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Homotopy Transfer and Self-Dual Schur Modules*

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Abstract

We consider the free 2-nilpotent graded Lie algebra \mathfrak{g} generated in degree one by a finite dimensional vector space V . We recall the beautiful result that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ of \mathfrak{g} with trivial coefficients carries a $GL(V)$ -representation having only the Schur modules V_λ with self-dual Young diagrams $\{\lambda : \lambda = \lambda'\}$ (each with multiplicity one) in its decomposition into $GL(V)$ -irreducibles. The homotopy transfer theorem due to Tornike Kadeishvili allows to equip the cohomology of the Lie algebra \mathfrak{g} with a structure of homotopy commutative algebra.

1 Homotopy algebras A_∞ and C_∞

We start by recalling the definition of homotopy algebra. For a pedagogical introduction to the subject we send the read to the textbook of J.-L. Loday and B. Valette [6].

DEFINITION 1 *A homotopy associative algebra, or A_∞ -algebra over \mathbb{K} is a \mathbb{Z} -graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)*

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

*satisfying the Stasheff identities **SI**(n) for $n \geq 1$*

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad \mathbf{SI}(n)$$

where the sum runs over all decompositions $n = r + s + t$.

Throughout the text we assume the Koszul sign rule $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. We define the shuffle product by the expression $(a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$ the sum running over shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$.

*A talk given by Todor Popov at the International Workshop “Supersymmetries & Quantum Symmetries”, held at BLTP of JINR, Dubna from July, 18th to July, 23th, 2011.

DEFINITION 2 (see e.g.[4]) *A homotopy commutative algebra C_∞ , or C_∞ -algebra is an A_∞ algebra $\{A, m_n\}$ with the additional condition: each operation m_n vanishes on shuffles*

$$m_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1. \quad (1)$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞ -algebra such that $m_n = 0$ for $n \geq 3$ is a super-commutative Differential Graded Algebra (DGA for short).

Morphism of two A_∞ -algebras A and B is a family of graded maps $f_n : A^{\otimes n} \rightarrow B$ for $n \geq 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq q \leq n} (-1)^S m_q(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_q})$$

where the sum is on all decompositions $i_1 + \dots + i_q = n$ and the sign on RHS is determined by $S = \sum_{k=1}^{q-1} (q-k)(i_k - 1)$. The morphism f is a *quasi-isomorphism of A_∞ -algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for $i \geq 1$. The identity morphism on A is the strict morphism f such that f_1 is the identity of A .

A morphism of C_∞ -algebras is a morphism of A_∞ -algebras with components vanishing on shuffles $f_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, 1 \leq p \leq n-1$.

2 Homotopy Transfer Theorem

LEMMA 1 (see e.g.[6]) *Every cochain complex (A, d) of vector spaces over a field \mathbb{K} has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A, d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \rightarrow A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \xrightleftharpoons[i]{p} H^n$ are chain homomorphisms (satisfying the additional conditions $hh = 0, hi = 0$ and $ph = 0$). With these choices done the complex $(H^\bullet(A), 0)$ is a deformation retract of (A, d)

$$h \circlearrowleft (A, d) \xrightleftharpoons[i]{p} (H^\bullet(A), 0), \quad pi = Id_{H^\bullet(A)}, \quad ip - Id_A = dh + hd. \quad (2)$$

Let now (A, d, μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d . The cochain complexes (A, d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$.

THEOREM 1 (Kadeishvili[4]) *Let (A, d, μ) be a (commutative) DGA over a field \mathbb{K} . There exists a A_∞ -algebra (C_∞ -algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$ -quasi-isomorphism $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$ such that the inclusion $f_1 = i : H^\bullet(A) \rightarrow A$ is a cocycle-choosing homomorphism of cochain complexes. The differential on $H^\bullet(A)$ is zero $m_1 = 0$ and m_2 is the associative operation induced by the multiplication on A . The resulting structure is unique up to quasi-isomorphism.*

3 Homology and cohomology of the Lie algebra \mathfrak{g}

Let \mathfrak{g} be the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \oplus \wedge^2 V$ generated by the finite dimensional vector space V over the ground field \mathbb{K} of characteristics zero. The Lie bracket on \mathfrak{g} reads

$$[x, y] := x \wedge y \quad \text{when } x, y \in V \quad \text{and} \quad [x, y] := 0 \quad \text{otherwise} .$$

We define the homology with trivial coefficients of the Lie algebra \mathfrak{g} through the Chevalley-Eilenberg complex $C_\bullet(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}, \partial_\bullet)$ having differential $\partial_n : \wedge^n \mathfrak{g} \rightarrow \wedge^{n-1} \mathfrak{g}$,

$$\partial_n(x_1 \wedge \dots \wedge x_n) = \sum_{i < j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_n \quad p > 0. \quad (3)$$

The homology $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra \mathfrak{g} is the homology space of the complex $C(\mathfrak{g})$

$$H_n(\mathfrak{g}, \mathbb{K}) := H_n(C_\bullet(\mathfrak{g})) , \quad H_n(C_\bullet(\mathfrak{g})) = \ker \partial_n / \text{im} \partial_{n+1} .$$

The differential ∂ is induced by the Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ of the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. It identifies a pair of degree 1 generators $e_i, e_j \in \mathfrak{g}_1$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \mathfrak{g}_2$. In greater details the chain degrees read

$$\wedge^n \mathfrak{g} = \wedge^n (V \oplus \wedge^2 V) = \bigoplus_{s+r=n} \wedge^s (\wedge^2 V) \otimes \wedge^r V \quad (4)$$

and differentials $\partial_{n=r+s} : \wedge^s (\wedge^2 V) \otimes \wedge^r V \rightarrow \wedge^{s+1} (\wedge^2 V) \otimes \wedge^{r-2} V$ are given by

$$\begin{aligned} \partial_n : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r &\mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} &\otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r . \end{aligned}$$

The differential ∂ commutes with the $GL(V)$ -action thus the homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ is also a $GL(V)$ -module; its decomposition into irreducible polynomial representations V_λ (the so called Schur modules) is given by the following beautiful result.

THEOREM 2 (Józefiak and Weyman[3], Sigg[7]) *The homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \wedge^2 V$ decomposes into irreducible $GL(V)$ -modules*

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\wedge^\bullet \mathfrak{g}, \partial_\bullet) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda \quad (5)$$

where the sum is over the self-dual Young diagrams $\{\lambda : \lambda = \lambda'\}$ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

By duality, one has the cochain complex $\text{Hom}_{\mathbb{K}}(C(\mathfrak{g}), \mathbb{K}) = (\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ which is a (super) commutative DGA. The cohomology $H^n(\mathfrak{g}, \mathbb{K})$ with trivial coefficients is calculated by the complex $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$

$$H^n(\mathfrak{g}, \mathbb{K}) := H^n(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet) .$$

Here the coboundary map $\delta^n : \wedge^n \mathfrak{g}^* \rightarrow \wedge^{n+1} \mathfrak{g}^*$ is transposed¹ to the differential ∂_{n+1}

$$\begin{aligned} \delta^n : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_1^* \wedge \dots \wedge e_r^* &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* &\otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_1^* \wedge \dots \wedge \dots \wedge e_r^* , \end{aligned} \quad (6)$$

it is (up to a conventional sign) a continuation of the dualization of the Lie bracket $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$ by the Leibniz rule.

¹In the presence of metric one has $\delta := \partial^*$ (see below).

PROPOSITION 1 ([2]) *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ is a homotopy commutative algebra. The C_∞ -algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ is generated in degree 1, i.e., in $H^1(\mathfrak{g}, \mathbb{K})$, by the operations m_2 and m_3 .*

Sketch of the proof. By Lemma 1 the commutative DGA $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ has a deformation retract $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ thus from the Kadeishvili homotopy transfer Theorem 1 follows that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a C_∞ -algebra.

To prove that the C_∞ -algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ is generated by m_2 and m_3 we will need convenient choice of the homotopy h , the projection p and the inclusion i in the deformation retract(2).

Let us choose a metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ on the vector space V and an orthonormal basis $\langle e_i, e_j \rangle = \delta_{ij}$. The choice induces a metric on $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. In presence of metric g the differential δ is identified with the adjoint of ∂ , $\delta \stackrel{g}{:=} \partial^*$ (see eq.(5)) while ∂ plays the role of homotopy. The deformation retract of the complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ takes the following form[7]

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}, \quad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \stackrel{g}{=} \partial.$$

Here the projection p identifies the subspace $\ker \delta \cap \ker \delta^*$ with $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\text{im} \delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

Due to the isomorphisms $H^n(\mathfrak{g}, \mathbb{K}) \cong H_n^*(\mathfrak{g}, \mathbb{K})$ (i.e., $\text{Tor}_n^{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})$ in the category of graded algebras [1]) induced by $V \stackrel{g}{\cong} V^*$ the theorem 2 implies the decomposition

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\bigwedge^\bullet \mathfrak{g}^*, \delta) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over the self-dual Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

We were able to show in [2] that with the use of the explicit expressions[5] for the operations $m_2(x, y) := p\mu(i(x), i(y))$ and $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$ one can generate all the elements in $H^\bullet(\mathfrak{g}, \mathbb{K})$ by the degree one elements $H^1(\mathfrak{g}, \mathbb{K})$. \square

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