivalent to (28). □

**Corollary 15.** Let  $v(x) \geq 0$  be a singular  $L$ -harmonic function on  $D$ . The functions  $v$  and  $v^*$  are comparable: there exists  $c > 0$  such that for all  $x \in D$

$$c^{-1}v^*(x) \leq v(x) \leq cv^*(x). \quad (29)$$

*Proof.* We use the Martin representations (26), (27), the Corollary 9 and the fact that  $l_D(x_0, Q) > \delta > 0$  for all  $Q \in \partial D$ . □

### 3.4 Perturbation formulas in the diffusion case

In the present article we exploit the perturbation formulas in the case of the singular operator  $L = \Delta^{\alpha/2} + b \cdot \nabla$ ,  $1 < \alpha < 2$ . In this short chapter we make a parenthesis and briefly discuss the case  $\alpha = 2$  and  $d \geq 3$ , corresponding to the diffusion operator

$$L = \frac{1}{2}\Delta + b \cdot \nabla$$

on  $\mathbb{R}^d$ . The potential theory for such diffusion generators was studied by Cranston and Zhao[16], and more recently by Ifra and Riahi[20], Kim and Song[27] and Luks[30]. Our methods allow to enrich this theory by some new perturbation formulas.

We suppose that  $b \in \mathcal{K}_d^1$  and we assume additionally that  $D$  is connected, i.e. it is a domain. Recall that Cranston and Zhao[16] worked under this condition and a complementary second condition  $|b|^2 \in \mathcal{K}_{d-1}^1$ ; Kim and Song[27] suppressed the condition on  $|b|^2$  and considered signed measures in the place of  $b$ .

**Proposition 16.** *Let  $L = \frac{1}{2}\Delta + b \cdot \nabla$  with  $b \in \mathcal{K}_d^1$ . Then the following perturbation formula for the  $L$ -Green function  $\tilde{G}_D$  holds if  $x, y \in \mathbb{R}^d, x \neq y$ .*

$$\tilde{G}_D(x, y) = G_D(x, y) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla_z G_D(z, y) dz. \quad (30)$$

*Proof.* Note that by [27, Theorem 6.2], we have the estimate

$$\tilde{G}_D(x, y) \leq C|x - y|^{2-d}, \quad x, y \in \mathbb{R}^d. \quad (31)$$

The proof of the Proposition is the same as the proof of [9, Lemma 12] in the case  $1 < \alpha < 2$ , with (31) replacing [9, Lemma 7].  $\square$

Let us mention that a perturbation formula for the  $L$ -Green function was proposed in [20], but under a restrictive assumption of boundedness of the Kato norm  $\|b\|$  of  $b$ . A simpler direct proof of the estimate (31) without using the precise estimates [27, Theorem 6.2] should be available.

Next we obtain a perturbation formula for the Martin kernel of Laplacians with a gradient perturbation.

**Proposition 17.** *Let  $L = \frac{1}{2}\Delta + b \cdot \nabla$  with  $b \in \mathcal{K}_d^1$ . Then the following perturbation formula for the  $L$ -Martin kernel  $\tilde{M}_D$  holds if  $x \in D$  and  $Q \in \partial D$ .*

$$\tilde{M}_D(x, Q) = \frac{1}{l_D(x_0, Q)} \left[ M_D(x, Q) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla_z M_D(z, Q) dz \right] \quad (32)$$

where  $l_D(x_0, Q)$  is a continuous function on  $\partial D$ , equal

$$l_D(x_0, Q) = M_D(x_0, Q) + \int_D \tilde{G}_D(x_0, z) b(z) \cdot \nabla_z M_D(z, Q) dz > 0.$$

*Proof.* We follow the proof of the Theorem 8 in the case  $\alpha = 2$ .  $\square$

The next perturbation formula concerns the  $L$ -Poisson kernel  $\tilde{P}_D(x, Q)$ .

**Proposition 18.** *Let  $L = \frac{1}{2}\Delta + b \cdot \nabla$  with  $b \in \mathcal{K}_d^1$ . Then the following perturbation formula for the  $L$ -Poisson kernel  $\tilde{P}_D$  holds if  $x \in D$  and  $Q \in \partial D$ .*

$$\tilde{P}_D(x, Q) = P_D(x, Q) + \int_D \tilde{G}_D(x, z) b(z) \cdot \nabla_z P_D(z, Q) dz. \quad (33)$$

*Proof.* Observe that by the formula (30) the function  $\tilde{G}_D$  has the same differentiability properties as the function  $G_D$ . In particular the inner normal derivative  $\frac{\partial \tilde{G}_D}{\partial n}(x, Q)$  exists for  $x \in D$  and  $Q \in \partial D$ . It is known (see [20, page 173]) and possible to prove by the Green formula that

$$\tilde{P}_D(x, Q) = \frac{\partial \tilde{G}_D}{\partial n}(x, Q).$$

The formula (33) then follows by differentiating of the formula (30) in the direction of the inner normal unit vector  $n$ . We omit the technical details.  $\square$

Let us finish this section by some remarks. The formula  $\tilde{P}_D(x, Q) = \frac{\partial \tilde{G}_D}{\partial n}(x, Q)$  implies, like in the Laplacian case, that the  $L$ -Martin and the  $L$ -Poisson kernels are related by the formula

$$\tilde{M}_D(x, Q) = \frac{\tilde{P}_D(x, Q)}{\tilde{P}_D(x_0, Q)}. \quad (34)$$

On the other hand, if we insert the formula  $M_D(x, Q) = \frac{P_D(x, Q)}{P_D(x_0, Q)}$  into (33), we obtain using (32)

$$\tilde{P}_D(x, Q) = P_D(x_0, Q) l_D(x_0, Q) \tilde{M}_D(x, Q).$$

Evaluating the last equation at  $x_0$  we obtain a formula for the function  $l_D(x_0, Q)$  intervening in the perturbation formula (32)

$$l_D(x_0, Q) = \frac{\tilde{P}_D(x_0, Q)}{P_D(x_0, Q)}$$

and another proof of the formula (34).

## 4 Relative Fatou Theorem for $L$ -harmonic functions

We prove in this section an important boundary property of  $L$ -harmonic functions: the Relative Fatou Theorem. As in the preceding sections, we consider a nonempty bounded  $\mathcal{C}^{1,1}$  open set  $D$ . Recall the Relative Fatou Theorem in the  $\alpha$ -stable case. It was proved in [31] for Lipschitz sets  $D$ .

**Theorem 19.** *Let  $g$  and  $h$  be two non-negative singular  $\alpha$ -harmonic functions on  $D$ , with Martin representations*

$$g(x) = \int_{\partial D} M_D(x, Q) d\mu^{(g)}(Q), \quad h(x) = \int_{\partial D} M_D(x, Q) d\mu^{(h)}(Q), \quad x \in D.$$

*Then, for  $\mu^{(h)}$ -almost all  $Q \in \partial D$ ,*

$$\lim_{x \rightarrow Q} \frac{g(x)}{h(x)} = f(x)$$

*where  $f$  is the density of the absolute continuous part of  $\mu^{(g)}$  in the decomposition  $\mu^{(g)} = f d\mu^{(h)} + \mu_{\text{sing}}^{(g)}$  with respect to the measure  $\mu^{(h)}$ , and  $x \rightarrow Q$  non-tangentially.*

Our objective in this section is to prove an analogous limit property for non-negative singular  $L$ -harmonic functions  $u$  and  $v$  on  $D$ .

If we denote the integral part of the perturbation formula (28) by

$$I_{v^*}(x) = \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla v^*(w) dw$$

then we have

$$u = u^* + I_{u^*}, \quad v = v^* + I_{v^*}$$

where  $u^*$  and  $v^*$  are singular  $\alpha$ -harmonic non-negative functions. We write

$$\frac{u(x)}{v(x)} = \frac{u^*(x)}{v^*(x)} \frac{1 + \frac{I_{u^*}(x)}{u^*(x)}}{1 + \frac{I_{v^*}(x)}{v^*(x)}} \quad (35)$$

The limit boundary behaviors of the quotients  $\frac{u(x)}{v(x)}$  and  $\frac{u^*(x)}{v^*(x)}$  will be related if we control the limit behavior of the quotients  $\frac{I_{u^*}(x)}{u^*(x)}$  and  $\frac{I_{v^*}(x)}{v^*(x)}$ . Thus we start with discussing the properties of the quotient  $\frac{I_h(x)}{h(x)}$  for a singular  $\alpha$ -harmonic non-negative function  $h$ .

**Lemma 20.** *Let the Martin representation  $h(x) = \int_{\partial D} M(x, Q) d\mu^{(h)}(Q)$  for some non-negative finite measure  $\mu$  on  $\partial D$ . Then, if  $Q \notin \text{supp}(\mu^{(h)})$*

$$\lim_{x \rightarrow Q} h(x) = 0$$

*and if  $Q \in \text{supp}(\mu^{(h)})$  and  $x \rightarrow Q$  non-tangentially*

$$\lim_{x \rightarrow Q} h(x) = +\infty.$$

*Proof.* The limit in the case  $Q \notin \text{supp}(\mu^{(h)})$  follows easily from the Martin representation of  $h$  and the Lebesgue theorem. In the case  $Q \in \text{supp}(\mu^{(h)})$  we use the following result of Wu [38].

Let  $f$  be a  $\Delta$ -harmonic function on  $D$ , corresponding via the Martin representation to a finite measure  $\mu = \mu^{(h)}$  on  $\partial D$ . If  $Q \in \text{supp}\mu$ , then

$$\liminf_{x \rightarrow Q} f(x) > 0,$$

provided  $x \rightarrow Q$  non-tangentially. We have, on  $D$  of class  $\mathcal{C}^{1,1}$

$$\begin{aligned} f(x) &= \int_{\partial D} P_D^\Delta(x, y) \mu(dy) \leq c \int_{\partial D} \frac{\delta_D(x)}{|x - y|^d} \mu(dy) \\ &= c \delta_D(x)^{1-\frac{\alpha}{2}} \int_{\partial D} \frac{\delta_D(x)^{\frac{\alpha}{2}}}{|x - y|^d} \mu(dy) \leq C \delta_D(x)^{1-\frac{\alpha}{2}} \int_{\partial D} M_D(x, y) \mu(dy) \end{aligned}$$

Consequently

$$h(x) \geq \frac{1}{C} \frac{f(x)}{\delta_D(x)^{1-\frac{\alpha}{2}}}$$

and the second part of the Lemma follows. □

**Lemma 21.** *The quotient  $\frac{I_h(x)}{h(x)}$  is bounded. More exactly, there exists  $c > 0$  such that*

$$c^{-1} \leq 1 + \frac{I_h(x)}{h(x)} \leq c. \quad (36)$$

*Proof.* Observe that by Corollary 15 and the formula (28), the quotient  $\frac{I_{v^*}(x)}{v^*(x)}$  is bounded. More exactly, there exists  $c > 0$  such that

$$c^{-1} \leq 1 + \frac{I_{v^*}(x)}{v^*(x)} \leq c.$$

As the function  $l_D(x_0, Q)$  is bounded, any singular  $\alpha$ -harmonic non-negative function  $h$  is of the form  $v^*$  for a singular  $L$ -harmonic non-negative function  $v$ . □

By (2), if we denote

$$J_h(x) = \int_D G_D(x, w) b(w) \cdot \nabla h(w) dw$$

then

$$I_h(x) \sim J_h(x)$$

In particular, by Lemma 21, the quotient  $J_h(x)/h(x)$  is bounded. We prove a much stronger property of this quotient in the following lemma.

**Lemma 22.** *Let  $h$  be a non-negative singular  $\alpha$ -harmonic function on  $D$ , with the Martin representation  $h(x) = \int_{\partial D} M_D(x, Q) d\mu^{(h)}(Q)$  for a finite measure  $\mu^{(h)}$  on  $\partial D$ . Then, when  $Q \in \text{supp}(\mu^{(h)})$  and  $x \rightarrow Q$  non-tangentially, we have*

$$\lim_{x \rightarrow Q} \frac{J_h(x)}{h(x)} = 0.$$

*Proof.* We will show that  $\frac{G_D(x, w)h(w)}{h(x)\delta_D(w)}$  is uniformly integrable in  $x \in D$  against the measure  $|b(w)|dw$ . Let  $\varepsilon > 0$ . Since  $J_h(x)/h(x)$  is bounded it suffices to show that there is  $\delta > 0$  such that

$$\int_F \frac{G_D(x, w)h(w)}{h(x)\delta_D(w)} |b(w)|dw \leq \varepsilon, \quad (37)$$

provided  $\lambda(F) < \delta$ . Here,  $\lambda$  denotes the Lebesgue measure on  $\mathbb{R}^d$ . First, we note that

$$\begin{aligned} & \int_F \frac{G_D(x, w)h(w)}{h(x)\delta_D(w)} |b(w)|dw \\ &= \int_F \int_{\partial D} \frac{G_D(x, w)M_D(w, Q)}{h(x)\delta_D(w)} d\mu^{(h)}(Q) |b(w)|dw \\ &= \int_{\partial D} \frac{M_D(x, Q)}{h(x)} \left( \int_F \frac{G_D(x, w)M_D(w, Q)}{M_D(x, Q)\delta_D(w)} |b(w)|dw \right) d\mu^{(h)}(Q). \end{aligned} \quad (38)$$

The function  $\frac{G_D(x, w)G_D(w, y)}{G_D(x, y)\delta_D(w)}$  is uniformly integrable in  $x, y \in D$  against  $|b(w)|dw$  (see the proof of [9, Lemma 11]). Hence, there exists  $\delta > 0$  such that for  $\lambda(F) < \delta$ ,

$$\int_F \frac{G_D(x, w)G_D(w, y)}{G_D(x, y)\delta_D(w)} |b(w)|dw < \varepsilon, \quad x, y \in D,$$



and consequently

$$\begin{aligned} \int_F \frac{G_D(x, w) M_D(w, Q)}{M_D(x, Q) \delta_D(w)} |b(w)| dw &= \int_F \lim_{D \ni y \rightarrow Q} \frac{G_D(x, w) G_D(w, y)}{G_D(x, y) \delta_D(w)} |b(w)| dw \\ &= \lim_{D \ni y \rightarrow Q} \int_F \frac{G_D(x, w) G_D(w, y)}{G_D(x, y) \delta_D(w)} |b(w)| dw \leq \varepsilon. \end{aligned}$$

Now, (37) follows from (38) and Martin representation of  $h$ . For  $Q \in \text{supp} \mu^{(h)}$ ,  $\lim_{D \ni x \rightarrow Q} h(x) = \infty$  from the Lemma 20. Hence, by uniform integrability,

$$\begin{aligned} \lim_{D \ni x \rightarrow Q} \frac{|J_h(x)|}{h(x)} &\leq c \lim_{D \ni x \rightarrow Q} \int_D \frac{G_D(x, w) h(w)}{h(x) \delta_D(w)} |b(w)| dw \\ &= c \int_D \lim_{D \ni x \rightarrow Q} \frac{G_D(x, w) h(w)}{h(x) \delta_D(w)} |b(w)| dw = 0. \end{aligned}$$

□

Now, we return to the Relative Fatou Theorem for  $L$ -harmonic functions. Let  $u$  and  $v$  be two non-negative singular  $L$ -harmonic functions on  $D$ . By Theorem 13, they have a Martin representation

$$u(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\mu(Q), \quad v(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\nu(Q), \quad x \in D$$

where  $\mu$  and  $\nu$  are two Borel finite measures concentrated on  $\partial D$ .

We decompose the measure  $\mu$  into its absolutely continuous and singular parts with respect to the measure  $\nu$

$$d\mu = f d\nu + d\mu_{\text{sing}}$$

with a non-negative function  $f \in L^1(\nu)$  and  $\nu(\text{supp}(\mu_{\text{sing}})) = 0$ .

**Theorem 23. (Relative Fatou Theorem)** *For  $\nu$ -almost every point  $Q \in \partial D$  we have*

$$\lim_{x \rightarrow Q} \frac{u(x)}{v(x)} = f(Q) \tag{39}$$

when  $x \rightarrow Q$  non-tangentially.

*Proof.* We will use the Relative Fatou Theorem for the singular  $\alpha$ -harmonic functions  $u^*$  and  $v^*$  defined according to (27).

Let  $Q \in \text{supp}(\nu) \setminus \text{supp}(\mu)$ . Then, if  $x \rightarrow Q$ ,  $v^*(x) \rightarrow \infty$  and  $u^*(x) \rightarrow 0$ , so  $\lim_{x \rightarrow Q} \frac{u^*(x)}{v^*(x)} = 0$ . The formulas (36) and (35) imply that in this case

$$\lim_{x \rightarrow Q} \frac{u(x)}{v(x)} = 0.$$

Let us consider the case  $Q \in \text{supp}(\nu) \cap \text{supp}(\mu)$ . As

$$\frac{d\mu(Q)}{l_D(x_0, Q)} = f \frac{d\nu(Q)}{l_D(x_0, Q)} + \frac{d\mu_{\text{sing}}(Q)}{l_D(x_0, Q)},$$

the Relative Fatou Theorem for the singular  $\alpha$ -harmonic functions  $u^*$  and  $v^*$  says that for  $\nu$ -almost every point  $Q \in \partial D$

$$\lim_{x \rightarrow Q} \frac{u^*(x)}{v^*(x)} = f(Q)$$

when  $x \rightarrow Q$  non-tangentially. The formula (39) then follows by the formula (35) and the Lemma 22.  $\square$

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