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# Homotopy Transfer and Self-Dual Schur Modules\*

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## Abstract

We consider the free 2-nilpotent graded Lie algebra  $\mathfrak{g}$  generated in degree one by a finite dimensional vector space  $V$ . We recall the beautiful result that the cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K})$  of  $\mathfrak{g}$  with trivial coefficients carries a  $GL(V)$ -representation having only the Schur modules  $V_\lambda$  with self-dual Young diagrams  $\{\lambda : \lambda = \lambda'\}$  (each with multiplicity one) in its decomposition into  $GL(V)$ -irreducibles. The homotopy transfer theorem due to Tornike Kadeishvili allows to equip the cohomology of the Lie algebra  $\mathfrak{g}$  with a structure of homotopy commutative algebra.

## 1 Homotopy algebras $A_\infty$ and $C_\infty$

We start by recalling the definition of homotopy algebra. For a pedagogical introduction to the subject we send the read to the textbook of J.-L. Loday and B. Valette [6].

**DEFINITION 1** *A homotopy associative algebra, or  $A_\infty$ -algebra over  $\mathbb{K}$  is a  $\mathbb{Z}$ -graded vector space  $A = \bigoplus_{i \in \mathbb{Z}} A^i$  endowed with a family of graded mappings (operations)*

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

*satisfying the Stasheff identities **SI**( $n$ ) for  $n \geq 1$*

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad \mathbf{SI}(n)$$

*where the sum runs over all decompositions  $n = r + s + t$ .*

Throughout the text we assume the Koszul sign rule  $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$ . We define the shuffle product by the expression  $(a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$  the sum running over shuffles  $Sh_{p,q}$ , i.e., over all permutations  $\sigma \in S_{p+q}$  such that  $\sigma(1) < \sigma(2) < \dots < \sigma(p)$  and  $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$ .

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\*A talk given by Todor Popov at the International Workshop “Supersymmetries & Quantum Symmetries”, held at BLTP of JINR, Dubna from July, 18th to July, 23th, 2011.

**DEFINITION 2** (see e.g.[4]) *A homotopy commutative algebra  $C_\infty$ , or  $C_\infty$ -algebra is an  $A_\infty$  algebra  $\{A, m_n\}$  with the additional condition: each operation  $m_n$  vanishes on shuffles*

$$m_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1. \quad (1)$$

In particular for  $m_2$  we have  $m_2(a \otimes b \pm b \otimes a) = 0$ , so a  $C_\infty$ -algebra such that  $m_n = 0$  for  $n \geq 3$  is a super-commutative Differential Graded Algebra (DGA for short).

Morphism of two  $A_\infty$ -algebras  $A$  and  $B$  is a family of graded maps  $f_n : A^{\otimes n} \rightarrow B$  for  $n \geq 1$  with  $\deg f_n = 1 - n$  such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq q \leq n} (-1)^S m_q(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_q})$$

where the sum is on all decompositions  $i_1 + \dots + i_q = n$  and the sign on RHS is determined by  $S = \sum_{k=1}^{q-1} (q-k)(i_k - 1)$ . The morphism  $f$  is a *quasi-isomorphism of  $A_\infty$ -algebras* if  $f_1$  is a quasi-isomorphism. It is strict if  $f_i = 0$  for  $i \geq 1$ . The identity morphism on  $A$  is the strict morphism  $f$  such that  $f_1$  is the identity of  $A$ .

A morphism of  $C_\infty$ -algebras is a morphism of  $A_\infty$ -algebras with components vanishing on shuffles  $f_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, 1 \leq p \leq n-1$ .

## 2 Homotopy Transfer Theorem

**LEMMA 1** (see e.g.[6]) *Every cochain complex  $(A, d)$  of vector spaces over a field  $\mathbb{K}$  has its cohomology  $H^\bullet(A)$  as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex  $(A, d)$  such that  $A^n \cong B^n \oplus H^n \oplus B^{n+1}$  where  $H^n$  is the cohomology and  $B^n$  is the space of coboundaries,  $B^n = dA^{n-1}$ . We choose a homotopy  $h : A^n \rightarrow A^{n-1}$  which identifies  $B^n$  with its copy in  $A^{n-1}$  and is 0 on  $H^n \oplus B^{n+1}$ . The projection  $p$  to the cohomology and the cocycle-choosing inclusion  $i$  given by  $A^n \xrightleftharpoons[i]{p} H^n$  are chain homomorphisms (satisfying the additional conditions  $hh = 0, hi = 0$  and  $ph = 0$ ). With these choices done the complex  $(H^\bullet(A), 0)$  is a deformation retract of  $(A, d)$

$$h \circlearrowleft (A, d) \xrightleftharpoons[i]{p} (H^\bullet(A), 0), \quad pi = Id_{H^\bullet(A)}, \quad ip - Id_A = dh + hd. \quad (2)$$

Let now  $(A, d, \mu)$  be a DGA, i.e.,  $A$  is endowed with an associative product  $\mu$  compatible with  $d$ . The cochain complexes  $(A, d)$  and its contraction  $H^\bullet(A)$  are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on  $A$  can be transferred to an  $A_\infty$ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract  $H^\bullet(A)$ .

**THEOREM 1** (Kadeishvili[4]) *Let  $(A, d, \mu)$  be a (commutative) DGA over a field  $\mathbb{K}$ . There exists a  $A_\infty$ -algebra ( $C_\infty$ -algebra) structure on the cohomology  $H^\bullet(A)$  and a  $A_\infty(C_\infty)$ -quasi-isomorphism  $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$  such that the inclusion  $f_1 = i : H^\bullet(A) \rightarrow A$  is a cocycle-choosing homomorphism of cochain complexes. The differential on  $H^\bullet(A)$  is zero  $m_1 = 0$  and  $m_2$  is the associative operation induced by the multiplication on  $A$ . The resulting structure is unique up to quasi-isomorphism.*

### 3 Homology and cohomology of the Lie algebra $\mathfrak{g}$

Let  $\mathfrak{g}$  be the 2-nilpotent graded Lie algebra  $\mathfrak{g} = V \oplus \wedge^2 V$  generated by the finite dimensional vector space  $V$  over the ground field  $\mathbb{K}$  of characteristics zero. The Lie bracket on  $\mathfrak{g}$  reads

$$[x, y] := x \wedge y \quad \text{when } x, y \in V \quad \text{and} \quad [x, y] := 0 \quad \text{otherwise} .$$

We define the homology with trivial coefficients of the Lie algebra  $\mathfrak{g}$  through the Chevalley-Eilenberg complex  $C_\bullet(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}, \partial_\bullet)$  having differential  $\partial_n : \wedge^n \mathfrak{g} \rightarrow \wedge^{n-1} \mathfrak{g}$ ,

$$\partial_n(x_1 \wedge \dots \wedge x_n) = \sum_{i < j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_n \quad p > 0. \quad (3)$$

The homology  $H_n(\mathfrak{g}, \mathbb{K})$  of the Lie algebra  $\mathfrak{g}$  is the homology space of the complex  $C(\mathfrak{g})$

$$H_n(\mathfrak{g}, \mathbb{K}) := H_n(C_\bullet(\mathfrak{g})) , \quad H_n(C_\bullet(\mathfrak{g})) = \ker \partial_n / \text{im} \partial_{n+1} .$$

The differential  $\partial$  is induced by the Lie bracket  $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$  of the graded Lie algebra  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ . It identifies a pair of degree 1 generators  $e_i, e_j \in \mathfrak{g}_1$  with one degree 2 generator  $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \mathfrak{g}_2$ . In greater details the chain degrees read

$$\wedge^n \mathfrak{g} = \wedge^n (V \oplus \wedge^2 V) = \bigoplus_{s+r=n} \wedge^s (\wedge^2 V) \otimes \wedge^r V \quad (4)$$

and differentials  $\partial_{n=r+s} : \wedge^s (\wedge^2 V) \otimes \wedge^r V \rightarrow \wedge^{s+1} (\wedge^2 V) \otimes \wedge^{r-2} V$  are given by

$$\begin{aligned} \partial_n : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r &\mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} &\otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r . \end{aligned}$$

The differential  $\partial$  commutes with the  $GL(V)$ -action thus the homology  $H_\bullet(\mathfrak{g}, \mathbb{K})$  is also a  $GL(V)$ -module; its decomposition into irreducible polynomial representations  $V_\lambda$  (the so called Schur modules) is given by the following beautiful result.

**THEOREM 2 (Józefiak and Weyman[3], Sigg[7])** *The homology  $H_\bullet(\mathfrak{g}, \mathbb{K})$  of the 2-nilpotent Lie algebra  $\mathfrak{g} = V \oplus \wedge^2 V$  decomposes into irreducible  $GL(V)$ -modules*

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\wedge^\bullet \mathfrak{g}, \partial_\bullet) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda \quad (5)$$

where the sum is over the self-dual Young diagrams  $\{\lambda : \lambda = \lambda'\}$  such that  $n = \frac{1}{2}(|\lambda| + r(\lambda))$ .

By duality, one has the cochain complex  $\text{Hom}_{\mathbb{K}}(C(\mathfrak{g}), \mathbb{K}) = (\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$  which is a (super) commutative DGA. The cohomology  $H^n(\mathfrak{g}, \mathbb{K})$  with trivial coefficients is calculated by the complex  $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$

$$H^n(\mathfrak{g}, \mathbb{K}) := H^n(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet) .$$

Here the coboundary map  $\delta^n : \wedge^n \mathfrak{g}^* \rightarrow \wedge^{n+1} \mathfrak{g}^*$  is transposed<sup>1</sup> to the differential  $\partial_{n+1}$

$$\begin{aligned} \delta^n : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_1^* \wedge \dots \wedge e_r^* &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* &\otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_1^* \wedge \dots \wedge \dots \wedge e_r^* , \end{aligned} \quad (6)$$

it is (up to a conventional sign) a continuation of the dualization of the Lie bracket  $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$  by the Leibniz rule.

<sup>1</sup>In the presence of metric one has  $\delta := \partial^*$  (see below).

**PROPOSITION 1 ([2])** *The cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K})$  of the 2-nilpotent graded Lie algebra  $\mathfrak{g} = V \otimes \bigwedge^2 V$  is a homotopy commutative algebra. The  $C_\infty$ -algebra  $H^\bullet(\mathfrak{g}, \mathbb{K})$  is generated in degree 1, i.e., in  $H^1(\mathfrak{g}, \mathbb{K})$ , by the operations  $m_2$  and  $m_3$ .*

**Sketch of the proof.** By Lemma 1 the commutative DGA  $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$  has a deformation retract  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$  thus from the Kadeishvili homotopy transfer Theorem 1 follows that the cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K})$  is a  $C_\infty$ -algebra.

To prove that the  $C_\infty$ -algebra  $H^\bullet(\mathfrak{g}, \mathbb{K})$  is generated by  $m_2$  and  $m_3$  we will need convenient choice of the homotopy  $h$ , the projection  $p$  and the inclusion  $i$  in the deformation retract(2).

Let us choose a metric  $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$  on the vector space  $V$  and an orthonormal basis  $\langle e_i, e_j \rangle = \delta_{ij}$ . The choice induces a metric on  $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$ . In presence of metric  $g$  the differential  $\delta$  is identified with the adjoint of  $\partial$ ,  $\delta \stackrel{g}{:=} \partial^*$  (see eq.(5)) while  $\partial$  plays the role of homotopy. The deformation retract of the complex  $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$  takes the following form[7]

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}, \quad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \stackrel{g}{=} \partial.$$

Here the projection  $p$  identifies the subspace  $\ker \delta \cap \ker \delta^*$  with  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ , which is the orthogonal complement of the space of the coboundaries  $\text{im} \delta$ . The cocycle-choosing homomorphism  $i$  is  $Id$  on  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$  and zero on coboundaries.

Due to the isomorphisms  $H^n(\mathfrak{g}, \mathbb{K}) \cong H_n^*(\mathfrak{g}, \mathbb{K})$  (i.e.,  $\text{Tor}_n^{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})$  in the category of graded algebras [1]) induced by  $V \stackrel{g}{\cong} V^*$  the theorem 2 implies the decomposition

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\bigwedge^\bullet \mathfrak{g}^*, \delta) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over the self-dual Young diagrams  $\lambda$  such that  $n = \frac{1}{2}(|\lambda| + r(\lambda))$ .

We were able to show in [2] that with the use of the explicit expressions[5] for the operations  $m_2(x, y) := p\mu(i(x), i(y))$  and  $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$  one can generate all the elements in  $H^\bullet(\mathfrak{g}, \mathbb{K})$  by the degree one elements  $H^1(\mathfrak{g}, \mathbb{K})$ .  $\square$

## References

- [1] H. Cartan. *Homologie et cohomologie d'une algèbre graduée*. Sèm. H. Cartan, **11**(1958), 1-20.
- [2] M. Dubois-Violette, T. Popov, *Young tableaux and homotopy commutative algebras*, to appear in Proceedings of "Lie Theory-9", Springer Proceedings in Mathematics(2012).
- [3] T. Józefiak, J. Weyman, *Representation-theoretic interpretation of a formula of D. E. Littlewood*, Math. Proc. Cambridge Phil. Soc., **103**(1988), 193-196.
- [4] T. Kadeishvili, *The  $A_\infty$ -algebra Structure and Cohomology of Hochschild and Harrison*, Proc. of Tbil. Math. Inst. **91**(1988), 19-27.
- [5] M. Kontsevich, Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Math. Phys. Stud., **21**(2000), Kluwer Acad. Publ., Dordrecht, 255-307.
- [6] J.-L. Loday, B. Valette, *Algebraic Operads*, (to appear)
- [7] S. Sigg, *Laplacian and homology of free 2-step nilpotent Lie algebras*, J. Algebra **185**(1996), 144-161.